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A FAMILY OF ACCRUED BENEFIT ACTUARIAL COST METHODS

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I. INTRODUCTION

TROWBRIDGE [6] has developed a family of aggregate actuarial cost methods without supplemental liability.[1] It would seem of interest to attempt to develop a corresponding family of accrued benefit actuarial cost methods with similar flexibility. The recent legislative activity in Canada on the subject of compulsory portable pensions and the discussion of related proposals in the United States have tended to sharpen the interest in accrued benefit actuarial cost methods.

In addition to this motivation for attempting such a development, the extensive discussion of pension funding in an economy characterized by inflationary tendencies at the Seventeenth International Congress of Actuaries has also served to create interest in flexible actuarial cost methods. Several papers presented at that Congress discussed proposals to defer pension funding. (See especially Ammeter [1] and Laing [4].)

In this paper no attempt will be made to pass judgment on the possible impact of deferred funding on the security of pension expectations or on its probable success in minimizing real pension costs. However, a family of accrued benefit actuarial cost methods will be developed which will permit the purchase of retirement income at either an accelerating or decelerating rate.

II. THE STATIONARY, UNIT BENEFIT MODEL

In discussing actuarial cost methods, use of the superb framework provided by Trowbridge's fundamental paper [5] is almost obligatory. In

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[1] In this paper an attempt will be made to utilize the actuarial cost (pension-funding) terminology developed by the Committee on Pension and Profit-sharing Terminology and reported in the *Journal of Insurance*, XXX (1963), 456–64. This terminology is used by McGill in his book *Fundamentals of Private Pensions* (2d ed.), which is now read by students preparing for Part 8 of the examinations of the Society.

this section, therefore, we will adopt his strict assumptions in order to concentrate attention on the family of actuarial cost methods being developed; that is, we shall assume that the population under study is stationary, with all entrants at age a and all retirements at age r. We will assume that pension benefits will be paid at an annual rate of one to all lives above age r. For reasons of convenience only, we shall adopt a continuous model rather than the discrete one used by Trowbridge.

We define $m(x)$, $a \leq x \leq r$ as the rate at which retirement income is being purchased at age x for a member of the pension group. We shall call the function denoted by $m(x)$ a pension purchase density function (p.p.d.f.). Our discussion will center on p.p.d.f.'s such that $m(x) \geq 0$, $a \leq x \leq r$, and

$$\int_a^r m(x)\,dx = 1\,.$$

Note the formal similarity between the p.p.d.f. and the probability density function for an absolutely continuous random variable.

We define the cumulative pension purchase function (c.p.p.f.), denoted by $M(x)$, as

$$M(x) = \int_a^x m(t)\,dt,\ a \leq x \leq r\,.$$

Certain properties of the cumulative pension purchase function are immediately apparent. Note that $M(x) = 0$, $x < a$, $M(r) = 1$, and $M(x)$ never decreases. It is not surprising, in view of the definition of the p.p.d.f., that there is a striking formal similarity between the c.p.p.f. and a cumulative distribution function of an absolutely continuous random variable.

One may define an actuarial cost method by specifying the p.p.d.f. The normal cost will be

$$\int_a^r l_x m(x)\,{}_{r-x|}\bar{a}_x dx\,,$$

and the accrued liability for active lives will be given by

$$\int_a^r l_x M(x)\,{}_{r-x|}\bar{a}_x dx\,.$$

Integration by parts will establish the equation of equilibrium,

$$\int_a^r l_x m(x)\,{}_{r-x|}\bar{a}_x dx + \delta\Big[\int_r^\infty l_x\bar{a}_x dx + \int_a^r l_x M(x)\,{}_{r-x|}\bar{a}_x dx\Big] = T_r\,.$$

We let $m(x)'$ denote the first derivative of $m(x)$. If $M(x)'' = m(x)' < 0$, we will say that the actuarial cost method defined by $m(x)$ results in decelerating funding at age x, and, if $M(x)'' = m(x)' > 0$, we will say that the method results in accelerating funding at age x. If $m(x)' < 0$, $a < x < r$, we will say that the actuarial cost method associated with $m(x)$ is a decelerating actuarial cost method, and, if $m(x)' > 0, a < x < r$, we will call the associated actuarial cost method an accelerating cost method. If $m_1(x)$ is associated with a decelerating cost method and $m_2(x)$ is associated with an accelerating cost method, a simple argument—strikingly similar to that used in Lidstone's theorem of life contingencies—will show that $M_1(x) > M_2(x)$, $a < x < r$.

Note that $m(x) = 1/(r - a)$ defines the individual accrued benefit actuarial cost method (unit credit funding). The actuarial cost method is characterized by $m(x)' = 0$, and correspondingly this method is neither an accelerating nor decelerating method.

Setting $m(x) = ({}_{r-a|}\bar{a}_a)/({}_{r-x|}\bar{a}_x \cdot \bar{a}_{a:\overline{r-a}|})$ defines a projected benefit method with individual level costs and a supplemental liability (entry age normal funding). It is easy to confirm that for this method

$$m(x)' = -D_x(\mu_x + \delta)/(\bar{N}_a - \bar{N}_r) < 0\,, \qquad a < x < r\,,$$

which implies that it is a decelerating cost method.

The fact that the entry age normal actuarial cost method is a decelerating actuarial cost method is an important, albeit somewhat belabored, actuarial fact. Certain important practical implications of this characteristic of the entry age normal actuarial cost method have been discussed recently by Griffin [3] (see especially the section of the paper entitled "Actuarial Liability" v. "Cost of Accrued Benefits").

If the p.p.d.f. has one mass point at age r—that is, if $m(x) = 0$ when $a \le x < r$ and $m(x) = 1$ when $x = r$—then the result is terminal funding. At the other extreme, if the p.p.d.f. has one mass point at age a—that is, if $m(x) = 1$ when $x = a$ and $m(x) = 0$ when $a < x \le r$—then the result is initial funding. Note that in these two cases the analogy is with a random variable with a degenerate distribution rather than with an absolutely continuous random variable. The c.p.p.f's in these cases become step functions with single jumps of height one at, respectively, $x = r$ and $x = a$.

There is no limit to the number of possible choices of $m(x)$, hence no limit to the number of actuarial cost methods. For example, setting $m(x) = e^{sx}/(e^{sr} - e^{sa})$ will define an actuarial cost method. If $s > 0$, this method will result in an accelerating cost method; if $s < 0$, this method will result in a decelerating one.

As $s \to 0$, we note that this p.p.d.f. approaches $1/(r - a)$, which is, of course, the p.p.d.f. associated with the accrued benefit actuarial cost method (unit credit funding).

A linear p.p.d.f. may be determined by selecting constants b and c that will satisfy the equation

$$\int_a^r (b+cx)\,dx = (r-a)[b+c(r+a)/2] = 1 ,$$

subject to the conditions $b + ca > 0$ and $b + cr > 0$.

In a paper on a practical subject, such as actuarial cost methods, a numerical example that illustrates the ideas being discussed is traditional. There seems to be no better way to develop an illustration on the subject than to start with the stationary population model used by Trowbridge [5]. The interest and mortality assumptions ($2\frac{1}{2}$ per cent interest and 1937 Standard Annuity Mortality) currently seem unrealistic but have been used to facilitate more direct comparison with earlier work on actuarial cost methods.

In Table 1 a continuous model is used, unlike the discrete model used in earlier illustrations using these assumptions. The trapezoidal rule and the usual actuarial approximations were used whenever it was necessary to approximate the value of an integral. Therefore, the equation of equilibrium will not be exactly satisfied by the approximations.

III. SOME PRACTICAL EXTENSIONS

In order to remove the unit benefit assumption made previously, we shall define $b(i, x_i)$ to be the symbol for the estimated annual pension payment rate commencing at age r to be paid to life i from among the group of n active lives in the pension group, when the estimate is made at attained age x_i for life i. Then the accrued liability for the active lives of a pension group, using an actuarial cost method defined by the p.p.d.f. $m(x)$, is

$$\sum_{i=1}^{n} b(i, x_i) M_i(x_i)\, {}_{r-x_i|}\bar{a}_{x_i} .$$

The c.p.p.f. is indexed with i to allow for possible variations in the age at entry among the individual members of the active group. Upon the establishment of the pension plan, a similar computation including pensions awarded to lives already beyond age r would yield the initial accrued liability. The corresponding normal cost would be

$$\sum_{i=1}^{n} b(i, x_i)\, m_i(x_i)\, {}_{r-x_i|}\bar{a}_{x_i} .$$

The p.p.d.f. is indexed with i to allow for possible individual variations in the age at entry.

TABLE 1

ILLUSTRATION OF THE PROPERTIES OF VARIOUS ACTUARIAL COST METHODS, USING A CONTINUOUS MODEL, UNIT ANNUAL BENEFIT RATES, A STATIONARY POPULATION, TROWBRIDGE'S SERVICE TABLE, AND 2½ PER CENT INTEREST

x	UNIT CREDIT	ENTRY AGE NORMAL*	EXPONENTIAL		Linear†	Terminal	Initial
			$s=0.1$†	$s=-0.1$*			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Pension Purchase Density Function, $m(x)$‡						
30......	0.029	0.135	0.003	0.103	0.018	0.000	1.000
35......	.029	.053	.005	.063	.021	0.000	0.000
40......	.029	.032	.008	.038	.024	0.000	0.000
45......	.029	.022	.014	.023	.027	0.000	0.000
50......	.029	.016	.023	.014	.030	0.000	0.000
55......	.029	.012	.038	.008	.033	0.000	0.000
60......	.029	.007	.063	.005	.036	0.000	0.000
65......	0.029	0.006	0.103	0.003	0.039	1.000	0.000
	Cumulative Pension Purchase Function, $M(x)$						
30......	0.000	0.000	0.000	0.000	0.000	0.000	1.000
35......	0.143	0.433	0.020	0.406	0.098	0.000	1.000
40......	0.285	0.635	0.054	0.652	0.211	0.000	1.000
45......	0.429	0.767	0.108	0.801	0.338	0.000	1.000
50......	0.571	0.862	0.199	0.892	0.481	0.000	1.000
55......	0.714	0.926	0.348	0.946	0.639	0.000	1.000
60......	0.857	0.969	0.594	0.980	0.812	0.000	1.000
65......	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Ultimate Normal Cost						
	7,756	6,278	9,470	6,208	8,170	11,584	4,881
	Ultimate Fund						
	273,661	332,788	204,535	336,147	257,307	118,290	389,759

* Decelerating actuarial cost method.

† Accelerating actuarial cost method.

‡ Col. 1: 1/35.
Col. 2: $D_x/(\bar{N}_{30}-\bar{N}_{65})$.
Col. 3: $0.1\, e^{0.1x}/(e^{6.5}-e^{3.0})$.
Col. 4: $0.1\, e^{-0.1x}/(e^{-3.0}-e^{-6.5})$.
Col. 5: $2x/3{,}325$.
Col. 6: $m(x)=1,\ x=65,\ =0$ elsewhere.
Col. 7: $m(x)=1,\ x=30,\ =0$ elsewhere.

Changes in the estimated annual rate of pension payment commencing at age r, along with other deviations from expected results, would cause the accrued liability, as determined by periodic valuations, to differ from that expected. The immediate method of adjusting for these gains and the computation of a special 10 per cent Treasury base for the losses, using the same methods that are currently used with the accrued benefit actuarial cost method, could be employed.

IV. SOME GENERAL CONSIDERATIONS

In Section II it was pointed out that accelerating funding, when pushed to the limit, results in terminal funding. It would unduly expand the scope of this technical paper to repeat the many persuasive arguments in favor of higher levels of pension funding than result from the use of the terminal method. Nevertheless, for the sake of completeness, it should be pointed out that accelerating funding might be used to permit retention of funds with the sponsoring company for a longer period of time. Of course, if the outlook is not as good for the sponsoring company as for the economy as a whole, it would be wise to invest outside the company (decelerating funding). According to Bronson [2, p. 76] decelerating funding takes the "calculated risk that future taxes will not be higher than today's rates. Or, again, that tomorrow's dollars will not be cheaper than those of today." Accelerating funding would involve taking the opposite chances. Accelerating funding also lessens the impact of withdrawals on estimated costs and thus makes projections of this perplexing factor less critical to the determination of the estimated required contribution.

The generalized concept of individual funding methods developed in this paper introduces flexibility into individual actuarial cost methods as Trowbridge's unfunded family of methods does for aggregate methods. This might become very important if vesting becomes compulsory. To determine an individual's benefit upon leaving his job, one would need only to know the p.p.d.f. and the worker's age at entry and at withdrawal. In fact, such regulations might set minimum funding by specifying a minimum p.p.d.f., and sponsoring organizations might in turn be guided by this minimum in selecting their p.p.d.f.'s.

REFERENCES

1. AMMETER, H. "Funding of Retirement Income," *Transactions of the Seventeenth International Congress of Actuaries*, III, 19–28.
2. BRONSON, D. C. *Concepts of Actuarial Soundness in Pension Plans.* Homewood, Ill.: Richard D. Irwin, Inc., 1957.
3. GRIFFIN, F. L. "Concepts of Adequacy in Pension Plan Funding," *TSA*, XVIII (1966), 46–63.

4. LAING, C. E. "Maintained Value Pensions," *Transactions of the Seventeenth International Congress of Actuaries*, III, 264–78.
5. TROWBRIDGE, C. L. "Fundamentals of Pension Funding," *TSA*, IV (1952), 17–43.
6. ———. "The Unfunded Present Value Family of Pension Funding Methods," *TSA*, XV (1963), 151–92.

DISCUSSION OF PRECEDING PAPER

CHARLES L. TROWBRIDGE:

Mr. Cooper and Dr. Hickman are to be congratulated on an interesting generalization of the accrued benefit actuarial cost method. Essentially they have retained the principle that a portion $m(x)$ of the ultimate pension should be funded at each age x but abandoned the principle that $m(x)$ is necessarily constant. By varying the mathematical form of $m(x)$, they produce initial funding, terminal funding, unit credit, and entry age normal as special cases.

Though they do not claim to have done so, it appears that they have in almost the same way generalized the entry age normal method. The principle behind entry age normal is that an amount $\bar{a}_r$ (for a unit of pension) is to be accumulated by level contributions from entry age a to retirement age r. If we retain the concept of contributions at each age $a \leq x \leq r$ but abandon the requirement that all be equal, we obtain a family of projected benefit cost methods of the entry age normal type. Actually, it is the *same family* as that so well outlined in the paper before us.

It appears that the family from the paper can be put into entry age normal dress by a single transformation. Define a function

$$n(x) = m(x) \cdot \frac{{}_{r-x|}\bar{a}_x}{\bar{a}_r} = m(x) \cdot \frac{D_r}{D_x}.$$

Then if $m(x)$ is the Cooper-Hickman pension *purchase* density function (p.p.d.f.), we might call its transposed form $n(x)$ the pension *funding* density function (p.f.d.f.).

$$\int_a^r n(x) \cdot \frac{D_x}{D_r}\, dx = 1 ,$$

and $n(x)$ can be viewed as the portion of the ultimate contribution (viewed as a unit payable at age r) to be made at age x.

If the first derivative $n'(x)$ is zero for $a \leq x \leq r$, $n(x)$ is a constant equal to $D_r/(\bar{N}_a - \bar{N}_r)$, and the traditional level form of entry age normal results. In this case

$$m(x) = n(x) \cdot \frac{D_x}{D_r} = \frac{D_x}{\bar{N}_a - \bar{N}_r} = {}_{r-a|}\bar{a}_a / {}_{r-x|}\bar{a}_x \cdot \bar{a}_{a:\overline{r-a|}},$$

as shown in the paper. This is *level funding* from an $n(x)$ viewpoint, just as it is a *decelerating pension purchase* from an $m(x)$ viewpoint.

If the first derivative $m'(x)$ is zero for $a \leq x \leq r$, $m(x)$ is a constant equal to $1/(r - a)$, but $n'(x) > 0$ and $n(x)$ is an increasing function of x, $[D_r/(r - a)D_x]$. Hence, traditional unit credit is accelerating funding from an $n(x)$ viewpoint, just as it is *level pension purchase* from an $m(x)$ viewpoint.

Perhaps Cooper and Hickman can develop this duality of their generalization more elegantly in their reply to the discussion.

CECIL J. NESBITT:

I heard of the paper before I saw a copy and wondered on what principle a new family of actuarial cost methods had been defined which did not fall under Taylor's generalized family of aggregate actuarial cost methods. The words "accrued benefit" in the title make clear the distinction. In the present paper, a modified unit credit funding principle is utilized rather than an aggregate funding principle.

On reading the paper, my thoughts went off in a tangential direction to wonder whether density and cumulative functions could be useful for life insurance premiums and reserves. A distinction arises immediately in that ordinarily the full face amount of insurance is available from issue and only the maturity value is deferred, as in the case of a pension benefit. This introduces complications, as will be seen.

To follow the paper's notations as closely as possible, let us consider a unit endowment insurance, issued at age a, maturing with a value of 1 at age r, having continuous premiums payable at annual rate $\pi(x)$ at age x and with reserve $V(x)$ at age x. Let $m(x)$ be a maturity fund density function (m.f.d.f.) and $M(x)$ a corresponding cumulative maturity fund function (c.m.f.f.). We take as our starting point the reserve $V(x)$, which is chosen to satisfy $V(a) = 0$, $V(r) = 1$, and the reserve equation

$$dV(x)/dx = \pi(x) + \delta V(x) - \mu_x[1 - V(x)] \, .$$

If we consider the reserve as accumulating under interest only, then we may take

$$V(x) = M(x)v^{r-x}$$

or

$$M(x) = V(x)(1 + i)^{r-x} \, .$$

It is evident that $M(a) = 0$, $M(r) = 1$, but it is not at all clear that $M(x)$ never decreases. We find next that

$$m(x) = (1 + i)^{r-x}[dV(x)/dx - \delta V(x)]$$
$$= (1 + i)^{r-x}\{\pi(x) - \mu_x[1 - V(x)]\}$$

or

$$\pi(x) = m(x)v^{r-x} + \mu_x[1 - V(x)] \, ,$$

which is interpretable immediately. For the level premium case, $\pi(x) = \bar{P}(\bar{A}_{a:\overline{r-a}|})$ and $V(x) = {}_{x-a}\bar{V}(\bar{A}_{a:\overline{r-a}|})$, and there would be a good chance that $m(x)$ remains positive.

For the reserve considered accumulating under benefit of both interest and survivorship, we would take

$$V(x) = M_1(x)(D_r/D_x)$$

or

$$M_1(x) = V(x)(D_x/D_r) .$$

Then

$$m_1(x) = (D_x/D_r)[dV(x)/dx - (\mu_x + \delta)V(x)] = (D_x/D_r)[\pi(x) - \mu_x]$$

and

$$\pi(x) = m_1(x)(D_r/D_x) + \mu_x .$$

For the level premium case,

$$m_1(x) = (D_x/D_r)[\bar{P}(\bar{A}_{a:\overline{r-a}|}) - \mu_x]$$

and at the higher ages might be negative.

We have emerged with a family of funding methods for endowment insurance with c.m.f.f. taken either as $V(x)(1 + i)^{r-x}$ or $V(x)(D_x/D_r)$, depending on whether we regard the reserve as accumulating under interest only or under both interest and survivorship. From the second of these, we can obtain the family in the paper if we now assume that $V(r) = \bar{a}_r$ and that $V(x)$ satisfies

$$dV(x)/dx = \pi(x) + (\mu_x + \delta)V(x) .$$

Also, we need here to take

$$M_1(x) = [V(x)/V(r)](D_x/D_r) .$$

One finds

$$m_1(x) = (D_x/D_r)[\pi(x)/V(r)]$$

or

$$\pi(x) = m_1(x)(D_r/D_x)V(r) ,$$

which is comparable to the authors' formulas.

If, however, we followed the here somewhat artificial approach of regarding the reserve as accumulating under interest only, then

$$\begin{aligned} M(x) &= [V(x)/V(r)](1 + i)^{r-x} , \\ m(x) &= (1 + i)^{r-x}[dV(x)/dx - \delta V(x)]/V(r) \\ &= (1 + i)^{r-x}[\pi(x) + \mu_x V(x)]/V(r) , \end{aligned}$$

and

$$\pi_x = m(x)V(r)v^{r-x} - \mu_x V(x)$$
$$= V(r)v^{r-x}[m(x) - \mu_x M(x)] .$$

On a discrete basis, it is possible that funding controlled by density functions may be made practical for either insurance or pension benefits. At any rate, the authors have introduced an interesting generalization that has been a pleasure to explore.

(AUTHORS' REVIEW OF DISCUSSION)

STEVEN L. COOPER AND JAMES C. HICKMAN:

Mr. Barnet Berin, in a very helpful personal letter to the authors, mildly objected to the title of this paper. He suggested that "A General Family of Actuarial Cost Methods" would have been a more appropriate title. However, it remained for Mr. Trowbridge to indicate specifically how easily some of the ideas developed in this paper may be modified to furnish additional insights into actuarial cost methods.

Trowbridge defined a pension funding density function (p.f.d.f.), which he denoted by $n(x)$. Any function which satisfies the following two conditions may serve as a p.f.d.f. and thus define an actuarial cost method:

$$n(x) \geq 0 , \qquad a \leq x \leq r , \tag{1}$$

$$\int_a^r [n(x)D_x/D_r]dx = 1 . \tag{2}$$

The p.f.d.f. is related to the pension purchase density function (p.p.d.f.), which is defined in the paper and denoted by $m(x)$, by

$$n(x) = [m(x)D_r]/D_x .$$

The basic equation of equilibrium for a stationary population with unit annual retirement benefits becomes, in terms of the p.f.d.f.,

$$\bar{a}_r\int_a^r l_x n(x)dx + \delta\left\{\int_r^\infty l_x\bar{a}_x dx + \bar{a}_r\int_a^r l_x\left[\int_a^x n(t)D_t dt/D_x\right]dx\right\} = T_r .$$

In this equation the first term represents the ultimate annual contribution rate, and the term in braces represents the ultimate fund.

In the short summary tabulation on page 64, the p.p.d.f. and the p.f.d.f. are exhibited for the common actuarial cost methods mentioned in the paper.

It is also of interest to examine the derivative of the p.f.d.f. We have

$$dn(x)/dx = (D_r/D_x)\{[dm(x)/dx] + m(x)(\mu_x + \delta)\} .$$

Using this equation and Trowbridge's comments concerning level p.f.d.f.'s, it is easy to see that, if the p.p.d.f. is level or increasing, the associated p.f.d.f. is increasing.

Let us also exhibit the derivative of the p.p.d.f. We have

$$dm(x)/dx = (D_x/D_r)\{[dn(x)/dx] - n(x)(\mu_x + \delta)\} .$$

Using this equation and Trowbridge's comments concerning level p.f.d.f.'s, it is easy to see that, if the p.f.d.f. is level or decreasing, the associated p.p.d.f. is decreasing.

Method	p.p.d.f. $m(x)$	p.f.d.f. $n(x)$
Entry age normal........	$D_x/(\bar{N}_a-\bar{N}_r)$, $a\le x\le r$	$D_r/(\bar{N}_a-\bar{N}_r)$, $a\le x\le r$
Unit credit....	$1/(r-a)$, $a\le x\le r$	$D_r/[(r-a)D_x]$, $a\le x\le r$
Terminal......	$\begin{cases}0, & a\le x<r\\ 1, & x=r\end{cases}$	0, $a\le x<r$ 1, $x=r$
Initial.........	$\begin{cases}1, & x=a\\ 0, & a<x\le r\end{cases}$	D_r/D_a, $x=a$ 0, $a<x\le r$

The authors acknowledge that they are somewhat overwhelmed by the fascinating extension of the application of density functions in the mathematics of life contingencies made by Professor Nesbitt. The authors had a much more modest extension in mind. It had occurrred to them that if $m(x)$ denotes a postretirement insurance purchased density function and $M(x)$ denotes a cumulative postretirement insurance purchased function, the general equation of equilibrium for a stationary population with unit insurance benefits on a pay-as-you-go basis before retirement and fully paid after retirement would be given by

$$\left[(l_a - l_r) + \int_a^r m(x)\, l_{x\ r-x|}\bar{A}_x dx\right] + \delta\left[\int_r^\infty l_x \bar{A}_x dx + \int_a^r M(x)\, l_{x\ r-x|}\bar{A}_x dx\right] = l_a .$$

The authors have little to offer in the way of extending Nesbitt's discussion other than to point out the insights into the nature of premiums that may be obtained by interpreting the expressions for $\pi(x)$, the annual premium rate.

The authors would also like to stress, with respect to the results that satisfy the differential equation

$$dV(x)/dx = \pi(x) + (\mu_x + \delta)V(x) ,$$

that Nesbitt's $\pi(x)$ corresponds to Trowbridge's p.f.d.f., $n(x)$.

When this paper was written, it was conjectured that the key idea would be of interest primarily for its utility as a unifying idea for teaching the properties of various actuarial cost methods. The authors want to thank Trowbridge and Nesbitt for showing how one simple idea can spawn several better ideas. The authors also want to thank the second author's son, Charles Hickman, for taking several days of his summer vacation to help develop the numerical example.