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SOME INSTANCES OF THE SUPERIORITY OF GEOMETRIC METHODS OVER ARITHMETIC METHODS OF INTERPOLATION AND EXTRAPOLATION

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INTRODUCTION

IN ORDER to explain the title and purpose of this paper, let us first consider some examples. Suppose tables of values are calculated at 3 per cent and $3\frac{1}{2}$ per cent, and an approximation at 4 per cent is desired. Many actuaries would use a formula that is tantamount to

$$f(4\%) \doteq 2f(3\tfrac{1}{2}\%) - f(3\%)\,. \tag{1}$$

Others might use the formula

$$f(4\%) \doteq \frac{[f(3\frac{1}{2}\%)]^2}{f(3\%)}. \tag{2}$$

The question arises as to which one gives the better approximation. This paper will show that, for most of the values desired by actuaries, the latter is superior.

Similarly, the approximation given by

$$f(4\%) \doteq (1+\Delta)^3 f(2\tfrac{1}{2}\%) \doteq 3f(3\tfrac{1}{2}\%) - 3f(3\%) + f(2\tfrac{1}{2}\%) \tag{3}$$

is usually not so good as that given by

$$f(4\%) = e^{\log f(4\%)} \doteq e^{(1+\Delta)^3 \log f(2\frac{1}{2}\%)} \doteq \frac{[f(3\frac{1}{2}\%)]^3}{[f(3\%)]^3} \times f(2\tfrac{1}{2}\%)\,. \tag{4}$$

For the purposes of this paper, formulas with the characteristics of (1) and (3) are designated as "arithmetic formulas," and those comparable with (2) and (4) are called "geometric formulas," because of their analogies with the general terms of arithmetic and geometric series.

It will be noted that geometric formulas can be obtained from arithmetic formulas by changing the coefficients to exponents and addition to multiplication.

It will be further noted that the use of geometric formulas for a particular function is identical to the use of arithmetic formulas for the logarithm of that function whenever the function is positive for the range considered.

This device should serve as an aid in extending the investigation of geometric formulas into the areas of osculatory interpolation and graduation. The scope of this paper is limited to the application of Newton's formula.

THEORY

Let us define the terms $R_{n+1}(x)$, $S_{n+1}(x)$, U_x, and V_x as follows:

$$R_{n+1}(x)=\left[(x-a)(x-b)\ldots(x-l)\,\frac{d^{n+1}}{d\xi^{n+1}}u_\xi\right]\left[\frac{1}{(n+1)!}\right]$$

$$S_{n+1}(x)=\left[(x-a)(x-b)\ldots(x-l)\,\frac{d^{n+1}}{d\theta^{n+1}}\log u_\theta\right]\left[\frac{1}{(n+1)!}\right]$$

$$U_x=u_a+(x-a)\underset{b}{\Delta}u_a+(x-a)(x-b)\underset{bc}{\Delta^2}u_a+\ldots$$
$$+(x-a)(x-b)\ldots(x-k)\underset{bc\ldots l}{\Delta^n}u_a\,,$$

and

$$V_x=\log u_a+(x-a)\underset{b}{\Delta}\log u_a+(x-a)(x-b)\underset{bc}{\Delta^2}\log u_a$$
$$+(x-a)(x-b)\ldots(x-k)\underset{bc\ldots l}{\Delta^n}\log u_a\,,$$

where ξ and θ are values in the interval including all the arguments involved, as defined on page 57 of Harry Freeman's *Finite Differences for Actuarial Students* (Cambridge University Press, 1962).

It is proved on pages 56 and 57 of the book by Freeman that

$$R_{n+1}(x)=u_x-U_x\,. \tag{5}$$

Similarly,

$$S_{n+1}(x)=\log u_x-V_x\,. \tag{6}$$

We will state the following nine theorems for instances when u_x is positive, the proofs of which are given in the Appendix:

THEOREM I: If $S_{n+1}(x)$ is positive and if

$$u_x[1-e^{-S_{n+1}(x)}]<R_{n+1}(x),$$

then

$$|u_x-e^{V_x}|<|u_x-U_x|\,.$$

THEOREM II: If $S_{n+1}(x)$ is positive and if

$$R_{n+1}(x) < u_x[e^{-S_{n+1}(x)} - 1],$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM III: If $S_{n+1}(x)$ is negative and if

$$R_{n+1}(x) < u_x[1 - e^{-S_{n+1}(x)}],$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM IV: If $S_{n+1}(x)$ is negative and if

$$u_x[e^{-S_{n+1}(x)} - 1] < R_{n+1}(x),$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM V: If $0 < S_{n+1}(x)$ and

$$u_x[1 - e^{-S_{n+1}(x)}] < |R_{n+1}(x)|,$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM VI: If $S_{n+1}(x) < 0$ and

$$u_x[e^{-S_{n+1}(x)} - 1] < |R_{+1}(x)|,$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM VII: If

$$|u_x[1 - e^{-S_{n+1}(x)}]| < |R_{n+1}(x)|,$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM VIII: If $0 < S_{n+1}(x)$ and

$$u_x[S_{n+1}(x)] < |R_{n+1}(x)|,$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

THEOREM IX: If

$$|S_{n+1}(x)| < 1$$

and

$$u_x[|S_{n+1}(x)| + (0.72)\, S_{n+1}^2(x)] < |R_{n+1}(x)|,$$

then

$$|u_x - e^{V_x}| < |u_x - U_x|.$$

The theorems might be paraphrased by saying that, for geometric interpolation or extrapolation to give a greater degree of accuracy than arithmetic interpolation or extrapolation, it is sufficient that u_x be positive throughout the range of arguments and that the premises in any one of the nine theorems be true.

Examination of the first four theorems reveals that additional conclusions which may be derived are that $R_{n+1}(x)$ is positive in Theorems I and IV and is negative in Theorems II and III. If this additional conclusion in each of these theorems is made a premise and combined with the converse of the second premise in each theorem, we deduce conclusions which are converse to those in the first four theorems. Additionally, we can also see that the first premises of the theorems are now additional conclusions.

This analysis reveals not only that the second premise in each of the first four theorems is sufficient but that it is necessary, if we ignore the trivial cases when either $R_{n+1}(x)$ or $S_{n+1}(x)$ is zero. This means, of course, that, if the second premise in any one of the first six theorems is not true, then arithmetic interpolation or extrapolation would be more accurate than geometric interpolation or extrapolation.

The premise in Theorem VII is also both necessary and sufficient; however, the second premise in each of Theorems VIII and IX is sufficient but not necessary, the purpose of these latter theorems being to set forth conditions not involving the exponential functions.

To determine that a geometrical formula gives a higher degree of accuracy than an arithmetical formula, it is sufficient for practical purposes when second and higher differences are ignored to show that

$$(u_x')^2 < (u_x)(u_x'') \, .$$

When this expression is true, when the range of arguments is sufficiently small that ξ can be deemed to equal θ, and when the difference intervals are sufficiently small that

$$1 - e^{-S_{n+1}(x)} \doteq S_{n+1}(x) \, ,$$

then the premises of either Theorem I or Theorem III are satisfied.

It can be shown without too much difficulty that the premises in at least one of the theorems are satisfied for the functions

$$(1+i)^n \, , \quad v^n \, , \quad a_{\overline{n}|} \, , \quad s_{\overline{n}|} \, , \quad a_x \, , \quad A_x \, , \quad \text{and} \quad P_x$$

whenever first differences only are used, differences are taken with respect to the interest rate, and the function is assumed to be a polynomial of degree higher than 2. Except for the functions $(1+i)^n$ and v^n, the algebra becomes involved when second and higher differences are used.

It is not surprising that geometric methods give exact values when differences are taken with respect to n for the functions $(1+i)^n$ and v^n. It is more interesting that arithmetic methods give better values for $s_{\overline{n}|}$ and $a_{\overline{n}|}$ when differences are taken with respect to n except for $s_{\overline{n}|}$ when n is such that $(1+i)^n$ is greater than or approximately equal to 2. The

TABLE 1

ILLUSTRATIONS OF ACTUAL AND EXTRAPOLATED VALUES

(Differences with Respect to the Interest Rate)

f (4%)	ACTUAL	EXTRAPOLATED FROM 3% AND 3½% VALUES		EXTRAPOLATED FROM 2½%, 3%, AND 3½% VALUES	
		Geometric	Arithmetic	Geometric	Arithmetic
$(1+i)^3$	1.12486	1.12494	1.12471	1.12486	1.12486
$(1+i)^{33}$	3.64838	3.65119	3.57155	3.64835	3.63767
$(1+i)^{100}$	50.50495	50.62295	43.16418	50.50380	47.73204
v^3	.88900	.88893	.88874	.88900	.88900
v^{33}	.27409	.27388	.26566	.27410	.27565
v^{100}	.01980	.01975	.01209	.01980	.02473
$s_{\overline{3}\|}$	3.12160	3.12163	3.12155	3.12160	3.12160
$s_{\overline{33}\|}$	66.20953	66.10756	65.60458	66.21335	66.14414
$s_{\overline{100}\|}$	1237.62370	1225.28223	1117.93558	1239.64521	1198.52043
$a_{\overline{3}\|}$	2.77509	2.77492	2.77466	2.77509	2.77510
$a_{\overline{33}\|}$	18.14765	18.10575	18.01462	18.14883	18.16513
$a_{\overline{100}\|}$	24.50500	24.20408	23.71195	24.54558	24.78367
$\ddot{a}_{10}$	22.8855	22.7319	22.4851	22.8986	22.9820
$\ddot{a}_{55}$	13.1156	13.0939	13.0637	13.1166	13.1215
$\ddot{a}_{90}$	3.3747	3.3741	3.3737	.33747	3.3747
A_{10}	.11979	.11860	.10964	.11975	.12344
A_{55}	.49555	.49429	.49078	.49559	.49624
A_{90}	.87020	.87000	.86975	.87020	.87021
P_{10}	.00523	.00521	.00509	.00523	.00526
P_{55}	.03778	.03775	.03770	.03778	.03779
P_{90}	.25786	.25786	.25785	.25789	.25789

reason for the latter is that when $(1+i)^n$ exceeds 2, the function $s_{\overline{n}|}$ tends to behave more like an exponential function than a linear one.

ILLUSTRATIONS

It is likely that most people are more interested in practice than in theory. Practical demonstrations are not only easier to follow but also give a visual indication of the magnitude of the superior degree of accuracy of geometric methods over arithmetic methods in various instances.

The middle two columns of Table 1 show values of various functions at

4 per cent interest as extrapolated from 3 per cent and $3\frac{1}{2}$ per cent tables. The last two columns of the table give values extrapolated from $2\frac{1}{2}$ per cent, 3 per cent, and $3\frac{1}{2}$ per cent values. (The functions using life contingencies were based on "United States White Males: 1959–61.")

Table 2 presents values of various functions at 4 per cent interest as extrapolated from values five and ten years less than the duration or age associated with the function.

TABLE 2

ILLUSTRATIONS OF ACTUAL AND EXTRAPOLATED VALUES
(Differences with Respect to the Duration or Age)

f (4%)	Actual	Geometric	Arithmetic
$(1+i)^{100}$	50.50495	50.50495	48.90344
$s_{\overline{15}\rceil}$	20.02359	26.61337	18.59589
$s_{\overline{30}\rceil}$	56.08494	58.24357	53.51374
$a_{\overline{30}\rceil}$	17.29203	17.95758	17.65383
$\ddot{a}_{55}$	13.1156	13.2927	13.1508
$\ddot{a}_{90}$	3.3747	3.4244	3.1741
A_{55}	.49555	.50320	.49420
A_{90}	.87020	.88052	.87792
P_{55}	.03778	.03784	.03610
P_{90}	.25786	.25713	.24034
$N_{55} \div 10{,}000$	12.2927	12.7643	10.7863
$N_{90} \div 100$	4.5585	6.5447	negative

CONCLUSION

Obviously, best results arise from the choice of an interpolation formula producing a curve that most closely follows that of the actual curve of the function being interpolated or extrapolated.

Many actuaries may find this paper useful for practical applications. Others may find it a stimulus for further investigation. There are many opportunities for further investigation of the subject and related aspects, and it is hoped there will be substantial discussion and further papers by those with more time and mathematical ability than the author.

APPENDIX

Whenever $S_{n+1}(x)$ is positive, we have, using formula (6),

$$e^{V_x} = e^{\log u_x - S_{n+1}(x)} = u_x e^{-S_{n+1}(x)} < u_x . \qquad (7)$$

Similarly, whenever $S_{n+1}(x)$ is negative,

$$u_x < e^{V_x} . \qquad (8)$$

Theorem I is easily proved by applying formulas (5) and (6) to

$$u_x[1 - e^{-S_{n+1}(x)}] < R_{n+1}(x),$$

which gives

$$u_x - u_x e^{V_x - \log u_x} < u_x - U_x ,$$

resulting in

$$u_x - e^{V_x} < u_x - U_x .$$

This expression and (7) give

$$U_x < e^{V_x} < u_x .$$

Hence,

$$U_x - u_x < e^{V_x} - u_x < 0$$

and

$$|u_x - e^{V_x}| < |u_x - U_x| .$$

Theorem II is similarly proved by deriving that

$$0 < u_x - e^{V_x} < U_x - u_x .$$

Theorem III is proved by using (8) to derive

$$u_x < e^{V_x} < U_x .$$

Theorem IV is proved by deriving that $0 < e^{V_x} - u_x < u_x - U_x$.

Theorem V is a corollary of Theorems I and II.

Theorem VI is a corollary of Theorems III and IV.

Similarly, Theorem VII is a corollary of Theorems V and VI.

Theorem VIII is derived from Theorem V by using the general relationship,

$$1 - e^{-y} < y ,$$

if y is not zero.

Theorem IX is obviously true for the case when $S_{n+1}(x)$ is positive by reference to Theorem VIII. It is proved from Theorem VI for the case when $S_{n+1}(x)$ is negative by showing that

$$u_x[e^{|S_{n+1}(x)|} - 1] < u_x[|S_{n+1}(x)| + 0.72 S_{n+1}^2(x)],$$

when

$$|S_{n+1}(x)| < 1 .$$