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THE STANDARD DEVIATION OF EXCESS LOSSES—ACTUARIAL NOTE

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A CHARACTERISTIC of the net premium for insurance of expected losses in excess of a certain retention limit is its combination, generally, of relatively small probabilities with a distribution of large amounts. While the excess-loss premium may be quite small, it has been pointed out that the resulting variance of excess losses is large enough to produce a standard deviation several times the size of the premium. Recent references to this point appear in "An Introduction to Collective Risk Theory and Its Application to Stop-Loss Reinsurance," by Paul Markham Kahn (*TSA*, XIV, 400), and in Robert C. Tookey's discussion; they in turn refer to Mr. Ammeter's comments.

It is the purpose of this note to give a derivation and evaluation of the standard deviation of the excess losses, using Dwight K. Bartlett's paper entitled "Excess Ratio Distributions in Risk Theory" (*TSA*, XVII, 435) as a point of departure. His assumption of the gamma function to represent the compound distribution of claim rate and amount is used, as are his technique and notation. This being so, the reader is referred to the parent paper for the background and definitions of symbols, except for the few added here. A similar extension of Newton L. Bowers' paper, "Expansion of Probability Density Functions as a Sum of Gamma Densities with Applications in Risk Theory" (*TSA*, XVIII, 125), is also developed to give the standard deviation of excess claims when moments beyond the second are used. In this instance Bowers' notation is used with slight modification.

The process is to first derive an expression for the second moment of the excess losses, $\phi_2(n)$, corresponding to Bartlett's first moment, $\phi(n)$. After adjusting these for β, the scale parameter of the gamma function, the variance and standard deviation are found in the same generalized dimension as expressed in Bartlett's Table 1.

$$\phi_2(n)=\int_n^\infty (x-n)^2\frac{x^\alpha e^{-x/\beta}}{\Gamma(\alpha+1)\beta^{\alpha+1}}\,dx$$

$$=\int_n^\infty (x^2-2xn+n^2)\frac{x^\alpha e^{-x/\beta}}{\Gamma(\alpha+1)\beta^{\alpha+1}}\,dx$$

$$\phi_2(n) = (a+1)(a+2)\beta^2 \int_{n/\beta}^{\infty} \frac{(x/\beta)^{a+2}e^{-x/\beta}}{\Gamma(a+3)}\, d\left(\frac{x}{\beta}\right)$$

$$-2(a+1)\beta n \int_{n/\beta}^{\infty} \frac{(x/\beta)^{a+1}e^{-x/\beta}}{\Gamma(a+2)}\, d\left(\frac{x}{\beta}\right)$$

$$+n^2 \int_{n\,\beta}^{\infty} \frac{(x/\beta)^{a}e^{-x/\beta}}{\Gamma(a+1)}\, d\left(\frac{x}{\beta}\right)$$

$$= (a+1)(a+2)\beta^2\left[1 - I\left(\frac{n}{\beta},\ a+2\right)\right]$$

$$-2(a+1)\beta n\left[1 - I\left(\frac{n}{\beta},\ a+1\right)\right] + n^2\left[1 - I\left(\frac{n}{\beta},\ a\right)\right],$$

where

$$I\left(\frac{n}{\beta},\ a\right) = \int_0^{n/\beta} \frac{(x/\beta)^{a}e^{-x/\beta}}{\Gamma(a+1)}\, d\left(\frac{x}{\beta}\right).$$

Since $a + 1 = X$, $n/\beta = Y$, and

$$\beta = \frac{n}{Y} = \frac{\mu_2^A + (\mu_1^A)^2}{\mu_1^A} = \frac{m_2}{m_1},$$

where m_n = nth moment of claims distribution about zero and $m_1 = \mu_1^A$,

$$\phi_2(n) = X(X+1)\left(\frac{n}{Y}\right)^2 [1 - I(Y, X+1)]$$

$$-2X\frac{n^2}{Y}[1 - I(Y, X)] + n^2[1 - I(Y, X-1)],$$

or

$$\phi_2(n)\left(\frac{Y}{n}\right)^2 = X(X+1)[1 - I(Y, X+1)]$$

$$-2XY[1 - I(Y,X)] + (Y)^2[1 - I(Y, X-1)],$$

or

$$\phi_2(n)\left(\frac{m_1}{m_2}\right)^2 = X\{(X+1)[1-I(Y,X+1)] - Y[1-I(Y,X)]\}$$

$$-Y\{X[1 - I(Y, X)] - Y[1 - I(Y, X-1)]\}$$

$$= X\phi'(n)\left(\frac{m_1}{m_2}\right) - Y\phi(n)\left(\frac{m_1}{m_2}\right), \qquad (1)$$

where

$$\phi'(n)\left(\frac{m_1}{m_2}\right) = (X+1)[1-I(Y,X+1)] - Y[1-I(Y,X)] \qquad (2)$$

and

$$\phi(n)\left(\frac{m_1}{m_2}\right) = X[1 - I(Y, X)] - Y[1 - I(Y, X-1)]. \qquad (3)$$

Letting $S(n)$ = standard deviation of excess losses,

$$S(n)\left(\frac{m_1}{m_2}\right)=\sqrt{(\text{variance})\left(\frac{m_1}{m_2}\right)^2}$$
$$=\sqrt{\phi_2(n)\left(\frac{m_1}{m_2}\right)^2-\left[\phi(n)\left(\frac{m_1}{m_2}\right)\right]^2}\,. \qquad (4)$$

Expression (3) is the function tabulated in Bartlett's Table 1, and expression (2) can be obtained from the table approximately by interpolation for the value $X + 1$, allowing at the same time for a modified limit of $Y/(X + 1)$. The approximation is fairly reliable in the central area of Bartlett's Table 1 but gets out of hand elsewhere. Consequently, Table 1 of this paper was prepared from basic calculations. Results in the form of ratios of standard deviation to excess-loss net premium show at a

TABLE 1

EXCESS LOSS COVERAGE

RATIO OF STANDARD DEVIATION OF EXCESS LOSS TO NET PREMIUM

VALUES OF $S(n)(m_1/m_2)/\phi(n)(m_1/m_2)$

(Based on Gamma Distribution for Aggregate Claims)

RETENTION LIMIT Y/X	GAMMA X								
	0.1	0.3	0.5	1.0	2.5	5.0	10.0	25.0	50.0
100%..	4.0	2.8	2.4	2.1	1.9	1.7	1.7	1.6	1.5
110....	4.0	2.8	2.5	2.2	2.1	2.0	2.0	2.2	2.5
120....	4.0	2.9	2.6	2.4	2.3	2.3	2.5	3.2	4.5
130....	4.1	3.0	2.7	2.5	2.5	2.7	3.2	4.9	8.7
140....	4.2	3.1	2.8	2.7	2.8	3.1	4.0	7.7	19.4
150....	4.2	3.2	3.0	2.8	3.0	3.6	5.0	12.7	47.0
160....	4.3	3.3	3.0	3.0	3.3	4.2	6.5	21.6	100+
170....	4.3	3.4	3.2	3.2	3.7	5.0	8.5	37.7	100+
180....	4.4	3.4	3.3	3.3	4.1	5.8	11.0	72.5	100+
190....	4.5	3.5	3.4	3.5	4.5	6.8	14.7	100+	100+
200....	4.5	3.6	3.5	3.7	5.0	8.1	19.1	100+	100+

NOTES.—

1. $\frac{m_1}{m_2}=\frac{\mu_1^A}{\mu_2^A+(\mu_1^A)^2}$ used in Bartlett's paper (*TSA*, XVII, 435).

2. $X=\lambda\mu_1^A\left[\frac{\mu_1^A}{\mu_2^A+(\mu_1^A)^2}\right]$ (also in Bartlett's paper).

$=$ Net premium for full coverage scaled down by the ratio $\frac{\mu_1^A}{\mu_2^A+(\mu_1^A)^2}$.

glance the relationship mentioned in the first paragraph and also are more readily interpolable than actual values.

Table 1 indicates a minimum ratio, for the gamma distribution, of about 1.5 and, in what may be the most useful area of Bartlett's Table 1, the standard deviation is from 2 to 6 times the excess-loss net premium. This bears out Tookey's remark that, if a "security loading" of one-half the standard deviation is added to the net premium, it may be tripled or even quadrupled. The trend of these ratios raises the question of whether the security loading should be uniformly one-half the standard deviation. Where X is small, say, under 0.5, some factor less than $\frac{1}{2}$ might be appropriate or even necessary, since here the excess-loss net premium is a substantial proportion of the total net premium. At the other extreme, larger factors up to 1.0 or even more might be appropriate to give the security-loaded premium some substance. A loading based on the size of the standard deviation does make some provision against the wide range of random variations inherent in excess-loss coverage. It should be mentioned here, however, that the sense of the word "security," appropriated from other contexts, does not include protection against the effects of war, pestilence, natural catastrophes, and so forth, which presumably would be limited by appropriate clauses in the reinsurance treaty.

Bowers' paper provides a method of allowing for additional moments, beyond the second, of the actual distribution, specifically the third to fifth moments. His notation is used in what follows except that, in order to facilitate matters, the gamma density in the form that he uses is expressed as a function of the variable and the constant α, so as to use derivatives of this function.

Let

$$f(x,\ \alpha-1)=\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \tag{5}$$

so that

$$\begin{aligned} f'(x,\ \alpha-1) &= f(x,\ \alpha-2)-f(x,\ \alpha-1)\ , \\ f''(x,\ \alpha-1) &= f(x,\ \alpha-3)-2f(x,\ \alpha-2)+f(x,\ \alpha-1)\ ,\ \text{etc.} \end{aligned} \tag{6}$$

Then, starting with Bowers' expression (5) and collecting terms involving A, B, and C, it reduces to the following alternative form:

$$f(z)=f(z,\ \alpha-1)-Af'''(z,\ \alpha+2)+Bf^{\text{iv}}(z,\ \alpha+3)-Cf^{\text{v}}(z,\ \alpha+4)\ , \tag{7}$$

from which

$$F(x) = \int_0^x f(z)\,dz \\ = \Gamma(x, a) - Af''(x, a+2) + Bf'''(x, a+3) \\ - Cf^{iv}(x, a+4) \tag{8}$$

an equivalent form of Bowers' expression (7).

Continuing with the stop-loss premium,

$$\pi(x) = \int_x^\infty (z-x)\,f(z)\,dz \\ = \int_x^\infty (z-x)\,[\,f(z, a-1) - Af'''(z, a+2) \\ + Bf^{iv}(z, a+3) - Cf^{v}(z, a+4)\,]\,dz\,. \tag{9}$$

The right-hand terms involving A, B, and C may be integrated by parts, and the left-hand term can be handled by substituting from the identity:

$$(z - x)\,f(z, a - 1) = af(z, a) - xf(z, a - 1)\,, \tag{10}$$

so that

$$\pi(x) = a[1 - \Gamma(x, a + 1)] - x[1 - \Gamma(x, a)] - Af'(x, a + 2) \\ + Bf''(x, a + 3) - Cf'''(x, a + 4)\,, \tag{11}$$

which is equivalent to Bowers' expression (8).

Following a similar procedure for the second moment of excess claims, we have

$$\pi_2(x) = \int_x^\infty (z-x)^2 f(z)\,dz \\ = \int_x^\infty (z-x)^2[\,f(z, a-1) - Af'''(z, a+2) \\ + Bf^{iv}(z, a+3) - Cf^{v}(z, a+4)\,]\,dz\,, \tag{12}$$

in which again the terms involving A, B, and C are integrated by parts and the left-hand term by use of the identity

$$(z - x)^2 f(z, a - 1) = a(a + 1)\,f(z, a + 1) \\ - 2\,axf(z, a) + x^2 f(z, a - 1)\,, \tag{13}$$

yielding

$$\pi_2(x) = a(a + 1)[1 - \Gamma(x, a + 2)] - 2ax\,[1 - \Gamma(x, a + 1)] \\ + x^2[1 - \Gamma(x, a)] + 2Af(x, a + 2) \\ - 2Bf'(x, a + 3) + 2Cf''(x, a + 4) \tag{14}$$

$$\begin{aligned}\pi_2(x) &= a\{(a+1)[1-\Gamma(x, a+2)] - x[1-\Gamma(x, a+1)]\} \\ &\qquad - x\{a[1-\Gamma(x, a+1)] - x[1-\Gamma(x, a)]\} \qquad (15) \\ &\qquad + 2Af(x, a+2) - 2Bf'(x, a+3) + 2Cf''(x, a+4) \\ &= a[\theta'(x)] - x[\theta(x)] + 2Af(x, a+2) - 2Bf'(x, a+3) \\ &\qquad + 2Cf''(x, a+4)\,, \qquad (16)\end{aligned}$$

where $\theta'(x)$ and $\theta(x)$ are the Bartlett forms of the stop-loss premiums analogous to those referred to earlier in expressions (2) and (3). The standard deviation of excess losses, σ_E, is given in the usual form, $\sqrt{\pi_2(x) - [\pi(x)]^2}$.

In illustration, the same example from Bartlett employed by Bowers is used here, including Bowers' values for $\pi(x)$, A, B, and C, with results shown in the following table:

No. Moments	Stop-Loss Premium $\pi(x)$	Standard Deviation of Excess Losses σ_E	Ratio $\sigma_E/\pi(x)$	$\pi(x)+\frac{1}{2}\sigma_E$
2.	0.554	1.344	2.43	1.23
3.	.562	1.249	2.22	1.19
4.	.541	1.237	2.29	1.16
5.	0.536	1.247	2.33	1.16

The direction of change in the standard deviation, as adjustments for the third to fifth moments are made, may or may not follow that of the premium, and hence the ratio $\sigma_E/\pi(x)$ also has its own pattern of variation. It is interesting that the function $\pi(x) + \frac{1}{2}\sigma_E$ decreases and tends to level off, in this particular example, as successive higher moment adjustments are made. In this case at least, the use of moments beyond the second is less conservative if this function were to be used as a security-loaded premium. The trend in other examples might be quite different, of course, depending on the relationships of A, B, and C and other elements of the formula.

DISCUSSION OF PRECEDING PAPER

DWIGHT K. BARTLETT, III:

Needless to say, I am flattered that Mr. Thomson found enough of interest or value in my paper entitled "Excess Ratio Distributions in Risk Theory" (*TSA*, XVII, 435) to use it as a point of departure for his interesting actuarial note. Until seeing Mr. Thomson's work, I had not appreciated how large the standard deviation of the typical excess ratio distribution would be in comparison with the mean of the distribution. Perhaps this explains, at least in part, why reinsurance companies have been so reluctant to push the marketing of stop-loss reinsurance.

Perhaps one way to get an intuitive feeling for the situation is to think of the distribution of losses on a single, one-year term, individual life insurance policy. If we assume an expected mortality rate of 1 per cent, the mean of the distribution of losses would be 1 per cent and the standard deviation would be 10 per cent, or 10 times the mean. If we consider a block of 10,000 such policies, the mean of the distribution of losses would be 100 deaths and the standard deviation 10 deaths, or one-tenth of the mean. Insurance becomes a viable institution only as a result of the sharing of risk between a large number of individual risk entities. A reinsurance company reinsuring other life companies on a stop-loss program would have a maximum number of 1,600 individual risk entities in its stop-loss program, since there are only 1,600 other life companies in the country. This may not be a sufficient number of individual risk entities to keep the standard deviation of their whole potential stop-loss reinsurance program within tolerable limits.

The main experience of life companies in stop-loss programs has been in the experience-rating of group policies. Apparently the companies have felt that they do have an adequate number of group policies to permit them to charge a risk-sharing charge or stop-loss premium which contains reasonably modest loadings for contingencies even though the standard deviation of excess losses on any one group policy is large in comparison with the mean value of excess losses.

ROBERT C. TOOKEY:

Mr. Thomson's actuarial note on the standard deviation of excess losses is certainly a welcome addition to our actuarial literature relating to collective risk theory. The paper focuses on the standard deviations of excess losses that result solely from random fluctuations. The problem of variations due to underwriting error (resulting in a true expected claim cost different from the assumed expected claim cost) was not within the scope of this paper.

Since the author referred to my discussion of Dr. Kahn's paper, it appeared appropriate to make a remark on the subject. I should like to reiterate a point that I raised in that discussion concerning the potential financial effect of underestimating expected claims. If we have variations within the universe (if we have subsets within our set, so to speak), what type of loading on the net premium is necessary to minimize the probability of losses to the reinsurance company resulting from simply charging a standard stop-loss premium on a somewhat substandard case? Perhaps the theory developed many years ago on experience-rating of groups would be useful. Could not a life insurance company with its own unique book of business be considered like a group life insurance case with its unique characteristics? A company's portfolio of risks would have certain unique attributes (such as its underwriting practice in regard to borderline risks, its nonmedical underwriting limits, the effect of quasi-group insurance) which would lead to an underlying expected mortality rate different from the average for the universe. What type of mathematical function would most accurately exhibit the distribution of these variations from the mean? In this area there may be room for still another actuarial note on collective risk theory.

(AUTHOR'S REVIEW OF DISCUSSION)

PAUL THOMSON:

My thanks to Messrs. Bartlett and Tookey for their thoughtful comments. Each one lends perspective to my actuarial note, and I appreciate their efforts.

Mr. Tookey raises an especially important point that perhaps should be re-emphasized; that is, that there are many other elements besides a tabular net-premium calculation, and an allowance for random variations, that need to be considered in setting a stop-loss premium. To name some, there are the subjective aspects of individual risks which he mentions. To some extent these unique characteristics may be allowed for empirically if the experience claim distributions of the prospect company are used to calculate tabular net premiums. This may be valid provided past distributions are reliable guides to a company's future distributions. Most likely they are not, which is undoubtedly one of the motivations of the stop-loss insurance concept. In other words, a large premium-loading is needed to cover many uncertainties. This leads back to an often expressed warning to the unwary—and I am glad of the chance to reiterate it—that the mathematics of a theory may often be only a preliminary tool which aids the actuary in finding the right ball park but which may leave him out in left field unless intangibles are evaluated with skill and discernment.