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AN APPROXIMATION TO THE DISTRIBUTION OF ANNUITY COSTS

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INTRODUCTION

THE purpose of this paper is to illustrate the use of a new approximation to determine contingency reserves against adverse mortality experience for a portfolio of life annuity contracts. This problem was discussed by Fretwell and Hickman [4] in 1964 and before that by Boermeester [1] in 1956 and Taylor [7] in 1952. The approaches all involved an individual risk theory model where losses on the portfolio are viewed as the sum of losses on the individual lives. The chief differences between the papers were in the methods of approximating the reserve required. Fretwell and Hickman stressed probability inequalities, Boermeester used a Monte Carlo approach, while Taylor fitted a Pearson Type III curve by equating its first three moments to those of the portfolio loss distribution. The method of estimating the required reserve in this paper is derived from that used by Finnish regulatory authorities to define the equalization reserve required of a company.

THE CORNISH-FISHER EXPANSION

At the 1966 ASTIN Colloquium, two Finnish actuaries, L. Kauppi and P. Ojantakanen, described their search for a simplified method to duplicate the results of the Esscher approximation as applied to a collective risk model. They obtained a formula which uses a measure of skewness to adjust a standardized normal variable. The normal distribution is then used to calculate probabilities. The method when applied to claim amount data used in connection with a Poisson distribution for the number of claims gave results very close to those of the Esscher approximation. Later they applied their method to data presented by Bohman and Esscher [3], where the assumption had been made that the number of claims followed a negative binomial distribution. In this case, the new method gave results which were superior to those of the Esscher approximation. The authors noted that their formula was the first two terms of a series devel-

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oped by Cornish and Fisher. The actual method of arriving at this series is not used in what follows but may be found in Kendall [6, p. 158].

The formula that will be used in this paper is the first four terms of this series of Cornish and Fisher and is

$$x_\epsilon = y_\epsilon + \frac{\gamma_1}{6}(y_\epsilon^2 - 1) + \left[\frac{\gamma_2}{24}(y_\epsilon^3 - 3y_\epsilon) - \frac{\gamma_1^2}{36}(2y_\epsilon^3 - 5y_\epsilon)\right]$$

$$+ \left[\frac{\gamma_3}{120}(y_\epsilon^4 - 6y_\epsilon^2 + 3) - \frac{\gamma_1\gamma_2}{24}(y_\epsilon^4 - 5y_\epsilon^2 + 2) \qquad (1)\right.$$

$$\left. + \frac{\gamma_1^3}{324}(12y_\epsilon^4 - 53y_\epsilon^2 + 17)\right].$$

Before the symbols are defined, we shall examine what the formula does. The symbols x_ϵ and y_ϵ are both values of random variables X and Y, whose means are zero and whose variances are one. The formula gives a series for a point x_ϵ, which is defined as the value such that $P[X > x_\epsilon] = \epsilon$ for the standardized variable X. The formula for x_ϵ is written in terms of y_ϵ, which is the value such that $P[Y > y_\epsilon] = \epsilon$, where Y has the standard normal distribution. For instance, for $\epsilon = 0.10$ a standard normal table shows $y_{0.10} = 1.282$; likewise $y_{0.05} = 1.645$ and $y_{0.01} = 2.326$.

The parameters, γ_1, γ_2, and γ_3, depend on the first five moments of X. These parameters are, as might be expected, equal to zero if X has a normal distribution, since x_ϵ should then equal y_ϵ. The parameters are defined by

$$\gamma_k = \frac{\chi_{k+2}^{(X)}}{\{\chi_2(X)\}^{k+2/2}}, \qquad (2)$$

where $\chi_k(X)$ is the kth cumulant of the random variable X. The first five cumulants written in terms of moments about the origin are given in the Appendix, along with other useful information about cumulants.

APPLICATION TO ANNUITY COSTS

We come now to the problem of constructing a model to use in calculating the distribution of the present value of all annuity payments to be paid to a closed block of annuitants. We shall view this present value as the sum of present values due to the various individual annuitants. In symbols, we have

$$Z = C_1 + C_2 + C_3 + \ldots + C_n ,$$

where Z is the total present value for the entire portfolio, and C_i is the present value associated with ith annuitant. This is the individual risk theory approach which has a long history in actuarial literature. Wooddy [8, p. 13] gives a discussion of the individual approach along with a bibliography on risk theory in general. We shall assume that the individual costs are independent random variables. From this it follows (see Property 2, Appendix) that the kth cumulant of Z is simply the sum of the kth cumulants of $C_1, C_2, C_3, \ldots, C_n$. Therefore, it is necessary only to calculate the cumulants of the present value of future annuity payments to an annuitant aged x.

Consider such an annuitant aged x and assume he is being paid a life annuity of 1 per year at the beginning of each year. Then the individual present value is given by

$$C = \ddot{a}_{\overline{T}|} ,$$

where T is the random variable equal to the year of death (measured from the present time) of the annuitant. In terms of a life table

$$P[T=t] = \frac{d_{x+t-1}}{l_x} \qquad t=1, 2, 3, \ldots .$$

To facilitate the calculation of the cumulants of C, we note that

$$C = \ddot{a}_{\overline{T}|} = \frac{1-v^T}{d} . \tag{3}$$

By Property 1 shown in the Appendix,

$$\chi_1(C) = \frac{1}{d} - \frac{1}{d}\chi_1(v^T)$$

and

$$\chi_k(C) = \left(\frac{-1}{d}\right)^k \chi_k(v^T) \qquad \text{for} \qquad k=2, 3, 4, \ldots . \tag{4}$$

We now use the formulas (10) for the cumulants in terms of the moments about the origin as given in the Appendix. To do this, we note that the kth moment about the origin for the random variable v^T is

$$\sum_{t=1}^{\infty} (v^t)^k P[T=t] = \sum_{t=1}^{\infty} v^{kt} \frac{d_{x+t-1}}{l_x} = A_x^{(k)} ,$$

where $A_x^{(k)}$ is the net single premium for whole life insurance at an interest rate of $i' = (1+i)^k - 1$. If $k = 1$, the standard notation, A_x, will be used. Then

$$\begin{aligned}
\chi_1(C) &= \frac{1}{d}[1 - A_x] \\
\chi_2(C) &= \frac{1}{d^2}[A_x^{(2)} - (A_x)^2] \\
\chi_3(C) &= \frac{1}{d^3}[-A_x^{(3)} + 3A_x^{(2)}A_x - 2(A_x)^3] \\
\chi_4(C) &= \frac{1}{d^4}[A_x^{(4)} - 3(A_x^{(2)})^2 - 4A_x^{(3)}A_x + 12A_x^{(2)}(A_x)^2 - 6(A_x)^4] \\
\chi_5(C) &= \frac{1}{d^5}[-A_x^{(5)} + 10A_x^{(3)}A_x^{(2)} + 5A_x^{(4)}A_x - 20A_x^{(3)}(A_x)^2 \\
&\qquad + 60A_x^{(2)}(A_x)^3 - 30(A_x^{(2)})^2A_x - 24(A_x)^5].
\end{aligned} \tag{5}$$

FRACTIONAL ANNUITIES

We now discuss the changes needed if the annuity is paid m times per year. Following Taylor [7, p. 112], we view the cost associated with any annuitant as the sum of

$$\ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{\overline{T-1}|}$$

representing amounts paid in years prior to the year of death plus a term for the cost in the year of death. The conditional expectation of this additional cost, given T, is

$$v^{T-1}\sum_{s=0}^{m-1}\left(\frac{1}{m} + a_{\overline{\frac{s}{m}}|}^{(m)}\right)\cdot P[S = s\,|\,T]. \tag{6}$$

The random variable S equals the number of fractional periods completed in the year of death. If deaths are assumed to occur uniformly in the year of death, then $P[S = s\,|\,T] = 1/m$ for $s = 0, 1, 2, \ldots, m-1$, independent of t. Using this assumption, the conditional expectation may be shown to equal

$$v^{T-1}\left[\frac{1}{m} + \frac{1}{i^{(m)}} - \frac{d}{i^{(m)}d^{(m)}}\right] = v^{T-1}\left[\frac{1}{d^{(m)}} - \frac{d}{i^{(m)}d^{(m)}}\right]. \tag{7}$$

Therefore, the total cost associated with an annuitant is

$$\begin{aligned}
C^{(m)} &= \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{\overline{T-1}|} + v^{T-1}\left[\frac{1}{d^{(m)}} - \frac{d}{i^{(m)}d^{(m)}}\right] \\
&= \frac{d}{d^{(m)}}\frac{1 - v^{T-1}}{d} + v^{T-1}\left[\frac{1}{d^{(m)}} - \frac{d}{i^{(m)}d^{(m)}}\right] = \frac{1}{d^{(m)}} - \frac{i}{i^{(m)}d^{(m)}}\,v^T.
\end{aligned} \tag{8}$$

Comparing this expression with the expression (3) for C, we note that

$$\chi_k[C^{(m)}] = \left[\frac{id}{i^{(m)}d^{(m)}}\right]^k \chi_k(C) \qquad \text{for} \qquad k = 2, 3, \ldots . \qquad (9)$$

Remembering the definition of the correction coefficients γ_i (2), we see that they are independent of the frequency of payment of the annuity. The second cumulant, the variance, is changed as indicated. The tables of cumulants of individual present values which follow in this paper were constructed on the assumption of an annual payment annuity so the standard deviation should be multiplied by $id/i^{(m)}d^{(m)}$ in the case of an annuity paid m times a year. This ratio is, however, very close to 1. For instance, for a monthly annuity valued at 4 per cent $id/i^{(m)}d^{(m)} =$ 1.00013. In an example which follows, this correction will be ignored.

The first cumulant, the mean, depends on the mode of payment and also on whether a payment is due at the valuation date or not. However, the mean of the random variable C is the traditional net single premium $\ddot{a}_x$. Therefore we shall use the standard adjustment of subtracting $(m - 1)/2m$ from $\ddot{a}_x$ to account for the fractional mode of payment and subtract an additional $1/m$ if the annuity is an "immediate annuity."

CUMULANTS OF INDIVIDUAL PRESENT VALUES BASED ON THE 1951 GROUP ANNUITY TABLE

We now give the results of the calculations of the first five cumulants on two mortality bases, the 1951 Group Annuity Table, Male and Female, and two interest bases, 3½ per cent and 4 per cent. It should be noted that accuracy requirements necessitated the use of recursive formulas for the cumulants. A copy of the computer program written in FORTRAN II language is available.

TWO EXAMPLES

We next exhibit the use of the method presented in this paper to calculate contingency reserves which are defined as differences between "required reserves" and the classical actuarial reserve. The "required reserve" is defined as the amount needed at present in the fund so that the probability is $1 - \epsilon$ that the fund will be sufficient to pay all annuitants. The required reserve thus depends on the value of ϵ.

The first example will be the one used by Boermeester [1] and later by Fretwell and Hickman [4]. The mortality basis used was the 1949 Annuity Table and the interest rate was 2½ per cent. Immediate annual annuities were used. A calculation was made to determine the cumulants

Age	Cumulants					
	Q	First	Second	Third	Fourth	Fifth
50.....	.006475	16.64396	25.34375	−130.39311	359.97123	8674.89575
51.....	.007187	16.29703	25.60608	−125.95417	277.86994	9199.39026
52.....	.007938	15.94703	25.80072	−120.80211	196.16738	9550.49854
53.....	.008731	15.59397	25.92923	−115.01298	116.16865	9728.87573
54.....	.009563	15.23780	25.99344	−108.66323	38.99468	9740.46313
55.....	.010436	14.87840	25.99674	−101.84259	− 34.35058	9596.90955
56.....	.011346	14.51563	25.94315	− 94.64327	−103.03086	9313.71082
57.....	.012298	14.14921	25.83841	− 87.17075	−166.32177	8910.28137
58.....	.013302	13.77889	25.68851	− 79.52948	−223.63616	8407.20850
59.....	.014379	13.40446	25.49906	− 71.81879	−274.50022	7824.79218
60.....	.015555	13.02591	25.27398	− 64.12384	−318.53294	7181.31049
61.....	.016866	12.64349	25.01533	− 56.51678	−355.41316	6492.93842
62.....	.018353	12.25775	24.72260	− 49.05428	−384.85743	5773.46521
63.....	.020068	11.86962	24.39289	− 41.78079	−406.61277	5035.00354
64.....	.022067	11.48044	24.02042	− 34.72721	−420.46194	4288.20392
65.....	.024418	11.09202	23.59693	− 27.91335	−426.26512	3543.16562
66.....	.027193	10.70668	23.11119	− 21.34331	−424.01847	2809.58801
67.....	.030112	10.32724	22.54914	− 15.00050	−413.97792	2097.45505
68.....	.032986	9.95342	21.92194	− 8.95394	−397.47955	1429.61714
69.....	.035943	9.58289	21.25522	− 3.30935	−376.30613	828.66349
70.....	.039303	9.21448	20.56622	1.85899	−351.72365	306.06353
71.....	.043183	8.84982	19.85418	6.52812	−324.28342	− 136.61518
72.....	.047476	8.49124	19.11463	10.69230	−294.52447	− 500.29370
73.....	.052084	8.13988	18.35100	14.34436	−263.36007	− 786.43306
74.....	.057077	7.79581	17.57279	17.47832	−231.86171	− 998.41791
75.....	.062427	7.45943	16.78795	20.09484	−200.92068	−1141.92509
76.....	.068347	7.13065	16.00692	22.19474	−171.34340	−1224.10358
77.....	.075132	6.81072	15.23460	23.77892	−143.60353	−1252.76947
78.....	.082687	6.50265	14.46851	24.85389	−117.90640	−1235.86031
79.....	.090946	6.20861	13.70879	25.44305	− 94.49819	−1182.31677
80.....	.099679	5.93025	12.95600	25.58575	− 73.56166	−1101.45496
81.....	.108706	5.66776	12.21334	25.33841	− 55.23340	−1002.71478
82.....	.117979	5.42036	11.48511	24.76542	− 39.54655	− 894.63284
83.....	.127437	5.18703	10.77454	23.92977	− 26.41023	− 784.06759
84.....	.137073	4.96649	10.08428	22.89132	− 15.65205	− 676.23790
85.....	.146852	4.75744	9.41608	21.70364	− 7.04695	− 574.76937
86.....	.156836	4.55835	8.77159	20.41405	− .35214	− 481.99529
87.....	.167120	4.36794	8.15189	19.06040	4.67965	− 399.08477
88.....	.177787	4.18526	7.55737	17.67183	8.28988	− 326.34550
89.....	.188919	4.00960	6.98789	16.27107	10.70873	− 263.52665
90.....	.200594	3.84048	6.44275	14.87640	12.15302	− 210.04070
91.....	.212555	3.67760	5.92048	13.50430	12.82279	− 165.13840
92.....	.225161	3.51938	5.42139	12.17401	12.88769	− 128.11162
93.....	.238524	3.36529	4.94517	10.89550	12.48419	− 98.05124
94.....	.252765	3.21490	4.49144	9.67618	11.72866	− 74.04072
95.....	.268025	3.06788	4.05988	8.52128	10.71988	− 55.18739
96.....	.284455	2.92394	3.65006	7.43411	9.54166	− 40.64045
97.....	.302223	2.78289	3.26146	6.41640	8.26600	− 29.60424
98.....	.321515	2.64452	2.89339	5.46862	6.95609	− 21.34800
99.....	.342526	2.50865	2.54484	4.59055	5.66956	− 15.21790
100.....	.365462	2.37492	2.21438	3.78221	4.46093	− 10.65685
101.....	.390538	2.24265	1.90022	3.04549	3.38127	− 7.23286
102.....	.417979	2.11029	1.60075	2.38599	2.47129	− 4.66543
103.....	.450096	1.97441	1.31683	1.81275	1.74552	− 2.81046
104.....	.489201	1.83398	1.05132	1.32849	1.18654	− 1.55822
105.....	.537605	1.68984	.80783	.93112	.76692	− .78647
106.....	.597619	1.54411	.58970	.61519	.46071	− .36083
107.....	.671554	1.39954	.39935	.37321	.24708	− .15411
108.....	.761722	1.25904	.23793	.19644	.10915	− .06461
109.....	.870434	1.12518	.10528	.07536	.03178	− .02486
110.....	1.000000	1.00000	.00000	.00000	.00000	.00000

Age	Cumulants					
	Q	First	Second	Third	Fourth	Fifth
50.....	.006475	15.76280	21.01081	−108.73854	362.70114	5226.01788
51.....	.007187	15.45338	21.32712	−105.92953	300.93156	5750.68335
52.....	.007938	15.14033	21.58693	−102.48101	237.98860	6161.37738
53.....	.008731	14.82361	21.79095	− 98.44581	174.92807	6453.44550
54.....	.009563	14.50318	21.94019	− 93.87824	112.68886	6626.26166
55.....	.010436	14.17890	22.03708	− 88.84570	52.15333	6683.64813
56.....	.011346	13.85060	22.08464	− 83.41936	− 5.90550	6632.75653
57.....	.012298	13.51800	22.08755	− 77.68413	− 60.77382	6484.20905
58.....	.013302	13.18082	22.05077	− 71.72604	−111.83161	6250.18567
59.....	.014379	12.83883	21.97900	− 65.62855	−158.53914	5943.32648
60.....	.015555	12.49201	21.87545	− 59.46409	−200.42915	5575.35516
61.....	.016866	12.14054	21.74165	− 53.29481	−237.07989	5156.90973
62.....	.018353	11.78492	21.57677	− 47.16985	−268.09882	4697.21954
63.....	.020068	11.42602	21.37769	− 41.12802	−293.11362	4204.57507
64.....	.022067	11.06512	21.13857	− 35.19615	−311.77722	3686.44016
65.....	.024418	10.70392	20.85109	− 29.39115	−323.80131	3150.17816
66.....	.027193	10.34468	20.50398	− 23.71538	−329.01152	2603.20721
67.....	.030112	9.99013	20.08309	− 18.15168	−327.46230	2053.67429
68.....	.032986	9.64001	19.59796	− 12.76769	−320.09744	1521.93950
69.....	.035943	9.29212	19.07209	− 7.66998	−308.26897	1029.45862
70.....	.039303	8.94533	18.52134	− 2.93603	−292.94421	588.57790
71.....	.043183	8.60120	17.94458	1.40706	−274.51915	202.89069
72.....	.047476	8.26202	17.33710	5.34838	−253.39910	− 126.08946
73.....	.052084	7.92894	16.70172	8.87335	−230.33733	− 396.95050
74.....	.057077	7.60204	16.04716	11.96684	−206.25725	− 609.81208
75.....	.062427	7.28174	15.38080	14.62023	−181.95369	− 767.04740
76.....	.068347	6.96800	14.71253	16.82454	−158.17216	− 872.54611
77.....	.075132	6.66206	14.04703	18.57206	−135.37672	− 931.27902
78.....	.082687	6.36690	13.38185	19.86174	−113.79847	− 948.78609
79.....	.090946	6.08470	12.71712	20.70772	− 93.72017	− 931.71553
80.....	.099679	5.81713	12.05341	21.13941	− 75.37927	− 887.39137
81.....	.108706	5.56448	11.39396	21.20267	− 58.98280	− 823.59010
82.....	.117979	5.32603	10.74314	20.95184	− 44.64751	− 747.64703
83.....	.127437	5.10086	10.10437	20.44149	− 32.37229	− 665.71681
84.....	.137073	4.88778	9.48053	19.72450	− 22.07380	− 582.70872
85.....	.146852	4.68555	8.87361	18.84910	− 13.61007	− 502.24370
86.....	.156836	4.49274	8.28558	17.85851	− 6.81264	− 426.85122
87.....	.167120	4.30812	7.71777	16.78785	− 1.49799	− 358.03117
88.....	.177787	4.13078	7.17087	15.66454	2.52198	− 296.48972
89.....	.188919	3.96006	6.64502	14.51038	5.43327	− 242.39254
90.....	.200594	3.79551	6.13979	13.34325	7.41660	− 195.55596
91.....	.212555	3.63686	5.65395	12.17954	8.64310	− 155.60333
92.....	.225161	3.48258	5.18808	11.03848	9.25890	− 122.15394
93.....	.238524	3.33215	4.74205	9.93058	9.38480	− 94.58895
94.....	.252765	3.18517	4.31570	8.86400	9.12601	− 72.24165
95.....	.268025	3.04132	3.90886	7.84487	8.57364	− 54.43060
96.....	.284455	2.90034	3.52129	6.87750	7.80674	− 40.48038
97.....	.302223	2.76202	3.15266	5.96466	6.89468	− 29.73803
98.....	.321515	2.62620	2.80241	5.10792	5.89968	− 21.58568
99.....	.342526	2.49268	2.46972	4.30810	4.87960	− 15.45505
100.....	.365462	2.36114	2.15331	3.56622	3.89023	− 10.84923
101.....	.390538	2.23090	1.85155	2.88502	2.98488	− 7.37299
102.....	.417979	2.10043	1.56293	2.27088	2.20781	− 4.76236
103.....	.450096	1.96634	1.28837	1.73351	1.57872	− 2.87557
104.....	.489201	1.82757	1.03074	1.27661	1.08697	− 1.59957
105.....	.537605	1.68496	.79368	.89921	.71180	− .80988
106.....	.597619	1.54058	.58058	.59710	.43326	− .37159
107.....	.671554	1.39719	.39398	.36408	.23542	− .15732
108.....	.761722	1.25766	.23520	.19262	.10542	− .06453
109.....	.870434	1.12458	.10427	.07428	.03117	− .02427
110.....	1.000000	1.00000	.00000	.00000	.00000	.00000

Age	Cumulants					
	Q	First	Second	Third	Fourth	Fifth
50.....	.003070	18.49553	19.00272	−117.65865	805.66077	−1525.22578
51.....	.003319	18.16364	19.40602	−115.81430	738.91290	−500.72512
52.....	.003597	17.82352	19.80307	−113.67951	669.88207	482.23052
53.....	.003908	17.47521	20.19166	−111.24339	598.93437	1412.82675
54.....	.004257	17.11874	20.56943	−108.49781	526.52197	2279.72998
55.....	.004648	16.75422	20.93374	−105.43680	453.17656	3071.28250
56.....	.005102	16.38176	21.28211	−102.06318	379.57642	3775.49646
57.....	.005637	16.00176	21.60844	−98.34060	306.09727	4380.75983
58.....	.006265	15.61484	21.90429	−94.21460	233.05357	4875.17841
59.....	.006997	15.22173	22.16075	−89.63688	160.90149	5247.33496
60.....	.007837	14.82321	22.36899	−84.57172	90.24640	5487.67444
61.....	.008788	14.42003	22.52190	−79.01180	21.88109	5590.61627
62.....	.009848	14.01288	22.61430	−72.97713	−43.31132	5555.93329
63.....	.011010	13.60228	22.64385	−66.51906	−104.42549	5389.94257
64.....	.012264	13.18857	22.61163	−59.72120	−160.61013	5105.83661
65.....	.013597	12.77180	22.52241	−52.69620	−211.13547	4722.85602
66.....	.014991	12.35176	22.38470	−45.58270	−255.42267	4264.69324
67.....	.016457	11.92788	22.21115	−38.54482	−293.03075	3757.52203
68.....	.018198	11.49961	22.01497	−31.74657	−323.62296	3225.48538
69.....	.020354	11.06852	21.79063	−25.23907	−346.81226	2679.22699
70.....	.023098	10.63743	21.52449	−19.04315	−362.13857	2126.89954
71.....	.026527	10.21059	21.19465	−13.15627	−369.08695	1576.19292
72.....	.030468	9.79273	20.77905	−7.58768	−367.37761	1039.26361
73.....	.034779	9.38646	20.27413	−2.40972	−357.58350	537.07738
74.....	.039413	8.99275	19.68815	2.29245	−340.93567	90.12575
75.....	.044309	8.61192	19.03271	6.45455	−318.93150	−287.33079
76.....	.049512	8.24360	18.32249	10.03296	−293.17823	−587.76062
77.....	.055108	7.88766	17.56953	13.00981	−265.09033	−810.36161
78.....	.061093	7.54449	16.78188	15.39028	−235.82396	−959.36283
79.....	.067459	7.21429	15.96727	17.19745	−206.39981	−1042.39093
80.....	.074146	6.89706	15.13289	18.47031	−177.67722	−1069.33661
81.....	.081114	6.59224	14.28673	19.26042	−150.37922	−1051.29695
82.....	.088374	6.29890	13.43699	19.62641	−125.05680	−999.45010
83.....	.095943	6.01602	12.59094	19.62841	−102.07070	−924.10022
84.....	.103904	5.74254	11.75523	19.32446	−81.62297	−834.19265
85.....	.112328	5.47768	10.93498	18.76725	−63.76812	−736.99555
86.....	.121295	5.22084	10.13437	18.00463	−48.45878	−638.23177
87.....	.130885	4.97161	9.35673	17.07961	−35.57316	−542.21507
88.....	.141188	4.72965	8.60476	16.03091	−24.93830	−452.04241
89.....	.152300	4.49480	7.88055	14.89329	−16.34640	−369.77858
90.....	.164331	4.26698	7.18558	13.69828	−9.56921	−296.65036
91.....	.177144	4.04625	6.52058	12.47536	−4.37145	−233.22007
92.....	.191099	3.83162	5.88804	11.25431	−.53754	−179.62909
93.....	.206341	3.62309	5.28899	10.05484	2.14601	−135.36562
94.....	.223029	3.42074	4.72421	8.89358	3.88136	−99.64485
95.....	.241336	3.22466	4.19421	7.78423	4.85595	−71.51005
96.....	.261451	3.03497	3.69921	6.73772	5.23993	−49.91728
97.....	.283581	2.85180	3.23918	5.76232	5.18448	−33.80407
98.....	.307953	2.67527	2.81377	4.86382	4.82092	−22.14347
99.....	.334812	2.50548	2.42231	4.04564	4.26075	−13.98458
100.....	.364429	2.34245	2.06379	3.30916	3.59639	−8.48135
101.....	.397100	2.18612	1.73677	2.65402	2.90268	−4.91118
102.....	.433150	2.03621	1.43944	2.07891	2.23847	−2.68689
103.....	.472930	1.89199	1.16972	1.58262	1.64690	−1.36251
104.....	.518156	1.75159	.92637	1.16456	1.15216	−.62444
105.....	.570545	1.61442	.70899	.82122	.75899	−.25220
106.....	.631813	1.48076	.51749	.54718	.46214	−.09103
107.....	.703676	1.35145	.35166	.33578	.25130	−.03599
108.....	.787851	1.22754	.21094	.17950	.11357	−.02175
109.....	.886054	1.11009	.09425	.07031	.03469	−.01388
110.....	1.000000	1.00000	.00000	.00000	.00000	.00000

GROUP ANNUITY TABLE FOR 1951—FEMALE, 4.0 PER CENT

Age	Cumulants					
	Q	First	Second	Third	Fourth	Fifth
50	.003070	17.41792	15.33623	−94.08325	652.18456	−2050.79510
51	.003319	17.12721	15.73819	−93.30998	607.85397	−1309.81458
52	.003597	16.82815	16.13921	−92.30308	561.02658	− 580.44204
53	.003908	16.52071	16.53744	−91.04904	511.90594	129.23157
54	.004257	16.20486	16.93084	−89.53627	460.77045	810.31731
55	.004648	15.88066	17.31709	−87.75456	407.97106	1453.23035
56	.005102	15.54816	17.69400	−85.70103	353.99321	2047.54758
57	.005637	15.20767	18.05602	−83.33796	299.06649	2583.06592
58	.006265	14.85974	18.39538	−80.61029	243.36396	3048.99973
59	.006997	14.50501	18.70376	−77.46646	187.18191	3434.25812
60	.007837	14.14417	18.97273	−73.86425	130.95537	3728.47366
61	.008788	13.77792	19.19529	−69.78531	75.30632	3923.61887
62	.009848	13.40686	19.36611	−65.23496	20.96967	4015.32724
63	.011010	13.03146	19.48234	−60.24673	− 31.25667	4004.11072
64	.012264	12.65202	19.54427	−54.88425	− 80.57331	3895.98517
65	.013597	12.26856	19.55561	−49.23932	−126.23945	3702.19638
66	.014991	11.88085	19.52362	−43.43022	−167.60296	3438.26151
67	.016457	11.48831	19.45960	−37.60219	−204.09346	3122.59036
68	.018198	11.09035	19.37553	−31.90440	−235.21291	2772.91068
69	.020354	10.68848	19.26601	−26.38293	−260.47673	2396.58002
70	.023098	10.28536	19.11785	−21.05553	−279.33721	1999.10040
71	.026527	9.88510	18.90974	−15.91710	−291.19043	1585.71300
72	.030468	9.49231	18.61993	−10.97341	−295.60568	1165.83012
73	.034779	9.10955	18.24385	− 6.29353	−292.84920	757.58972
74	.039413	8.73783	17.78819	− 1.96334	−283.77428	380.13618
75	.044309	8.37753	17.26299	1.94762	−269.51623	48.27515
76	.049512	8.02836	16.68142	5.38666	−251.36252	− 228.35450
77	.055108	7.69025	16.05434	8.32462	−230.48656	− 445.84764
78	.061093	7.36366	15.38896	10.75435	−207.87924	− 604.88285
79	.067459	7.04884	14.69227	12.68552	−184.44896	− 709.37428
80	.074146	6.74586	13.97088	14.14327	−160.99192	− 765.63107
81	.081114	6.45425	13.23231	15.16523	−138.20799	− 781.48485
82	.088374	6.17315	12.48439	15.79687	−116.66053	− 765.35629
83	.095943	5.90162	11.73412	16.08662	− 96.75112	− 725.40481
84	.103904	5.63868	10.98804	16.08240	− 78.74149	− 669.02090
85	.112328	5.38361	10.25125	15.82866	− 62.75749	− 602.46470
86	.121295	5.13585	9.52794	15.36649	− 48.82775	− 530.88194
87	.130885	4.89503	8.82158	14.73354	− 36.90669	− 458.34547
88	.141188	4.66087	8.13501	13.96432	− 26.89454	− 387.96048
89	.152300	4.43322	7.47052	13.09034	− 18.65160	− 321.97723
90	.164331	4.21204	6.82981	12.14081	− 12.01120	− 261.92448
91	.177144	3.99743	6.21386	11.14360	− 6.79227	− 208.73491
92	.191099	3.78842	5.62540	10.12729	− 2.82595	− 162.94093
93	.206341	3.58505	5.06573	9.11145	.06332	− 124.44318
94	.223029	3.38742	4.53590	8.11300	2.04672	− 92.84777
95	.241336	3.19564	4.03670	7.14632	3.28708	− 67.55324
96	.261451	3.00985	3.56868	6.22334	3.93584	− 47.82673
97	.283581	2.83020	3.13207	5.35360	4.13061	− 32.86880
98	.307953	2.65684	2.72685	4.54430	3.99369	− 21.86701
99	.334812	2.48987	2.35264	3.80041	3.63122	− 14.03892
100	.364429	2.32937	2.00874	3.12489	3.13333	− 8.66512
101	.397100	2.17528	1.69400	2.51897	2.57482	− 5.11333
102	.433150	2.02735	1.40689	1.98287	2.01625	− 2.85635
103	.472930	1.88488	1.14558	1.51677	1.50370	− 1.48412
104	.518156	1.74603	.90905	1.12140	1.06525	− .70132
105	.570545	1.61021	.69710	.79450	.71010	− .29463
106	.631813	1.47773	.50978	.53185	.43732	− .11062
107	.703676	1.34942	.34707	.32789	.24048	− .04275
108	.787851	1.22634	.20856	.17609	.10995	− .02286
109	.886054	1.10956	.09335	.06930	.03402	− .01355
110	1.000000	1.00000	.00000	.00000	.00000	.00000

of the individual annuity costs at age 65 on this basis, and they were found to be

$$\chi_1(C) = 11.4960\,,$$

$$\chi_2(C) = 31.3938\,,$$

$$\chi_3(C) = -\ 40.7438\,,$$

$$\chi_4(C) = -\ 803.695\,,$$

$$\chi_5(C) = 6949.52\,.$$

The portfolio of lives in the Boermeester example consisted of nine lives, each age 65, with a yearly income of 1, and a tenth life, also age 65, with income I_{10}. The cumulants for the portfolio using Properties 1 and 2 in the Appendix are given by

$$\chi_k(Z) = (9 + I_{10}^k)\chi_k(C)\,.$$

From these the mean, standard deviation, and correction coefficients γ_1, γ_2, and γ_3 may be calculated. Table 1 gives the results of these calculations for the case $I_{10} = 10$.

We shall restrict ourselves to three values of ϵ, $\epsilon = 0.01$, $\epsilon = 0.05$, and $\epsilon = 0.10$, and shall evaluate the various coefficients which appear in the Cornish-Fisher expansion and which depend only on ϵ (Table 2).

Table 3 compares the results of the Cornish-Fisher method with those of several other methods. Most of the figures are from Fretwell and Hickman

TABLE 1

Mean	218.424
Standard deviation	58.4972
γ_1	− .2054
γ_2	− .6870
γ_3	1.0147

TABLE 2

USEFUL CONSTANTS FOR CORNISH-FISHER EXPANSION

	$\epsilon = .01$	$\epsilon = .05$	$\epsilon = .10$
y_ϵ	2.3263	1.6449	1.2816
$(y_\epsilon^2 - 1) \div 6$	.7353	.2843	.1071
$(y_\epsilon^3 - 3y_\epsilon) \div 24$	.2338	−.0202	−.0725
$-(2y_\epsilon^3 - 5y_\epsilon) \div 36$	−.3763	−.0188	.0611
$(y_\epsilon^4 - 6y_\epsilon^2 + 3) \div 120$	−.0015	−.0493	−.0346
$-(y_\epsilon^4 - 5y_\epsilon^2 + 2) \div 24$	−.1762	.1753	.1464
$(12y_\epsilon^4 - 53y_\epsilon^2 + 17) \div 324$	.2519	−.1190	−.1163

[4, p. 56], except those in the columns labeled "Rounding Method," which are from Boermeester [2, p. D311].

TABLE 3

REQUIRED RESERVE PER UNIT OF ANNUITY INCOME

	Rounding Method		Normal Distribution			Pearson Type III			Cornish-Fisher		
I_{10}/ϵ	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
1...	14.4	13.8	15.62	14.42	13.76	15.52	14.38	13.76	15.48	14.38	13.76
2...	14.5	13.9	15.77	14.53	13.85	15.66	14.49	13.84	15.60	14.47	13.85
5...	15.2	14.5	16.92	15.35	14.48	16.65	15.25	14.45	16.36	15.21	14.50
10...	16.2	15.4	18.66	16.58	15.44	18.19	16.39	15.38	17.56	16.35	15.50
25...	17.7	16.7	21.15	18.34	16.81	20.45	18.06	16.71	19.45	18.00	16.91
50...	18.6	17.5	22.56	19.35	17.58	21.75	19.02	17.48	20.58	18.95	17.70

As another illustration, let us consider a pension plan with ten retired lives aged 65, 67, 69, 71, 73, 75, 77, 80, 84, and 92. Each of the retired lives receives a monthly income of $100 except the annuitant aged 80, who receives $200 a month. We shall calculate the "required reserve" on a valuation basis of the 1951 Group Annuity Table Male at 4 per cent interest. Using a monetary unit of $100 per month, we construct the following worksheet:

CONTRIBUTION TO PORTFOLIO CUMULANT

Age	Income	First	Second	Third	Fourth	Fifth
65	1	10.70	20.85	−29.39	− 323.80	3150.18
67	1	9.99	20.08	−18.15	− 327.46	2053.67
69	1	9.29	19.07	− 7.67	− 308.27	1029.46
71	1	8.60	17.94	1.41	− 274.52	202.89
73	1	7.93	16.70	8.87	− 230.34	− 396.95
75	1	7.28	15.38	14.62	− 181.95	− 767.05
77	1	6.66	14.05	18.57	− 135.38	− 931.28
80	2	11.63	48.21	169.12	−1206.07	−28396.52
84	1	4.89	9.48	19.72	− 22.07	− 582.71
92	1	3.48	5.19	11.04	9.26	− 122.15
Totals	11	80.45	186.95	188.14	−3000.60	−24760.46

Mean $80.45-11(m-1)/2m=75.41$ for $m=12$

Standard deviation.........	13.67
γ_1......................	.0736
γ_2......................	− .0859
γ_3......................	− .0518

REQUIRED RESERVE

ϵ	x_ϵ	Units	Dollars
0.01	2.360	107.67	$129,200
0.05	1.669	98.23	117,900
0.10	1.297	93.14	111,800

These required reserves should be compared with the classical actuarial reserve of $90,500, which is the expected value of future annuity payments.

In conclusion, I would like to stress that the method used in this paper can be applied to other problems. In any problem concerning the distribution of a random variable where the cumulants can be calculated, either by summing cumulants for the individual costs as done in this paper or directly as in collective risk models, it will be possible to use the Cornish-Fisher expansion. Some problems that come to mind are (1) the distribution of one year costs on a group life contract or reinsurance contract; (2) the distribution of the present value of all future claims of an insurance portfolio.

I would like to take this opportunity to thank my colleague, Professor D. A. Jones, who made many helpful suggestions during the preparation of this paper and who, moreover, did the programming so that the tables of cumulants could be presented with the paper.

APPENDIX

CUMULANTS

The following is a brief summary of information on cumulants used in the main body of the paper. Additional detail may be found in Kendall [6, pp. 60–68].

We recall that the moment generating function for a random variable X is defined by

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} dF(x).$$

This function has two nice properties. The first is that the moments of X about the origin, μ_k', are given by

$$\mu_k' = M_X^{(k)}(0),$$

where $M_X^k(0)$ is the kth derivative of $M_X(t)$ evaluated at $t = 0$. The second nice property is that if $X_1, X_2, \ldots, X_n$ are independent random variables, the moment generating function of $Z = X_1 + X_2 + \ldots + X_n$ is given by

$$M_Z(t) = M_{X_1}(t) \cdot M_{X_2}(t) . \;\ldots\; . M_{X_n}(t),$$

assuming, of course, that all the $M_{X_i}(t)$ exist. Thus the moments of Z may be found in terms of the moments of the X_i. Unfortunately, the formulas are not handy. This is because the derivatives of products of the various $M_{X_i}(t)$ functions do not have simple forms.

These properties of the moment generating function motivate the definition of cumulants.

Definition: If X has a moment generating function, $M_X(t)$, then by the cumulant generating function of X we will mean the function

$$C_X(t) = \ln M_X(t) .$$

Further, the kth cumulant of X, denoted by $\chi_k(X)$, is given by

$$\chi_k(X) = C_X^{(k)}(0) .$$

Example: Let X be normal with mean μ and variance σ^2. Then its moment generating function is

$$M_X(t) = \exp\{\mu t + \tfrac{1}{2}\sigma^2 t^2\} .$$

Therefore, the cumulant generating function is

$$C_X(t) = \mu t + \tfrac{1}{2}\sigma^2 t^2 .$$

It is easily verified that $\chi_1(X) = \mu$, $\chi_2(X) = \sigma^2$, and $\chi_k(X) = 0$ for $k \geq 3$.

Property 1: If X is a random variable with a moment generating function and $Y = aX + b$, for constants a and b, then

$$\chi_1(Y) = a\chi_1(X) + b \quad \text{and}$$

$$\chi_k(Y) = a^k\chi_k(X) \quad \text{for} \quad k \geq 2 .$$

Proof:

$$C_Y(t) = \ln M_Y(t) = \ln E[e^{tY}] = \ln E[e^{atX+bt}] = \ln (e^{bt}\cdot E[e^{atX}])$$
$$= \ln (e^{bt}\cdot M_X(at)) = bt + C_X(at).$$

Thus, $C_Y^{(1)}(t) = b + aC_X^{(1)}(at)$, so, by setting $t = 0$, we have

$$\chi_1(Y) = b + a\chi_1(X) .$$

Further, $C_Y^{(k)}(t) = a^k C_X^{(k)}(at)$ for $k \geq 2$, which, when t is set equal to zero, gives

$$\chi_k(Y) = a^k\chi_k(X) \quad \text{for} \quad k \geq 2 .$$

Property 2: If $X_1, X_2, \ldots, X_n$ are independent random variables, each with a moment generating function, and $Z = X_1 + X_2 + \ldots + X_n$, then

$$\chi_k(Z) = \sum_{i=1}^{n} \chi_k(X_i).$$

Proof: The hypothesis implies that

$$M_Z(t) = M_{X_1}(t)\cdot M_{X_2}(t)\cdot \ldots \cdot M_{X_n}(t).$$

Therefore

$$C_Z(t) = \ln M_Z(t) = \ln M_{X_1}(t) + \ln M_{X_2}(t) + \ldots + \ln M_{X_n}(t)$$

so

$$C_Z(t) = C_{X_1}(t) + C_{X_2}(t) + \ldots + C_{X_n}(t).$$

If we now take the kth derivative of both sides of this last equation and set $t = 0$, we have Property 2.

Formulas relating cumulants to moments about the origin can be found by writing the Maclaurin expansions of each of the functions $M(t)$ and $C(t)$. Thus

$$M(t) = 1 + \mu t' + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \ldots .$$

$$C(t) = \chi_1 t + \chi_2 \frac{t^2}{2!} + \chi_3 \frac{t^3}{3!} + \ldots .$$

Further, we can write

$$C(t) = \ln M(t) = \ln\left[1 + \left(\mu t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \ldots\right)\right].$$

The expression on the right can be written as a power series in t by recalling that

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots .$$

Thus, we have two power series expansions for $C(t)$, one with coefficients in terms of the moments about the origin. Equating the coefficients of the various powers of t in these two expressions gives the following formulas for the first five cumulants.

$$\begin{aligned} \chi_1 &= \mu \\ \chi_2 &= \mu_2' - (\mu)^2 \\ \chi_3 &= \mu_3' - 3\mu_2'\mu + 2(\mu)^3 \\ \chi_4 &= \mu_4' - 4\mu_3'\mu + 12\mu_2'(\mu)^2 - 3(\mu_2')^2 - 6(\mu)^4 \\ \chi_5 &= \mu_5' - 5\mu_4'\mu + 20\mu_3'(\mu)^2 - 60\mu_2'(\mu)^3 \\ &\quad + 30\mu(\mu_2')^2 - 10\mu_2'\mu_3' + 24(\mu)^5 . \end{aligned} \tag{10}$$

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