TRANSACTIONS OF SOCIETY OF ACTUARIES
1968 VOL. 20 PT. 1 NO. 56 AB

VOL. XX, PART I MEETINGS No. 56A and 56B

TRANSACTIONS

APRIL, 1968

ANNUITY APPROXIMATIONS

SIMPLIFIED METHODS OF CALCULATING INDIVIDUAL AND GROUPED RESERVES FOR (a) DEATH BENEFITS AND (b) TEMPORARY LIFE ANNUITIES

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GENERAL

The Tail Wags the Dog

Even in these days of computers it will often be found that in the valuation-statistical area of group annuities some 80 per cent of the data are simple to handle by orthodox methods; the other 20 per cent may cause as much as 90 per cent of the work. The tail really does wag the dog. Perhaps group annuities is not alone in that respect. This is particularly true of the after-retirement death benefits. Those of the cash refund type are the most troublesome to value with any degree of precision, mainly because of the fractional terms involved. But "certain and life" annuities can also be troublesome, since for valuation they require two-dimension tabulation, some groups of which may include only very small amounts of data.

Both types of death benefit (as distinct from the annuity benefit) usually represent a small part of the whole liability. In cash refund cases the death benefits seldom represent more than 3 per cent; for certain and life they might rise to something like 10 per cent, depending upon the term certain. These percentages diminish, of course, with duration. It does not seem right that death benefits should cause so much work.

Approximations Inevitable but Differences Not Necessarily Serious

If the situation is to be improved, it does not seem likely that some new variations of orthodox actuarial processes will be forthcoming which would produce exact results; some approximations are inevitable. The "errors" of such approximations are not necessarily serious; they relate only to a small part of the total liability. If the death benefit represents

3 per cent of the whole value, for example, an error of, say, 5 per cent in the value of the death benefit is only 5 per cent of 3 per cent, or 1.5 per 1,000 of the total liability.

How Exact Is "Exact"?

One could go to the extreme and say that no calculation relating to the future is exact; it can only be an estimate. Actuaries have with justification become accustomed to treating as exact any calculation produced by orthodox formulas on a given mortality table. Practical considerations, however, have forced us to stretch that conception of exactness just a little. I have no fault to find with that. In the section entitled "Examples of Inexactness Commonly Tolerated," I give a few examples, not for the purpose of showing how inexact some "orthodox" calculations are but, rather, to demonstrate a relatively higher general level of accuracy in the approximations that I propose to develop.

Basis of Illustrative Calculations

Except where otherwise indicated all calculations in this study are based on the Prudential's Beta mortality table (the Prudential 1950 Group Annuity Valuation Table with ages rated down half a year). All annuities are assumed to be payable monthly in advance. It is the Prudential's practice to make allowance for generation mortality by a system of age ratings to the Beta table; the principles illustrated can be applied to either a generation or a "static" mortality foundation. There is no reason to believe that the principles I propose to demonstrate could not be applied with equal success to any other modern mortality table for retired lives.

Examples of Inexactness Commonly Tolerated

a) We all know how difficult it is to value data involving many combinations of joint ages. So the law of Gompertz is brought to the rescue—whether or not it fits the underlying mortality table. We value the joint and survivorship annuity $a_{\overline{xy}}$ as $a_x + a_y - a_z$, where z is derived from a rather imaginary table of uniform seniority.

Some comparative values are shown in Table 1. Where substantial cash refund benefits attach to survivorship annuities, the additional conservatism inherent in the calculation of the death benefit reserve, in conjunction with the above, would lead to an assumed value of about 102.5 per cent of the true one.

b) A cash refund death benefit can be regarded as commencing at n and diminishing continuously by 1 per annum until it expires at the end of n

years. If the applicable rates of mortality were such that the C function of the commutation tables remained constant at all ages over the n years, i.e., that corresponding to age s, the value of the benefit could be regarded to all intents and purposes as $(n^2/2)\ (C_s/D_x)$, where x is the entry age.

The C function, of course, is not constant; it varies, and it is quite troublesome in practice to make proper allowance for the different values. For the purposes of group valuation, it is common to assume that at all relevant ages the maximum value of C applies. Other expedients are to use the value of C for a few years older than retirement age—for example, $x + 3$ or $x + 5$. The most common retirement age is (male) 65, and cash refund death benefits on valuation data seem for the most part to expire within two years. After proper allowance is made for the diminishing

TABLE 1

VALUE OF IMMEDIATE ANNUITY OF 1 PER ANNUM*

Annuity and Valuation Ages	True (1)	Approximate (2)	(2)÷(1) (3)
Life—M65; F65	15.250	15.532	1.018
Life—M65; F60	16.399	16.628	1.014
10 years certain and life—M65; F65	15.389	15.527	1.009
10 years certain and life—M65; F60	16.520	16.618	1.006

* Calculations are based on the Prudential Beta Table, 3½ per cent, and all annuities are assumed payable monthly in advance.

"weights" of the death benefit, we find that at 3 per cent the "maximum C" expedient overstates the value by more than 20 per cent and the "$x + 3$" expedient by about 11 per cent. In the final result these death benefit errors would mean that the total (annuity plus death benefit) reserve is too large by about 2 per cent or 1.1 per cent.

c) In the relatively common case of a certain and life annuity, the valuation practices are not as exact as they might appear. On a valuation date of December 31, the terms certain expiring in any year are assumed to expire in the middle of that year. We thus require valuation factors catering for terms certain of ½, 1½, 2½ years, and so on. The factor for term $(n - \frac{1}{2})$ is usually found by taking the arithmetic mean of those at n and $(n - 1)$. As, however, the value of the death benefit depends very much on n^2, and not on n, a first-difference interpolation is inappropriate. The value is overstated by 1.23 per cent if $n = 5.0$ and by 4 per cent if $n = 3.0$.

d) When individual values are being calculated—for example, purchase rates for cash refund benefits—it is troublesome to work with fractional values; the duration is sometimes taken to the next-higher integer. If, for example, the true value is $(n - \frac{1}{2})$ and we use n, we are overvaluing the death benefit by more than 20 per cent if $n = 5.0$ and by more than 40 per cent if $n = 3.0$.

INDIVIDUAL CALCULATIONS

The "Crucial Function"

First of all, let us take a look at factors which have to be calculated individually, for example, purchase rates. We may learn something from them which will simplify the problems involved in group valuation. Cash refund annuities are not very different from certain and life; the corpus of

TABLE 2

VALUES OF CRUCIAL FUNCTION AT $3\frac{1}{2}$ PER CENT
COMPARISON OF FORMULA WITH ORTHODOX VALUES

Age	n = 5		n = 10	
	Formula	Orthodox	Formula	Orthodox
50	0.0802	0.0763	0.3206	0.3209
55	.1205	.1206	0.4821	0.4889
60	.1812	.1778	0.7249	0.7210
65	.2725	.2729	1.0900	1.0903
70	.4097	.4181	1.6389	1.5888
75	0.6161	0.6022	2.4644	2.1962

the benefits is identical for the same term. The only difference is due to interest—cash refund benefits are paid in one sum on death, whereas the others take the form of an annuity certain. First of all, the value of the death benefit has been examined, that is, the value of all benefits less the value of the corresponding life annuity. In the case of the certain and life annuity, this reduces to $\ddot{a}_{\overline{n}|}^{(12)} - \ddot{a}_{x:\overline{n}|}^{(12)}$, that is, the excess of an annuity certain over a temporary life annuity for the same term. We may refer to this as the "Crucial Function."

It has been found convenient to examine the value of the death benefit closely at a single valuation interest rate and then to see what different relationships hold when other interest rates are involved. My primary calculations, for both certain and life and cash refund, have been based on $3\frac{1}{2}$ per cent. As Table 2 shows, the value of the Crucial Function at $3\frac{1}{2}$ per cent—here designated ${}_{0.035}K_{x:n}$—can be represented closely by the formula $0.00005426n^2(1.085)^x$.

Applicability of the Above and Later Formulas

In selecting the elements of the suggested formulas, care has been taken to get a good fit where the coverage is between 60 and 70—the really important ages for death benefits. My own inclination would be to restrict the application of the methods to interest rates between 2 per cent and $5\frac{1}{2}$ per cent and to cases in which x does not exceed 75 and $x + n$ does not exceed 80. The approximations owe much to the fact that for most ages C_x is an increasing quantity. Once it begins to diminish, it becomes more difficult to devise a method which combines accuracy with simplicity. But, even in the extreme case of 75 at entry and a ten-year term, the

TABLE 3

CERTAIN AND LIFE ADJUSTMENT FACTORS (CLAF)*
OTHER INTEREST RATES

i LESS THAN 0.035		*i* GREATER THAN 0.035			
Z	CLAF	Z	CLAF	Z	CLAF
0.01	1.0065	0.01	0.99354	0.11	0.93123
.02	1.0130	.02	.98712	.12	.92522
.03	1.0196	.03	.98075	.13	.91924
.04	1.0262	.04	.97442	.14	.91330
.05	1.0329	.05	.96813	.15	.90740
.06	1.0396	.06	.96188	.16	.90154
.07	1.0464	.07	.95567	.17	.89572
.08	1.0532	.08	.94950	.18	.88994
.09	1.0600	.09	.94337	.19	.88419
0.10	1.0669	0.10	0.93728	0.20	0.87848

* The CLAF is found by entering the above table for Z, where Z is the product of n and the difference between i and 0.035.

formula produces a value which is only 12.2 per cent more than the true one. When added to the age-75 life annuity value of 7.8713, the formula result becomes 102.7 per cent of the correct one—an error which is far from outrageous when judged by the standards illustrated in the section entitled "Examples of Inexactness Commonly Tolerated."

Certain and Life Annuities: Other Rates of Interest

To arrive at the value of the certain and life death benefit for an interest rate other than $3\frac{1}{2}$ per cent, it is only necessary to multiply the $3\frac{1}{2}$ per cent Crucial Function by the certain and life adjustment factor (CLAF) shown in Table 3. The given values of the CLAF form a geometric series with a common ratio of 1.0065.

As will be seen from Table 4, the formula values give, for all practical purposes, a close approximation to the original.

There is no reason to believe that the formula method is not capable of general application. Table 5 shows some examples of how the formula value compares with the orthodox for intermediate ages, terms, and interest rates.

TABLE 4

COMPARISON OF FORMULA WITH ORTHODOX VALUES
CERTAIN AND LIFE ANNUITIES—INTEREST RATES OTHER THAN 3½ PER CENT

AGE	2½ PER CENT		4½ PER CENT		5½ PER CENT	
	Formula	Orthodox	Formula	Orthodox	Formula	Orthodox
	$n=5$					
50..........	0.083	0.079	0.078	0.074	0.075	0.071
55..........	.124	.125	.117	.116	.113	.112
60..........	.187	.184	.175	.172	.170	.166
65..........	.281	.282	.264	.264	.255	.255
70..........	.423	.432	.397	.405	.384	.392
75..........	0.636	0.622	0.596	0.583	0.577	0.565
	$n=10$					
50..........	0.342	0.343	0.300	0.300	0.282	0.281
55..........	0.514	0.522	0.452	0.458	0.424	0.430
60..........	0.773	0.770	0.679	0.676	0.637	0.634
65..........	1.163	1.164	1.022	1.022	0.958	0.960
70..........	1.749	1.694	1.536	1.492	1.440	1.402
75..........	2.629	2.340	2.310	2.063	2.165	1.941

TABLE 5

Age	n	i	Formula	Orthodox
63...........	2	0.0275	0.037	0.036
63...........	8	.0300	0.608	0.606
63...........	8	.0500	0.548	0.546
59...........	6	.0400	0.236	0.233
59...........	1	.0275	0.007	0.005
68...........	10	.0550	1.223	1.214
68...........	2	.0275	0.056	0.057
73...........	3	.0300	0.190	0.193
73...........	2	0.0500	0.082	0.083

Individual Values for Cash Refund Cases

It would be possible to proceed exactly as for certain and life cases, that is, compile a table of 3½ per cent values and then adjust for other rates of interest. But we do not really need another foundation table like ${}_{0.035}K_{x:n}$. The cash refund values bear a close resemblance to the certain and life ones. We can use the same 3½ per cent Crucial Function and develop new adjustment factors (CRAF) to adjust to cash refund and to the required interest basis.

TABLE 6

CASH REFUND ANNUITIES

AGE	2½ PER CENT		3½ PER CENT		4½ PER CENT		5½ PER CENT	
	Formula*	Orthodox	Formula*	Orthodox	Formula*	Orthodox	Formula*	Orthodox
	$n=5$							
55.....	0.132	0.133	0.130	0.130	0.128	0.128	0.126	0.126
60.....	.199	.195	.196	.192	.193	.189	.189	.186
65.....	.299	.299	.294	.294	.290	.290	.285	.285
70.....	0.450	0.458	0.443	0.451	0.435	0.443	0.428	0.436
	$n=10$							
55.....	0.561	0.570	0.544	0.551	0.527	0.533	0.510	0.516
60.....	0.844	0.839	0.818	0.812	0.793	0.785	0.767	0.759
65.....	1.269	1.270	1.230	1.228	1.192	1.188	1.154	1.150
70.....	1.908	1.852	1.850	1.792	1.792	1.737	1.735	1.684

* Formula value: Crucial Function multiplied by CRAF.

The following formula for CRAF has been found to give satisfactory results in practice:

$$\text{CRAF} = 1.0320 + 0.022n(1 - 16i) .$$

In Table 6 values calculated by CRAF are compared with orthodox (exact) ones. Table 7 shows some further examples of how the formula value compares with the orthodox for intermediate ages, terms, and interest rates.

Practical Use of Formula Methods, as Illustrated

In theory, practical approximations like these should not be necessary; computers are supposed to turn out complete sets of actuarial functions at all required interest rates. It does not always work out that way; the conditions now prevailing in the group annuity business are apt to get us involved with fractional ages and a remarkable variety of interest rates. Formulas of the type shown are not difficult to construct; apart from the greater accuracy that they give for fractional terms, they should help to cut down on the drudgery of making laborious calculations for small amounts of benefit.

VALUATION IN GROUPS

Reduction in Number of Valuation Groups

The business of group annuities differs from that of other branches in that separate valuations may be needed for dividend purposes for, perhaps, some hundreds of individual contracts. An elaborate two-dimension

TABLE 7

Age	n	i	Formula	Orthodox
57	7	0.0275	0.311	0.311
57	7	.0500	.296	.295
62	7	.0275	.468	.461
62	7	.0500	.444	.437
68	7	.0275	.763	.771
68	7	.0500	.725	.733
62	3	.0275	.082	.080
62	3	.0500	.080	.078
68	3	.0275	.134	.137
68	3	0.0500	0.131	0.134

tabulation—for example, for certain and life annuities—may not be out of place for the entire annual statement valuation. But, when the figures are broken up by contract, the work of preparing, proving, and valuing the statistics is apt to become out of proportion to the amounts involved. It would be a big help if we could devise a means whereby we could cut down on separate tabulations for all combinations of age and term. Even in a computerized operation this could be a saving. The system described below achieves this and produces an accurate result, proceeding from the main file to the valuation summaries without any intermediate calculation.

Adaptation of Crucial Function to Valuation Purposes

Let us think, for the present, in terms of certain and life annuities. In a 1967 valuation, for example, a life may be age 63 and n, the unexpired term, 7 years. The Crucial Function gives the value of this death benefit as $(7)^2[K(1+r)^{63}]$. A year later the age is 64, and the term is down to six years; the formula gives $(6)^2[K(1 + r)^{64}]$. We can simplify the valuation work by proceeding as follows:

Let x equal valuation year *minus* birth year. Then $n^2K(1 + r)^x = [n^2K(1 + r)^{2000-\text{birth year}}] \div [(1 + r)^{2000-\text{valuation year}}]$.

The 1967 reserve is thus $(7)^2[K(1 + r)^{96}] \div (1 + r)^{33} = (7)^2 [K(1 + r)^{63}]$.

The 1968 reserve then becomes $(6)^2[K(1 + r)^{96}] \div (1 + r)^{32} = (6)^2[K(1 + r)^{64}]$.

It will be noted that part of the expression, $K(1 + r)^{96}$, is common to both valuation years—and, of course, to all lives born in 1904 in all valuation years so long as the death benefit lasts. This means that we can multiply it by the amount of annuity and record the result as a valuation constant at the commencement of the annuity.

The correcting adjustment depends only on the valuation year and is common to all valuation ages; it can, therefore, be applied at the end to the total valuation results. An illustrative example showing the same cases in two valuations is given in Appendix I. It may look rather formidable, but the same format could cover many valuation ages and terms instead of the few shown.

The method is continuous in form; it is basically simple. It is only necessary to record at entry a constant quantity which is the product of the annuity and $K(1 + r)^{2000-\text{birth year}}$. At all subsequent valuations, then, such constants can be grouped by year of expiry—that is, independently of age—and the reserves found from a single-dimension tabulation.

The numbers used in the illustration are quite imaginary; it is interesting to note that the "errors" are of the order of 4 in 1,000 of the death benefit reserve or 1 in 10,000 of the total reserve.

Cash refund annuities would be valued in exactly the same way, using their own Crucial Function. As such benefits are usually small, it might be satisfactory to make some arbitrary adjustment and use, say, 110 per cent of the Crucial Function designed to fit certain and life annuities.

Another Use for the Crucial Function

The facilities of the Crucial Function can also be used to cut down on the two-dimension tabulations required for the valuation of temporary annuities. Reverting to the basic formula, $Kn^2(1 + r)^x = \ddot{a}_{\overline{n}|}^{(12)} - \ddot{a}_{x:\overline{n}|}^{(12)}$, so that $\ddot{a}_{x:\overline{n}|}^{(12)} = \ddot{a}_{\overline{n}|}^{(12)} - Kn^2(1 + r)^x$.

The value of a temporary annuity of S is thus $S\ddot{a}_{\overline{n}|}^{(12)} - SKn^2(1 + r)^x$. As we did above for installment refund annuities, we can bring the latter part of the expression into a form more convenient for group valuation: $SKn^2(1 + r)^x = (n)^2[SK(1 + r)^{2000-\text{birth year}}] \div [(1 + r)^{2000-\text{valuation year}}]$. We therefore record at entry a constant of $SK(1 + r)^{2000-\text{birth year}}$ and make each year a single adjustment, applicable to all ages, to produce the result. The procedure is illustrated in Appendix II. Again, the results are exact enough for all practical purposes.

It may be convenient to use for temporary annuities the same Crucial Function values as for certain and life death benefits. That is legitimate if the "fit" is a good one. But the Crucial Function element in the certain and life case is an *addition* to the reserve; in temporary annuities it serves, on the other hand, to *reduce* the reserve. Any conservatism built into the Crucial Function values for certain and life benefits would operate adversely if the same values are used for temporary annuities.

CONCLUSION

General Observations

The methods described above are unorthodox in the sense that the results are not exactly what would be produced by applying actuarial processes to the basic rates of mortality. There does not seem to be any reason, however, why derived actuarial functions such as annuity or reserve values should not themselves be graduated so long as (*a*) there are practical advantages to be gained by doing so and (*b*) the results remain close to the original. The values of the Crucial Function were found by trial and error; the work was not oppressive. For the basis illustrated above, a few trials showed that a suitable value for $(1 + r)$ would be about 1.085. That particular quantity was chosen for convenience because its powers were available from compound interest tables; some other quantity might have led to a slightly better over-all fit. There is very little difference between monthly and continuous annuities, to which the n^2 element is more directly appropriate. A more nearly correct monthly equivalent would have been much more difficult to work with, there would have been compensating changes in the Crucial Function, and little accuracy would have been gained in the final results.

APPENDIX I

SPECIMEN STATISTICS AND VALUATION INSTALLMENT REFUND ANNUITIES

Assumptions

Valuation age = valuation year *minus* year of birth

Outstanding term = n = expiry year *minus* valuation year

Valuation basis = Beta table at 3½ per cent interest

Crucial Function = $0.00005426\, n^2(1.085)^x$

$E = 0.00005426(1.085)^{2000-\text{year of birth}}$

$G = (1.085)^{2000-\text{valuation year }(y)}$

$G\ 1967 = (1.085)^{33} = 14.76323$

$G\ 1968 = (1.085)^{32} = 13.60666$

STATISTICS

LINE		CASE NO. 1	2	3	4	5	6	TOTAL
1....	Year of birth (male)	1900	1900	1900	1905	1905	1905	....
2....	S=amount of annuity	100	150	200	200	150	100	900
3....	Year of expiry	1971	1972	1973	1971	1972	1973	....
4....	$S \times E$..........	18.943	28.415	37.886	25.196	18.897	12.598	141.935

VALUATIONS

Year of Birth (5)	Valuation Age x (6)	Annuity S (7)	$\ddot{a}_x^{(12)}$ (8)	(7)×(8) (9)	Valuation Year (y)	Year of Expiry (10)	n (11)	n^2 (12)	Σ SE (13)	(12)×(13) (14)	(14) ÷ Gy (15)
1900..	67	450	10.5479	4,746.56	1967	1971	4	16	44.139	706.22	......
1905..	62	450	12.3707	5,566.82	1967	1972	5	25	47.312	1,182.80	......
						1973	6	36	50.484	1,817.42	......
Total	....	900	....	10,313.38	.....	.....	....	....	141.935	3,706.44	251.06
1900..	68	450	10.1935	4,587.08	1968	1971	3	9	44.139	397.25	......
1905..	63	450	12.0045	5,402.02	1968	1972	4	16	47.312	756.99	......
						1973	5	25	50.484	1,262.10	......
Total	....	900	....	9,989.10	.....	.....	....	....	141.935	2,416.34	177.59

VALUATION SUMMARIES

	1967	1968
Life annuity reserve................................	10,313.38	9,989.10
Death benefit reserve................................	251.06	177.59
Total reserve, formula method.................	10,564.44	10,166.69
Reserve calculated by orthodox formulas and processes..	10,565.44	10,167.99

APPENDIX II

SPECIMEN STATISTICS AND VALUATION TEMPORARY LIFE ANNUITIES

Assumptions

Valuation age = valuation year *minus* year of birth
Outstanding term = n = expiry year *minus* valuation year
Valuation basis = Beta table at $3\frac{1}{2}$ per cent interest
Crucial Function = $0.00005426\, n^2(1.085)^x$
$E = 0.00005426(1.085)^{2000-\text{year of birth}}$
$G = (1.085)^{2000-\text{valuation year }(y)}$
$G\ 1967 = (1.085)^{33} = 14.76323$
$G\ 1968 = (1.085)^{32} = 13.60666$

STATISTICS

LINE		CASE NO.						TOTAL
		1	2	3	4	5	6	
1....	Year of birth (male)	1910	1910	1910	1905	1905	1905	....
2....	S=amount of annuity	100	150	200	200	150	100	900
3....	Year of expiry	1973	1974	1975	1973	1974	1975	....
4....	$S\times E$	8.378	12.567	16.757	25.196	18.897	12.598	94.393

VALUATIONS

Year of Expiry (5)	n (6)	Annuity S (7)	$a_{\overline{n}\vert}^{(12)}$ (8)	(7)×(8) (9)	Valuation Year (y)	Σ SE (10)	n^2 (11)	(10)×(11) (12)	(12)÷Gy (13)
1973......	6	300	5.4290	1,628.70	1967	33.574	36	1,208.66	....
1974......	7	300	6.2299	1,868.97	1967	31.464	49	1,541.74	....
1975......	8	300	7.0036	2,101.08	1967	29.355	64	1,878.72	....
Total....	.....	900	....	5,598.75	......	94.393	....	4,629.12	313.56
1973......	5	300	4.6002	1,380.06	1968	33.574	25	839.35	....
1974......	6	300	5.4290	1,628.70	1968	31.464	36	1,132.70	....
1975......	7	300	6.2299	1,868.97	1968	29.355	49	1,438.40	....
Total....	.....	900	....	4,877.73	......	94.393	....	3,410.45	250.65

VALUATION SUMMARIES

	1967	1968
Annuity certain portion............................	5,598.75	4,877.73
Less deduction for death risk........................	313.56	250.65
Formula reserve, as above.......................	5,285.19	4,627.08
Reserve calculated by orthodox formulas and processes..	5,287.10	4,629.25