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COLLECTIVE RISK RESULTS

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I. INTRODUCTION

THIS paper will derive some formulas for obtaining the ruin function and the distribution of total claims of collective risk theory. These results will be illustrated by five examples, of which two show the use of the ruin function in studying retention levels, two relate to the size of "acceptable" adverse fluctuation, and one involves the calculations of net premiums for stop-loss reinsurance treaties. Although these results can be expressed simply, their derivations are based on the mathematical theory of stochastic processes, so a brief discussion of this subject is included. The derivations appear as an appendix.

II. THE COLLECTIVE RISK STOCHASTIC PROCESS

The subject of stochastic processes arose from the desire to build mathematical models for certain natural processes. The random movements of small particles (called "Brownian motion") were analyzed mathematically by L. Bachelier [1] as early as 1900. The study of Brownian motion was greatly enhanced by the work of Norbert Wiener beginning in 1923 [16], and hence the stochastic model is usually called the "Wiener stochastic process." This process has played an important role in quantum physics and in some statistical problems. Actuaries can take pride in the early recognition (1903) by F. Lundberg [10] of the value of looking at the ensemble of risks. His papers began the study of the collective risk stochastic process.

The subjects of probability and statistics consider one, two, and possibly many random variables. The distribution functions for the random variables are prime areas of study. Functions of one or more random variables are new random variables, and their distributions are considered. For example, the distribution of the sample mean $\bar{x} = (X_1 + X_2 + \ldots + X_n)/n$ is discussed in Hoel [8] on pages 138–46. The distribution of the maximum of X_1 and X_2 could be evaluated as another example. The study of stochastic processes involves collections of infinitely many random variables. Sometimes these are indexed by the integers, that is, X_1, X_2, X_3, ... More frequently, they are indexed by a parameter in an interval of numbers. Thus $\{X_t, 0 \leq t \leq T\}$ can be used to denote an infinite collection of random variables, one for each point between 0 and

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T, inclusive. The interval $0 \leq t \leq T$ may represent time. Prime attention is given to calculating distributions of new random variables defined as functions of all the random variables. For example, if

$$\{\underset{0 \leq t \leq T}{\text{maximum}}\ X(t)\}$$

denotes the largest value of the $X(t)$'s between 0 and T, an interesting question is the determination of the distribution of

$$\{\underset{0 \leq t \leq T}{\text{maximum}}\ X(t)\}.$$

Let us now consider the risk business of an insurance company from the collective risk viewpoint. By describing this as a stochastic process, it will be seen that it has characteristics which are also common to stochastic processes used in the theory of queues, dams, storage, statistics, physics, and other fields. Therefore, results developed for these other fields may apply to collective risk problems. This description will be quite brief. The definitive work on the subject is by Harald Cramér [7].

Let us consider an insurance operation in which a number of policyholders have paid in premiums to provide certain benefits in the event certain things happen, such as death, sickness, fire, and so forth.

Let $P(z)$ be the distribution for a claim; that is, $P(z)$ is the probability that, if a claim occurs, it will be less than or equal to z. We will assume that $P(0) = 0$, which rules out nonpositive claims. Such claims occur when a life annuity terminates and a reserve is released. We will let the variable t be *operational* time. This means, for example, that, if past records indicate 30 claims per year, then $t = 30$ will correspond to one calendar year. $N(t)$ is the random number of claims in time t. We will assume that claims occur in such a way that $N(t)$ has a Poisson distribution with mean t. The reader can consult Dr. Paul Kahn's paper [9] relative to this assumption.

If p_1 = average claim amount, then in time t the insurance company would charge $p_1 t$ as the *aggregate* net risk premium, plus an *aggregate* security loading of λt. We also assume that the insurance company begins with a risk reserve of size u. At time t, the risk reserve $U(t)$ is given by

$$U(t) = u + (p_1 + \lambda)t - \sum_{i=1}^{N(t)} X_i,$$

where the X_i's are independent random variables, representing the claims. Each X_i has the distribution $P(z)$. The symbol

$$\sum_{i=1}^{N(t)} X_i$$

represents the *aggregate* claims up to time t. Since the upper limit of summation, $N(t)$, is itself a random variable, this sum is a random *number* (number of claims) of random variables (claims). The expression

$$\{\underset{0\leq t\leq T}{\text{minimum}}\ U(t)\}$$

will refer to the smallest value of $U(t)$ over the time range $0 \leq t \leq T$.

We wish to calculate

$$F(x, t) = P\left[\sum_{i=1}^{N(t)} X_i \leq x\right],$$

the distribution of total claims during time t,

$$\psi(u, T) = P[\{\underset{0\leq t\leq T}{\text{minimum}}\ U(t)\} < 0],$$

the probability of ruin on or before time T, and $\psi(u)$, the probability of eventual ruin.

To make these calculations, let

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

for each time point t. Since for any fixed constant T there are an infinite number of points between 0 and T, the collection of random variables $\{Y(t), 0 \leq t \leq T\}$ is a stochastic process. This process is sometimes called the "compound Poisson process." If each X_i could assume only the values 1 or 0, it would be called the "Poisson process." The name "Poisson" is appropriate because of the distribution of the random variable $N(t)$. Because the X_i's can be more arbitrary in value, the process is called the "compound Poisson process." This process has several special properties. First, it has independent increments. An increment is a difference $Y(t + s) - Y(s)$, for $s \geq 0$, $t > 0$. It is a random variable. To say that the process has independent increments means that such jumps in total claims in different time spans are independent random variables. Furthermore, the increments are stationary, which means that, for $t > 0$ and $s \geq 0$, the distribution of $Y(t + s) - Y(s)$ depends on t but not on s.

Let

$$Z(t) = (p_1 + \lambda)t - \sum_{i=1}^{N(t)} X_i.$$

Then $Z(t)$ represents the random gain at time t. The stochastic process $\{Z(t), 0 \leq t \leq T\}$ again has stationary, independent increments.

III. RUIN FUNCTION APPROXIMATIONS

John Wooddy's *Study Note on Risk Theory* [17] points out that ruin theory may be applied in determining retention limits and that u "may be an outside limit on the size of adverse fluctuation which management is willing to contemplate on a given line or block of business." This helps put "ruin" theory in a more constructive light.

A complete knowledge of $\psi(u)$ and $\psi(u, T)$ would be ideal in examining these problems. Section IV will be devoted to new methods of obtaining these functions. When it is impossible to derive the complete functions, approximations can be very useful. Recently Mr. D. K. Bartlett [2] and Dr. Newton Bowers [6] approximated the density of $F(x, t)$ by a sum of gamma densities. The author has now derived the mean and variance for the distribution function

$$\psi^*(u) = \begin{cases} 1 - \psi(u), & u \geq 0 \\ 0, & u < 0 \end{cases}$$

and will use these results in a future paper devoted to approximating the density of $\psi^*(u)$. As a distribution function, $\psi^*(u)$ involves the random variable Z, defined by

$$Z = \left\{ \underset{0 \leq t < \infty}{\text{maximum}} \left[\sum_{i=1}^{N(t)} X_i - t(p_1 + \lambda) \right] \right\}.$$

Roughly speaking, Z is the maximum excess of claims over income examined at each time point of very long time periods. But the reason for mentioning these results is that it was my plan to use them in a generalized Chebyshev's inequality to derive a bound for $\psi(u)$. The reader may recall that Chebyshev's inequality says that, if X is a random variable with mean μ and variance σ^2, then for $k > 0$, $P(|X - \mu| \geq k) \leq \sigma^2/k^2$. A generalization of this inequality was used, but the results were not nearly as good as Lundberg's bound on the ruin function:

$$\psi(u) \leq e^{-Ru}, \tag{1}$$

where R is the only positive root of the equation

$$1 + (p_1 + \lambda)s - \int_0^\infty e^{sy} dP(y) = 0 \tag{2}$$

(see ref. [7]). Here the Stieltjes integral,

$$\int_0^\infty e^{sy} dP(y),$$

reduces to

$$\int_0^{\infty} e^{\theta y} p(y) dy$$

if $P(y)$ has a derivative $p(y)$ for all y values and to

$$\sum_{i=1}^{n} e^{\theta y_i} \Delta P(y_i)$$

if $P(y)$ is a pure step function.

To illustrate ruin function approximations, consider two distributions from reference [4], which show the effects of reinsurance. In each case we want to know what initial reserve u would be needed to have $\psi(u) < .01$. The u's involved are now large enough to use the asymptotic formula on page 45 of reference [7]:

$$\psi(u) \sim Ce^{-Ru}, \tag{3}$$

where

$$C = \frac{\lambda}{q'(R) - 1 - \lambda}, \quad q(\theta) = \int_0^{\infty} e^{\theta y} dP(y), \quad q'(R) = \frac{d}{d\theta} q(\theta)|_{\theta = R},$$

and R is from equation (2).

Example 1

$$\begin{aligned} P_1(z) &= 0, z < 2, \\ &= .3, 2 \le z < 5, \\ &= .5, 5 \le z < 10, \\ &= .8, 10 \le z < 20, \\ &= 1.0, 20 \le z. \end{aligned}$$

Then $p_1 = E(X) = 8.6$ (\$8,600, since we are using \$1,000 units). If we let $\lambda = .3p_1 \doteq 2.6$, then $R \doteq .035$ from the equation $1 + 11.2s - .3e^{2s} - .2e^{5s} - .3e^{10s} - .2e^{20s} = 0$. Here $C = .2465$, and $\psi(\$91{,}429) \doteq .01$.

Example 2

$$\begin{aligned} P_2(z) &= 0, z < 2, \\ &= .3, 2 \le z < 5, \\ &= .5, 5 \le z < 10, \\ &= .8, 10 \le z < 20, \\ &= .85, 20 \le z < 30 \\ &= .90, 30 \le z < 40, \\ &= .95, 40 \le z < 50, \\ &= 1.00, 50 \le z. \end{aligned}$$

Here $p_1 = 11.6$. If we let $\lambda = .3p_1 \doteqdot 3.5$, then $R \doteqdot .0175$. The value of $C = .2372$, and $\psi(\$180{,}857) \doteqdot .01$.

IV. TRANSFORMS OF THE RUIN FUNCTIONS

In statistics a function

$$\int_{-\infty}^{\infty} e^{\theta z} dP(z)$$

of $P(z)$ and a parameter θ is used sometimes to capture the moments. This is called the "moment generating function" (see ref. [8], pp. 84, 96). Notice that our Stieltjes integral is a symbol which reduces to

$$\int_{-\infty}^{\infty} e^{\theta z} p(z)\, dz$$

if $P(z)$ has a derivative $p(z)$ for all z values and to

$$\sum_{i=0}^{\infty} e^{\theta z_i} f(z_i)$$

if $P(z)$ is a pure step function.

We know from reference [8] (page 108) that $P(z)$ is uniquely determined by its moment-generating function. This allows for instant inversion of the moment-generating function in known cases. For example, if the moment-generating function of some variable Z is found to be $e^{(1/2)\theta^2}$, then Z is a normal variable with mean 0 and variance 1. When the risks are all positive,

$$\int_{-\infty}^{\infty} e^{\theta z} dP(z)$$

reduces to

$$\int_{0}^{\infty} e^{\theta z} dP(z).$$

If $\theta = -\alpha$, for $\alpha > 0$, the above would be the Laplace-Stieltjes transform of $P(z)$. The process of inverting the transform yields $P(z)$. See the Appendix for a discussion of Laplace transforms. A readily available reference to Laplace transforms (with tables) is *Theory and Problems of Laplace Transforms* [12]. A book describing approximate methods for inverting Laplace transforms has been written recently by Drs. R. Bellman, R. Kalaba, and J. Lockett [5].

In my earlier paper [4], Laplace-Stieltjes transforms for $\psi(u)$ and $\psi(u, T)$ were given. These were obtained by using some results of G. Baxter and M. Donsker [3] which applied to stochastic processes with stationary independent increments. Some formulas for inverting the transforms to obtain $\psi(u)$ and $\psi(u, T)$ are now given. In each case, the inversion can be with respect to ordinary Laplace transforms rather than Laplace-Stieltjes transforms.

The purpose of the Baxter and Donsker paper [3] was to derive an expression for the double Laplace transform of the distribution of the expression

$$\{\underset{0\leq t\leq T}{\text{maximum}}\ Y(t)\}$$

for all stochastic processes which have stationary, independent increments. This expression was rather complex but could be simplified if restricted to one specific process which had stationary, independent increments. Thus considering the collective risk process (which has stationary, independent increments) gave a simpler expression for the transform which could be inverted, in some cases.

It should be remarked that in stochastic processes it frequently happens that one can obtain the Laplace-Stieltjes or Laplace transform of the distribution more easily than the distribution itself. This was the case in the Baxter and Donsker paper. Nevertheless, obtaining the transforms is a significant step.

We will use the notation $I_a\{f(a)\}$ for the inverse Laplace transform of $f(a)$. The following theorem gives an expression for $\psi(u)$ in terms of the inverse Laplace transform of an expression involving λ, p_1, and the Laplace-Stieltjes transform of $P(z)$, namely,

$$\int_0^\infty e^{-az}dP(z).$$

In the applications this last quantity is

$$\int^\infty e^{-az}p(z)\,dz$$

or

$$\sum_{i=0}^{\infty} e^{-az_i}f(z_i),$$

and one performs the inversion for the particular case involved (see Example 3, for instance). The proof of the theorem is in the Appendix. This theorem says that one way to find $\psi(u)$ is to invert a certain Laplace transform.

Theorem 1

If $u \geq 0$,

$$\psi(u) = 1 - \lambda I_a\left[\frac{1}{\int_0^\infty e^{-az}dP(z) - 1 + a(p_1+\lambda)}\right]. \qquad (4)$$

Example 3

$P(z) = 1 - e^{-Az}, z \geq 0, A > 0$. Then

$$\int_0^\infty e^{-az} dP(z) = \int_0^\infty e^{-az} A e^{-Az} dz = \frac{A}{a+A},$$

and $p_1 = 1/A$. Hence

$$\psi(u) = 1 - \lambda I_a \left[\frac{1}{A/(a+A) - 1 + a(1/A + \lambda)} \right].$$

The quantity within brackets may be rearranged as

$$\frac{1}{\lambda}\left[\frac{1}{a} - \frac{1}{1+\lambda A} \cdot \frac{1}{a + \lambda A^2/(1+\lambda A)}\right],$$

which is in easy form to calculate the inverse transform. See the Appendix for the inversion of Laplace transforms, including this example. Such inversion produces the result

$$\psi(u) = \frac{1}{1+\lambda A} e^{-[\lambda A^2/(1+\lambda A)]u}, \qquad u \geq 0.$$

The parameter A allows one to approximate various realistic claim distributions. Graphical examples are given in Example 5. For now, let us compare the initial reserves needed to hold $\psi(u) = .01$, or $.05$, or $.1$ when A equals 1 and .1, corresponding to mean claim amounts of $1,000 and $10,000.

We repeat that u may be regarded as an acceptable limit on adverse fluctuation on a given block of business. We will assume $\lambda = kp_1$. Since $p_1 = 1/A$, the above formula for $\psi(u)$ may be rewritten as

$$\psi(u) = \frac{1}{1+k} e^{-kA/(1+k)u}.$$

For $A = 1$, we have the following tabulation (in $1,000 units):

$\psi(u) = .01$		$\psi(u) = .05$		$\psi(u) = .1$	
u	k	u	k	u	k
18.800	.3	11.833	.3	8.831	.3
26.483	.2	16.846	.2	12.695	.2

For $A = .1$, the u's are all 10 times larger.

The following theorem utilizes another paper which refers to stochastic processes with stationary independent increments (Takács [13], especially pp. 371, 375). It gives an expression for $\psi(u)$ in terms of the convolutions of a distribution related to $P(z)$. The proof of the theorem is in the Appendix.

Theorem 2

Let $H_0^*(x) = 1$ if $x \geq 0$ and 0 if $x < 0$. Let

$$H^*(x) = \frac{1}{p_1}\int_0^x [1 - P(y)]\,dy \text{ for } x \geq 0 \text{ and } 0 \text{ for } x < 0;$$

$$H_1^*(x) = H^*(x);$$

$$H_n^*(x) = \int_0^x H_{n-1}^*(x - z)\,dH^*(z) \text{ for } x \geq 0 \text{ and } 0 \text{ for } x < 0;\ n \geq 1.$$

Then for $u \geq 0$,

$$\psi(u) = 1 - \frac{\lambda}{p_1 + \lambda}\sum_{n=0}^{\infty}\left(\frac{p_1}{p_1 + \lambda}\right)^n H_n^*(u). \tag{5}$$

This is *remarkably similar* to the well-known formula for $F(x, t)$:

$$F(x, t) = \sum_{n=0}^{\infty}\frac{t^n e^{-t}}{n!}P_n^*(x), \tag{6}$$

where $P_n^*(x)$ is the nth convolution of the claim distribution. Also note that the coefficients of the H_n^*'s in equation (5) are the probabilities of a geometric distribution.

The following properties of $H_n^*(u)$ are helpful:

a) $H_n^*(u)$ is continuous for all u, $n \geq 1$.
b) For fixed u, $\lim_{n\to\infty} H_n^*(u) = 0$.
c) For fixed n, $\lim_{u\to\infty} H_n^*(u) = 1$.

If we assume $\lambda = kp_1$, $0 < k < 1$, and $H_j^*(u) = 1$, $1 \leq j \leq n(p_1,u)$, $\psi(u) \doteqdot [1/(1 + k)]^{n+1}$ with error $\leq [1/(1 + k)]^{n+1}$. The integer n depends on p_1 and u, so that the expression does depend on p_1 as well as u. The above overstates the true probability of ruin. For small, but meaningful, values of $\psi(u)$, this expression with $p_1 = 1$ gives results which closely agree with the table in reference [4] for $\psi(u)$ for $P(z) = 0$, $z < 1$, and $P(z) = 1, z \geq 1$.

Using pages 417 and 418 of reference [9] and formula (5), one can rederive the expression for $\psi(u)$ in Example 3.

Example 4

Consider the following two distributions:

$$P_1(z) = 0, z < 10, \quad P_M(z) = 0, z < 10M,$$
$$= 1, z > 10; \quad = 1, z \geq 10M.$$

(The risk amounts are measured in \$1,000 units.) For $P_1(z)$, $H_j^*(u) = 1$ for $j \leq .1u$. For $P_M(z)$, $H_j^*(u) = 1$ for $j \leq u/10M$.

Assume that $\lambda = .3p_1$. Then for $P_1(z)$ and the sample value $u = 100$, $\psi(100) = (1/1.3)^{11} = .0556$ with error $\leq .0556$, whereas for $P_M(z)$, $\psi(M \times 10^2) = .0556$ with error $\leq .0556$. Thus increasing the average claim by a factor of M only requires a like multiplication of the initial capital to preserve the same probability of ruin.

The following theorem obtains $\psi(u, T)$ by an iterated inversion of a two-dimensional Laplace transform involving λ, p_1, $P(z)$, and a root of an equation. The proof of the theorem is in the Appendix. This theorem says that one way to find $\psi(u, T)$ is to perform two inversions of Laplace transforms. Inversion with respect to w gives a function of T, and inversion with respect to z gives $\psi(u, T)$.

Theorem 3

Assume that $u > 0$ and $T > 0$. Then

$$\psi(u,T) = 1 - I_z\left\{\frac{1}{z} I_w\left[\frac{1 - z/y(w)}{w + 1 - \int_0^\infty e^{-zx} dP(x) - z(p_1 + \lambda)}\right]\right\}, \quad (7)$$

where $y(w)$ is the only nonnegative solution of

$$w = (p_1 + \lambda) y(w) + \int_0^\infty e^{-y(w)x} dP(x) - 1, \qquad w \geq 0. \quad (8)$$

V. THE DISTRIBUTION OF TOTAL CLAIMS AND A STOP-LOSS PREMIUM

This section will give a formula which in some cases yields the exact distribution of total claims. In these cases one can obtain the exact value for the net premium for a stop-loss reinsurance treaty. Other risk situations may be approximated by these methods. However, when approximations are needed, as is usually the case, the method of Bartlett [2] and Bowers [6] probably is superior.

Theorem 4 says that one way to obtain $F(x, T)$ is to invert the Laplace transform of an inverse Laplace transform involving $P(z)$ and a root of an equation. One also has to allow a variable δ to approach 0 through positive values ($\lim_{\delta \to 0+}$). The proof is in the Appendix.

Theorem 4

Assume that $x > 0$, $T > 0$. Then

$$F(x,T) = \lim_{\delta \to 0+} I_z\left\{\frac{1}{z} I_w\left[\frac{1 - z/y(w)}{w + 1 - \int_0^\infty e^{-zx} dP(x)}\right]\right\}, \quad (9)$$

where $y(w)$ is the only nonnegative solution of

$$w=\frac{\delta}{T}y(w)+\int_0^\infty e^{-y(w)x}dP(x)-1\,, \qquad w\geq 0\,. \qquad (10)$$

Example 5

$P(z) = 1 - e^{-Az}$, $z \geq 0$, $A > 0$. This is the exponential distribution. Here $p_1 = 1/A$ ($1,000). By choosing A properly, one can approximate various distributions. For example, Chart I shows the distribution of Example 2 and the approximating curves $P(z) = 1 - e^{-z/11.6}$ and $P(z) = 1 - e^{-z/9.375}$. The value 9.375 was computed by forcing $P(15)$ to equal .8. A better fit could be achieved by appropriately choosing A, B, C, and D in $P(z) = 1 - Ae^{-Bz} - Ce^{-Dz}$. With some patience, one could extend the following results to this case.

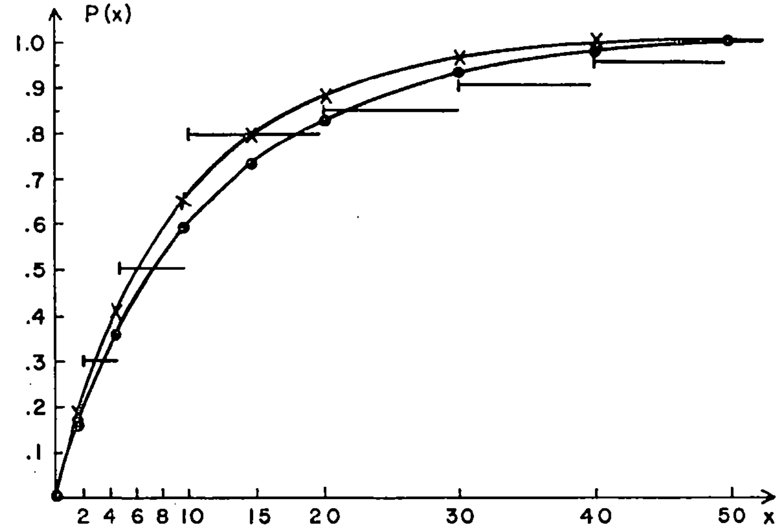

CHART I

As explained in the Appendix,

$$F(x,t)=e^{-t}\left[1+\int_0^x e^{-uA}tA\sum_{y=0}^{\infty}\frac{(tAu)^y}{(y+1)!y!}\,du\right]. \qquad (11)$$

For the case $A = 1$, an alternate derivation of formula (11) appears in formulas (2.1) and (10.6) of Kahn's paper [9], where one sets

$$P^{0*}(x) = 0,\ x<0$$
$$= 1,\ x\geq 0\,,$$

as Cramér does in reference [7] (page 21). When $A \neq 1$, one can use a scale change. That is,

$$F(x, t) = P[\,Y(t) \leq x\,]$$
$$= P[\,Y(t)\,A \leq xA\,].$$

Now

$$Y(t)\,A = \sum_{i=1}^{N(t)} X_i A\,.$$

Let $W_i = X_i A$. Then each W_i will have the exponential distribution with unit mean. Substituting Ax for x in formula (10.6) of reference [9] will give, after some simplification, formula (11).

We will now use formula (4.1) of Kahn's paper to compute the net premium for a stop-loss reinsurance treaty. The total net risk premium is $tp_1 = t/A$ for the calendar period for which t is the expected number of claims. Let u be a percentage of t/A. Then

$$\pi\left(u\frac{t}{A}\right) = \int_{u(t/A)}^{\infty}\left(x - u\frac{t}{A}\right)\left[\frac{d}{dx}F(x, t)\right]dx$$

$$= \int_{u(t/A)}^{\infty}\left(x - u\frac{t}{A}\right) e^{-t-Ax} tA \sum_{y=0}^{\infty}\frac{(tAx)^y}{(y+1)!\,y!}\,dx \qquad (12)$$

$$= \frac{te^{-t(1+u)}}{A}\sum_{y=0}^{\infty}\frac{t^y}{y!}\left[\frac{(ut)^{y+1}}{(y+1)!} + \left(1 - \frac{ut}{y+1}\right)\sum_{k=0}^{y}\frac{(ut)^k}{k!}\right].$$

This uses the fact that

$$\frac{1}{(y)!}\int_{ut}^{\infty} w^y e^{-w} dw = 1 - \Gamma(ut, y+1) = e^{-ut}\sum_{k=0}^{y}\frac{(ut)^k}{k!}.$$

$\Gamma(ut, k+1)$ is the incomplete gamma function (see ref. [6]).

Table 1 gives $\pi(ut/A)$ for various values of the parameters. $\pi(ut/A)$ equals the net premium charged to cover aggregate claims above $\$(ut/A) \times 10^3$. Table 1 was computed from equation (12) using an IBM 1620 computer. The mathematical error analysis of the series is difficult, but partial sums using thirty-one, thirty-two, thirty-three, thirty-four, and thirty-five terms indicated that the remaining terms of the series were dominated by geometric series with various common ratios. The truncation errors were then less than

$$a_{35}\sum_{n=1}^{\infty} r^n = a_{35}\left(\frac{r}{1-r}\right),$$

and several are indicated in parentheses under the premiums. The factor $[1 - ut/(y + 1)]$ in formula (12) is negative for the preliminary terms, but convergence is quite rapid *after* it turns positive. For these t values, it is believed that partial sums using forty-five terms would have reduced each error to less than \$1, but present facilities made this difficult. Of greater importance than decreasing the error bounds is enlarging the tables to include greater values of t. Perhaps some reader will do this. The decreasing pattern (as a function of t) for $u = 1.30$ and 1.40 is consistent with Bartlett's Table 1 (see ref. [2], p. 445).

TABLE 1

NET STOP-LOSS PREMIUMS

t	u = 1.00	1.10	1.20	1.30	1.40
	$A = .4$ (Average Claim = \$2,500)				
16	\$5,620	\$3,978	\$2,734	\$1,827	\$1,187
18	5,962	4,120	2,753	1,779	1,114
20	6,278	4,240	2,754	1,722	1,038
	$A = .1$ (Average Claim = \$10,000)				
16	\$22,478	\$15,910	\$10,936	\$7,306	\$4,750
18	23,847 (6)*	16,482	11,010	7,116	4,455
20	25,111 (53)*	16,958 (47)*	11,015	6,888	4,151

* Truncation error.

From equation (12), we see that $\pi(ut/A)$ varies with $1/A$. Therefore a table for $A = 1$ can be derived from the table for $A = 10$ by multiplying each entry by .1.

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APPENDIX

LAPLACE TRANSFORMS

Laplace transforms are similar to moment-generating functions. If one knows the transform of a function $F(t)$, one knows a great deal about $F(t)$. In fact, for the functions considered in most cases, there is a unique correspondence between the functions and their transforms. The process of "undoing" the transform to obtain $F(t)$ is called "inverting the transform." Reference [12] contains tables of functions and their transforms. Such tables can be read either way. That is, given a function, one can read its transform, or, given a transform, one can deduce what function produced it. This is visual inversion.

If $F(t)$ is a function of t for $t > 0$, its Laplace transform is the integral

$$\int_0^\infty e^{-\alpha t} F(t)\,dt\,,$$

for α a real number. It is denoted by $f(\alpha)$. For example, if $F(t) = t$, $t > 0$, then $f(\alpha) = 1/\alpha^2$, $\alpha > 0$. The process of inversion consists of finding $F(t)$ from $f(\alpha)$. For example, if $f(\alpha) = 1/(\alpha - A)$, $\alpha > 0$, we see from a table of transforms that $F(t) = e^{At}$.

Transforms and their inverses have the linearity property, which is most convenient. That is,

$$\int_0^\infty e^{-\alpha t}[\,aF_1(t) + bF_2(t)\,]\,dt = a f_1(\alpha) + b f_2(\alpha)\,,$$

and, using $I_\alpha[f(\alpha)]$ for the inverse Laplace transform of $f(\alpha)$, $I_\alpha[af_1(\alpha) + bf_2(\alpha)] = aF_1(t) + bF_2(t)$. Motivated by Example 3, we record the example

$$I_\alpha\left[a\left(\frac{1}{\alpha}\right) + b\left(\frac{1}{\alpha + c}\right)\right] = a + b e^{-ct}\,.$$

Laplace-Stieltjes transforms are of the form

$$\int_0^\infty e^{-\alpha t} d\beta(t) = f(\alpha)\,.$$

Knowing $\beta(t)$, one can compute $f(\alpha)$, and theoretically one can find $\beta(t)$ from $f(\alpha)$. However, such tables are hard to find; hence this paper has reduced all the transforms to regular Laplace transforms. This was done by the integration by parts formula for Stieltjes integrals (see ref. [15], page 160). This relates Laplace-Stieltjes and Laplace transforms by the formula:

$$\int_0^\infty e^{-\alpha t} d\beta(t) = \lim_{T\to\infty} e^{-\alpha t}\beta(t)\Big|_0^T + \alpha\int_0^\infty e^{-\alpha t}\beta(t)\,dt\,.$$

In the applications of this paper,

$$\lim_{T\to\infty} e^{-\alpha T}\beta(T) = 0\,.$$

Theorems 1, 3, and 4 contain the symbol

$$\int_0^\infty e^{-\alpha z} dP(z)\,,$$

but in the applications this is merely

$$\int_0^\infty e^{-\alpha z} p(z)\,dz$$

or

$$\sum_{i=0}^{\infty} e^{-az_i} f(z_i),$$

and one performs the operations of the theorems for the particular case involved (see Example 5, for instance).

Proof of Theorem 1.—By integration by parts

$$\int_0^{\infty} e^{-au} d_u \psi^*(u) = -\psi^*(0) + a\int_0^{\infty} e^{-au}\psi^*(u)\,du,$$

where

$$\psi^*(u) = \lim_{T\to\infty} P\left[\!\left[\left\{\underset{0\le t\le T}{\text{maximum}}\left[\sum_{i=1}^{N(t)} X_i - t(p_1+\lambda)\right]\right\} < u\right]\!\right].$$

Since the quantity in brackets equals 0 for $t = 0$, the maximum is ≥ 0; hence $\psi^*(0) = 0$. Dividing by a, we have from Corollary 3 (see ref. [4]), that

$$\int_0^{\infty} e^{-au}\psi^*(u)\,du = \frac{\lambda}{\int_0^{\infty} e^{-az} dP(z) - 1 + a(p_1+\lambda)}.$$

Inversion gives $\psi^*(u)$, and then for $u > 0$, $\psi(u) = 1 - \psi^*(u)$. Since $\psi(u)$ is continuous for $u > 0$ and continuous from the right at $u = 0$, the conclusion holds for $u = 0$ also. (It gives the usual $p_1/(p_1 + \lambda)$ value.)

The function $\psi^*(u)$ of reference [4] does not agree with $1 - \psi(u)$ at $u = 0$ because references [3] and [4] applied to distributions continuous from the left rather than from the right. Since $\psi(u)$ is continuous for $u > 0$, there is no difference in results for $u > 0$.

The $F(x, t)$ of Theorem 4 refers to

$$P\left[\sum_{i=1}^{N(t)} X_i < x\right]$$

rather than to the usual

$$P\left[\sum_{i=1}^{N(t)} X_i \leq x\right].$$

These two expressions are identical if the $F(x, t)$ is continuous but will differ at discontinuity points. This should cause no practical difficulty. For example, the $F(x, t)$ of equation (11) is continuous for $x \geq 0$.

Proof of Theorem 2.—$H^*(x)$ is a distribution function, and

$$a p_1 \int_0^{\infty} e^{-ax} dH^*(x) = \left[1 - \int_0^{\infty} e^{-ax} dP(x)\right], \qquad a \geq 0.$$

Substituting this in equation (4) gives

$$\psi(u) = 1 - \lambda I_a \left\{ \frac{1}{a(p_1+\lambda)\left[1 - \frac{p_1}{p_1+\lambda}\int_0^\infty e^{-ax} dH^*(x)\right]} \right\}.$$

We now use the power series for $(1-z)^{-1}$, $|z| < 1$, which is $(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots$.

By properties of the Laplace transform of a convolution, we have

$$\left[\int_0^\infty e^{-ax} dH^*(x)\right]^k = \int_0^\infty e^{-ax} dH_k^*(x).$$

Combining these two facts, we obtain

$$\psi(u) = 1 - \frac{\lambda}{p_1+\lambda} I_a \left\{ \left[1 + \frac{p_1}{p_1+\lambda}\int_0^\infty e^{-ax} dH^*(x) + \left(\frac{p_1}{p_1+\lambda}\right)^2 \int_0^\infty e^{-ax} dH_2^*(x) + \ldots \right] \Big/ a \right\}.$$

By integration by parts, and the fact that $H_k^*(0) = 0$ for $k \geq 1$,

$$\psi(u) = 1 - \frac{\lambda}{p_1+\lambda} I_a \left[\frac{1}{a} + \frac{p_1}{p_1+\lambda}\int_0^\infty e^{-ax} H^*(x)\,dx + \left(\frac{p_1}{p_1+\lambda}\right)^2 \int_0^\infty e^{-ax} H_2^*(x)\,dx + \ldots \right].$$

Term-by-term inversion gives equation (5), since

$$I_a\left[\int_0^\infty e^{-ax} H_k^*(x)\,dx\right] = H_k^*(x).$$

Proof of Theorem 3.—One begins with Corollary 2 of reference [4], inverts with respect to w, uses integration by parts, inverts with respect to z, and then uses the fact that $\psi^*(0, T) = 0$.

Proof of Theorem 4.—One begins with equation (12) of reference [4], inverts with respect to w, uses integration by parts, inverts with respect to z, and then uses the fact that

$$\lim_{\delta \to 0+} \sigma(-\delta, T, \delta/T) = \lim_{\delta \to 0+} P\left[\!\left[\left\{ \underset{0 \leq t \leq T}{\text{maximum}} \left[\sum_{i=1}^{N(t)} X_i - \frac{t\delta}{T} \right] \right\} < -\delta \right]\!\right] = 0.$$

Calculations of Example 5.—As explained earlier, a short derivation of equation (11) can be obtained by using several formulas from Kahn's paper and a scale change. However, it is desirable to illustrate obtaining $F(x, T)$ through formulas (9) and (10), and the following derivation does this: $P(z) = 1 - e^{-Az}$, $z \geq 0$. Here

$$y(w) = \left[1 + w - \frac{\delta}{T} A + \sqrt{\left(-1 - w + \frac{\delta}{T} A\right)^2 + 4w\frac{\delta}{T} A} \right] \Big/ \frac{2\delta}{T}.$$

Note that

$$\lim_{\delta\to 0+} y(w) = +\infty .$$

Hence a justifiable interchange of the limit and inversion operators gives

$$F(x, T) = I_z\left\{\frac{1}{z} I_w\left[\frac{1}{w+1-A/(z+A)}\right]\right\}$$

$$= e^{-T} I_z\left[\frac{1}{z} e^{-TA/(z+A)}\right]$$

$$= e^{-T} I_z\left[\left(1+\frac{A}{z}\right)\frac{1}{z+A} e^{-TA/(z+A)}\right]$$

$$= e^{-T}\left[e^{-xA}J_0(2\sqrt{-TAx})+A\int_0^x e^{-uA}J_0(2\sqrt{-TAu})\,du\right]$$

(mainly by formulas [83] and [13] of ref. [12])

$$= e^{-T}\left\{e^{-xA}I_0(2\sqrt{TAx}) + A\int_0^x e^{-uA}I_0(2\sqrt{TAu})\,du\right\},$$

where $J_0(z)$ and $I_0(z)$ are Bessel functions (see ref. [14]). If we use the series for I_0 (see ref. [14], page 372),

$$I_0(2\sqrt{TAx}) = \sum_{r=0}^{\infty}\frac{(TAx)^r}{r!r!},$$

an integration by parts, and the uniform convergence of the series,

$$F(x, T) = e^{-T}\left[1+\int_0^x e^{-uA}du\,I_0(2\sqrt{TAu})\right]$$

$$= e^{-T}\left[1+\int_0^x e^{-uA}TA\sum_{y=0}^{\infty}\frac{(TAu)^y}{(y+1)y!y!}\,du\right].$$

This last simplification was motivated by formula (52) of Takâcs [13].

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DISCUSSION OF PRECEDING PAPER

HARRY M. SARASON:

Mathematics is an aid to thinking. As Professor Beekman indicated, the mathematics of collective risk theory is an aid to thinking in many practical fields—and that word "practical" reminds us that thinking is an aid to action or to a decision to abstain from action.

The mathematics of "statistics" is shortest when we are dealing with just one statistical class. The simplest statistical mathematics to understand is the mathematics of the statistical class subdivided into so many subclasses that there is only one individual per subclass—the mathematics of "probability." The paper by Louis Levinson (see References) is pertinent here.

There are many underlying causes to consider in forecasting mortality ranges and possibilities—shortage of skilled health workers in this country; pollution of our air, water, food, and soil; depletion of our food nutrients and soil; cigarette and liquor advertising; discoveries of and improvements in health practices; and so on. Max Weinstein recently reported on a worsening of mortality for male retirees, in contrast to previous trends.

In our "predictions" we also recognize "acts of God"—catastrophes and epidemics—and some of us, at least, recognize acts of God for individual lives. We also realize our own ignorance: all probability calculations and all forecasts of foreseeable possibilities are based upon partial ignorance. We are too ignorant even to evaluate our own ignorance, but the wider and deeper we search for facts, the more we reduce our ignorance.

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(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. BEEKMAN:

The author thanks Mr. Sarason for his interesting and useful discussion. His references bring various thoughts to mind.

For example, the paper by Mr. Green reminds one that the collective risk model we used assumes a Poisson distribution of claims which assumes that the probability of more than one claim in a small time interval is approximately zero.

Some results have been obtained allowing multiple claims in the model. (See [11].) Also see the references to the Polya model in H. Bohman and F. Esscher, "Studies in Risk Theory with Numerical Illustrations concerning Distribution Functions and Stop Loss Premiums. I," *Skandinavisk Aktuarietidskrift*, 1963, pp. 173–225.

However, without the Poisson distribution of claims, the resulting stochastic process would no longer have independent increments. (See H. Hurwitz and Mark Kac, "Statistical Analysis of Certain Types of Random Functions," *Annals of Math. Statistics*, XV [1944], 175–81.) That means that the results of Baxter and Donsker and of Takács could not be applied.

The paper by Mr. Boermeester reminds one that it is possible to approximate stochastic processes by Monte Carlo techniques; this may provide future results.