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VARIABLE LIFE INSURANCE VIEWED VIA THE "ADDITIONS" CONCEPT

JAMES J. MURPHY AND RODNEY R. ROHDA

THE paper "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance" by Messrs. Fraser, Miller, and Sternhell (*TSA*, XXI, 343) presents one approach to variable life insurance, and the discussions which follow it explore some alternative approaches. Each plan is defined by formulas which relate changes in the face amount during each year to that year's investment performance.

It is the purpose of this paper to present alternative formulas for variable life insurance. These formulas were derived by viewing this insurance as a combination of (1) a conventional basic policy with a level, though nonguaranteed, death benefit and (2) additional coverage which is dependent upon the investment performance. Net level premiums and reserves for the basic policy are calculated by using an assumed interest rate. However, the reserve for the basic policy is maintained in the separate account with complete participation in the investment performance achieved through the "additions" concept.

Each year interest at the assumed rate must be credited to the basic policy's reserve. The difference between the actual investment earnings and those required by the basic policy's reserve is used to purchase positive or negative "additional" insurance. Reserves for this additional insurance are also maintained in the separate account, where they must earn interest at the assumed rate. Thus these additions also participate directly in the investment performance.

At all times the total death benefit equals the sum of the face amount for the level basic policy plus the total positive or negative additions purchased to that date. If the total amount of additions is negative, the total coverage is less than that for the level basic plan. The remainder of this note demonstrates how formulas for variable life insurance may be translated into this concept. All symbols used are as defined in the New York Life paper.

The basic face-amount formula for the New York Life design is

$$F_t = F_{t-1} \cdot \frac{{}_{t-1}V_x + P_x/F_{t-1}}{{}_{t-1}V_x + P_x} \cdot \frac{1 + i'_t}{1 + i} . \qquad (1)$$

This can be rewritten as follows:

$$F_t = \frac{(F_{t-1} \cdot {}_{t-1}V_x + P_x) \cdot (1 + i'_t)}{({}_{t-1}V_x + P_x) \cdot (1 + i)} . \tag{2}$$

By subtracting F_{t-1} from both sides of formula (2), we get an expression for the new "additions" purchased in year t:

$$F_t - F_{t-1} = \frac{(F_{t-1} \cdot {}_{t-1}V_x + P_x) \cdot (1 + i'_t) - F_{t-1} \cdot ({}_{t-1}V_x + P_x) \cdot (1 + i)}{({}_{t-1}V_x + P_x) \cdot (1 + i)} . \tag{3}$$

The split of the policy into the basic level face and the previous "additions" balance is shown by adding and subtracting ${}_{t-1}V_x$ in both terms of the numerator and P_x in the second term of the numerator:

$$F_t - F_{t-1} =$$

$$\frac{[(F_{t-1} - 1) \cdot {}_{t-1}V_x + ({}_{t-1}V_x + P_x)](1 + i'_t) - [(F_{t-1} - 1) \cdot {}_{t-1}V_x + ({}_{t-1}V_x + P_x) + (F_{t-1} - 1) \cdot P_x] \cdot (1 + i)}{({}_{t-1}V_x + P_x) \cdot (1 + i)} \tag{4}$$

By regrouping the terms of the numerator in formula (4), we obtain the "additions" formula, for the New York Life design:

$$F_t - F_{t-1} =$$

$$\frac{[({}_{t-1}V_x + P_x) + (F_{t-1} - 1) \cdot {}_{t-1}V_x] \cdot (i'_t - i) - (F_{t-1} - 1) \cdot P_x \cdot (1 + i)}{({}_{t-1}V_x + P_x) \cdot (1 + i)} . \tag{5}$$

Formula (5) shows that the amount of "additions" credited in each year is based on the deviation of actual from required investment experience on the total policy reserve. This reserve consists of the initial reserve for the level basic policy plus the previous terminal reserve on the additions balance. These "additions" differ from traditional dividend additions in two ways. First, they are "premium-paying additions" based on the plan and issue age of the basic policy. Each year, the excess investment performance is used first to pay the premium for all previously credited additions with required interest to the end of that year and then to provide for the new (positive or negative) additions.

The second difference is that both the deaths and the survivors of year t receive the new "addition." Beneficiaries receive the full amount, and a premium-paying reserve is set up for the survivors. This can be seen more clearly if the denominator is replaced by the equivalent expression $q_{x+t-1} + p_{x+t-1} \cdot {}_tV_x$. The total death benefit available is the same for all who entered the year and fully reflects the investment experience of that year and all prior years.

The "additions" concept can also be applied to alternative approaches to variable life insurance. For example, consider the plan proposed by Guy L. Fairbanks, Jr., in his discussion of the New York Life paper. The basic face-amount formula for his alternative is

$$F_t = F_{t-1} \cdot \frac{1 + i'_t}{1 + i} - \frac{P_x}{P_{x+t}} \cdot \left(\frac{1 + i'_t}{1 + i} - 1 \right). \tag{6}$$

Subtracting F_{t-1} from both sides and combining terms we get an expression for the new "addition" purchased in year t,

$$F_t - F_{t-1} = \left(F_{t-1} - \frac{P_x}{P_{x+t}} \right) \cdot \left(\frac{1 + i'_t}{1 + i} - 1 \right). \tag{7}$$

Multiplying the right side by A_{x+t-1}/A_{x+t-1} and rearranging terms, we get

$$F_t - F_{t-1} = \frac{[F_{t-1} \cdot A_{x+t-1} - (P_x/P_{x+t}) \cdot A_{x+t-1}](i'_t - i)}{A_{x+t-1} \cdot (1 + i)}. \tag{8}$$

In the New York Life actuaries' review of Mr. Fairbanks' discussion it was proved that

$$P_x = \left(\frac{P_x}{P_{x+t-1}} - \frac{P_x}{P_{x+t}} \right) \cdot A_{x+t-1} + \frac{P_x}{P_{x+t}} \cdot vq_{x+t-1}. \tag{9}$$

This can be rearranged to obtain an expression for $(P_x/P_{x+t}) \cdot A_{x+t-1}$:

$$\begin{aligned} \frac{P_x}{P_{x+t}} \cdot A_{x+t-1} &= \frac{P_x}{P_{x+t-1}} \cdot A_{x+t-1} + \frac{P_x}{P_{x+t}} \cdot vq_{x+t-1} - P_x \\ &= P_x \cdot \ddot{a}_{x+t-1} - P_x + \frac{P_x}{P_{x+t}} \cdot vq_{x+t-1}. \end{aligned} \tag{10}$$

Substituting formula (10) into formula (8), we obtain

$$F_t - F_{t-1} = \frac{[F_{t-1} \cdot A_{x+t-1} - P_x \cdot \ddot{a}_{x+t-1} + P_x - (P_x/P_{x+t}) \cdot vq_{x+t-1}] \cdot (i'_t - i)}{A_{x+t-1} \cdot (1 + i)}. \tag{11}$$

By adding and subtracting $A_{x+t-1} \cdot (i'_t - i)$ in the numerator and using the prospective definition for ${}_{t-1}V_x$, we can split the policy into the basic level plan plus the previous additions balance which gives the "additions" formula for the Fairbanks' design:

$$F_t - F_{t-1} = \frac{\{[{}_{t-1}V_x + P_x - (P_x/P_{x+t}) \cdot vq_{x+t-1}] + (F_{t-1} - 1) \cdot A_{x+t-1}\} \cdot (i'_t - i)}{A_{x+t-1} \cdot (1 + i)}. \tag{12}$$

Again this formula demonstrates how the new "addition" in each year is based on the deviation of actual from required investment performance on the total initial reserve in the separate account. However, this is a paid-up addition similar to the traditional dividend addition. As before, this addition is credited to both the deaths and survivors as demonstrated by substituting the equivalent expression $q_{x+t-1} + p_{x+t-1} \cdot A_{x+t}$ in the denominator. The beneficiaries receive the full amount and a paid-up reserve is established for the survivors.

CONCLUSION

This paper presents an alternative view of the basic structure of variable life insurance. It is shown as a combination of a basic policy with a level, though nonguaranteed, death benefit, and positive (or negative) additions purchased through deviations of the actual investment performance from that assumed in the basic policy. Differences between the various approaches to variable life insurance are then due to differences in the type of "additions" purchased. This is illustrated by the two plans examined above. For the New York Life plan, the additions are premium-paying and are based on the same plan of insurance as the level basic policy. The alternative proposed by Mr. Fairbanks uses traditional paid-up, though variable, additions.

DISCUSSION OF PRECEDING PAPER

WILSON H. SCOTT:

We are in debt to the authors of this paper for continuing the flow of information on this fascinating subject. This discussion retains the "additions" theme but on a paid-up additions basis.

In his discussion of the New York Life paper, Mr. Nagler developed the following general formula for excess insurance designs using only the separate account:

$$F_t = 1 + \frac{({}_{t-1}V_x + P_x)(i'_t - i) + (F_{t-1} - 1)\ {}_{t-1}R_x(1 + i'_t)}{q_{x+t-1}(1 - {}_tR_x) + {}_tR_x},$$

where ${}_tR_x$ is the reserve per \$1 of excess death benefit in year t.

This equation can be rearranged as follows:

$$F_t - 1 = \frac{(F_{t-1} - 1)\ {}_{t-1}R_x(1 + i)}{q_{x+t-1} + p_{x+t-1} \cdot {}_tR_x} + \frac{[{}_{t-1}V_x + P_x + (F_{t-1} - 1)\ {}_{t-1}R_x](i'_t - i)}{q_{x+t-1} + p_{x+t-1} \cdot {}_tR_x}.$$

The common denominator effects a distribution of ${}_tR_x$ to survivors per \$1 to deaths. In the second term, this distribution is applied to the year's excess interest on the composite initial reserve to purchase a new paid-up addition. In the first term, this distribution is applied to the initial reserve for existing additions, improved with tabular interest, to produce the tabular death benefit. For three variable life designs, Table 1 substitutes the appropriate value for ${}_tR_x$ to determine the tabular death benefit in year t from prior additions.

The choice of ${}_tR_x$ can be said to determine the character of the paid-up additions. Thus Mr. Walker's additions provide level coverage for life. The paid-up additions implied by the New York Life method provide benefits for life which decrease each year. Mr. Booth's method can be regarded as producing paid-up additions with increasing death benefits for life.

JEROME GOLDEN:

In their paper Messrs. Murphy and Rohda referred to the "additions" under the New York Life plan as premium-paying additions. As Mr. Nagler pointed out in his discussion of the Fraser, Miller, and Sternhell paper, the New York Life plan can also be viewed as producing paid-up

additions which decrease from year to year. Thus, in contrast to the Fairbanks plan, which involves the purchase of level paid-up whole life insurance, the New York Life plan involves the purchase of decreasing paid-up whole life insurance.

This discussion was prepared to present an algebraic expression for the pattern of such decreases. This pattern, which is a function of the policy year in which the additions are earned and the assumed interest rate, does not depend on the actual investment performance. The pattern of decrease is applicable to both positive and negative additions.

TABLE 1

Author	Value of ${}_tR_x$	Tabular Death Benefit in Year t from Prior Additions
Fraser Miller Sternhell	${}_tV_x$	$(F_{t-1}-1)\left(1-\frac{P_x}{{}_{t-1}V_x+P_x}\right)$
Walker	A_{x+t}	$F_{t-1}-1$
Booth	1	$(F_{t-1}-1)(1+i)$

The face amount as of the end of policy year t under the New York Life plan can be expressed as the sum of a, b, and c, where

(a) is the basic face amount, such as \$1,000;
(b) is the amount of outstanding insurance as of the end of policy year t under the decreasing paid-up insurance earned in prior policy years, and
(c) the additions earned in the immediately preceding policy year.

Symbolically, this can be written as

$$F = 1{,}000 + \sum_{s=1}^{t-1}(\mathrm{DF}_s^t) + \mathrm{DF}_t^t, \tag{1}$$

where

DF_s^t = Amount of outstanding insurance as of the end of policy year t under the decreasing paid-up insurance earned in year s

$$= \mathrm{DF}_s^s \cdot \prod_{r=s}^{t-1}\left(\frac{{}_rV}{{}_rV+P}\right), \tag{2}$$

and where

DF_s^s = Additions earned in policy year s as of the end of policy year s

$$= (Z_s - 1)\cdot F_s^*, \tag{3}$$

where

F_s^* = Face amount as of the end of policy year s if the net investment return in the separate account in policy year s were equal to the assumed interest rate (i.e., $Z_s = 1$)

$$= \frac{F_{s-1}{}^*{}_{s-1}V + P}{{}_{s-1}V + P}, \qquad F_1^* = 1{,}000\,.$$

The derivations of these formulas are available for anyone who is interested.

One observation that I would like to add is that, although our actuarial literature on variable life insurance is filled with splendid actuarial formulas, I would hope that we actuaries are spending as much time on the words that go with these formulas, so that the field force, the home office personnel, the regulatory authorities, and, most importantly, the buying public, can fully understand the variable life insurance products we are designing.

(AUTHORS' REVIEW OF DISCUSSION)

JAMES J. MURPHY AND RODNEY R. ROHDA:

We are grateful to Messrs. Scott and Golden for their discussions. Our paper's basic thesis is that variable life insurance may be viewed as a combination of a level, nonguaranteed death benefit and positive (or negative) "additions" which arise from investment performance. It is interesting to note that both discussions retain that basic concept. It would appear, then, that none of our readers disagree with this concept.

Both discussions present an interesting alternative formulation of the additions for the New York Life design. Their approach is very reasonable and will appeal to those who find the terms "premium-paying" and "addition" incompatible. The choice between these alternatives may well depend on which leads to a better understanding of how VLI works.

That brings us to Mr. Golden's final observation on words vs. formulas. It is our hope that the development of the formulas presented in the paper and discussions will provide a strong base for developing simple descriptions for variable life insurance. It was that hope that provided the impetus for writing our paper.