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Optimizing CPPI Investment Strategy for Life Insurance Companies: A Risk-Reward Analysis

By Aymeric Kalife and Saad Mouti

ABSTRACT

Individualized constant proportion portfolio insurance (iCPPI) products are attractive alternatives to traditional unit-linked products because the former offer a guaranteed minimum return, such as variable annuities. They also offer high potential returns whilst limiting the downside risk by implementing a dynamic allocation strategy between high-risk and risk-free assets tailored to the risk appetite of the beneficiary. But the performance evaluation of iCPPI products should not rely on the unrealistic assumptions of continuous market price variation and continuous rebalancing of asset allocations. We adopt a more general and realistic pricing jump model and examine several dynamic strategies and put options to mitigate the risk that the value of the product will fall below the guaranteed minimum (so-called "gap risk").

With rising life expectancies, current provisions for retirement may not be sufficient for people to secure an acceptable standard of living after retirement. To achieve sufficiently high investment returns together with low risks over the long term, customers' funds should remain invested in risky assets as well as in safer bonds over an extended period well into retirement. The design of long-term investment products should also reflect the requirements and risk appetites of individual investors.

From the point of view of the provider as well, iCPPI products provide an attractive alternative to many traditional retail long-term investment products and offer a guaranteed minimum return for several key reasons:

- They lower exposure to volatility and extreme market price movements along with slightly lower returns.
- They have lower costs.
- They require lower regulatory capital.

Besides their price transparency, open time horizon, and no early redemption penalty, CPPI products generally offer a wide range of alternative investments for the risky asset and the flexibility to add other guarantees such as ratchets.

The CPPI investment strategy provides a minimum guaranteed return, the *floor* (usually defined as the discounted value of the final capital guarantee), and aims to maintain a risk asset exposure equal to a constant multiple of the *cushion* (defined as the excess value of the fund above the floor) at all times. The capital guarantee at maturity and the multiplier are customized to the customer's risk appetite, usually between three and six (which may be constant or not, depending on the contract).

However, implementation comes with many concrete challenges, as raised in section 1. The rebalancing of the asset allocation can be made only at discrete times. There are transaction costs, and risky asset prices may jump. There is likely to be a difference between the realized return compared to the hypothetical value of a CPPI strategy computed under traditional unrealistic theoretical conditions of continuous price movements, unfettered zero-cost trading, and continuous rebalancing. In particular, there is a non-zero probability for the value of the fund to fall below the guaranteed floor, called the "gap risk," as illustrated by the impact of introducing discontinuous jump processes in the modeling within the risky asset dynamics.

Section 2 deals with concrete strategies that at least partially mitigate such gap risk through a dynamically risk-adjusted multiplier and the use of put options.

SECTION 1. CPPI MANAGEMENT: FROM THEORY TO PRACTICE

CPPI Mechanism Basics

Consider that at time t a risky asset (e.g., a share) with price S and a risk-free asset (e.g., a Treasury bond) with price B returns a constant rate r. The CPPI fund is invested in these two assets so that part of its value—the floor F_t—is guaranteed, whilst the excess value above the floor—the cushion C_t, which equals $V_t - F_t$—remains exposed to the risky asset price fluctuations. At any time, the exposure to the risky asset is kept at a constant multiple m of the cushion, that is, $m \times C_t$ (where m is usually held in practice between 3 and 6, implying that the asset manager borrows dynamically to buy the risky asset or may in practice buy the non-risky part only close to the expiration of the contract).

The risky asset S is defined by the usual lognormal continuous-time diffusion equation with drift μ and volatility σ;

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad \frac{dB_t}{B_t} = rdt$$

$$dV_t = m(V_t - F_t)\frac{dS_t}{S_t} + (V_t - m(V_t - F_t))rdt$$

$$V_t = F_t + (V_0 - F_0)\exp\left(\left(m(\mu - r) + r - \frac{m^2\sigma^2}{2}\right)t + m\sigma W_t\right)$$

This makes the portfolio value V independent on the path followed by the underlying S, while the probability to touch the floor is zero.[1]

The cushion C_t is then also lognormally distributed:

$$\frac{dC^t}{C_t} = (m\mu + (1-m)r)dt + m\sigma dW_t$$

$$C_t = C_0 \exp\left(\left(m(\mu - r) + r - \frac{m^2\sigma^2}{2}\right)t + m\sigma W_t\right)$$

However, such statistical assumptions are unrealistic and not consistent with market practice. Two alternatives are studied to remedy this: modeling in a discrete-time framework and using discontinuous jump processes (such as the Kou model)

Discrete-Time CPPI

A sequence of equidistant points in the interval [0, T] is defined, between which the portfolio asset allocation is updated. The first time the portfolio value touches the floor is defined by the following formula:

$$t_s = \min\{t_k \epsilon \Theta | V_{tk} - F_{tk} \leq 0\}$$

The probability of touching the floor now becomes greater than zero, assuming the portfolio has not breached the floor up to time t_k. The probability of breaching the floor at time t_{k+1} is that of a downside jump in the risky asset of more than about $1/m$, as evidenced below:

$$V^{\tau}_{t_{k+1}} - F_{t_{k+1}} = \begin{cases} (V_{tk} - F_{tk})\left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{N}}\right) & if\ V_{tk} - F_{tk} > 0 \\ (V_{tk} - F_{tk})e^{r\frac{T}{n}} & \text{if } V_{tk} - F_{tk} \leq 0 \end{cases}$$

Assuming the breach of the floor did not occur until t_k,

$$V^{\tau}_{t_{k+1}} > F_{t_{k+1}} \Leftrightarrow \left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{N}}\right) > 0$$

$$\Leftrightarrow \frac{S_{t_{k+1}}}{S_{t_k}} > \frac{m-1}{m}e^{r\frac{T}{N}}$$

As the interest rate return is close to zero over one day, we get the following result:

$$\frac{S_{t_{k+1}}}{S_{t_k}} - 1 > -\frac{1}{m}$$

Backtesting on three rebalancing frequencies (daily, weekly and monthly), over Q1 2006 to Q3 2007 S&P 500 index in Figure 1, illustrates that the CPPI strategy under daily rebalancing performs better than the weekly and monthly ones within bearish markets. We tested 10,000 simulation paths using the Black & Scholes model with a three-month realized volatility, a constant asset return m = 8%, a risk-free rate r = 4%, a duration of five years and 10 basis points (bps) transaction costs. This result reflects how highly responsive daily rebalancing is to decreasing the risk exposure, which prevents the bond floor from being breached and thus ensures the capital guarantee at maturity (as illustrated by fatter left tails in Figure 2). On the other hand, the 5 percent and 0.5 percent quantiles in Figure 3 show that the CPPI with m = 6 has a larger right tail. It performs better than the other two in a bullish market even though the mean return is similar to CPPI with m = 3.

Figure 1
Performance Depending On Multiplier vs. Buy and Hold Strategy

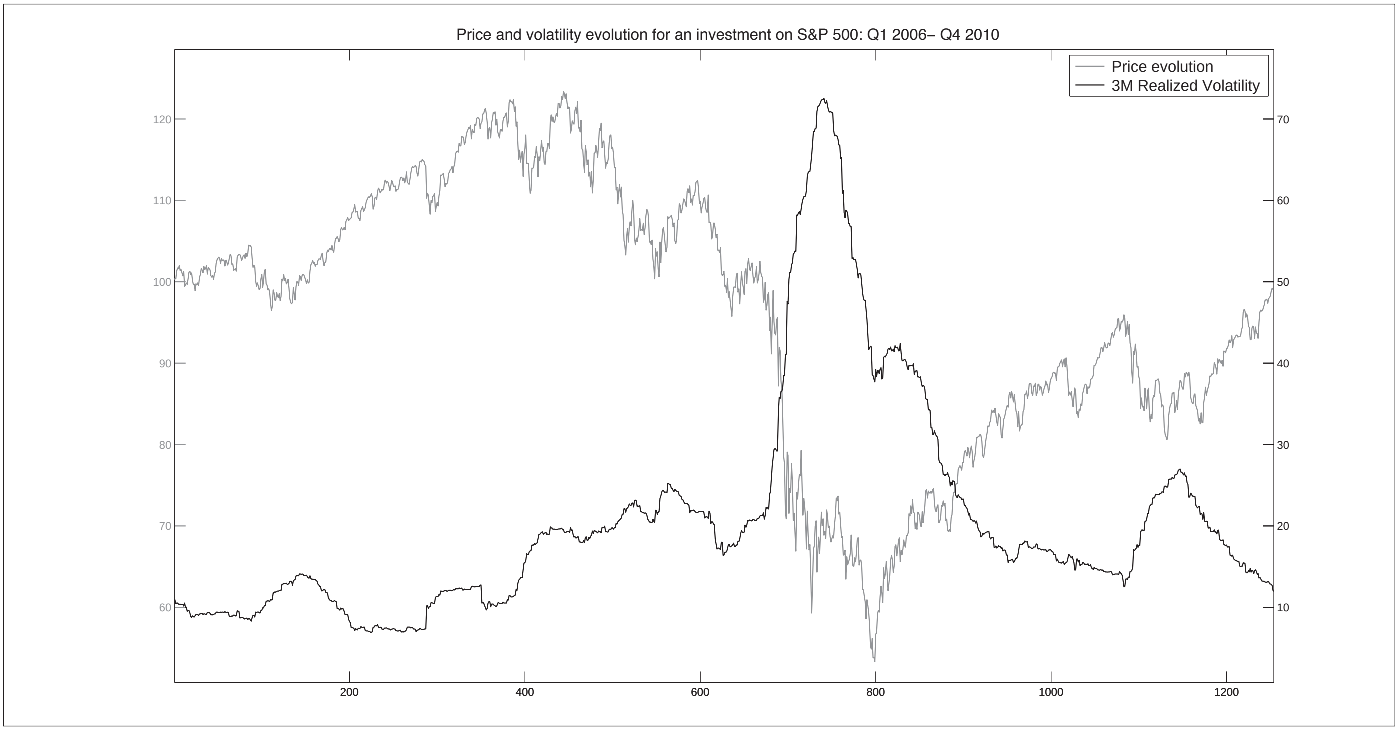


Figure 2
Statistical Metrics Depending On Multiplier and Rebalancing Frequency

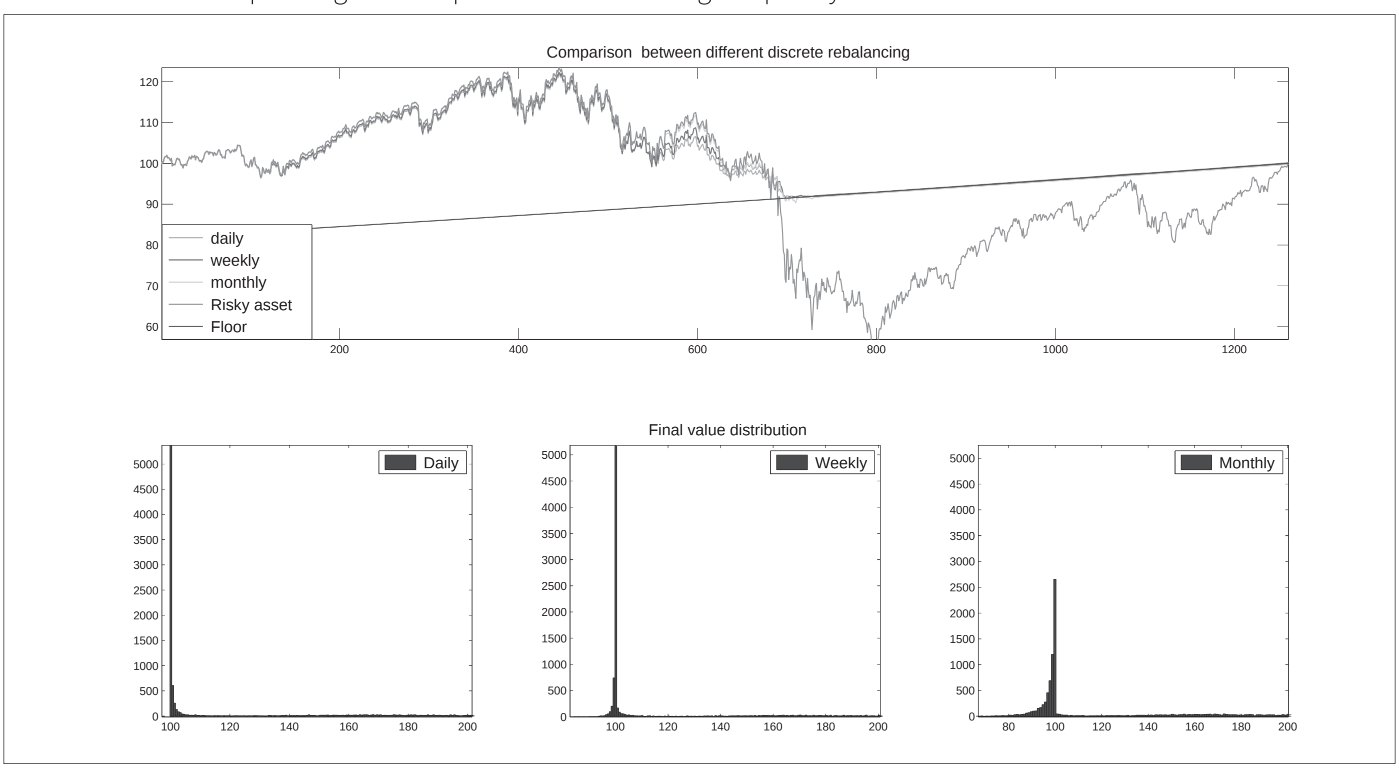


Figure 3
Statistical Metrics Depending On Multiplier and Rebalancing Frequency

	CPPI with *m* = 3			CPPI with *m* = 6		
	Daily	Weeky	Monthly	Daily	Weeky	Monthly
Man	123.31	122.39	119.75	124.10	124.87	125.01
Std-dev	31.58	32.66	36.86	42.62	43.88	48.10
95% quantile	100.48	99.98	97.01	99.99	99.13	89.69
99.5% quantile	100.02	99.88	91.47	99.98	95.20	74.26
5% quantile	194.37	195.23	197.94	216.51	218.50	225.46
0.5% quantile	266.47	284.07	282.58	294.49	293.75	311.46
Rebalancing cost	0.91	0.44	0.26	0.78	0.46	0.31

However, using a constant volatility and lognormal distribution modeling is not consistent with empirically observed jumps during extreme market moves. They are likely to breach the bond floor. Jumps are thus added in the next section.

Jump Modeling

For computational tractability, we chose the double exponential Kou model.[2] The Kou model introduces jumps into the stochastic process for stock returns as a set of random Poisson processes. The Kou model is defined as follows:

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW + d\left(\Sigma_{i=1}^{N_t} e^{Y_i} - 1\right)$$

where W is a standard Brownian motion, N is the added (Poisson) jump process, where the jump sizes {Y1, Y2, . . .} are independent and identically distributed (iid) random variables with a common asymmetric double exponential density and

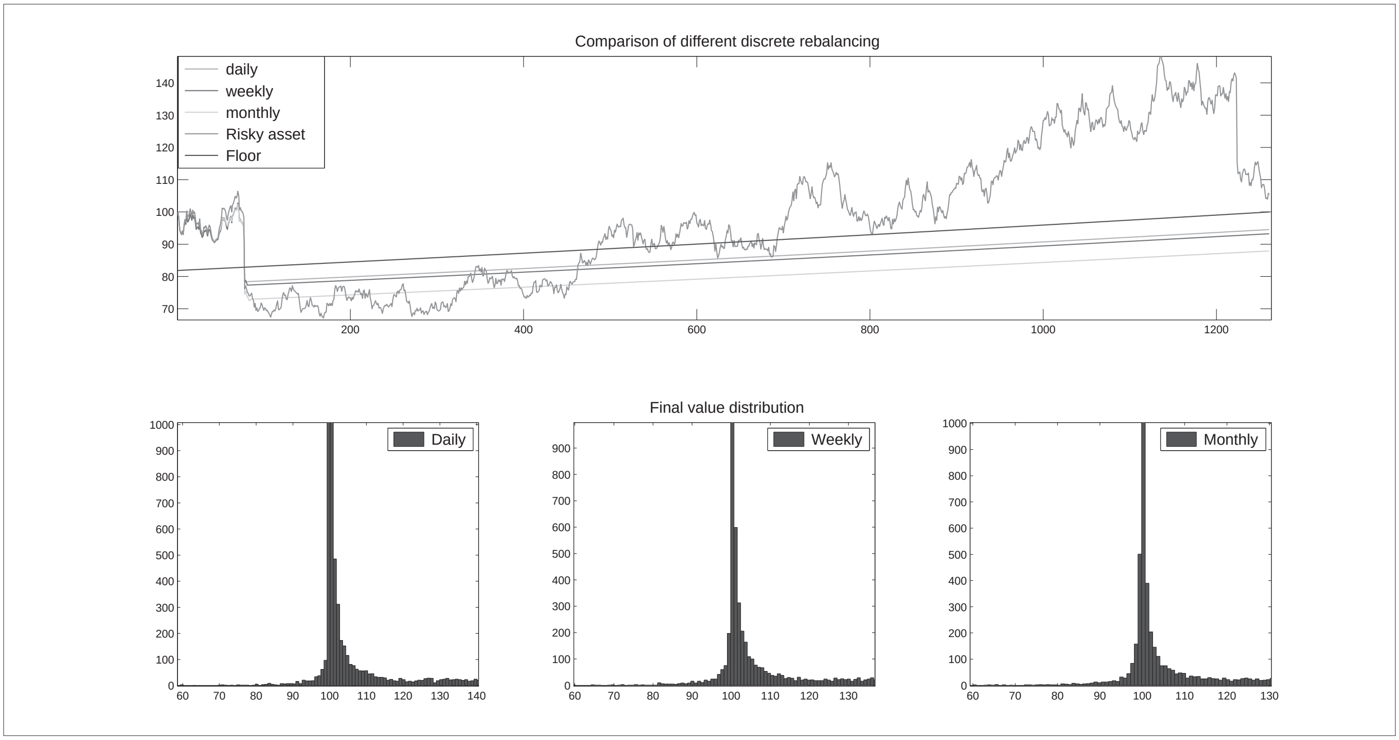
$$f(y) = (1 - p)\gamma^+ e^{-\gamma^+ y} 1_{y \geq 0} + p\gamma^- e^{-\gamma^- y} 1_{y<0}$$

γ^+/γ^- are the intensity of positive/negative jumps, and $(1 - p)$ and p are the likelihood of positive and negative jumps, respectively. The calibration has been carried out by minimizing the quadratic error on options prices with a one-month maturity and strikes from 80 percent to 110 percent of the underlying. The strategy results are shown in Figures 4 and 5.

Figure 4
Statistical Metrics Depending On Multiplier and Rebalancing Frequency

	Kou model		
	Daily	Weeky	Monthly
Man	146.28	147.10	147.57
Std-dev	52.84	52.93	53.11
95% quantile	92.19	92.21	92.03
99.5% quantile	59.38	59.08	59.23
5% quantile	238.13	238.67	239.41
0.5% quantile	349.41	350.92	350.37
Rebalancing cost	0.92	0.45	0.26

Figure 5
Simulation and Distribution of the Three Rebalancing Frequencies Under the Kou Model



The results in Figure 6 demonstrate that, whereas the probability of breaching the floor (the gap risk) significantly decreases to negligible under the traditional unrealistic assumption of continuous price movements (B&S in the figure) as the rebalancing frequency increases to daily, that is no longer the case under more realistic discontinuous modeling assumptions (here the Kou model), even with continuous rebalancing frequency.

Figure 6
Probability Of Breaching The Floor Depending On Asset Dynamics Modeling And Rebalancing Frequency

Model	Frequency	Prob(Breach Floor)
B&S	Monthly	9.07×10^{5}
	Weekly	1.2×10^{10}
	Daily	~
Kou	Continuos	0.00410

Section 2 deals with concrete strategies that at least partially mitigate gap risk through a dynamically risk-adjusted multiplier and the use of put options.

SECTION 2: MITIGATING THE DOWNSIDE RISK (GAP RISK)

Adjusting The Multiplier To Market Conditions

The manager usually sets the multiplier at the beginning of the period. Still, the probability of breaching the floor may surge in a market crash, or the manager might miss the subsequent market recovery. Thus, the multiplier needs to be adjusted according to current market conditions.

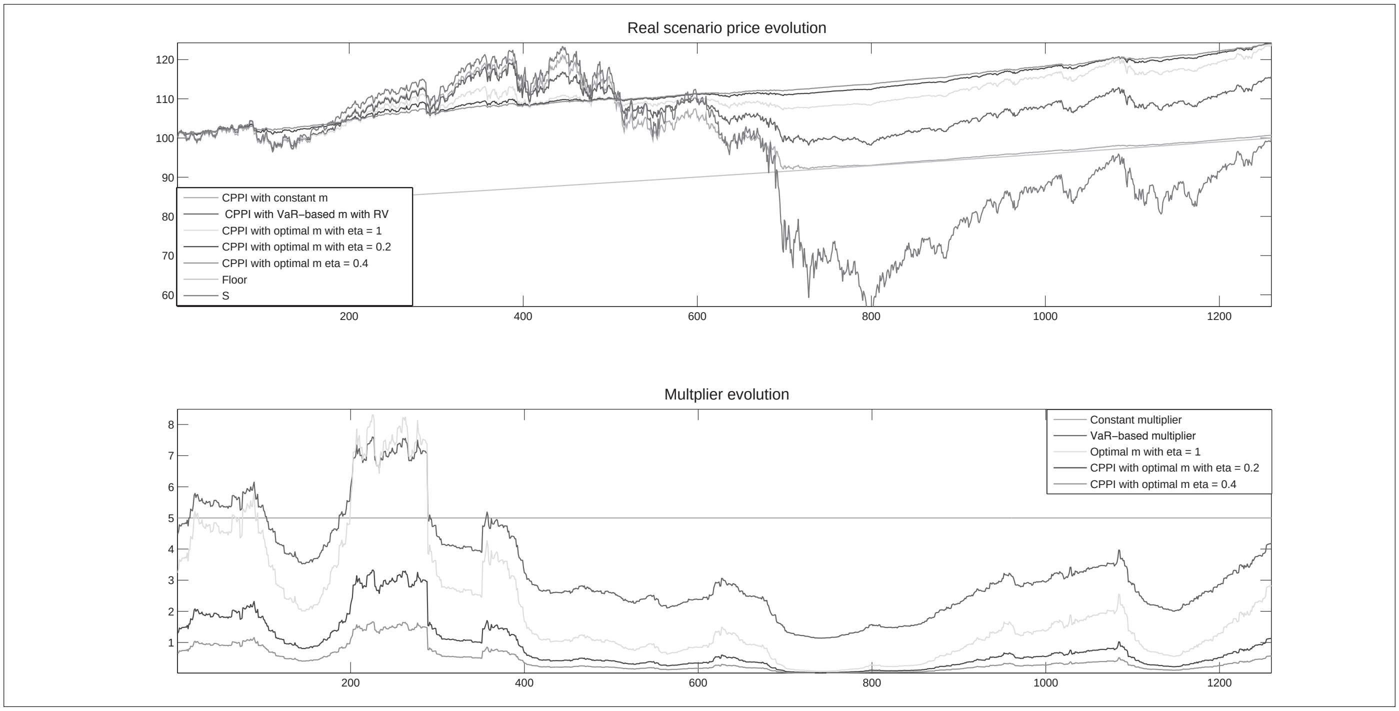
A first approach to defining a dynamic multiplier is the choice of an "optimal" m^* (for instance, using optimal certainty equivalent returns with hyperbolic absolute risk aversion utilities and log-normal distribution[3]). m^* is defined by the following formula:

$$m^* = \eta \frac{(\mu - r)}{\sigma^2}$$

where η is the sensitivity of the investor's risk tolerance to the level of wealth.

An alternative is a value-at-risk (VaR)–based multiplier where investors choose the confidence level according to their tolerance for tail risks.[4] m_t is defined as follows:

Figure 7
Comparison of Different Multipliers (VaR-based with p = 99.5% vs. the Optimal One with γ = 0.2, 0.4 and Based On Realized Volatility)



$$m_t = \frac{1}{1 - \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right)(T-t) - z_p \sigma \sqrt{T-t}\right)}$$

These two approaches offer an interesting alternative to the constant multiplier, which lacks flexibility depending on market conditions. Based on backtesting of data from 2006 to 2011 (Figure 7), the VaR-based multiplier performs better than the "optimal" one in bullish and recovery markets. In contrast, during bear markets, using the "optimal" multiplier (through m < 1) helps keep a relatively higher cushion (but misses the recovery as it makes no provision for high leverage).

To allow for a higher level of participation in the market recovery, the multiplier is adjusted with a modified volatility estimator. This is done either through a short-term exponentially weighted moving average (EWMA) realized volatility (λ = 0.94) or an estimator based on implied volatility of the strike consistent with the latest market returns. For example, if the underlying jumped 5 percent downward, the implied volatility with strike 95 percent would be chosen. This adjustment would enable the model to capture more of the upside return when markets rebound. For example, reinvesting in the risky asset in Q3 2009 in the backtest results in higher returns, as illustrated with the stock's rising ongoing performance shown in Figure 8.

Finally, the fixed frequency rebalancing may be switched to a trigger rebalancing when the multiplier is out of a specific range chosen by the portfolio manager, as illustrated by the stock's higher performance in Figure 9. On average, the rebalancing frequency becomes every other day, which is consistent with the usual practice in CPPI asset management—while the cost of rebalancing is cut by half in comparison to a daily rebalancing (that is, as low as weekly or monthly).

Adjusting the multiplier dynamically allows it to be more reactive to market conditions and explicitly dependent on the investor's risk aversion. However, it is still exposed to the downside risk in case of sudden jumps (a "black swan" event such as a market crash of 20 percent in one day) where options may be useful to hedge such gap risks.

Hedging Gap Risk

A simple hedging strategy for the CPPI can be constructed using short maturity put options. Touching the bond floor is mathematically equivalent to the cushion becoming negative. Assuming the event has not occurred up to time t_k, the gap risk is defined by

Figure 8
Comparison Between Dynamic Multiplier Based on RV and on IV Through Backtesting

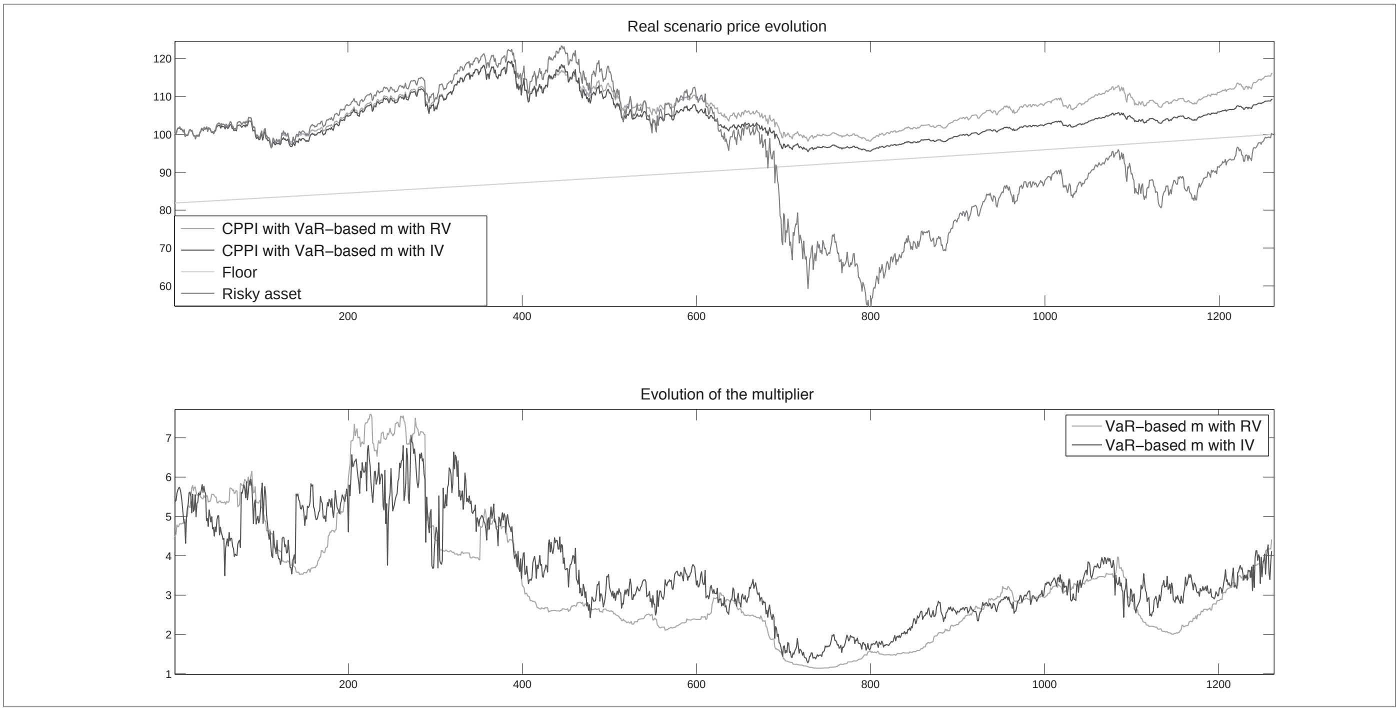


Figure 9
Comparison Between Trigger Rebalancing vs Fixed Frequency Rebalancing

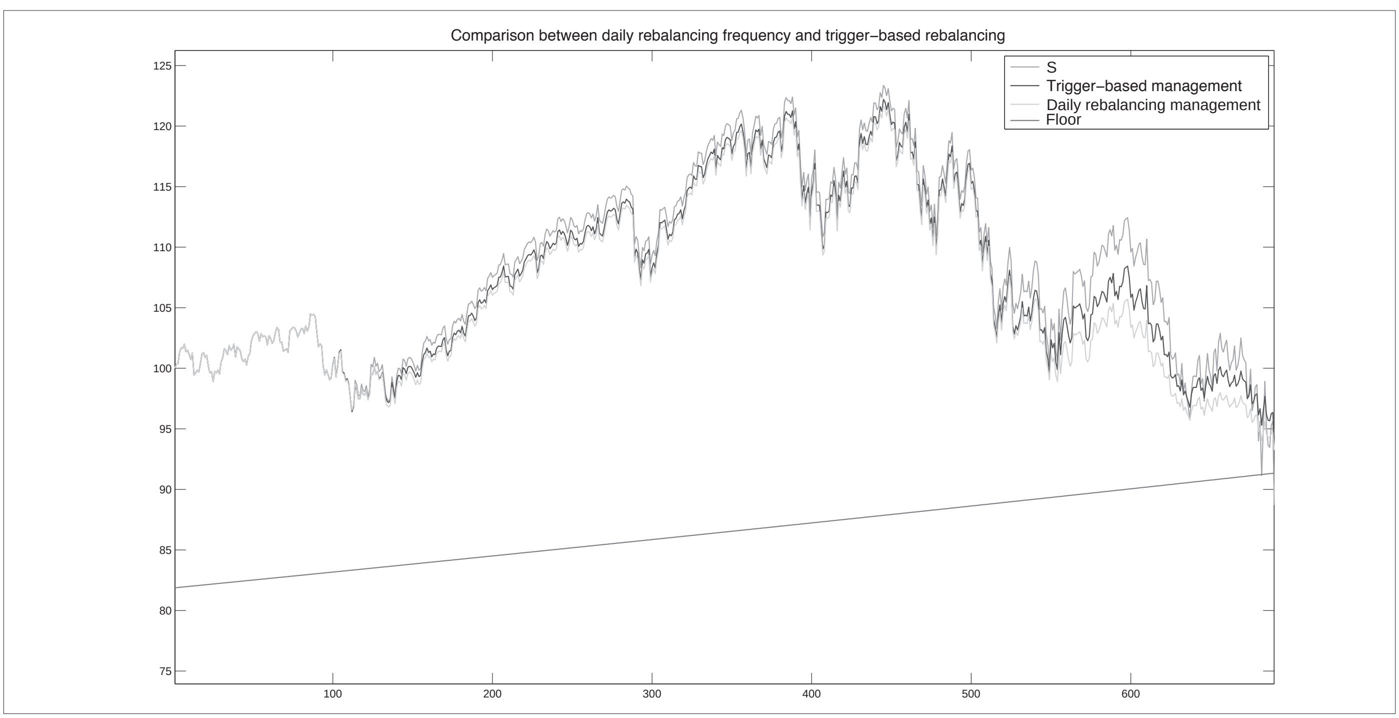
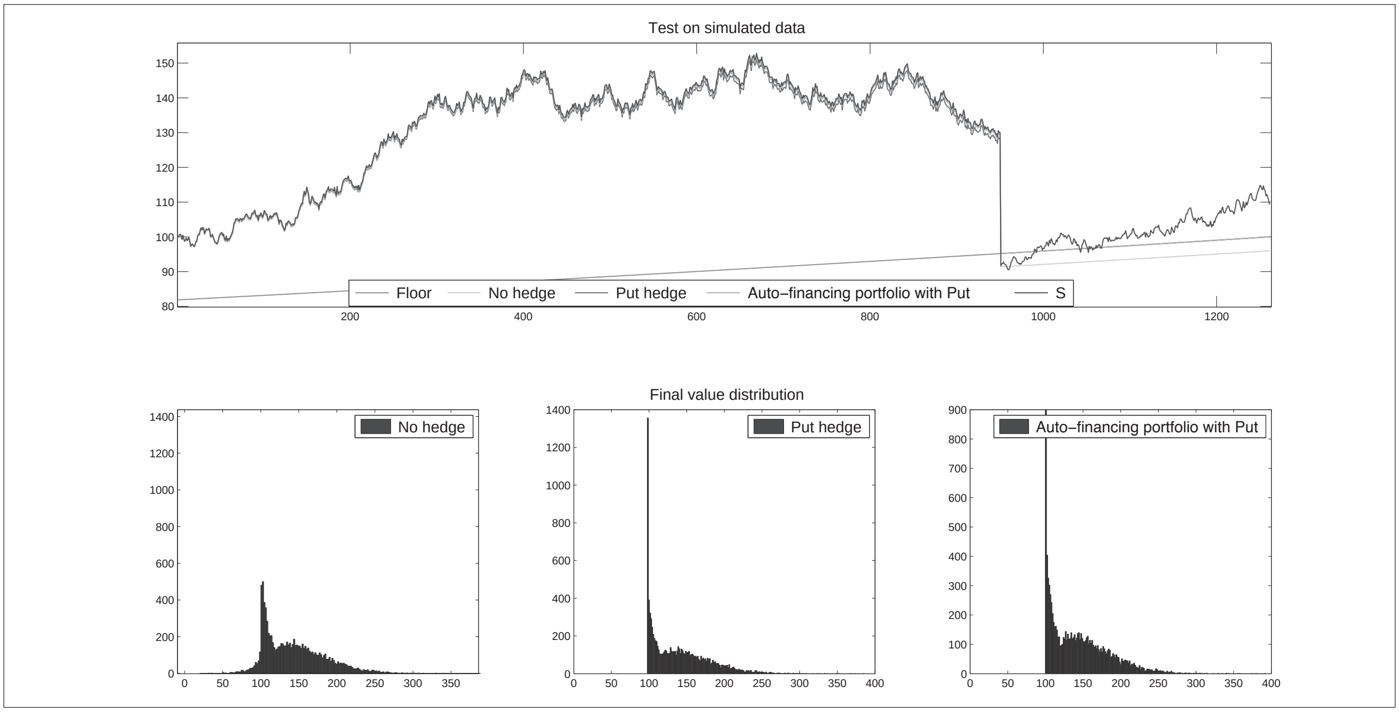


Figure 10
Comparison Between No Hedging and Put Hedging



$$C_{t_{k+1}} < 0 \Leftrightarrow m\frac{S_{t_{k+1}}}{s_{t_k}} - (m-1)e^{r\frac{T}{N}} < 0$$

This risk can be hedged by buying put options at each rebalancing period with a strike price of

$$\left(1-\frac{1}{m}e^{r\frac{T}{N}}S_{t_k}\right)$$

and with maturity equal to the CPPI rebalancing frequency. To hedge the whole portfolio, the manager needs a number of puts equal to

$$m\frac{C_{t_k}}{S_{t_k}}$$

which is the risky asset exposure. The discounted payoff in this case is then

$$e^{-r\frac{T}{N}}C_{tk}\left((m-1)e^{r\frac{T}{N}} - m\frac{S_{t_{k+1}}}{S_{t_k}}\right)^+$$

While the hedging cost is

$$\text{Cost}_{t_k} = m\frac{C_{t_k}}{S_{t_k}}\mathbb{E}^{\mathbb{Q}}\left[\left(\left(1-\frac{1}{m}\right)e^{r\frac{T}{N}}S_{t_k} - S_{t_{k+1}}\right)^+\right]$$

We observe the following impacts of hedging with puts:

- The guarantee is ensured, and the manager no longer holds the risk of breaching the floor. However, once the put is exercised and the floor recovered, the manager needs to monetize that option to keep the guarantee until maturity.

- In terms of profit and loss distributions, the CPPI distribution with put option hedging is a truncation of the classic CPPI where losses are cut (left tail limited by the guarantee).

CONCLUSION

In this article we have presented a study of the CPPI as an insurance contract, a review of its theory and practice as well as its modeling and hedging issues for a risk/return/cost perspective. The main conclusions are as follows:

- Jump modeling is an essential element of CPPI modeling. It allows the model to measure the non-zero probability of breaching the floor.

- Correctly choosing and adjusting the multiplier dynamically significantly reduces the downside risk according to a VaR indicator. The multiplier decreases in periods of market turmoil, reducing the risk exposure, and increases during periods of market recovery.

- Hedging the gap risk is possible through normal put options. ■

Professor Aymeric Kalife, AQ, is the head of savings and variable annuities at AXA Group and an associate professor in finance at Paris Dauphine University. He can be reached at *aymeric.kalife.fr@gmail.com*.

Saad Mouti completed his Ph.D. in Financial and Actuarial Mathematics at Pierre and Marie Curie University (Paris 6, France) in 2017. His current research interests include sustainable responsible investing, market impact on options, volatility modeling and forecasting.

ENDNOTES

1 F. Black and R. Jones. 1987. "Simplifying portfolio insurance." *Journal of Portfolio Management* 14(1): 48–51.

2 S.G. Kou. 2002. "A jump-diffusion model for option pricing." *Management Science* 48(8): 1086–1101.

3 Jacques Pézier. 2011. "Rationalization of investment preference criteria." ICMA Centre Discussion Papers in Finance DP2011-12. University of Reading.

4 C. Jiang, Y. Ma and Y. An. 2009. "The effectiveness of the VaR-based portfolio insurance strategy: An empirical analysis." *International Review of Financial Analysis* 18(4): 185–197.

5 P. Tankov. 2010. "Pricing and hedging gap risk." *Journal of Computational Finance* 13(3): 33–59.