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A RUIN FUNCTION APPROXIMATION

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ABSTRACT

This paper derives a formula for approximating the ruin function of collective risk theory. This formula is simple to calculate and is based on the distribution of claims and the security loading. It is partially based on several papers in the *Transactions* dealing with approximating the distribution of total claims in collective risk theory. The derivations for the formula are given in the Appendix.

Use of the ruin function in setting retention limits is explained. The following questions are answered, for example. If the retention level is set at $\$X$, what fund is needed in order that the adverse fluctuation will stay below its value with high probability, say, 0.99? Various retention levels and funds are compared for practical distributions. If the upper limit on adverse fluctuation is $\$Y$, what is the corresponding retention level?

Use of the ruin function in deciding the amount of initial capital for a new line of business is discussed.

INTRODUCTION

IN COLLECTIVE risk theory the ruin function is of great importance. We will emphasize this briefly with several examples. But, if we grant its importance, it is necessary to know how to calculate it. My recent paper [3] contains some new ways in which to calculate it. These and earlier methods, however, are hard to perform in most cases and impossible to perform in many practical cases. It is therefore very worthwhile to study approximation methods. We will derive an approximation to the ruin function, using some recent results of Mr. D. K. Bartlett [1] and Dr. Newton L. Bowers [4]. We will then compare values obtained through the approximation method with the exact values in one well-known example and then give several values of the ruin function for a practical example.

Let us consider two examples in which the ruin function can be used. Assume that you work for a company that is contemplating entering a

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new line of insurance, say, accident and health insurance. Your company would like to set aside some capital for this venture. How much is needed in order that there is a 99 per cent chance that you will not have to dip into your other funds? Provided you are willing to make several assumptions, this question can be answered by using the ruin function.

As a second example, assume that you are involved in setting a retention limit and are interested in the following questions. If the retention level is set at $\$X$, what fund is needed in order that the adverse fluctuation will stay below its value with high probability, say, 0.99? Or, if the upper limit on adverse fluctuation is $\$Y$, what is the corresponding retention level? Again the fundamental tool for answering these questions is the ruin function.

Let us now describe the ruin function in words and symbols. The ruin function computes the chance that the initial reserve plus premiums minus claims ever turns negative.

I. THE RUIN FUNCTION

We will repeat part of the description in my recent paper [3]. Let $P(z)$ be the distribution for a claim; that is, $P(z)$ is the probability that, if a claim occurs, it will be less than or equal to z. We will assume that $P(0) = 0$, which rules out nonpositive claims. Such claims occur when a life annuity terminates and a reserve is released. We will let the variable t be *operational* time. This means, for example, that if past records indicate thirty claims per year, then $t = 30$ will correspond to one calendar year. $N(t)$ is the random number of claims in time t. We will assume that claims occur in such a way that $N(t)$ has a Poisson distribution with mean t. The reader can consult Dr. Paul Kahn's paper [6] relative to this assumption. If p_1 is the average claim amount, then in time t the insurance company would charge $p_1 t$ as the *aggregate* net risk premium, plus an *aggregate* security loading of λt. We also assume that the insurance company begins with a risk reserve of size u. At time t, the risk reserve $U(t)$ is given by

$$U(t) = u + (p_1 + \lambda)t - \sum_{i=1}^{N(t)} X_i\,, \tag{1}$$

where the X_i's are independent random variables, representing the claims. Each X_i has the distribution $P(z)$. The symbol

$$\sum_{i=1}^{N(t)} X_i$$

represents the *aggregate* claims up to time t. The expression

$$\underset{0 \le t \le T}{\text{minimum}}\ U(t)$$

will refer to the smallest value of $U(t)$ over the time range $0 \leq t \leq T$. The ruin function is

$$\psi(u) = \lim_{T\to\infty} P[\underset{0\leq t\leq T}{\text{minimum}}\ U(t) < 0] .$$

This represents the probability that the risk reserve eventually becomes negative.

Let us explain the use of $\psi(u)$ in the two examples. In the first example, the capital set aside is u. It is determined by setting the expression for $\psi(u) = 0.01$ and solving for u. In the second example, the retention level equals the upper limit of $P(z)$. The adverse fluctuation equals the excess of total claims over total gross premiums. It will exceed a level u with probability $\psi(u)$. Holding u fixed, one can vary the retention level so that $\psi(u)$ stays below an appropriate level, say, 0.01.

$\psi(u)$ is not a probability distribution function (see Hoel [5], pp. 23, 37), but it is simply related to one, namely,

$$\psi^*(u) = \begin{cases} 1 - \psi(u) , & u \geq 0 \\ 0 , & u < 0 . \end{cases} \tag{2}$$

As a distribution function, $\psi^*(u)$ involves the random variable Z defined by

$$Z = \underset{0\leq t<\infty}{\text{maximum}}\left[\sum_{i=1}^{N(t)} X_i - t(p_1 + \lambda)\right] . \tag{3}$$

The proof of this is in the Appendix.

Roughly speaking, Z is the maximum excess of claims over income examined at each time point of very long time periods. We will now quote the formulas for the mean and variance of $\psi^*(u)$.

II. MOMENTS OF $\psi^*(u)$

The introduction has revealed how $\psi(u)$ depends on $P(z)$ and λ. It is therefore to be expected that the mean value of $\psi^*(u)$ does too. The proofs of the following theorems are in the Appendix.

Theorem 1.—If $E(X^2)$ is the second moment about the origin of the claim distribution and λ is the security loading, the mean of $\psi^*(u)$ is denoted and given by the formula

$$E(Z) = \frac{E(X^2)}{2\lambda} . \tag{4}$$

One would expect the variance of Z to depend on higher moments of the claim distribution. This is the case.

Theorem 2.—If $E(X^3)$ is the third moment about the origin of the claim distribution, and λ is the security loading,

$$\text{Var}(Z) = \frac{E(X^3)}{3\lambda} + [E(Z)]^2 . \tag{5}$$

As the first example of these calculations, assume that $P(z) = 1 - e^{-z}$, $z \geq 0$. Then $E[X^k] = k!$. Assume $\lambda = 0.3$. Then $E(Z) = 3.333$, and $\text{Var}(Z) = 17.778$. (We are using \$1,000 units.)

As the second example, assume that

$$\begin{aligned} P(z) &= 0 && \text{for} && z < 2 \\ &= 0.3 && \text{for} && 2 \leq z < 5 \\ &= 0.5 && \text{for} && 5 \leq z < 10 \\ &= 0.8 && \text{for} && 10 \leq z < 20 \\ &= 1.0 && \text{for} && 20 \leq z . \end{aligned}$$

Then $E(Z) = 22.4$ and $\text{Var}(Z) = 746.6$. These figures are based on $\lambda = 0.3E(X) = 2.6$, $E(X^2) = 116.2$, and $E(X^3) = 1927.4$.

As a third example, assume that

$$\begin{aligned} P(z) &= 0 && \text{for} && z < 2 \\ &= 0.3 && \text{for} && 2 \leq z < 5 \\ &= 0.5 && \text{for} && 5 \leq z < 10 \\ &= 0.8 && \text{for} && 10 \leq z < 20 \\ &= 0.85 && \text{for} && 20 \leq z < 30 \\ &= 0.90 && \text{for} && 30 \leq z < 40 \\ &= 0.95 && \text{for} && 40 \leq z < 50 \\ &= 1.00 && \text{for} && 50 \leq z . \end{aligned}$$

Here $\lambda = 0.3E(X) = 3.5$, $E(X^2) = 306.2$, and $E(X^3) = 11{,}527.4$. Using these figures, we obtain $E(Z) = 43.7$ and $\text{Var}(Z) = 3{,}007.5$.

III. APPROXIMATION OF $\psi^*(u)$

The papers by Bartlett and Bowers tell how to approximate a density function in terms of gamma densities. We will now apply the preceding results to such an approximation. Since Bowers' paper refers to the approximation of any density (not just to that of aggregate losses, as Bartlett's paper does), we will refer to Bowers' paper throughout. We assume, as Bowers does [4, p. 127], that $\psi^*(u)$ has a density function.

Theorem 3.—The distribution function $\psi^*(u)$ has a jump of $1 - p_1/(p_1 + \lambda)$ at $u = 0$, and for $u > 0$ has the approximate form

$$\psi^*(u) = \Gamma(\beta u, a) = \int_0^{\beta u} \frac{w^{a-1}e^{-w}}{\Gamma(a)} dw , \tag{6}$$

where $\beta = E(Z)/\text{Var}(Z)$ and $a = [E(Z)]^2/\text{Var}(Z)$.

We will now illustrate this with Example 1. There $\beta = 0.187$, $\alpha = 0.625$. To compare with the known values of 0.99, 0.95, and 0.90 from reference [3], we will let $u = 18.800$, 11.833, and 8.831 thousand dollar units, respectively. To find $\psi^*(18.8) = \Gamma(3.516, 0.625)$, one can use the references to incomplete gamma functions cited by Bowers. We used reference [8]. Some explanation of that table is appropriate. By the use of the letters of that reference, several transformations reveal that the "areas" are values of

$$\Gamma\left(\frac{4}{a_3^2} + \frac{2}{a_3}t, \frac{4}{a_3^2}\right).$$

Since $4/a_3^2$ will always be too large for us, we repeatedly used a recursion relation of Bowers:

$$\Gamma(x, \alpha - 3) = \Gamma(x, \alpha + 1) + \frac{x^\alpha e^{-x}}{\Gamma(\alpha + 1)} + \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} + \frac{x^{\alpha-2} e^{-x}}{\Gamma(\alpha - 1)} + \frac{x^{\alpha-3} e^{-x}}{\Gamma(\alpha - 2)}.$$

Thus, setting $4/a_3^2 = 4.625$, we obtained $a_3 = 0.93$, and, setting $4/a_3^2 + (2/a_3)t = 3.516$, we obtained $t = -0.5157$. Using these values of a_3 and t produced $\Gamma(3.516, 4.625) = 0.3428$. We next computed $(3.516)^{3.625}e^{-3.516}/\Gamma(4.625)$, using the fact that $\Gamma(4.625) = 3.625(2.625)(1.625)\Gamma(1.625)$ and that $\Gamma(1.625) = 0.897$ (reference [7]). The quotient value obtained was 0.2042. The other terms had values of 0.2105, 0.1572, and 0.0727 and were easy to calculate. Thus $(3.516)^{2.625}e^{-3.516}/\Gamma(3.625) = 3.625/3.516$ $(0.2042) = 0.2105$. Adding these five numbers gave $\psi^*(18.8) = \Gamma(3.516, 0.625) = 0.9874$, remarkably close to the exact value. Similarly, $\psi^*(11.833) = 0.9505$, and $\psi^*(8.831) = 0.9039$. This provides a rough check on the accuracy of the approximation and implies that this method is probably quite accurate.

We will now compare the initial capitals needed to hold $\psi(u) \doteq 0.01$ in Examples 2 and 3. This will illustrate the effect of raising the retention limit. For Example 2, $\beta = 0.03$, $\alpha = 0.67$, and $\psi^*(125) = 0.989$. For Example 3, $\beta = 0.015$, $\alpha = 0.635$, and $\psi^*(250) = 0.989$. Thus, increasing the retention level from $20,000 to $50,000 requires $125,000 additional initial capital to hold the probability of ruin to 0.01. The approach to 0.99 probability becomes quite slow. For example, with a fund of $200,000, one already has a probability of 0.98.

If we felt that $200,000 was an appropriate upper bound on adverse fluctuation, we could have solved for the retention level, by trying distributions with upper limits varying from $20,000 to $50,000, until we found one such that $\psi(200) \fallingdotseq 0.01$. By using the previous distributions as guides, we are led to the following distribution:

$$\begin{aligned} P(z) &= 0 && \text{for} \quad z < 2 \\ &= 0.3 && \text{for} \quad 2 \leq z < 5 \\ &= 0.5 && \text{for} \quad 5 \leq z < 10 \\ &= 0.8 && \text{for} \quad 10 \leq z < 20 \\ &= 0.87 && \text{for} \quad 20 \leq z < 30 \\ &= 0.94 && \text{for} \quad 30 \leq z < 40 \\ &= 1.00 && \text{for} \quad 40 \leq z\,. \end{aligned}$$

Again, we assume $\lambda = 0.3p_1$, or 3.15.

If the reader consults the author's earlier paper [3], the asymptotic formula (3) for $\psi(u)$ produced much lower values for the u's necessary to hold $\psi(u) \fallingdotseq 0.01$ in Examples 2 and 3, namely, \$91,429 and \$180,857. One reason for these differences will now be given. The u's probably were not large enough to use the asymptotic formula with great accuracy, and the results were too low. This statement is based on a study of Table VII on page 45 of Harald Cramèr's paper "Collective Risk Theory," the 1955 *Jubilee Volume of Försäkringsaktiebolaget Skandia.* That table shows 98 per cent accuracy for the asymptotic formula only for a u value 40 times p_1. This emphasizes the advantage of this paper's approximation method, since it applies for all $u \geq 0$.

APPENDIX

We will now give the proofs of the three theorems and of equation (3).

PROOF OF EQUATION 3

Assume that $u \geq 0$:

$$\begin{aligned} \psi^*(u) &= 1 - \psi(u) \\ &= 1 - \lim_{T\to\infty} P[\underset{0\leq t\leq T}{\text{minimum}}\ U(t) < 0]\,. \end{aligned}$$

But

$$1 = \lim_{T\to\infty} P[\underset{0\leq t\leq T}{\text{minimum}}\ U(t) < 0] + \lim_{T\to\infty} P[\underset{0\leq t\leq T}{\text{minimum}}\ U(t) \geq 0]\,.$$

Hence

$$\begin{aligned} \psi^*(u) &= \lim_{T\to\infty} P[\underset{0\leq t\leq T}{\text{minimum}}\ U(t) \geq 0] \\ &= \lim_{T\to\infty} P\left\{\underset{0\leq t\leq T}{\text{minimum}} \left[u + (p_1 + \lambda)t - \sum_{i=1}^{N(t)} X_i\right] \geq 0\right\} \\ &\qquad \text{by the definition of } U(t) \end{aligned}$$

$$= \lim_{T\to\infty} P\left\{\underset{0\le t\le T}{\text{maximum}} \left[\sum_{i=1}^{N(t)} X_i - t(p_1+\lambda)\right] < u\right\}$$

by inspection of the sample graphs

$$= P\left\{\underset{0\le t<\infty}{\text{maximum}} \left[\sum_{i=1}^{N(t)} X_i - t(p_1+\lambda)\right] < u\right\}$$

by the monotonicity of the sets.

(See Wilks [10], p. 13.)

PROOF OF THEOREM 1

Since $Z \geq 0$,

$$E(Z) = \int_0^\infty u d_u P_Z(u) = \int_0^\infty u d_u \psi^*(u) .$$

(As in reference [3], the Stieltjes integral

$$\int_0^\infty u d_u P_Z(u)$$

reduces to

$$\int_0^\infty u p(u) du ,$$

if $P_Z(u)$ has a derivative $p(u)$ for all u values, and to

$$\sum_{i=1}^{n} u_i \Delta P_Z(u_i) ,$$

if $P_Z(u)$ is a pure step function.)

By corollary 3 of reference [2],

$$\int_0^\infty e^{-au} d_u \psi^*(u) = a\lambda \Big/ \left[\int_0^\infty e^{-ax} dP(x) - 1 + a(p_1+\lambda)\right] . \quad (7)$$

Using the Stieltjes version of Theorem 14 of Widder [9], p. 358,

$$E[Z] = -\frac{d}{da}\int_0^\infty e^{-au} d_u \psi^*(u)\,|_{a=0}$$

$$= \int_0^\infty e^{-au} u d_u \psi^*(u)\,|_{a=0} .$$

Differentiating the right-hand side of equation (7) produces the form 0/0 when $a = 0$. We then apply L'Hôpital's rule (see reference [9], p. 260) twice to obtain equation (4).

PROOF OF THEOREM 2

$$\text{Var}(Z) = E(Z^2) - [E(Z)]^2 .$$

$$E(Z^2) = \int_0^\infty u^2 d_u \psi^*(u)$$

$$= \frac{d^2}{da^2} \int_0^\infty e^{-au} d_u \psi^*(u) \,|_{a=0} .$$

Differentiating the right-hand side of equation (7) twice produces the form 0/0 when $a = 0$. We then apply L'Hôpital's rule three times and do much elementary algebra to obtain equation (5).

PROOF OF THEOREM 3

The first part is well known since $\psi(0) = p_1/(p_1 + \lambda)$. Following Bowers [4, p. 127], if $X = \beta Z$, with $E(X) = \text{Var}(X)$, then $\beta E(Z) = \beta^2 \text{Var}(Z)$ or $\beta = E(Z)/\text{Var}(Z)$.

In Bowers' notation, $a = E(X) = \beta E(Z) = [E(Z)]^2/\text{Var}(Z)$.

We also have

$$P[Z \leq u] = P[\beta Z \leq \beta u] = \int_0^{\beta u} f(x) dx ,$$

where we assume $\psi^*(u)$ has a density $f(x)$. Using the first term of series (2) in reference [4], $f(x) = x^{a-1}e^{-x}/\Gamma(a)$, for a as above. This completes the proof.

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