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AN ANALYSIS OF CONTRIBUTIONS TO SURPLUS

ROBERT H. JORDAN

ABSTRACT

This paper sets forth the development of formulas for analyzing the sources of changes in the excess of the asset share over the cash value, such excess being considered the surplus. The change in surplus is found to consist of four factors. Three bear a strong resemblance to the three factors of the contribution dividend formula. The fourth is a statement of the contribution to surplus during a year that results from the existence of a surplus at the beginning of the year.

The most likely use of the methods and formulas developed in the paper is in the analysis of unsatisfactory surplus patterns which occur when testing new rate-cash value-dividend structures. However, the reasons for unsatisfactory surplus development that may exist in the testing of present rate structures can become better understood through this method of analysis or a variant of it.

INTRODUCTION

ACTUARIAL literature abounds with references to the use of asset shares as an aid or guide in setting dividends.[1] In some cases, reference is made to the effect on dividends of each element of experience involved in the asset share calculation. Usually, however, asset share results, whether they are the accumulations themselves or their present values or the yearly changes in them, are examined as a whole. The relationship of the dividend formula and its experience elements on the one hand and the asset share formula and its experience elements on the other hand does not seem to have been examined in detail. In particular, there does not seem to be any discussion of how to go about analyzing unsatisfactory asset shares to determine what causes them to be unsatisfactory.

The development of computer programs for calculating asset shares

[1] Among them are the following: Robert T. Jackson, "Some Observations on Ordinary Dividends," *TSA*, XI, 764; J. B. Maclean and E. W. Marshall, *Actuarial Studies, No. 6;* Harwood Rosser, "A Present Value Approach to Profit Margins and Dividends," *TSA*, III, 187; and Walter G. Bowerman, "Contribution of Dividends," *T.I.C.A.*, IX, I, 78.

has resulted in an abundance of asset share results becoming available to many actuaries in their study of contemplated dividend scales and rate structures. Such calculations may be, and usually are, carried to very long durations. The results of such calculations may show systematic departures from what is desired or anticipated. Unless the reasons for such departures are established, misgivings and misunderstandings about the results must be expected. In the absence of correction for these departures, the usefulness of asset shares is bound to be lessened.

PURPOSE

The purpose of this paper is to show that the reasons for asset shares' departing from desired values can be determined through an analysis of the annual increments in surplus. This method has been designed for the case in which dividends are calculated by the conventional three-factor dividend formula.

The increment will be shown to consist of four factors. Three of them bear a strong resemblance to the factors of the dividend formula; the fourth reflects the increment in surplus caused by the existence of a surplus at the beginning of the year for which the analysis is being made. If a termination dividend is allowed (making the dividend formula a four-factor formula, in effect), a fifth factor is needed for this analysis.

The value of this form of analysis derives from the following:

1. By means of it one can examine the amount of the contribution to surplus caused by each experience assumption and can relate that contribution to the objective that has been set for it (which may, of course, have been set without any attempt at great precision).
2. The formulas of the elements of the surplus contribution become clear. On occasion it may develop that one element has consistently been causing a negative contribution when it had been assumed to be producing a positive or zero contribution; this result is usually caused by the fact that the formula for the contribution is not what intuition may have led one to expect.
3. The contribution to surplus from each contributing element and, hence, the total contribution can be set with great precision for the long term. This obviates the need to calculate extended tables of asset shares since the results will have been "controlled."

DEVELOPMENT OF THE FOUR FACTORS OF THE CONTRIBUTION TO SURPLUS WHEN CONTINUOUS RESERVES ARE HELD

For participating insurance the classical use of asset shares is to demonstrate that a rate-cash value-dividend structure is equitable by showing that sufficient assets are generated by the class of business being studied to carry out the obligations that attach to it; in this paper, the need for surplus will be considered one of the obligations. In evaluating a rate-cash value-dividend structure, it seems safe to say that the determination of what is equitable is a highly individualistic matter. The determination depends on many factors about which decisions must be made before the calculation can be made, but two have particular importance.

First, it is necessary to establish a philosophy that can be reflected in the calculations concerning the assistance older policies, having adequate surplus, must give to new contracts (which is referred to sometimes as "how expenses will be allocated" and sometimes as "how interest will be allocated"). Second, surplus objectives must be set: When must the asset share exceed the cash value? What progress of surplus is desired thereafter?

The development that follows does not depend upon the decision regarding assistance provided by older policies to new business. If that decision affects only the allocation of expenses, the formulas are unaffected, although the values represented by the symbols will be affected. If that decision has an impact on the allocation of investment income, it is possible that the asset share formula will be different from the conventional formula; one such variation of the asset share formula is illustrated under the section headed "Asset Funds."

The development that follows can apply to all policy years. However, as a practical matter, it is most effective if applied only where a suitable surplus has been (or, in the opinion of the analyst, should have been) developed, and this paper studies only that situation. In other words, this method is intended primarily for use after the cessation of the inevitable interplay among (*a*) amortization of heavy initial expense; (*b*) dividend expense charges designed to give smoothness and the desired "steepness"; and (*c*) the availability of "select mortality gains."

Accordingly, a definition of the surplus objective is needed. Therefore, for purposes of this paper, attainment of the following objectives will be considered necessary for a rate-cash value-dividend structure to be equitable:

1. At the end of a years or at the end of the premium-paying period, if earlier, this duration being expressed as a, the asset share is related to the cash value in accordance with the following formula:

$$ {}_{a}\mathrm{AS}_{x}^{p} = (1 + {}_{a}k_{x}^{p}) \cdot {}_{a}\mathrm{CV}_{x}^{p} \, , $$

   where ${}_{a}\mathrm{AS}_{x}^{p}$ is the asset share for plan p for issue age x at the end of a years and ${}_{a}\mathrm{CV}_{x}^{p}$ is the cash value for plan p for issue age x at the end of a years. k should usually be in the vicinity of, say, 0.05, but, as the symbol indicates, it can vary by issue age and plan.

2. If $(1 + {}_tk_x^p) = {}_t\mathrm{AS}_x^p / {}_t\mathrm{CV}_x^p$, then for $t > a$, $(1 + {}_tk_x^p)$ should be essentially equal to the value for year a.

Why, one may well ask, should variation in $(1 + {}_ak_x^p)$ be allowed by plan and age at issue? For purposes of a discussion such as this, it is not necessary to recognize such variation. To state that there may be variation in $(1 + {}_ak_x^p)$ merely reflects the practical significance of competition and other judgment factors that must be weighed in dealing with the on-the-job problem. It may well be that, as results develop, some re-evaluation will occur in the number of years defined by a, the values of $(1 + {}_ak_x^p)$, or both.

In order to keep symbolic complexity to a minimum, it will be assumed in the formulas developed later in the paper that there are no variations by plan, or by amount, and so forth, in the asset share and dividend rates of interest, mortality, and withdrawal, although it is recognized that some variation does occur in practice; the adjustments to the formulas needed to give effect to such variations are easily made. Also to reduce complexity, the formulas are developed only for the Ordinary Life plan. The adjustments needed for other plans, or during any paid-up period, can be readily developed.

It is convenient at this point to establish the following relationships and definitions:

$$p_{x+t-1}^{V} \cdot {}_t\bar{V}(A_x) \doteq {}_{t-1}\bar{V}(A_x) \cdot (1 + i^V) + \bar{P}_x^d \cdot (1 + i^V) - q_{x+t-1}^{V}\left(1{,}000 + \frac{\bar{P}_x^d}{2}\right)\left(1 + \frac{i^V}{2}\right), \tag{1}$$

$$ {}_tD_x = (\bar{L}_x - {}_tE_x^D)(1 + i^D) + (i^D - i^V)[{}_{t-1}\bar{V}(\bar{A}_x) + \bar{P}_x^d] + (q_{x+t-1}^{V} - q_{x+t-1}^{D})[1{,}000 - {}_t\bar{V}(A_x)], \tag{2}$$

$$ {}_t\mathrm{AS}_x = \left(\frac{1}{1 - q_{[x]+t-1}^{\mathrm{AS}} - wq_{[x],t-1}^{\mathrm{AS}}}\right)\Big[({}_{t-1}\mathrm{AS}_x + \mathrm{GP}_x - {}_tE_x^{\mathrm{AS}})(1 + i^{\mathrm{AS}}) - q_{[x]+t-1}^{\mathrm{AS}}\left(1{,}000 + \frac{\mathrm{GP}_x}{2}\right)\left(1 + \frac{i^{\mathrm{AS}}}{2}\right) - {}_tD_x - {}_t\mathrm{CV}_x \cdot wq_{[x]\ t-1}^{\mathrm{AS}}\Big], \tag{3}$$

where

${}_t\mathrm{CV}_x$ = tth-year cash value per \$1,000 face amount;
${}_t\bar{V}(\bar{A}_x)$ = continuous terminal reserve at the end of the tth year for a continuous assurance of \$1,000;
$\bar{P}_x^d = \bar{P}(\bar{A}_x) \cdot d/\delta$;

i^V, i^D, i^{AS} = valuation, dividend distribution, and asset share accumulation interest rates, respectively;

$q^V_{x+t}, q^D_{x+t}, q^{AS}_{[x]+t}$ = valuation, dividend, and asset share mortality rates, respectively, for policy year $t+1$;

${}_tD_x$ = tth-year dividend (payable in full upon death) (formula given is only one of many that are in use);

GP_x = gross premium per \$1,000 face amount;

$\bar{L}_x$ = loading per \$1,000 face amount $= GP_x - \bar{P}^d_x$;

${}_tE^D_x, {}_tE^{AS}_x$ = tth-year expense charges in the dividend formula and in the asset share, respectively;

$wq^{AS}_{[x],t}$ = rate of voluntary withdrawal for policy year $t+1$.

The following functions are those required by the development that follows:

$$F^V_x = 1{,}000\cdot\frac{i^V}{2} + \frac{\bar{P}^d_x}{2}\left(1+\frac{i^V}{2}\right);$$

$$F^D_x = 1{,}000\cdot\frac{i^D}{2} + \frac{\bar{P}^d_x}{2}\left(1+\frac{i^D}{2}\right);$$

$$F^{AS}_x = 1{,}000\cdot\frac{i^{AS}}{2} + \frac{GP_x}{2}\left(1+\frac{i^{AS}}{2}\right).$$

If ${}_tAS_x = {}_tCV_x + {}_t\bar{S}_x$, where ${}_t\bar{S}_x$ is the surplus per \$1,000 face amount at the end of year t, then the increase in surplus in year t for a plan where all values are calculated on a continuous basis may be expressed as

$${}_t\bar{C}_x = {}_t\bar{S}_x - {}_{t-1}\bar{S}_x = ({}_tAS_x - {}_tCV_x) - {}_{t-1}\bar{S}_x\,. \tag{4}$$

It will be assumed that ${}_tCV_x = {}_t\bar{V}(\bar{A}_x)$ for all $t \geqq \alpha$. It is not material what reserve method is used, although it will be assumed that renewal net premiums are level.

Then, dropping the reference to the issue age from the symbols,

$${}_t\bar{C} = {}_tAS - {}_tV(\bar{A}) - {}_{t-1}\bar{S}\,, \tag{5}$$

$$\begin{aligned}{}_t\bar{C} = \left(\frac{1}{1-q^{AS}_{t-1}-wq^{AS}_{t-1}}\right)\Big[&({}_{t-1}AS + GP - {}_tE^{AS})(1+i^{AS}) \\ &- q^{AS}_{t-1}\left(1{,}000+\frac{GP}{2}\right)\left(1+\frac{i^{AS}}{2}\right) - {}_tV(A)\cdot wq^{AS}_{t-1} - {}_tD\Big] \\ &- {}_t\bar{V}(\bar{A}) - {}_{t-1}\bar{S}\,.\end{aligned} \tag{6}$$

Appendix A shows that, substituting the expression given in formula (2) for ${}_tD$ and simplifying, results in

$$ {}_t\bar{C} \doteq \frac{1}{(1 - q_{t-1}^{\mathrm{AS}} - wq_{t-1}^{\mathrm{AS}})} \begin{cases} (i^{\mathrm{AS}} - i^D)[{}_{t-1}\bar{V}(\bar{A}) + \bar{P}^d + \bar{L}] \\ \qquad + {}_tE^D \cdot i^D - {}_tE^{\mathrm{AS}} \cdot i^{\mathrm{AS}} \\ + (q_{t-1}^D - q_{t-1}^{\mathrm{AS}})[1{,}000 - {}_t\bar{V}(\bar{A})] \\ \qquad + q_{t-1}^V \cdot F^V - q_{t-1}^{\mathrm{AS}} \cdot F^{\mathrm{AS}} \\ + {}_tE^D - {}_tE^{\mathrm{AS}} \\ + {}_{t-1}\bar{S}(i^{\mathrm{AS}} + q_{t-1}^{\mathrm{AS}} + wq_{t-1}^{\mathrm{AS}}) \,. \end{cases} \tag{7} $$

This relationship applies at any duration for which ${}_{t-1}\mathrm{CV} = {}_{t-1}\bar{V}(\bar{A})$ and ${}_t\mathrm{CV} = {}_t\bar{V}(\bar{A})$, as noted previously.

ANALYSIS OF THE ELEMENTS OF THE CONTRIBUTION TO SURPLUS

It is convenient to consider formula (7) in the following way:

$$ {}_t\bar{C} \doteq \frac{1}{(1 - q_{t-1}^{\mathrm{AS}} - wq_{t-1}^{\mathrm{AS}})} ({}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E + {}_t\bar{C}^S) \,. \tag{8} $$

${}_t\bar{C}^I$ can for most practical purposes be taken as $(i^{\mathrm{AS}} - i^D)[{}_{t-1}\bar{V}(\bar{A}) + \bar{P}^d + \bar{L}]$, the sum of the other elements usually being negligible. This formula is of the form expected and is parallel to the excess interest element of the dividend formula. As is customarily thought, if $i^{\mathrm{AS}} > i^D$, there is a regular contribution to surplus from interest, which is nearly proportionate to the initial reserve. If $i^{\mathrm{AS}} = i^D$, there is no contribution to surplus.

${}_t\bar{C}^M$ can be taken as $(q_{t-1}^D - q_{t-1}^{\mathrm{AS}})[1{,}000 - {}_t\bar{V}(\bar{A})]$, if the difference between asset share mortality and valuation mortality is equal to or less than the difference between current mortality and 1958 C.S.O. mortality, since in that case the sum of the last two terms is small at all but very high attained ages, as shown in Table 1.

For this element, too, the expected result applies, that is, if $q_{t-1}^D = q_{t-1}^{\mathrm{AS}}$, then ${}_t\bar{C}^M \doteq 0$. If some conservatism is involved in setting q_{t-1}^D, ${}_t\bar{C}^M$ will be positive. If the conservatism increases with age, then at higher attained ages ${}_t\bar{C}^M$ may take on values large enough to cause asset shares to increase more rapidly than desired. If an attempt is made to bring the asset share back to the "proper" level through a relatively simple adjustment of the dividend expense charges, only a temporary alleviation of the difficulty is likely, since the values of ${}_t\bar{C}^M$ can range over a wide set of values when q_{t-1}^D is "conservative."

${}_t\bar{C}^E$ is also unremarkable in form. The contribution to surplus that develops from this source, however, can be remarkable, depending on the dividend expense charges employed. Too literal an acceptance of the oversimplification "the loading return element of the dividend is arbitrarily determined so that the over-all objective (however expressed) may be achieved" can backfire. While this statement of the nature of the loading return element of the dividend realistically describes the situation to the date on which the "equity objective" is to be reached, that is, before the end of a years, reliance on it thereafter is not recommended. If the longer-term dividend expense charges are determined arbitrarily, it is a foregone conclusion that inequities will develop. It does not seem that a practical set of arbitrary expense charges can be developed that

TABLE 1*

VALUES OF $q_{t-1}^{V} \cdot F^{V} - q_{t-1}^{AS} \cdot F^{AS}$ FOR 1958 C.S.O. 2½ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20	\$ 0.00	\$−0.04	\$−0.15	\$−0.29
30	− .05	− .16	− .31	−0.95
40	−0.16	−0.33	−0.95	−2.77

* In this and subsequent tables, $i^{AS} = 0.04$ and q_{t-1}^{AS} are on the 1955–60 Ultimate Basic Table (male and female combined).

will deal satisfactorily across widely differing plans, issue ages, and attained ages. As one finds that a given arbitrary adjustment, much needed for some reason in one area, gives unsuitable results in some other area, one is tempted to add another arbitrary adjustment, and so on.

From the points of view of consistency of surplus development and simplicity of operation, it would seem best for the "ultimate" dividend expense charges to be exactly equal to or slightly larger than the "ultimate" asset share expense charges. The ideal, of course, would be to have the same set of ultimate dividend expense charge factors for all plans and all series. If variations in average size by series are reflected in developing the per \$1,000 asset share expense charges, or if significantly different later duration commissions or service fees are paid, it would be entirely consistent to reflect such variations in the dividend scales of the various series as experience develops.

This approach anticipates that charges or credits in the dividend scale for such items as inadequate settlement option rates, inadequate disability income rates, and the like, should be reflected directly in the dividend scale with appropriate fund accounting for such charges. If, instead, the charge is effected through an unidentified increase in the scale of expense charges without fund accounting, it may be impossible later to determine the extent of the funding that has occurred, with consequent difficulty in determining what must be done to give equitable treatment to the policyholders involved.

${}_t\bar{C}^S$, the fourth factor, represents the contribution to surplus in year t resulting from the existence of a surplus at the beginning of year t. Note that in the development no reference was made to the possible values of ${}_{t-1}\bar{S}$, so that the formula applies equally well to both negative and positive values of ${}_{t-1}\bar{S}$. The symmetry of the function is pleasing, and its simplicity makes obvious the general reasoning explanation of its existence and meaning. When $i^{AS} \doteq 4$ per cent and $wq^{AS} \doteq 2$ per cent, ${}_{t-1}\bar{S}$ changes by about 6 per cent of itself each year, there being an additional increment based on q_{t-1}^{AS}. For most plans and ages at issue the compound annual increase in the reserve for $t \geqq 20$ is 6 per cent or less, which suggests that it is appropriate for the asset share to increase at a rate of 6 per cent or less each year at this stage of the life of a policy. Therefore, if the desired surplus at $t = 20$ of about 5 per cent of the reserve is achieved, it is possible (1) to set each dividend experience element exactly equal to its corresponding asset share experience element, which causes ${}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E \doteq 0$; (2) to discontinue calculating asset shares for subsequent durations and still have certainty that equity will be achieved at later durations; and (3) to be satisfied that a suitable surplus will be generated at later durations.

Let us suppose now that for some reason it is found that a negative surplus develops at $t = 20$ for a given plan and issue age where $20 \geqq \alpha$. Let us also suppose that it is determined that the "deficit" at that point should not be allowed to affect future determinations of equity, that is, the deficit is either to be absorbed by other policyholders or is to be considered spurious (the cause cannot be uncovered). Then future "equity" can be assured by controlling the relationship of the experience elements in the dividend scale relative to those in asset shares so that the sum of ${}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E \doteq 0$. This result is, of course, most easily and satisfactorily achieved if the value of each ${}_t\bar{C}^j$ is maintained at or very close to zero.

If longer-duration asset shares are calculated in which negative values of ${}_t\bar{S}$ are left in the calculation unadjusted, it is quite possible for ${}_t\mathrm{AS}$ to become less than ${}_{t-1}\mathrm{AS}$, even if ${}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E$ had been zero for many years. In such a case, it is apparent that the asset shares themselves will not be of much value in determining whether rough justice is being meted out to policyholders of the various plans and durations. However, an examination of the ${}_t\bar{C}^j$ may throw some light on the situation.

In the complex case where close parallelism among the elements of the dividend scale and the asset share calculation does not exist, that is, where arbitrary adjustments have been made to the dividend scale in one or more areas, and where a "deficit" is encountered at some point which is to be ignored in determining future dividends, then the real power of the use of ${}_t\bar{C}^j$ can manifest itself. In cases such as this, the expense charge in the dividend will probably have complex characteristics which result in unusual loading returns. By analyzing ${}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E$, both in its components and its total, one can learn not only whether further deficits will be caused by departures from the concept of equity being employed but also in just what way the results depart from what is desired. Such an analysis is, of course, a good guide to the solution to the problem.

It should be noted that no comment has been made, nor is one necessary, in this discussion of the method of determining the underlying experience factors. This method applies equally well whether only ultimate, or both select and ultimate, mortality rates are used for dividend mortality charge purposes. Expense rates in the asset share may involve any desired degree of spreading of acquisition expenses into renewal years. A comment is in order, however, on the fact that, if spreading were done over say, twenty-five or thirty years and the basic equity objectives were set at the end of twenty years, a discontinuity would necessarily develop, either in the dividends or in ${}_t\bar{C}^j$, at the point where the spreading ended.

THE FOUR FACTORS WHEN CURTATE RESERVES ARE HELD

In this section we will examine the four factors for the case where curtate rather than continuous functions are used to calculate both reserves and cash values. In this case the set of relationships corresponding to formulas (1), (2) and (3) become

$$p_{t-1}^{V}\cdot {}_tV = {}_{t-1}V\cdot(1+i^V) + P\cdot(1+i^V) - q_{t-1}^{V}\cdot 1{,}000\,, \tag{9}$$

$$\begin{aligned} {}_tD = (L - {}_tE^D)(1+i^D) &+ (i^D - i^V)({}_{t-1}V + P) \\ &+ (q_{t-1}^{V} - q_{t-1}^{D})(1{,}000 - {}_tV)\,, \end{aligned} \tag{10}$$

$$\begin{aligned} {}_t\mathrm{AS} = \frac{1}{(1 - q_{t-1}^{\mathrm{AS}} - wq_{t-1}^{\mathrm{AS}})}\cdot\Big[&({}_{t-1}\mathrm{AS} + \mathrm{GP} - {}_tE^{\mathrm{AS}})(1+i^{\mathrm{AS}}) \\ &- q_{t-1}^{\mathrm{AS}}\Big(1{,}000 + \frac{\mathrm{GP}}{2}\Big)\Big(1 + \frac{i^{\mathrm{AS}}}{2}\Big) - {}_t\mathrm{CV}\cdot wq_{t-1}^{\mathrm{AS}} - {}_tD\Big]\,. \end{aligned} \tag{11}$$

A set of substitutions and manipulations similar to those made in Appendix A results in

$$
{}_tC = \frac{1}{(1 - q_{t-1}^{AS} - wq_{t-1}^{AS})} \begin{cases} (i^{AS} - i^{D})({}_{t-1}V + P + L) \\ \qquad + {}_tE^{D} \cdot i^{D} - {}_tE^{AS} \cdot i^{AS} \\ + (q_{t-1}^{D} - q_{t-1}^{AS})(1{,}000 - {}_tV) - q_{t-1}^{AS} \cdot F^{AS} \\ + {}_tE^{D} - {}_tE^{AS} \\ + {}_{t-1}S(i^{AS} + q_{t-1}^{AS} + wq_{t-1}^{AS}) \,. \end{cases} \tag{12}
$$

If we now designate the major expressions inside the large bracket as ${}_tC^j$, it will be observed that the form of each ${}_tC^j$ is the same as the form of the corresponding ${}_t\bar{C}^j$ except for ${}_tC^M$. The difference in the form of ${}_tC^M$

TABLE 2

VALUES OF $q_{t-1}^{AS} \cdot F^{AS}$ FOR 1941 C.S.O. $2\frac{1}{2}$ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20........	\$0.09	\$0.31	\$0.92	\$ 2.61
30........	.28	0.80	2.11	6.07
40........	0.72	1.83	4.91	13.57

is attributable to the fact that the curtate reserve formula does not make allowance for immediate payment of claims and for pro rata refund of the annual premium at death (or its equivalent, if premiums are considered to be paid continuously). Since the increase in cash value is not reduced by this charge, the net result, as compared with the continuous formula, is a reduction in the contribution to surplus. The value of $q_{t-1}^{AS} \cdot F^{AS}$, for which there is no corrective element in the formula for ${}_tC^M$, is not insignificant, as shown in Table 2. For the Ordinary Life plan with a typical scale of current participating premium rates, with $i^{AS} = 4$ per cent and mortality on the 1955–60 Ultimate Basic Table (males and females combined), values of $q_{t-1}^{AS} \cdot F^{AS}$ are as shown in Table 2.

The size and the amount of variation in these values suggest that there can be considerable difficulty in attempting to offset them through adjustment of other elements of the dividend scale. For example, suppose that $i^{AS} - i^{D} = 0.15$ per cent. Then the interest contribution, $(i^{AS} - i^{D})({}_{t-1}V + P + L)$, where values are on the 1941 C.S.O. $2\frac{1}{2}$ per cent basis, would be as shown in Table 3.

The difficulty of providing for the difference between the values in Tables 2 and 3 through a special charge in the loading return portion of the dividend is obvious.

It may be thought that a solution to this problem is to use a reserve value in the mortality factor of the dividend formula that anticipates the extra benefits, such as ${}_t\bar{V}(\bar{A}_x)$. This adjustment will not solve the problem. The difference between the mortality elements of the dividends under the two formulas is

$$\begin{aligned}(q^V_{t-1} - q^D_{t-1})(1{,}000 - {}_tV) - (q^V_{t-1} - q^D_{t-1})[1{,}000 - {}_t\bar{V}(A)] \\ = (q^V_{t-1} - q^D_{t-1})[{}_t\bar{V}(\bar{A}) - {}_tV] \\ \doteq (q^V_{t-1} - q^D_{t-1})\left[{}_tV\left(1 + \frac{1}{2\bar{a}}\right) - {}_tV\right] \\ \doteq (q^V_{t-1} - q^D_{t-1}) \cdot \frac{{}_tV}{2\bar{a}}\,.\end{aligned}$$ [2]

TABLE 3

VALUES OF ${}_tC^I$ FOR 1941 C.S.O. 2½ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20.........	\$0.42	\$0.55	\$0.69	\$0.84
30.........	.67	0.82	0.99	1.13
40.........	0.91	1.07	1.21	1.33

TABLE 4

VALUES OF $(q^V_{t-1} - q^D_{t-1}) \cdot {}_tV/2\bar{a}$ FOR 1941 C.S.O. 2½ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20.........	\$0.03	\$0.06	\$0.14	\$0.48
30.........	.06	.15	0.44	1.22
40.........	0.14	0.40	1.02	2.24

[2] See C. W. Jordan, Jr., *Life Contingencies* (2d ed.; Chicago: The Society of Actuaries, 1967), p. 115.

Values of this function on the 1941 C.S.O., $2\frac{1}{2}$ per cent basis, with q_{t-1}^{D} on the 1955–60 Ultimate Basic Table (males and females combined) are shown in Table 4.

The values of $(q_{t-1}^{V} - q_{t-1}^{D})\cdot {}_tV/2\bar{a}$ are not as large as, nor do they follow the pattern of, the "error" that it is desired to eliminate, namely, $q_{t-1}^{AS}\cdot F^{AS}$, so that another method of adjustment must be sought.

An approach having superficial appeal would be to use continuous reserve and net premium values in the dividend formula in place of the curtate values, while the progress of asset shares is tested against the cash

TABLE 5

VALUES OF $q_{t-1}^{V} \cdot F^{V} - q_{t-1}^{AS} \cdot F^{AS}$ FOR 1941 C.S.O. $2\frac{1}{2}$ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20.........	\$0.07	\$0.07	\$0.09	\$0.32
30.........	.05	.04	.15	.37
40.........	0.01	0.04	0.07	−0.29

values actually allowed, which are, of course, on the curtate basis. It can be shown that the resulting value of ${}_tC$ can be expressed as

$$\frac{1}{(1-q_{t-1}^{AS}-wq_{t-1}^{AS})}\left\{\begin{aligned}&(i^{AS}-i^{D})[{}_{t-1}\bar{V}(\bar{A})+\bar{P}^{d}+\bar{L}]\\&\qquad\qquad+{}_tE^{D}\cdot i^{D}-{}_tE^{AS}\cdot i^{AS}\\&+(q_{t-1}^{D}-q_{t-1}^{AS})[1{,}000-{}_t\bar{V}(\bar{A})]\\&\qquad\qquad+q_{t-1}^{V}\cdot F^{V}-q_{t-1}^{AS}\cdot F^{AS}\\&+{}_tE^{D}-{}_tE^{AS}\\&+{}_{t-1}S(i^{AS}+q_{t-1}^{AS}+wq_{t-1}^{AS})\\&-\frac{1}{2\bar{a}}[{}_{t-1}\bar{V}(\bar{A})(1+i^{AS})-{}_t\bar{V}(\bar{A})(1-q_{t-1}^{AS})]\,.\end{aligned}\right. \tag{13}$$

As before it is usually safe to ignore $({}_tE^{D}\cdot i^{D} - {}_tE^{AS}\cdot i^{AS})$ as being negligible; for 1941 C.S.O. issues the sum of $q_{t-1}^{V}\cdot F^{V} - q_{t-1}^{AS}\cdot F^{AS}$ appears to be small enough to ignore also (see Table 5). However, the sum shown on the last line of formula (13), as shown in Table 6, is not negligible and varies in as complex a way as does the "error" we sought to eliminate. This approach appears to be unsatisfactory.

It appears that the best solution to this problem is to introduce into the dividend formula a charge such as $q_{t-1}^{D} \cdot F^{D}$, values of which are shown in Table 7. By reducing the dividend by this amount, there will be produced very close to an exact offset to the negative contribution to surplus caused by reflection in the asset shares of immediate payment of claims and a pro rata refund of premium at death. As a result the dividend scale expense charge formula for a series of policies on the curtate basis can be more consistent with those for a series using continuous functions in calculations of reserves and cash values.

TABLE 6

VALUES OF $(1/2\ddot{a})\,[{}_{t-1}\bar{V}(1+i^{AS}) - {}_{t}\bar{V}(1-q_{t-1}^{AS})]$ FOR 1941 C.S.O. $2\frac{1}{2}$ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20	$-0.08	$-0.04	$0.20	$ 1.07
30	.10	0.40	1.25	4.01
40	0.47	1.24	3.48	10.04

TABLE 7

VALUES OF $q_{t-1}^{D} \cdot F^{D}$ FOR 1941 C.S.O. $2\frac{1}{2}$ PER CENT ISSUES

YEAR	ISSUE AGE			
	25	35	45	55
20	$0.09	$0.29	$0.87	$ 2.46
30	.26	0.75	1.99	5.71
40	0.67	1.71	4.62	12.77

ASSET FUNDS

Another interesting variation of the formula for ${}_{t}\bar{C}$ arises where, in the accumulation of the asset share, interest is credited on the initial reserve (or a similar function) rather than on the previous asset share plus premium less expenses. Such an accumulation will be referred to herein as an "asset fund" to distinguish it from the conventional asset share. One explanation for use of this technique is that, if the interest rate used is that earned on interest-bearing liabilities, the resulting aggregate of interest credits in asset funds, including those asset funds only implicitly calculated, will be equal to the actual total interest earnings of the company. The formula for the accumulation of the asset fund is obtained by replacing

$$i^{\mathrm{AS}}\left[{}_{t-1}\mathrm{AS} + \mathrm{GP} - {}_tE^{\mathrm{AS}} - \tfrac{1}{2}\left(1{,}000 + \frac{\mathrm{GP}}{2}\right)\cdot q_{t-1}^{\mathrm{AS}}\right]$$

by

$$i^{\mathrm{AF}}\left[{}_{t-1}\bar{V}(\bar{A}) + \bar{P}^{d} - \tfrac{1}{2}\left(1{,}000 + \frac{\mathrm{GP}}{2}\right)\cdot q_{t-1}^{\mathrm{AF}}\right] .$$

This substitution results in the following formula:

$$
{}_t\bar{C}^{\mathrm{AF}} = \frac{1}{(1 - q_{t-1}^{\mathrm{AF}} - wq_{t-1}^{\mathrm{AF}})}\left\{\begin{aligned}
&[{}_{t-1}\bar{V}(\bar{A}) + \bar{P}^{d}](i^{\mathrm{AF}} - i^{D}) \\
&\qquad\qquad - i^{D}(\bar{L} - {}_tE^{D}) \\
&+ (q_{t-1}^{D} - q_{t-1}^{\mathrm{AF}})[1{,}000 - {}_t\bar{V}(\bar{A})] \\
&\qquad\qquad + q_{t-1}^{V}\cdot F^{V} - q_{t-1}^{\mathrm{AF}}\cdot F^{\mathrm{AF}} \\
&+ {}_tE^{D} - {}_tE^{\mathrm{AF}} \\
&+ {}_{t-1}S(q_{t-1}^{\mathrm{AF}} + wq_{t-1}^{\mathrm{AF}}) ,
\end{aligned}\right. \tag{14}
$$

where

$$F^{\mathrm{AF}} = \left[1{,}000\,\frac{i^{\mathrm{AF}}}{2} + \frac{\mathrm{GP}}{2}\left(1 + \frac{i^{\mathrm{AF}}}{2}\right)\right] .$$

In this instance $i^{D}(\bar{L} - {}_tE^{D})$ is not negligible and $q_{t-1}^{V}\cdot F^{V} - q_{t-1}^{\mathrm{AF}}\cdot F^{\mathrm{AF}}$ may be large enough for 1958 C.S.O. issues to require recognition. The striking difference in the formula, however, rests in the fact that i^{AF} does not appear in ${}_t\bar{C}^{S}$, reflecting the fact that interest is not credited or charged on the surplus, whether it be positive or negative. The asset fund builds up faster than the asset share in the early durations; its growth at later durations is slower (assuming that a surplus is achieved in both cases). It may be worth noting that, in the case of asset funds, if there is spreading of any of the acquisition expenses over any part of the life of the policy, the annuity value used as a divisor in the spreading formula should be based on 0 per cent interest, since interest is not accumulated in the fund calculation on the expense charges.

MISCELLANEOUS SOURCES OF GAIN

As a general rule it does not seem desirable to reflect in the asset share (or asset fund) assumptions sources of gain that are not predictable or the generation of which is only remotely related to the development of the funds which the asset shares represent. Some examples of such sources of gain are capital gains on common stock; release of redundant, strengthened reserves on old disability policies; gains from supplementary contracts; and so forth. The difficulties attendant upon attempts to reflect such gains in asset share calculations need not be elaborated on here. It seems sufficient to say that attempts to reflect such gains directly in the asset shares are almost certainly going to fall of their own weight. Perhaps a reasonable approach to take to this problem is to consider such gains as available to help all business indirectly. Such an approach will permit the use of values of ${}_a(1 + k_x^p)$ that are less than those that would be allowed in the absence of such gains. So long as ${}_a(1 + k_x^p) > 1$ for all plans and ages and the subsequent ${}_tC \geqq 0$, equity can be considered to have been accomplished if the over-all company surplus objectives continue to be maintained.

TERMINATION DIVIDENDS

If $(1 + {}_ak_x^p) > 1$ and ${}_t\bar{S}$ is regularly increasing, as anticipated by the foregoing discussion, serious consideration will undoubtedly be given to the payment of a termination dividend, certainly at least to those policyholders whose policies persist until endowment maturity (including attainment of the terminal age of the valuation mortality table). It is apparent that, in raising this point, the questions of maintenance of a reserve for unusual catastrophes and the contributions of various generations of policyholders to such a reserve are bound to come up; that, however, is beyond the scope of this discussion.

If termination dividends are paid upon all terminations, the formula for the contribution to surplus must be modified by a fifth element:

$$ {}_tC^{\mathrm{TD}} = -\,{}_t\mathrm{TD}\left[q_{t-1}^{\mathrm{AS}}\cdot\left(1+\frac{i^{\mathrm{AS}}}{2}\right)+wq_{t-1}^{\mathrm{AS}}\right]\cdot\frac{1}{(1-q_{t-1}^{\mathrm{AS}}-wq_{t-1}^{\mathrm{AS}})}\,. \qquad (15) $$

Presumably ${}_t\mathrm{TD}$ will be determined in such a way that the resulting ${}_t\bar{S}$ will still be regularly increasing. This result will be accomplished if the sum of ${}_t\bar{C}$, formula (8), plus ${}_tC^{\mathrm{TD}}$ is greater than zero.

WITHDRAWAL RATES

All the formulas that have been developed here anticipate the use of a withdrawal rate in the calculation throughout. It is, of course, true that in many actuarial papers and in the *Study Notes* distributed to students, it is stated that it is not considered sound to use an assumed profit on lapse to reduce nonparticipating premiums. It is also stated that sound practice calls for increased gross premiums where tests indicate probable significant loss on lapse. Applying this reasoning to participating insurance would result in the conclusion that, when the asset share exceeds the cash value, the lapse rate should not be included in the asset share calculation. The application of this reasoning to participating insurance is not universally accepted, however.

The form of ${}_t\bar{C}^S$ and the survivorship factor, $1/(1 - q_{t-1}^{AS} - wq_{t-1}^{AS})$, make clear the form of the contribution to surplus caused by introduction of the withdrawal rate at durations where ${}_t\bar{S} > 0$. Roughly, the increase in surplus due to inclusion of the lapse rate in the calculation may be expressed as $wq_{t-1}^{AS}({}_t\bar{C}^I + {}_t\bar{C}^M + {}_t\bar{C}^E + {}_t\bar{C}^S)$.

One point of view considers this contribution to surplus as a "miscellaneous gain" from a somewhat unpredictable source that should not be reflected directly. In part this attitude stems from knowledge that the compounding of even a very low rate of withdrawal on a positive surplus can result in later surpluses that appear unreasonable.

Another point of view considers that later duration withdrawal rates are reasonably predictable and at a low level and, therefore, may properly be included in the calculation. This approach in effect considers that, as they are looked upon negatively in early duration calculations, they should be looked upon positively in later calculations; in other words, it should be a "two-way street." One point that can be made in favor of this view is that, by including withdrawal rates, one source of unreflected miscellaneous gain is eliminated, thereby reducing the extent to which the absolute values resulting from the calculations must be adjusted mentally.

The development of the formulas in this paper is not materially affected by the decision in this matter. If it is decided not to reflect withdrawal rates, it is only necessary to remove wq_{t-1}^{AS} from the formula for ${}_t\bar{C}^S$ and to replace $(1 - q_{t-1}^{AS} - wq_{t-1}^{AS})$ with $(1 - q_{t-1}^{AS})$.

FORMULA VARIATIONS

It is not held out that formula (2) is the only possible three-factor dividend formula or that it is the best. There are many variations of the dividend formula in actual use; their impact on the form of ${}_t\bar{C}^j$ can easily be ascertained and the analysis carried out with the ${}_t\bar{C}^j$ calculated accordingly. Formula (3) assumes a pro rata refund of premium on death and payment of the full dividend for the current year at death. It is recognized that other assumptions are possible and that use of them will have an impact on the form of ${}_t\bar{C}^j$. An asset share formula with federal income tax reflected directly is developed in Appendix B.

SUMMARY

The analytic approach developed here was devised for the purpose of examining the increase in surplus in asset share calculations from one duration to the next, so that the reason for the observed progress of surplus in numerous calculations might be understood better than it has been. A natural outgrowth of such an analysis is an examination of the dividend formula, and/or the experience assumptions used in it, as both can be involved if there is an unsatisfactory pattern of development of surplus. One potential advantage that might have been derived from this analysis in other times, if the ${}_tC^j$ were structured suitably, was assurance that the surplus would progress as desired for long periods after the time at which asset share calculations were stopped. Now that a great many asset shares can be computed very quickly, this advantage is negligible; on the other hand, where there are many asset share values available, there is a greater possibility to discover that something is amiss. It is hoped that an analysis of this type will be helpful if that possibility develops.

APPENDIX A

$$
{}_t\bar{C} = \frac{1}{1 - q^{AS}_{t-1} - wq^{AS}_{t-1}} \left\{ \begin{array}{l} ({}_{t-1}AS + GP - {}_tE^{AS})(1 + i^{AS}) \\ \quad - q^{AS}_{t-1}\left[\left(1{,}000 + \frac{GP}{2}\right)\left(1 + \frac{i^{AS}}{2}\right)\right] \\ - {}_tCV \cdot wq^{AS}_{t-1} - (\bar{L} - {}_tE^{D})(1 + i^{D}) \\ \quad - (i^{D} - i^{V})[{}_{t-1}\bar{V}(\bar{A}) + \bar{P}^{d}] \\ - (q^{V}_{t-1} - q^{D}_{t-1})[1{,}000 - {}_t\bar{V}(\bar{A})] \end{array} \right. \qquad \text{(A1)}
$$

$$
- {}_t\bar{V}(\bar{A}) - {}_{t-1}\bar{S} .
$$

$$
{}_t\bar{C} = \frac{1}{1 - q^{AS}_{t-1} - wq^{AS}_{t-1}} \left\{ \begin{array}{l} [{}_{t-1}\bar{V}(\bar{A}) + {}_{t-1}\bar{S} + \bar{P}^{d} + \bar{L} - {}_tE^{AS}] \\ \quad \times (1 + i^{AS}) - 1{,}000 \cdot q^{AS}_{t-1} \\ - q^{AS}_{t-1} \cdot F^{AS} - {}_t\bar{V}(\bar{A}) \cdot wq^{AS}_{t-1} \\ \quad - (\bar{L} - {}_tE^{D})(1 + i^{D}) \\ - (i^{D} - i^{V})[{}_{t-1}V(\bar{A}) + \bar{P}^{d}] \\ \quad - q^{V}_{t-1}[1{,}000 - {}_t\bar{V}(A)] \\ + q^{D}_{t-1}[1{,}000 - {}_t\bar{V}(\bar{A})] \\ \quad - {}_t\bar{V}(\bar{A})(1 - q^{AS}_{t-1} - wq^{AS}_{t-1}) \\ - {}_{t-1}\bar{S}(1 - q^{AS}_{t-1} - wq^{AS}_{t-1}) . \end{array} \right. \qquad \text{(A2)}
$$

$$
{}_t\bar{C} = \frac{1}{1 - q_{t-1}^{\mathrm{AS}} - wq_{t-1}^{\mathrm{AS}}}
\begin{cases}
(i^{\mathrm{AS}} - i^{D})[{}_{t-1}V(A) + \bar{P}^{d} + \bar{L}] \\
\qquad\qquad + {}_tE^{D}\cdot i^{D} - {}_tE^{\mathrm{AS}}\cdot i^{\mathrm{AS}} \\
+ (q_{t-1}^{D} - q_{t-1}^{\mathrm{AS}})[1{,}000 - {}_tV(A)] \\
\qquad\qquad - q_{t-1}^{\mathrm{AS}}\cdot F^{\mathrm{AS}} \\
+ {}_tE^{D} - {}_tE^{\mathrm{AS}} \\
+ {}_{t-1}\bar{S}(i^{\mathrm{AS}} + q_{t-1}^{\mathrm{AS}} + wq_{t-1}^{\mathrm{AS}}) \\
+ [{}_{t-1}\bar{V}(A) + \bar{P}^{d}](1 + i^{V}) \\
\qquad - q_{t-1}^{V}[1{,}000 - {}_t\bar{V}(\bar{A})] - {}_t\bar{V}(\bar{A})\,.
\end{cases}
\tag{A3}
$$

$$
{}_t\bar{C} \doteqdot \frac{1}{1 - q_{t-1}^{\mathrm{AS}} - wq_{t-1}^{\mathrm{AS}}}
\begin{cases}
(i^{\mathrm{AS}} - i^{D})[{}_{t-1}V(A) + \bar{P}^{d} + \bar{L}] \\
\qquad\qquad + {}_tE^{D}\cdot i^{D} - {}_tE^{\mathrm{AS}}\cdot i^{\mathrm{AS}} \\
+ (q_{t-1}^{D} - q_{t-1}^{\mathrm{AS}})[1{,}000 - {}_tV(A)] \\
\qquad\qquad + q_{t-1}^{V}\cdot F^{V} - q_{t-1}^{\mathrm{AS}}\cdot F^{\mathrm{AS}} \\
+ {}_tE^{D} - {}_tE^{\mathrm{AS}} \\
+ {}_{t-1}\bar{S}(i^{\mathrm{AS}} + q_{t-1}^{\mathrm{AS}} + wq_{t-1}^{\mathrm{AS}})\,.
\end{cases}
\tag{A4}
$$

APPENDIX B

ASSET SHARES REFLECTING PHASE I FEDERAL INCOME TAX DIRECTLY

Many mutual life insurance companies are taxed only in Phase I and have large enough taxable investment income so that the lower tax rate on the first $25,000 of taxable investment income and the $250,000 limitation on certain deductions are of minor consequence. It may be of some benefit to examine the form of the asset share formula if the working of the Phase I portion federal income tax law is reflected in the interest credited in the accumulation. The following development is a "rough" approach in which refinement has been ignored. Implicit in this approach, for example, are at least the following assumptions: the current interest rate and the five-year-average rate are the same; the deduction is calculated by applying the "adjusted reserves rate" to the adjusted terminal reserve; there is no tax exempt interest; and policy years and calendar years coincide. Notwithstanding all this, it is believed that the resulting formula is indicative of what will result even when refinement is introduced into the development and that it will provide a base for further analysis of this subject.

Let

I^{BT} = Net investment income per \$1,000 of insurance for a particular plan-age-duration grouping after investment expenses but before Phase I federal income tax.

I^{AT} = I^{BT} after Phase I federal income tax.

D = Phase I deduction for federal income tax purposes (the "policyholder's share").

0.48 = Rate of federal income tax on taxable investment income (assuming no surcharge).

i^{BT} = Company's net earned interest rate on assets before federal income tax.

i^{V} = Valuation interest rate.

i^{AR} = The "adjusted reserves rate" = i^{BT} (under the assumptions being used).

AP = Adjustment percentage = $(1 + 10i^{V} - 10i^{BT})$.

Then

$$I^{BT} = i^{BT}\left[({}_{t-1}\text{AS} + \text{GP} - {}_{t}E^{AS}) - \tfrac{1}{2}q_{t-1}^{AS}\left(1{,}000 + \frac{\text{GP}}{2}\right)\right],$$

$$D = i^{AR}\cdot{}_{t}\bar{V}(\bar{A})\cdot\text{AP} = i^{BT}\cdot{}_{t}\bar{V}(\bar{A})\cdot\text{AP},$$

$$\begin{aligned} I^{AT} &= I^{BT} - (0.48)(I^{BT} - D) \\ &= (0.52)I^{BT} + 0.48D \\ &= (0.52)\cdot i^{BT}\left[({}_{t-1}\text{AS} + \text{GP} - {}_{t}E^{AS}) - \tfrac{1}{2}q_{t-1}^{AS}\left(1{,}000 + \frac{\text{GP}}{2}\right)\right] \\ &\qquad + (0.48)\cdot i^{BT}\cdot{}_{t}\bar{V}(\bar{A})\cdot\text{AP}, \end{aligned}$$

and the resulting asset share formula is

$$\begin{aligned} {}_{t}\text{AS} = \Big[&({}_{t-1}\text{AS} + \text{GP} - {}_{t}E^{AS})[1 + (0.52)(i^{BT})] + (0.48)\cdot i^{BT}\cdot{}_{t}\bar{V}(A)\cdot\text{AP} \\ &- q_{t-1}^{AS}\left(1{,}000 + \frac{\text{GP}}{2}\right)\left(1 + \frac{0.52i^{BT}}{2}\right) \\ &- {}_{t}\text{CV}\cdot wq_{t-1} - D_{t}\Big] \div (1 - q_{t-1}^{AS} - wq_{t-1}^{AS}). \end{aligned}$$

This is the conventional asset share formula (3) with interest rate equal to $(0.52)\cdot i^{BT}$ and an additional, additive element equal to $(0.48)\cdot i^{BT}\cdot{}_{t}\bar{V}(\bar{A})\cdot\text{AP}$. The net result is, of course, to credit a greater amount of interest at the time when the asset share is less than the reserve and less interest when the asset share is greater than the reserve than would the use of a single after-tax rate. The result is consistent with the tax facts of life.