

# TRANSACTIONS

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## AN UPPER BOUND ON THE STOP-LOSS NET PREMIUM—ACTUARIAL NOTE

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### ABSTRACT

One type of reinsurance contract which has attracted some attention recently is stop-loss reinsurance. In its simplest form, the reinsurer agrees to pay all losses of the insurer in excess of an agreed limit. This Note concerns itself with a simple upper bound on this premium. The bound depends only on the mean and variance of the distribution of total claims. After a proof that is strongly reminiscent of that of the Chebyshev inequality, tables are given comparing the upper bound with the net premium calculated under certain distribution assumptions and with net premiums calculated as part of the work of Bohman and Esscher. The Note is concluded with a very brief review of related inequalities which hold for stop-loss premiums where claim-limit maximums and coinsurance features are included.

SEVERAL recent papers [1-5] have discussed the approximation of the stop-loss net premium. This Note concerns itself with a simple upper bound on this premium. The upper bound is expressed in the following theorem. We note that, if  $X$  is the random variable of total claims, then  $\Pi(z)$  is the net premium for stop-loss reinsurance for losses in excess of an amount  $z$ .

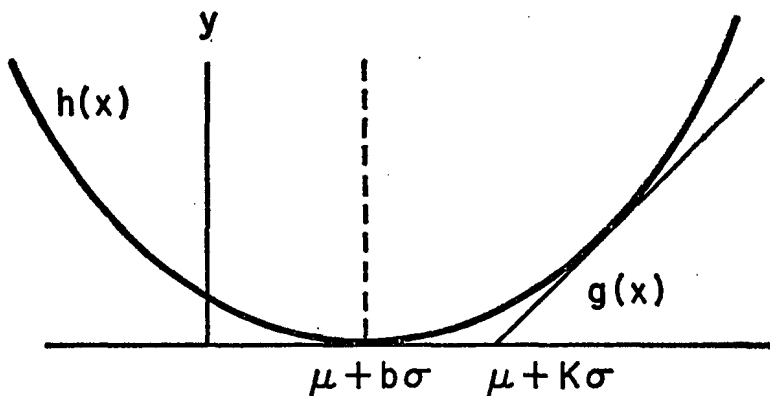
**THEOREM:** Let  $X$  be a random variable with mean  $\mu$ , variance  $\sigma^2$ , and distribution function  $F(x)$ . Then if  $z = \mu + K\sigma$ , we have

$$\Pi(z) = \int_z^{\infty} (x - z) dF(x) < \frac{\sigma}{2} \cdot \frac{1}{K + \sqrt{1 + K^2}}. \quad (1)$$

**PROOF:** Let

$$g(x) = \begin{cases} x - z = x - \mu - K\sigma & x > z \\ 0 & x < z \end{cases}$$

and let  $h(x) = a[x - (\mu + b\sigma)]^2$ .



If  $a$  and  $b$  are chosen so that  $g(x) \leq h(x)$  for all  $x$ , as in the above diagram, then  $E[g(X)] \leq E[h(X)]$ . But we observe that

$$E[g(X)] = \int_z^{\infty} (x - z) dF(x) = \Pi(z),$$

while

$$E[h(X)] = a \int_{-\infty}^{\infty} (x - \mu - b\sigma)^2 dF(x) = a\sigma^2(1 + b^2),$$

so that  $\Pi(z) \leq a\sigma^2(1 + b^2)$ . Now, for  $g(x) \leq h(x)$  to hold for all  $x$ , it is sufficient to require that (1)  $b < K$  and that (2)  $g(x) = h(x)$  at exactly one  $x$ , where  $x > z$ , that is, the graph of  $g(x)$  is tangent to the graph of  $h(x)$  at one point, with  $x > z$  in addition to  $x = \mu + b\sigma$ . The equation  $g(x) = h(x)$  has a single solution if the discriminant of the quadratic  $h(x) - g(x)$  is zero. This leads to the condition that

$$a = \frac{1}{4\sigma(K - b)^2},$$

so that

$$\Pi(z) < \frac{\sigma}{4} \frac{1 + b^2}{K - b}. \tag{2}$$

This inequality holds for all  $b < K$ . To make the inequality as sharp as possible, we choose  $b$  to minimize the right side of formula (2). This turns out to be equivalent to solving  $b^2 - 2bK - 1 = 0$ , subject to  $b < K$ . The solution is

$$b = K - \sqrt{1 + K^2}. \tag{3}$$

Substituting this into formula (2) gives our result that

$$\Pi(z) < \frac{\sigma}{2} \cdot \frac{1}{K + \sqrt{1 + K^2}}.$$

To show that the inequality developed is the "best possible" involving just the first two moments, we demonstrate a distribution of  $X$  for which equality holds.

Consider a particular value of  $K$ . Assume  $X$  is a discrete random variable taking on just two values, the two values where  $g(x) = h(x)$ . From formula (3) we have  $\mu + b\sigma = \mu + \sigma(K - \sqrt{1 + K^2})$  as one of the two values of  $x$ , where  $g(x) = h(x)$ . The other is the point where  $h'(x) = 1$ . This can be shown to be  $x = \mu + \sigma(K + \sqrt{1 + K^2})$ . If we assign probabilities as

$$\Pr [X = \mu + \sigma(K - \sqrt{1 + K^2})] = \frac{(K + \sqrt{1 + K^2})^2}{1 + (K + \sqrt{1 + K^2})^2}$$

and

$$\Pr [X = \mu + \sigma(K + \sqrt{1 + K^2})] = \frac{1}{1 + (K + \sqrt{1 + K^2})^2},$$

it can be verified that  $E[X] = \mu$  and  $\text{Var} [X] = \sigma^2$  as needed. Further

$$\Pi(z) = \int_z^\infty (x - z) dF(x)$$

or, equivalently, in the discrete case

$$\begin{aligned} \Pi(z) &= \sum_{x_i > z} (x_i - z) P(X = x_i) \\ &= \frac{\{[\mu + \sigma(K + \sqrt{1 + K^2})] - (\mu + \sigma K)\}}{1 + (K + \sqrt{1 + K^2})^2} \\ &= \frac{\sigma}{2} \cdot \frac{1}{K + \sqrt{1 + K^2}}, \end{aligned}$$

as given in the theorem. Therefore for any value of  $K$  we can demonstrate a distribution of  $X$  for which equality holds, and thus the inequality as stated cannot be improved without additional information regarding the distribution of  $X$ .

We now compare this upper bound with the net premium under certain distribution assumptions on  $X$ , the amount of total claims (see Table 1). These distributions are illustrative of the effect of using different distributions to evaluate stop-loss premiums. The normal distri-

bution is a symmetric distribution, while the other two are positively skewed. The Pareto distribution chosen has finite variance but infinite third moment [ $f(x) = 3x^{-4}$  for  $x > 1$ ]. No suggestion that the Pareto distribution, in particular, be used to estimate stop-loss premiums is intended.

For a final comparison, we show results for part of the study by Bohman and Esscher [3] (see Table 2). Stop-loss net premiums are compared with the corresponding bound developed above. The particular case used is for 100 expected claims where the number of claims follows a particular ( $k = 20$ ) member of the negative binomial family of distributions. The individual claim-size distribution is what the authors labeled as Life Insurance B, being based on data from a Swedish life company between 1957 and 1961. The claim amounts were scaled so that the expected size of a single claim would be one. Thus, for this example,  $E(X) = 100$ . The authors indicate that  $\text{Var}(X) = (67.947)^2$ .

The disparity between the upper bound and the stop-loss premium given by Bohman and Esscher is not too excessive for stop-loss limits not more than 3 standard deviations above the mean.

TABLE 1  
STOP-LOSS NET PREMIUM  
(As Proportion of Standard Deviation)

Stop-Loss Level ( $K$ )	Upper Bound	Normal Distribution	Pearson III $\mu_1 = \sigma^2$	Pareto Distribution
0.0	0.5000	0.3989	0.3907	0.2566
0.5	.3090	.1978	.2184	.1545
1.0	.2071	.0833	.1165	.1031
1.5	.1514	.0293	.0598	.0737
2.0	.1180	.0085	.0297	.0553
2.5	.0963	.0020	.0144	.0430
3.0	0.0811	0.0004	0.0068	0.0344

TABLE 2

Stop-Loss Limit	$K$	Stop-Loss Premium, Bohman-Esscher	Upper Bound
100 . . . . .	0	20.99	33.97
167.9 . . . . .	1	8.42	14.07
235.9 . . . . .	2	4.680	8.020
303.8 . . . . .	3	3.035	5.513
371.8 . . . . .	4	1.740	4.182
507.7 . . . . .	6	0.1741	2.812

The usual stop-loss reinsurance contract covers total claims in excess of a lower limit but has a claim-limit maximum. Often a coinsurance feature is also included—that is, the reinsurer pays only a percentage of claims in excess of the lower limit up to the maximum. Let us study what changes result in the inequality presented above when maximum-limit and coinsurance features are included. We will indicate what the statement of the inequality becomes with these changes and hint at the proof of these new statements. For this purpose, let  $K$  be the lower limit in standard measure, that is, the lower limit is  $\mu + K\sigma$ ; let  $\mu + K'\sigma$  be the upper limit beyond which no additional reinsurance payments are made; and let  $100 c$  be the percentage of claims paid in excess of the lower limit.

CASE 1. Coinsurance, but no maximum claim limit. In this case it is easy to see that

$$\Pi(z) < \frac{c\sigma}{2(K + \sqrt{1 + K^2})}.$$

This benefit is just  $c$  times the benefit with no coinsurance, so that the upper bound is  $c$  times the upper bound established in the theorem. Further, the example following the theorem can be adjusted to show this result is “best possible.”

For the two other cases we shall assume that coinsurance is included. When coinsurance is not present,  $c$  is replaced by 1.

CASE 2. Maximum claim limit with  $K' > K + \sqrt{1 + K^2}$ . The bound developed in Case 1,  $c\sigma/2(K + \sqrt{1 + K^2})$ , is clearly an upper bound for the modified benefit since a benefit with a maximum claim limit is less expensive than one with no maximum. Again the example following the proof of the theorem shows that this bound is “best possible,” involving only the mean and variance when the upper limit,  $\mu + K'\sigma$ , is greater than  $\mu + (K + \sqrt{1 + K^2})\sigma$ .

CASE 3. Maximum claim limit with  $K' \leq K + \sqrt{1 + K^2}$ . The inequality can in this case be improved to be

$$\Pi(z) < \frac{c\sigma(K' - K)}{1 + (K')^2}.$$

The proof for this result follows lines similar to that of the main theorem. However, the condition the parabola  $h(x)$  must satisfy is that it go through the point  $[\mu + K'\sigma, c\sigma(K' - K)]$  rather than be tangent to the graph of  $g(x)$  at one point  $x > z$ . This leads to the changed form of the inequality.

The author would like to thank the reviewer of the paper for the sug-

gestion to include a discussion of coinsurance and maximum benefit limits and to indicate more refined bounds in these cases.

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## ADDENDUM

I have received a considerably more elegant proof of the theorem of this note from Hilary Seal, and I would like to present it as a very interesting addition to the paper.

Let

$$I = \int_z^{\infty} (x - z) dF(x) = \sigma \int_K^{\infty} (y - K) dF_0(y)$$

and

$$J = \int_0^z (x - z) dF(x) = \sigma \int_{-\mu/\sigma}^K (y - K) dF_0(y),$$

where  $F_0(y)$  is the equivalent of  $F(x)$  when  $y = (x - \mu)/\sigma$  and where  $z = \mu + K\sigma$ . Then

$$I + J = \sigma \int_{-\mu/\sigma}^{\infty} (y - K) dF_0(y) = -\sigma K,$$

while

$$\begin{aligned} I - J &= \sigma \int_K^{\infty} (y - K) dF_0(y) - \sigma \int_{-\mu/\sigma}^K (y - K) dF_0(y) \\ &= \sigma \int_{-\mu/\sigma}^{\infty} |y - K| dF_0(y). \end{aligned}$$

Hence

$$\begin{aligned}
 (I - J)^2 &= \sigma^2 \left[ \int_{-\mu/\sigma}^{\infty} |y - K| dF_0(y) \right]^2 \\
 &\leq \sigma^2 \int_{-\mu/\sigma}^{\infty} dF_0(y) \int_{-\mu/\sigma}^{\infty} (y - K)^2 dF_0(y) \\
 &= \sigma^2 \int_{-\mu/\sigma}^{\infty} (y^2 - 2yK + K^2) dF_0(y) \\
 &= \sigma^2(1 + K^2) .
 \end{aligned}$$

The above inequality is an application of the well-known Cauchy Schwarz inequality.

Thus  $I - J \leq \sigma\sqrt{1 + K^2}$ , and since  $I + J = -\sigma K$ , we have

$$I \leq \frac{\sigma}{2} (\sqrt{1 + K^2} - K) = \frac{\sigma}{2} \left( \frac{1}{\sqrt{1 + K^2} + K} \right).$$

