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ANALYSIS OF BASIC ACTUARIAL THEORY FOR FIXED PREMIUM VARIABLE BENEFIT LIFE INSURANCE

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ABSTRACT

This paper presents an analysis of the basic actuarial theory for life insurance policies which have (1) fixed premiums, (2) the entire reserve held in a separate account, the assets of which would be invested primarily in common stocks, and (3) benefits adjusted to reflect the investment performance of the separate account in such a manner that the policyowners would bear the entire investment risk and the life insurance company would not share any part of the investment risk. Policies satisfying these three basic objectives are referred to as fixed premium variable benefit policies.

Face amounts under such policies are adjusted by a simple method that satisfies the requirement that the reserve per dollar of actual face amount at the end of each policy year for a fixed premium variable benefit policy be exactly the same as that for a corresponding fixed benefit policy. It is shown that the actual face amount applicable at the end of any policy year is equal to the actual face amount applicable at the end of the preceding policy year multiplied by a Y factor, representing the adjustment to reflect the fact that a fixed premium is payable, and a Z factor, representing the adjustment to reflect the relationship between the actual net annual investment return on the separate account during the policy year and the interest rate assumed in the calculation of net annual premiums and reserves.

There is no need to change present statutory minimum nonforfeiture and reserve standards in order to accommodate fixed premium variable benefit policies as long as such standards are interpreted as being applicable per dollar of actual face amount. Similarly, cash surrender and nonforfeiture values per dollar of actual face amount can be illustrated in policy forms for these policies in exactly the same way as those presently illustrated for regular fixed benefit policies. Some other problem areas that are discussed are grace period, reinstatement, policy loans, dividend options, and settlement options.

The paper clearly indicates that it is possible to develop actuarially

sound fixed premium variable benefit life insurance policies. These policies would offer the public the opportunity of buying a life insurance product that reflects the investment performance of reserves invested in equities but that has practically all the characteristics of regular fixed benefit life insurance policies. The paper was written in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of fixed premium variable benefit policies and of equity-based variable life insurance policies that reflect various alternative approaches.

I. INTRODUCTION

THE authors of this paper were assigned the problem of determining the kind of variable life insurance policy that they would recommend if there were no statutory or regulatory problems to take into account at either the state or federal level. This paper presents the results of some of the basic actuarial research that was done at the New York Life Insurance Company in connection with this problem.

The first step in this research consisted of a review of all the available literature describing the different types of variable life insurance policies that have been introduced in foreign countries, where life insurance companies do not face the same statutory and regulatory problems that they do in the United States. After a thorough review of this literature, the authors agreed that they were not completely satisfied with any of the variable life insurance products currently being sold in other countries.

It was then decided to approach this problem on a purely theoretical basis, keeping in mind the following basic objectives:

1. An attempt would be made to develop the basic actuarial theory for life insurance policies with variable benefits and with fixed premiums.
2. The life insurance company would hold the reserves for these life insurance policies in a separate account, the assets of which would be invested primarily in common stocks.
3. The benefits payable under these life insurance policies would be appropriately adjusted to reflect the investment performance of the separate account, so that the policyowners would bear the entire investment risk with respect to the investment performance of the separate account and the life insurance company would not share any part of this investment risk.

It should be noted that the basic actuarial theory for variable life insurance policies, under which all premiums and benefits are expressed in terms of units instead of dollars, is, of course, exactly the same as the basic actuarial theory for corresponding fixed dollar life insurance policies, under which all premiums and benefits are expressed in terms of dollars. The basic problem with this type of variable life insurance policy is the fact that premiums would have to vary in accordance with variations in the unit value of the separate account.

As far as the authors could determine, nothing has yet been published with respect to the basic actuarial theory underlying variable benefit life insurance policies under which fixed premiums (in terms of dollars) are payable, the entire reserve is invested in a separate account, and the benefits vary to reflect the investment performance of the separate account in such a manner that the life insurance company does not share any part of the investment risk. The basic actuarial theory for variable life insurance policies of this type (hereinafter referred to as "fixed premium variable benefit" policies) is developed in Section II of this paper.

Section III discusses the changes required in the basic actuarial concepts underlying the standard valuation and nonforfeiture laws in order to accommodate fixed premium variable benefit policies.

Section IV discusses the policy-form problems involved in illustrating actual cash-surrender values and nonforfeiture benefits for fixed premium variable benefit policies.

Section V discusses some other areas where changes in existing statutory requirements would be desirable in order to accommodate fixed premium variable benefit policies. The specific areas discussed are grace period, reinstatement, policy loans, dividend options, and settlement options.

Section VI discusses some possible variations in the basic concepts underlying fixed premium variable benefit policies. The particular variations discussed are a combination of fixed benefits and variable benefits in the same policy, options to vary premiums within prescribed limits, and guarantee of minimum benefits for appropriate extra premium.

Section VII presents the conclusion.

This paper does not cover any of the possible regulatory requirements that may be introduced at the federal level by the Securities and Exchange Commission in connection with fixed premium variable benefit policies.

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II. DEVELOPMENT OF BASIC ACTUARIAL THEORY

Our objective in this section is to develop a method for determining the death benefits under a fixed premium variable benefit life insurance policy which will result in the entire investment risk being borne by the policyowners.

Basic Theory for Fixed Premium Variable Benefit Life Insurance Policy Using Traditional Assumptions

In order to illustrate the basic concepts as simply as possible, we will first consider a level annual premium whole life insurance policy using traditional functions, that is, with premiums payable at the beginning of the policy year and death benefits payable at the end of the policy year of death.

We begin with the familiar equation of equilibrium showing the relationship between successive terminal reserves under a fixed premium fixed benefit whole life insurance policy for a face amount of $1.

$$({}_{t-1}V_x + P_x)(1 + i) = q_{x+t-1}(1 - {}_tV_x) + {}_tV_x \,, \qquad (1)$$

where

${}_{t-1}V_x$ = Terminal reserve at the end of policy year $t - 1$ for a whole life policy issued at age x.

P_x = Net level annual premium for a whole life policy issued at age x

i = Interest rate assumed in the calculation of the net annual premium and reserves.

q_{x+t-1} = Rate of mortality at attained age $x + t - 1$.

${}_tV_x$ = Terminal reserve at the end of policy year t for a whole life policy issued at age x.

Let us now consider a fixed premium variable benefit whole life insurance policy with an initial face amount of $1 and with the same fixed net level annual premium as that for the corresponding fixed premium fixed benefit whole life insurance policy. Let us assume that the reserves for this policy will be invested in a separate account and that the face amount of this policy will be adjusted annually to reflect the investment performance of the separate account, so that the entire investment risk is borne by the policyowners. We will further require that the reserve per $1 of face amount at the end of each policy year for the fixed premium variable benefit whole life insurance policy be the same as that for the corresponding fixed premium fixed benefit whole life insurance policy. The counterpart of equation (1) for a policy of this type is

$$[F_{t-1}({}_{t-1}V_x) + P_x](1 + i'_t) = q_{x+t-1}[F_t - F_t({}_tV_x)] + F_t({}_tV_x) \,, \qquad (2)$$

where

i'_t = Actual net annual investment return on the separate account during the tth policy year, including realized and unrealized appreciation and depreciation.

F_{t-1} and F_t = Face amounts at the end of the $(t-1)$st and tth policy years, respectively. It should be noted that the initial face amount $F_0 = 1$.

Equation (2) can be rewritten as follows:

$$[F_{t-1}({}_{t-1}V_x) + P_x](1 + i_t') = F_t[q_{x+t-1}(1 - {}_tV_x) + {}_tV_x] . \tag{3}$$

The expression in brackets on the right side of equation (3) can be seen, by referring to equation (1), to be equal to $({}_{t-1}V_x+P_x)(1+i)$. Substituting in equation (3), we obtain

$$[F_{t-1}({}_{t-1}V_x) + P_x](1 + i_t') = F_t[({}_{t-1}V_x + P_x)(1 + i)] . \tag{4}$$

Solving for F_t, we obtain

$$F_t = \left[\frac{F_{t-1}({}_{t-1}V_x) + P_x}{{}_{t-1}V_x + P_x}\right]\left(\frac{1 + i_t'}{1 + i}\right), \tag{5}$$

which can be rewritten as

$$F_t = F_{t-1}\left(\frac{{}_{t-1}V_x + P_x/F_{t-1}}{{}_{t-1}V_x + P_x}\right)\left(\frac{1 + i_t'}{1 + i}\right). \tag{6}$$

It is seen that the face amount F_t at the end of policy year t can be obtained by multiplying the face amount F_{t-1} at the end of policy year $t - 1$ by two factors, which we will call Y_t and Z_t, where

$$Y_t = \frac{{}_{t-1}V_x + P_x/F_{t-1}}{{}_{t-1}V_x + P_x} ; \tag{7}$$

$$Z_t = \frac{1 + i_t'}{1 + i} . \tag{8}$$

Thus, in terms of these factors we have

$$F_t = F_{t-1}Y_tZ_t . \tag{9}$$

It can be seen that the determination of the actual face amounts under a fixed premium variable benefit policy involves the use of a recursion process. Thus, for the first policy year, we have

$$F_0 = 1 ; \tag{10}$$

$$Y_1 = \frac{{}_0V_x + P_x/F_0}{{}_0V_x + P_x} = \frac{P_x}{P_x} = 1 ; \tag{11}$$

$$Z_1 = \frac{1 + i_1'}{1 + i} ; \tag{12}$$

$$F_1 = F_0Y_1Z_1 = \frac{1 + i_1'}{1 + i} ; \tag{13}$$

and for the second policy year, we have

$$Y_2 = \frac{{}_1V_x + P_x/F_1}{{}_1V_x + P_x}\,; \tag{14}$$

$$Z_2 = \frac{1 + i_2'}{1 + i}\,; \tag{15}$$

$$F_2 = F_1 Y_2 Z_2\,. \tag{16}$$

To obtain the face amounts for the third and subsequent policy years, the process is continued, making repeated use of equations (7), (8), and (9).

While the above development was for a whole life policy, it should be apparent that the derivation can be readily extended to any of the standard forms of insurance, such as term, endowment, and limited payment life plans. Furthermore, the derivation can be extended to (*a*) plans in which the net premiums are not level but the dollar amount of each net premium is determined in advance in accordance with a specified schedule (e.g., a modified whole life plan or a plan under a modified reserve method) and (*b*) plans in which the face amount under the corresponding fixed benefit policy varies from year to year in accordance with a specified schedule.

Note that after a policy becomes paid up

$$Y_t = \frac{{}_{t-1}V + 0/F_{t-1}}{{}_{t-1}V + 0} = \frac{{}_{t-1}V}{{}_{t-1}V} = 1\,, \tag{17}$$

and the face amounts change each year only according to the Z_t factors.

Illustrative Results

Table 1 illustrates the calculation of face amounts of insurance for the first three policy years under a fixed premium variable benefit whole life policy with an initial face amount of \$1,000 issued to a male age 55, under various levels of investment performance. The net level annual premium and the reserves are based on the 1958 C.S.O. Table, 3 per cent interest, and traditional functions. Illustrative calculations are shown for constant net annual investment returns of 0, 3, 6, and 9 per cent on the separate account, including realized and unrealized appreciation and depreciation.

Table 2 summarizes the results of calculations similar to those in Table 1 carried out to the last age of the mortality table. The columns headed "Constant" show the face amounts of insurance if actual investment performance were at the rates indicated for every policy year.

TABLE 1

ILLUSTRATIVE CALCULATIONS FOR FIRST THREE POLICY YEARS FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED TO A MALE AGE 55

(Net Level Premiums and Reserves Based on 1958 C.S.O. Table, 3 Per Cent Interest, and Traditional Functions)

	NET ANNUAL INVESTMENT PERFORMANCE OF SEPARATE ACCOUNT			
	$i'_t=0\%$	$i'_t=3\%$	$i'_t=6\%$	$i'_t=9\%$
	First Policy Year			
1. Net level annual premium per $1,000	$ 39.09	$ 39.09	$ 39.09	$ 39.09
2. Initial face amount $=1,000\ F_0$	1,000	1,000	1,000	1,000
3. 1,000 (1) ÷ (2)	39.09	39.09	39.09	39.09
4. Terminal reserve per $1,000 end of prior year	0	0	0	0
5. (3) + (4)	39.09	39.09	39.09	39.09
6. (1) + (4)	39.09	39.09	39.09	39.09
7. Factor $Y_1=(5)\div(6)$	1.0000	1.0000	1.0000	1.0000
8. Factor $Z_1=(1+i'_1)\div 1.03$	0.9709	1.0000	1.0291	1.0583
9. Face amount under fixed premium variable benefit policy $=1,000\ F_1=1,000\ F_0Y_1Z_1=(2)\ (7)\ (8)$	971	1,000	1,029	1,058
	Second Policy Year			
1. Net level annual premium per $1,000	$ 39.09	$ 39.09	$ 39.09	$ 39.09
2. Face amount end of prior year $=1,000\ F_1=(9)_{pr}$	971	1,000	1,029	1,058
3. 1,000 (1) ÷ (2)	40.26	39.09	37.99	36.95
4. Terminal reserve per $1,000 end of prior year	27.62	27.62	27.62	27.62
5. (3) + (4)	67.88	66.71	65.61	64.57
6. (1) + (4)	66.71	66.71	66.71	66.71
7. Factor $Y_2=(5)\div(6)$	1.0175	1.0000	0.9835	0.9679
8. Factor $Z_2=(1+i'_2)\div 1.03$	0.9709	1.0000	1.0291	1.0583
9. Face amount under fixed premium variable benefit policy $=1,000\ F_2=1,000\ F_1Y_2Z_2=(2)\ (7)\ (8)$	959	1,000	1,041	1,084
	Third Policy Year			
1. Net level annual premium per $1,000	$ 39.09	$ 39.09	$ 39.09	$ 39.09
2. Face amount end of prior year $=1,000\ F_2=(9)_{pr}$	959	1,000	1,041	1,084
3. 1,000 (1) ÷ (2)	40.76	39.09	37.55	36.06
4. Terminal reserve per $1,000 end of prior year	55.28	55.28	55.28	55.28
5. (3) + (4)	96.04	94.37	92.83	91.34
6. (1) + (4)	94.37	94.37	94.37	94.37
7. Factor $Y_3=(5)\div(6)$	1.0177	1.0000	0.9837	0.9679
8. Factor $Z_3=(1+i'_3)\div 1.03$	0.9709	1.0000	1.0291	1.0583
9. Face amount under fixed premium variable benefit policy $=1,000\ F_3=1,000\ F_2Y_3Z_3=(2)\ (7)\ (8)$	948	1,000	1,054	1,110

TABLE 2

ILLUSTRATIVE FACE AMOUNTS FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED TO A MALE AGE 55

(Net Level Premiums and Reserves Based on 1958 C.S.O. Table, 3 Per Cent Interest, and Traditional Functions)

END OF POLICY YEAR	NET ANNUAL INVESTMENT PERFORMANCE OF SEPARATE ACCOUNT							
	$i'=0\%$		$i'=3\%$		$i'=6\%$		$i'=9\%$	
	Constant	Simulated	Constant	Simulated	Constant	Simulated	Constant	Simulated
1	$971	$ 968	$1,000	$1,035	$1,029	$1,043	$1,058	$1,086
2	959	991	1,000	1,070	1,041	943	1,084	1,021
3	948	1,036	1,000	904	1,054	1,145	1,110	1,152
4	937	838	1,000	908	1,067	1,100	1,137	1,033
5	926	846	1,000	920	1,080	1,174	1,165	1,257
6	915	888	1,000	1,059	1,093	960	1,194	1,264
7	904	909	1,000	1,068	1,106	991	1,224	1,231
8	893	812	1,000	1,030	1,120	1,097	1,255	1,185
9	883	908	1,000	924	1,134	1,229	1,287	1,136
10	873	881	1,000	841	1,148	967	1,320	1,391
11	863	904	1,000	981	1,162	1,245	1,355	1,209
12	853	876	1,000	976	1,177	1,260	1,391	1,511
13	843	820	1,000	1,138	1,192	1,194	1,428	1,478
14	834	811	1,000	1,047	1,207	1,220	1,466	1,580
15	825	797	1,000	988	1,222	1,220	1,505	1,536
16	816	974	1,000	1,057	1,237	1,288	1,545	1,717
17	807	873	1,000	1,030	1,252	1,420	1,586	1,600
18	799	716	1,000	898	1,268	1,293	1,628	1,597
19	791	802	1,000	1,090	1,284	1,305	1,672	1,494
20	783	782	1,000	877	1,300	1,162	1,717	1,704
21	775	831	1,000	1,131	1,316	1,396	1,763	1,741
22	767	806	1,000	950	1,332	1,114	1,811	1,797
23	760	833	1,000	1,112	1,348	1,398	1,860	1,668
24	753	863	1,000	952	1,364	1,400	1,911	2,042
25	746	737	1,000	1,028	1,381	1,630	1,963	2,030
26	739	732	1,000	1,008	1,398	1,587	2,016	2,072
27	732	617	1,000	927	1,415	1,512	2,071	2,260
28	725	723	1,000	1,078	1,432	1,396	2,128	1,808
29	719	733	1,000	1,050	1,449	1,325	2,186	1,839
30	713	819	1,000	1,046	1,466	1,519	2,245	2,322
31	707	793	1,000	987	1,483	1,275	2,306	2,410
32	701	651	1,000	1,080	1,500	1,619	2,369	2,422
33	695	667	1,000	959	1,517	1,429	2,433	2,519
34	690	706	1,000	1,030	1,535	1,565	2,499	2,088
35	685	727	1,000	1,064	1,553	1,591	2,567	2,608
36	680	704	1,000	1,032	1,571	1,750	2,637	2,857
37	675	600	1,000	1,043	1,589	1,503	2,709	2,492
38	670	667	1,000	969	1,607	1,584	2,783	2,889
39	665	629	1,000	1,055	1,625	1,704	2,859	2,993
40	660	698	1,000	910	1,643	1,735	2,937	3,054
41	655	693	1,000	1,026	1,662	1,576	3,018	2,533
42	650	695	1,000	1,064	1,681	1,546	3,102	3,068
43	645	587	1,000	970	1,700	1,532	3,189	3,755
44	640	563	1,000	947	1,720	1,835	3,279	3,488
45	635	722	1,000	1,070	1,740	1,959	3,373	2,973

Of course, under realistic market conditions the rate would fluctuate considerably over the term of the policy. In order to reflect realistic conditions, a simulation program written in FORTRAN IV for the IBM 1130 computer was developed to produce stock market cycles which resemble what happens in the real world. The simulated stock market performance was developed as the product of three factors:

1. A *trend factor*, which is simply a regular interest accumulation at the assumed underlying net annual investment return.
2. A *cycle factor*, which behaves like the market cycles found in the real world and varies randomly for each simulation.
3. A *random factor*, which is independent of the trend or cycle factors.

The cycle and random factors were designed so that their effect tends to average 100 per cent over a given simulation.

The results under simulated market conditions, based on underlying net annual investment returns of 0, 3, 6, and 9 per cent, are shown in Table 2 under the columns headed "Simulated." It should be understood that the results presented here relate to a single simulation for each underlying net annual investment return with a trend factor based on the rate indicated and that these results do not represent the average of a large number of such simulations.

It is interesting to see what the results would have been for fixed premium variable benefit policies under actual market conditions over a long period of time, if such policies had been issued in the past. Accordingly, in Table 3 we have illustrated the face amounts of insurance under hypothetical fixed premium variable benefit policies issued in July, 1915, if the separate account had been fully invested in common stocks experiencing the performance level (including dividend yields) of Standard and Poor's Composite 500. Results are illustrated for whole life, twenty-pay life, and twenty-year endowment policies issued to a male at ages 25 and 55.

Based on the performance of the stock market over the last fifty-odd years, the results under the hypothetical fixed premium variable benefit policies shown in Table 3 are quite dramatic, especially for policies in force at the longer durations. For the two policies which could possibly be in force in 1968—the whole life and twenty-pay life policies issued at age 25—the face amount in 1968 is over 13 times the initial face amount of $1,000 for the whole life policy and over 22 times the initial face amount for the twenty-pay life policy. There are only occasional points during the period of coverage in this particular illustration where the face amount drops below the initial face amount of $1,000.

In order to illustrate results under rather adverse market conditions,

TABLE 3

ILLUSTRATIVE FACE AMOUNTS FOR FIXED PREMIUM VARIABLE BENEFIT POLICIES WITH INITIAL FACE AMOUNT OF $1,000 ISSUED IN JULY, 1915, WITH SEPARATE ACCOUNT INVESTED IN STANDARD AND POOR'S COMPOSITE 500

(Net Level Premiums and Reserves Based on 1958 C.S.O. Table, 3 Per Cent Interest, and Traditional Functions)

Policy Year Ending in:	Whole Life		20-Pay Life		20-Year Endowment	
	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue
1916....	$1,179	$1,179	$1,179	$1,179	$1,179	$1,179
1917....	1,075	1,066	1,078	1,069	1,081	1,071
1918....	936	928	939	931	942	932
1919....	1,246	1,243	1,248	1,244	1,250	1,245
1920....	1,021	1,010	1,025	1,014	1,029	1,016
1921....	869	861	872	863	875	865
1922....	1,192	1,190	1,194	1,190	1,196	1,191
1923....	1,139	1,130	1,142	1,133	1,145	1,135
1924....	1,296	1,281	1,300	1,286	1,304	1,289
1925....	1,594	1,565	1,601	1,575	1,607	1,580
1926....	1,790	1,738	1,802	1,756	1,811	1,765
1927....	2,115	2,028	2,135	2,059	2,149	2,075
1928....	2,629	2,484	2,662	2,538	2,685	2,565
1929....	3,752	3,484	3,811	3,585	3,853	3,637
1930....	2,657	2,416	2,709	2,509	2,746	2,557
1931....	1,700	1,522	1,738	1,591	1,765	1,627
1932....	782	696	801	729	814	747
1933....	1,321	1,199	1,349	1,243	1,369	1,268
1934....	1,146	1,039	1,170	1,078	1,188	1,100
1935....	1,469	1,336	1,500	1,384	1,522	1,412
1936....	2,080	1,884	2,153	1,987	....	....
1937....	2,200	1,970	2,326	2,147	....	....
1938....	1,625	1,438	1,755	1,620	....	....
1939....	1,574	1,385	1,724	1,591	....	....
1940....	1,356	1,187	1,504	1,388	....	....
1941....	1,402	1,226	1,568	1,447	....	....
1942....	1,200	1,048	1,355	1,250	....	....
1943....	1,672	1,464	1,897	1,750	....	....
1944....	1,836	1,598	2,108	1,945	....	....
1945....	2,123	1,834	2,469	2,278	....	....
1946....	2,588	2,213	3,053	2,817	....	....
1947....	2,271	1,917	2,722	2,511	....	....
1948....	2,306	1,926	2,803	2,586	....	....
1949....	2,230	1,843	2,748	2,535	....	....
1950....	2,700	2,209	3,370	3,109	....	....
1951....	3,457	2,793	4,377	4,038	....	....
1952....	3,973	3,162	5,110	4,714	....	....
1953....	3,911	3,061	5,111	4,714	....	....
1954....	4,922	3,792	6,532	6,025	....	....
1955....	6,921	5,239	9,333	8,609	....	....

TABLE 3—*Continued*

Policy Year Ending in:	Whole Life		20-Pay Life		20-Year Endowment	
	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue
1956....	7,805	5,793	10,702	9,872	....	....
1957....	7,546	5,492	10,521	9,705	....	....
1958....	7,396	5,281	10,479	9,666	....	....
1959....	9,359	6,562	13,471	12,426	....	....
1960....	8,479	5,834	12,399	11,437	....	....
1961....	10,069	....	14,952	....	....	....
1962....	8,689	....	13,102	....	....	....
1963....	10,191	....	15,594	....	....	....
1964....	12,095	....	18,781	....	....	....
1965....	12,222	....	19,260	....	....	....
1966....	11,832	....	18,915	....	....	....
1967....	13,280	....	21,533	....	....	....
1968....	13,560	....	22,300	....	....	....

Table 4 shows what would have happened if fixed premium variable benefit policies similar to those illustrated in Table 3 had been issued in August, 1929, just prior to a major stock market crash. It will be noted that for all but a few of the years from issue to 1942 the face amounts are less than the initial face amount of $1,000. However, from 1943 on, the face amounts exceed $1,000, reaching the $5,700–$12,500 range in 1968 for the plans and issue ages illustrated.

Alternative Unit Value Approach

It is, of course, possible to express the basic actuarial theory underlying a fixed premium variable benefit life insurance policy in an alternative manner by using number of units of face amount and unit values of the separate account in which the reserves are invested.

Let

u_0 = Unit value of separate account on the effective date of the policy;

u_t = Unit value of separate account at the end of the tth policy year.

Let us assume that unit values of the separate account are adjusted to reflect the actual net investment return on the separate account (i.e., i'_t

TABLE 4

ILLUSTRATIVE FACE AMOUNTS FOR FIXED PREMIUM VARIABLE BENEFIT POLICIES WITH INITIAL FACE AMOUNT OF $1,000 ISSUED IN AUGUST, 1929, WITH SEPARATE ACCOUNT INVESTED IN STANDARD AND POOR'S COMPOSITE 500

(Net Level Premiums and Reserves Based on 1958 C.S.O. Table, 3 Per Cent Interest, and Traditional Functions)

Policy Year Ending in:	Whole Life		20-Pay Life		20-Year Endowment	
	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue	Age 25 at Issue	Age 55 at Issue
1930......	$ 682	$ 682	$ 682	$ 682	$ 682	$ 682
1931......	569	580	565	577	562	575
1932......	467	485	460	479	455	476
1933......	821	871	804	855	791	847
1934......	717	754	704	742	693	736
1935......	960	1,011	943	995	928	986
1936......	1,371	1,431	1,349	1,413	1,330	1,402
1937......	1,337	1,374	1,321	1,364	1,306	1,358
1938......	1,008	1,023	999	1,020	990	1,018
1939......	943	955	936	953	928	951
1940......	916	927	909	925	902	923
1941......	929	942	922	939	915	937
1942......	820	832	814	829	808	827
1943......	1,169	1,191	1,159	1,184	1,150	1,180
1944......	1,286	1,303	1,277	1,298	1,268	1,295
1945......	1,559	1,567	1,550	1,567	1,541	1,566
1946......	1,654	1,645	1,648	1,654	1,641	1,656
1947......	1,516	1,491	1,514	1,507	1,509	1,513
1948......	1,592	1,552	1,593	1,576	1,589	1,586
1949......	1,541	1,489	1,545	1,520	1,543	1,533
1950......	1,901	1,822	1,936	1,904	....	....
1951......	2,434	2,306	2,528	2,486	....	....
1952......	2,631	2,456	2,797	2,750	....	....
1953......	2,460	2,261	2,676	2,631	....	....
1954......	3,153	2,858	3,503	3,444	....	....
1955......	4,516	4,026	5,135	5,048	....	....
1956......	4,898	4,282	5,711	5,614	....	....
1957......	4,611	3,952	5,510	5,416	....	....
1958......	4,823	4,056	5,899	5,798	....	....
1959......	5,907	4,873	7,388	7,262	....	....
1960......	5,535	4,474	7,081	6,961	....	....
1961......	6,495	5,147	8,489	8,345	....	....
1962......	5,550	4,308	7,410	7,284	....	....
1963......	6,684	5,090	9,104	8,949	....	....
1964......	7,403	5,524	10,287	10,111	....	....
1965......	7,738	5,657	10,967	10,779	....	....
1966......	6,735	4,823	9,730	9,563	....	....
1967......	8,059	5,660	11,856	11,653	....	....
1968......	8,370	5,760	12,539	12,324	....	....

during the tth policy year) in relation to the interest rate of i assumed for the calculation of net annual premiums and reserves, so that

$$u_1 = u_0\left(\frac{1+i_1'}{1+i}\right); \tag{18}$$

$$u_t = u_{t-1}\left(\frac{1+i_t'}{1+i}\right). \tag{19}$$

Now, let

X_0 = Initial number of units of face amount;

X_t = Number of units of face amount at the end of the tth policy year.

Let us now consider a fixed premium variable benefit whole life insurance policy with an initial face amount of \$1 and with a fixed net level annual premium of P_x. The initial face amount can be expressed as $\$1 = F_0 = X_0 u_0$.

The face amount at the end of the $(t-1)$st policy year can be expressed as

$$F_{t-1} = X_{t-1}u_{t-1}, \tag{20}$$

and the face amount at the end of the tth policy year can be expressed as

$$F_t = X_t u_t. \tag{21}$$

Substituting equations (20) and (21) in equation (6), we obtain

$$X_t u_t = X_{t-1}u_{t-1}\left(\frac{{}_{t-1}V_x + P_x/X_{t-1}u_{t-1}}{{}_{t-1}V_x + P_x}\right)\left(\frac{1+i_t'}{1+i}\right). \tag{22}$$

Substituting the value of u_t from equation (19) in equation (22), we obtain

$$X_t u_{t-1}\left(\frac{1+i_t'}{1+i}\right) = X_{t-1}u_{t-1}\left(\frac{{}_{t-1}V_x + P_x/X_{t-1}u_{t-1}}{{}_{t-1}V_x + P_x}\right)\left(\frac{1+i_t'}{1+i}\right). \tag{23}$$

Dividing both sides of equation (23) by $u_{t-1}[(1+i_t')/(1+i)]$, we obtain

$$X_t = X_{t-1}\left(\frac{{}_{t-1}V_x + P_x/X_{t-1}u_{t-1}}{{}_{t-1}V_x + P_x}\right). \tag{24}$$

Equation (24) defines the recursion process required to determine the change in number of units from the end of the $(t-1)$st policy year to the end of the tth policy year under a fixed premium variable benefit policy. It should be noted that this change in number of units actually takes place at the beginning of the tth policy year when the net annual premium P_x is placed in the separate account and reflects the fact that a fixed premium is payable under this policy. In order to keep the number of units of face amount constant from year to year, premiums would have to vary in accordance with variations in the unit values of the separate account.

If equations (20) and (24) are referred to, it can be seen that the number of units of face amount will decrease at the beginning of the tth policy year if $F_{t-1} = X_{t-1}u_{t-1}$ is greater than 1. On the other hand, the number of units of face amount will increase at the beginning of the tth policy year if $F_{t-1} = X_{t-1}u_{t-1}$ is less than 1.

Comparison of Two Methods

The complete symmetry between the two methods of analyzing the basic actuarial theory underlying a fixed premium variable benefit policy can now easily be demonstrated. Substituting F_{t-1} for $X_{t-1}u_{t-1}$ in equation (24), we obtain

$$X_t = X_{t-1}\left(\frac{{}_{t-1}V_x + P_x/F_{t-1}}{{}_{t-1}V_x + P_x}\right). \tag{25}$$

Referring to equation (7), we can see that the expression in parentheses in equation (25) is equal to Y_t. Substituting, we obtain

$$X_t = X_{t-1}Y_t\,; \tag{26}$$

$$Y_t = \frac{X_t}{X_{t-1}}. \tag{27}$$

Referring to equations (8) and (19), we can see that

$$u_t = u_{t-1}Z_t\,; \tag{28}$$

$$Z_t = \frac{u_t}{u_{t-1}}. \tag{29}$$

The roles of the Y_t and Z_t factors in the basic equation (9), that is,

$$F_t = F_{t-1}Y_tZ_t\,, \tag{9}$$

can now be clearly expressed as follows:

a) The role of the Y_t factor is to adjust the number of units of face amount (and hence the face amount) at the beginning of the tth policy year to reflect properly the fact that a fixed premium of P_x is payable at that time.

b) The role of the Z_t factor is to adjust the face amount so as to reflect the effect of the change in unit values from the beginning of the tth policy year to the end of the tth policy year.

It should be noted, however, that there is no need to refer to number of units or unit values in determining actual face amounts under a fixed premium variable benefit policy. Actual face amounts under such a policy can be determined solely from the Y_t and Z_t factors defined in equations (7) and (8).

Extension of Basic Theory to Policy with Net Annual Premiums and Reserves Based on Continuous Functions

Thus far we have considered a fixed premium variable benefit policy where the net annual premiums and terminal reserves are based on traditional functions. Formulas analogous to equations (7), (8), and (9) for determining the face amounts under a fixed premium variable benefit policy with net annual premiums and reserves based on continuous functions are presented below.

We will once again use a whole life policy to illustrate the basic formulas. The two bases for net annual premiums and terminal reserves using continuous functions commonly found in practice follow:

a) A "semi-continuous" basis, where the net annual premium is

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x}$$

and the terminal reserve at the end of the tth policy year is

$${}_tV(\bar{A}_x) = \bar{A}_{x+t} - P(\bar{A}_x)\ddot{a}_{x+t} .$$

b) A "fully continuous" basis, where the net annual premium is

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) = \frac{d}{\delta}\frac{\bar{A}_x}{\bar{a}_x}$$

and the terminal reserve at the end of the tth policy year is

$${}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} .$$

Both of the above bases reflect the assumption that the face amount is payable at the moment of death. Thus it is apparent that the face amount payable at death under a fixed premium variable benefit policy with net annual premiums and reserves calculated on basis (*a*) or (*b*) should vary continuously during the policy year in accordance with the net investment performance of the separate account from the beginning of the policy year to the moment of death.

Consider first a fixed premium variable benefit whole life policy where net annual premiums and reserves are calculated on basis (*a*) described above. Let F^a_{t-1} be the face amount payable at the end of the $(t-1)$st policy year and F^a_{t-1+f} be the face amount payable a fraction of a year f later. It can be shown that if we define Y and Z factors as follows:

$$Y^a_t = \frac{{}_{t-1}V(\bar{A}_x) + P(\bar{A}_x)/F^a_{t-1}}{{}_{t-1}V(\bar{A}_x) + P(\bar{A}_x)} , \tag{30}$$

$$Z^a_{t-1:f} = \frac{1 + i'_{t-1:f}}{(1+i)^f} , \quad 0 < f \le 1 , \tag{31}$$

where $i'_{t-1:f}$ is the actual net investment return from the beginning of the tth policy year to a fraction of a year f later (i.e., to the moment of death), then

$$F^a_{t-1+f} = F^a_{t-1} Y^a_t Z^a_{t-1:f}\,, \quad 0 < f < 1\,. \tag{32}$$

Similarly, under basis (b), let F^b_{t-1} be the face amount payable under a fixed premium variable benefit whole life policy at the end of the $(t-1)$st policy year and F^b_{t-1+f} be the face amount payable a fraction of a year f later. Then, if we define Y and Z factors as follows:

$$Y^b_t = \frac{{}_{t-1}\bar{V}(\bar{A}_x) + [(d/\delta)\bar{P}(\bar{A}_x)]/F^b_{t-1}}{{}_{t-1}\bar{V}(\bar{A}_x) + (d/\delta)\bar{P}(\bar{A}_x)}\,, \tag{33}$$

$$Z^b_{t-1:f} = \frac{1 + i'_{t-1:f}}{(1+i)^f}\,, \qquad 0 < f < 1\,, \tag{34}$$

it can be shown that

$$F^b_{t-1+f} = F^b_{t-1} Y^b_t Z^b_{t-1:f}\,, \quad 0 < f < 1\,. \tag{35}$$

It will be noted that

$$Z^a_{t-1:f} = Z^b_{t-1:f}\,.$$

As is true in the case of traditional functions, the above formulas were derived on the assumption that the reserve per \$1 of face amount at the end of each policy year for the fixed premium variable benefit whole life policy is the same as that for the corresponding fixed premium fixed benefit whole life policy.

III. CHANGES REQUIRED IN BASIC ACTUARIAL CONCEPTS UNDERLYING STANDARD VALUATION AND NONFORFEITURE LAWS

The basic actuarial concept underlying a fixed premium variable benefit policy is that the face amount of the policy is adjusted to reflect the investment performance of a separate account, based on the assumption that reserves for this policy are held in the separate account. The particular method of adjusting the face amount that was derived in Section II involved the additional requirement that the reserve per dollar of face amount at the end of each policy year for a fixed premium variable benefit policy be exactly the same as that for a corresponding fixed premium fixed benefit policy. Keeping this relationship in mind, we will analyze the basic actuarial concepts underlying the standard valuation and nonforfeiture laws in order to determine what changes are required in these concepts in order to accommodate a fixed premium variable benefit policy.

Standard Valuation Law

The basic concept underlying the standard valuation law is to provide a test for solvency by specifying the minimum reserve standards that life insurance companies can use in calculating the reserves they are required to hold in order to provide for future liabilities in connection with their in-force life insurance policies. Thus, for individual life insurance policies, the present statutory minimum reserve standards involve a conservative mortality table, a maximum interest rate, and a prospective valuation method.

In considering the application of this concept to fixed premium variable benefit policies, let us first examine the net level annual premium reserves for such policies from a prospective standpoint, using the particular assumption that future investment performance of the separate account will be at the assumed interest rate i used for calculation of net annual premiums and reserves. Under this assumption, the net level annual premium reserve for a fixed premium variable benefit whole life policy at duration t will reflect future face amounts $F_{t+1}, F_{t+2}, \ldots$, which can be calculated using formulas (7), (8), and (9) given in Section II of this paper. (Note that all Z_t factors are equal to 1 under the assumption that actual net annual investment performance of the separate account will be at the assumed interest rate i.) These face amounts will not be level except in the special case where F_t is equal to the initial face amount.

Therefore, the present value of future benefits can be expressed as

$$A'_{x+t} = \frac{1}{D_{x+t}} \sum_{j=0}^{\omega-x-t-1} (C_{x+t+j})(F_{t+j+1}) , \tag{36}$$

where the commutation functions are computed at the assumed interest rate i and the future face amounts $F_{t+1}, F_{t+2}, \ldots$, are calculated as follows:

$$F_{t+j+1} = F_{t+j} Y_{t+j+1} . \tag{37}$$

The present value of the fixed net level annual premiums P_x under the fixed premium variable benefit whole life policy is clearly $P_x \ddot{a}_{x+t}$, where $\ddot{a}_{x+t}$ is computed at the assumed interest rate i. The prospective reserve for the fixed premium variable benefit whole life policy is therefore equal to

$$A'_{x+t} - P_x \ddot{a}_{x+t} . \tag{38}$$

In Section II, however, it was indicated that the reserve for the fixed premium variable benefit whole life policy can be expressed as $F_t({}_tV_x)$. Since ${}_tV_x = A_{x+t} - P_x \ddot{a}_{x+t}$, it is therefore apparent that

$$F_t({}_tV_x) = F_t A_{x+t} - F_t P_x \ddot{a}_{x+t} . \tag{39}$$

This is also an expression for the prospective reserve on a fixed premium variable benefit whole life policy, and it seems to be quite different from the expression given in equation (38) above. Expression (39) involves a level benefit F_t rather than the series of nonlevel benefits defined by equation (37) and a net level annual premium of F_tP_x rather than P_x.

It is therefore essential to demonstrate that expressions (38) and (39) are equivalent, that is, that

$$\begin{aligned} A'_{x+t} - P_x\,\ddot{a}_{x+t} &= F_t({}_tV_x) \\ &= F_tA_{x+t} - F_tP_x\,\ddot{a}_{x+t}\,. \end{aligned} \tag{40}$$

A proof of equation (40) is presented in Appendix A. As also noted in Appendix A, this proof can be generalized to cover plans generally, rather than just whole life; any assumption as to future investment performance; and reserves computed according to methods other than the net level annual premium method (e.g., the commissioners reserve valuation method).

It can therefore be stated that $F_t({}_tV)$ is, in general, the correct terminal reserve for a fixed premium variable benefit policy, since it automatically takes into account the present values, under any level of actual investment performance, of the future benefits that will be payable and of the actual fixed net premiums that will be payable, based on the particular interest rate assumption and valuation method chosen. This result is to be expected in light of the fact that the company does not bear any investment risk under a fixed premium variable benefit policy.

Since $F_t({}_tV)$ has been shown to be the correct terminal reserve for a fixed premium variable benefit policy, the question of a proper minimum reserve standard for such policy is essentially the question of a proper standard for ${}_tV$. Since this factor is the same under a fixed premium variable benefit policy as that under a corresponding fixed benefit policy it, appears logical that it be subject to the same minimum standard. In this connection, it should be noted that any other approach would mean that, when actual investment performance was at the assumed rate and benefits under a fixed premium variable benefit policy were therefore the same as those under a corresponding fixed benefit policy, the minimum reserve for the fixed premium variable benefit policy would be different from that for the corresponding fixed benefit policy. It seems that any such situation would be quite anomalous.

It therefore appears that there is no need to change present statutory minimum reserve standards in order to accommodate fixed premium variable benefit policies, as long as such standards are interpreted as being applicable per dollar of actual face amount under the fixed premium variable benefit policy.

On the assumption that statutory minimum reserve standards are satisfied, life insurance companies should have the same choice of assumptions for actual reserves under fixed premium variable benefit policies as they presently have for fixed benefit policies. Such a choice will be important from the standpoint of product design, because benefits and reserves under a fixed premium variable benefit policy depend not only on the actual investment performance of the separate account but also on the reserve assumptions that determine the level and incidence of the net premiums that are deposited in the separate account.

This is illustrated in Tables 5 and 6, which show actual face amounts and terminal reserves under various levels of actual investment performance for a $1,000 initial face amount fixed premium variable benefit whole life policy issued to a male age 55. Reserves and net premiums, based on traditional functions, are computed on the basis of the 1958 C.S.O. table with assumed interest rates of either $2\frac{1}{2}$ or 3 per cent and on either the net level premium method or the commissioners reserve valuation method.

The actual face amounts illustrated in Table 5 indicate the following relationships:

a) For a given mortality basis, reserve method, and actual net investment performance, the lower the assumed interest rate the higher the actual face amount.

b) For a given mortality basis, assumed interest rate, and actual net investment performance, actual face amounts under the commissioners reserve valuation method are higher than those under the net level premium method, if actual investment performance is poorer than that according to the assumed interest rate, and are lower than those under the net level premium method if actual investment performance is better than that according to the assumed interest rate. This reflects the fact that, given the same mortality basis and assumed interest rate, funds (i.e., net premiums) are deposited into the separate account relatively later under the commissioners reserve valuation method than they are under the net level premium method. This is advantageous if actual net investment performance is worse than that assumed but disadvantageous if it is better than that assumed.

The actual reserves, per $1,000 *initial* face amount, illustrated in Table 6 indicate the following relationships:

a) For a given mortality basis, reserve method, and actual net investment performance, the lower the assumed interest rate the higher the actual reserve.

b) For a given mortality basis, assumed interest rate, and actual net investment performance, it is interesting to note that actual reserves under the commissioners reserve valuation method are not always lower than those under the net level premium method. As illustrated in Table 6, actual reserves under the

TABLE 5

ACTUAL FACE AMOUNTS FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED TO A MALE AGE 55

(Net Premiums and Reserves Based on 1958 C.S.O. Table, Net Level Premium or Commissioners Reserve Valuation Method, $2\frac{1}{2}$ or 3 Per Cent Interest and Traditional Functions)

Net Annual Investment Performance of Separate Account i' (Per Cent)	End of Policy Year	Net Level Premium Method		Commissioners Method	
		$2\frac{1}{2}$ Per Cent	3 Per Cent	$2\frac{1}{2}$ Per Cent	3 Per Cent
0	5	$ 937	$ 926	$ 947	$ 937
	10	892	873	902	884
	15	851	825	861	836
	20	815	783	824	794
	25	782	746	791	757
	30	753	713	762	724
	35	728	685	737	696
	40	706	660	715	671
	45	686	635	695	646
3	5	1,013	1,000	1,011	1,000
	10	1,023	1,000	1,021	1,000
	15	1,033	1,000	1,031	1,000
	20	1,043	1,000	1,041	1,000
	25	1,053	1,000	1,051	1,000
	30	1,063	1,000	1,061	1,000
	35	1,073	1,000	1,071	1,000
	40	1,082	1,000	1,078	1,000
	45	1,092	1,000	1,087	1,000
6	5	1,095	1,080	1,079	1,067
	10	1,177	1,148	1,158	1,133
	15	1,266	1,222	1,244	1,204
	20	1,360	1,300	1,335	1,279
	25	1,461	1,381	1,432	1,358
	30	1,567	1,466	1,533	1,439
	35	1,677	1,553	1,638	1,524
	40	1,793	1,643	1,747	1,609
	45	1,918	1,740	1,866	1,701
9	5	1,181	1,165	1,150	1,136
	10	1,356	1,320	1,315	1,286
	15	1,563	1,505	1,511	1,461
	20	1,804	1,717	1,739	1,661
	25	2,086	1,963	2,004	1,892
	30	2,416	2,245	2,311	2,157
	35	2,796	2,567	2,664	2,456
	40	3,237	2,937	3,072	2,797
	45	3,764	3,373	3,558	3,200

TABLE 6

ACTUAL TERMINAL RESERVES FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED TO A MALE AGE 55

(Net Premiums and Reserves Based on 1958 C.S.O. Table, Net Level Premium or Commissioners Reserve Valuation Method, 2½ or 3 Per Cent Interest and Traditional Functions)

Net Annual Investment Performance of Separate Account i' (Per Cent)	End of Policy Year	Net Level Premium Method		Commissioners Method	
		2½ Per Cent	3 Per Cent	2½ Per Cent	3 Per Cent
0	5	$ 135	$ 128	$ 112	$ 107
	10	252	239	236	223
	15	350	330	339	320
	20	428	401	421	396
	25	489	458	486	457
	30	534	498	533	499
	35	568	528	570	532
	40	603	559	607	566
	45	686	635	695	646
3	5	146	138	120	114
	10	290	273	267	253
	15	425	400	406	383
	20	547	512	531	499
	25	659	614	646	603
	30	753	698	743	690
	35	837	771	828	764
	40	924	847	916	843
	45	1,092	1,000	1,087	1,000
6	5	158	149	128	121
	10	333	314	303	286
	15	521	488	490	461
	20	713	666	681	638
	25	914	848	880	819
	30	1,110	1,024	1,073	992
	35	1,308	1,197	1,266	1,165
	40	1,531	1,392	1,484	1,356
	45	1,918	1,740	1,866	1,701
9	5	170	161	136	129
	10	384	361	344	325
	15	643	601	595	559
	20	946	880	888	828
	25	1,306	1,206	1,232	1,141
	30	1,712	1,568	1,618	1,488
	35	2,180	1,979	2,060	1,877
	40	2,764	2,489	2,610	2,358
	45	3,764	3,373	3,558	3,200

commissioners reserve valuation method can be higher than those under the net level premium method at the longer policy durations if actual net investment performance is worse than that assumed.

It should be kept in mind, however, that on a per $1,000 *actual* face amount basis, relationships between reserves computed according to various assumptions and methods for a fixed premium variable benefit policy are exactly the same as those under a corresponding fixed benefit policy, because reserves per dollar of actual face amount under a fixed premium variable benefit policy are the same as those per dollar of face amount under a corresponding fixed benefit policy. This means that, for a given mortality basis and assumed interest rate, reserves based on the commissioners reserve valuation method for a fixed premium variable benefit policy are always lower, per $1,000 actual face amount, than net level premium reserves per $1,000 actual face amount.

Standard Nonforfeiture Law

The basic concept underlying the standard nonforfeiture law is to specify minimum cash-surrender values for life insurance policies. These statutory minimum cash-surrender values may be considered to represent rough approximations to retrospective asset share accumulations that reflect the actual incidence of expenses, that is, the significantly higher level of expenses during the first policy year than during subsequent policy years.

Actually, the standard nonforfeiture law defines minimum cash-surrender values prospectively as equal to the present value of future benefits less the present value of future adjusted premiums. Using a whole life policy issued at age x for illustrative purposes, the adjusted premium $(\mathrm{AP})_x$ can be expressed as follows:

$$(\mathrm{AP})_x = \frac{A_x + I_x}{\ddot{a}_x} = P_x + \frac{I_x}{\ddot{a}_x}, \tag{41}$$

where A_x is the present value at issue of future benefits and I_x is the initial expense deficit specified in the law. The adjusted premium $(\mathrm{AP})_x$ may be considered as the sum of the net annual premium P_x and an additional amount $I_x/\ddot{a}_x$ required to amortize the specified initial expense deficit over the entire premium paying period.

The minimum cash-surrender value ${}_t(\mathrm{MCV})_x$ at duration t for a whole life policy issued at age x is defined in the standard nonforfeiture law as

$${}_t(\mathrm{MCV})_x = A_{x+t} - (\mathrm{AP})_x\ddot{a}_{x+t}\,. \tag{42}$$

Referring to equation (41), equation (42) can be expressed as follows:

$$ {}_t(\mathrm{MCV})_x = A_{x+t} - P_x\ddot{a}_{x+t} - (I_x/\ddot{a}_x)\ddot{a}_{x+t}\,; \tag{43} $$

$$ {}_t(\mathrm{MCV})_x = {}_tV_x - (I_x/\ddot{a}_x)\ddot{a}_{x+t}\,. \tag{44} $$

If we let ${}_tU_x$ represent the unamortized portion of the initial expense deficit at duration t, we obtain

$$ {}_tU_x = (I_x/\ddot{a}_x)\ddot{a}_{x+t} \tag{45} $$

and

$$ {}_t(\mathrm{MCV})_x = {}_tV_x - {}_tU_x\,. \tag{46} $$

In considering the basic problem of how to specify statutory minimum cash-surrender values for a fixed premium variable benefit life insurance policy, it is clear that the reserve part of equation (46) does not present any problem, since it has already been demonstrated that $F_t({}_tV_x)$ represents the appropriate reserve at the end of the tth policy year. The basic problem, therefore, is the definition of the unamortized portion of the initial expense deficit, that is, the definition of the fixed premium variable benefit analogue of ${}_tU_x$ in equation (46).

After a thorough analysis of various possible methods of handling this basic problem, it appeared to us that, from a practical point of view, one method was far superior to any of the other possible methods. Under the proposed method, the unamortized initial expense deficit per dollar of actual face amount at the end of each policy year for a fixed premium variable benefit policy would be exactly the same as that for a corresponding fixed premium fixed benefit policy. Under this proposed method, the statutory minimum cash-surrender value for a fixed premium variable benefit life insurance policy would be

$$ (F_t)\,{}_t(\mathrm{MCV})_x = F_t({}_tV_x) - F_t({}_tU_x)\,. \tag{47} $$

In other words, under the proposed method the statutory minimum cash-surrender value per dollar of actual face amount under a fixed premium variable benefit policy would be exactly the same as the statutory minimum cash-surrender value per dollar of face amount under a corresponding regular fixed benefit policy.

It is apparent that the proposed method of specifying minimum cash-surrender values would be quite easy to implement in the development of appropriate legislation and would not involve the recalculation of presently published minimum cash-surrender values, since they would continue to be applicable on a per $1,000 actual face amount basis for a fixed premium variable benefit policy.

Appendix B compares the proposed method of specifying minimum cash-surrender values with several alternative methods that were considered.

Since the method adopted for the definition of statutory minimum cash-surrender values for a fixed premium variable benefit policy will have significant effects on the methods that life insurance companies can adopt for the illustration of actual cash-surrender values and nonforfeiture values in life insurance policy forms, this problem will be considered in Section IV of this paper.

IV. PROBLEMS INVOLVED IN ILLUSTRATING ACTUAL CASH-SURRENDER VALUES AND NONFORFEITURE BENEFITS IN POLICY FORMS

In this section, we shall examine the problems involved in illustrating cash-surrender values and nonforfeiture benefits in policy forms for a fixed premium variable benefit policy if our proposed method of defining statutory minimum cash-surrender values is adopted.

Extension of the concept underlying our proposed method of defining minimum cash-surrender values would mean that the fixed premium variable benefit policy form could show a table of cash-surrender values per $1,000 of actual face amount. This table would look exactly like the table of cash-surrender values per $1,000 of face amount which appears in a fixed benefit policy.

In similar fashion, the fixed premium variable benefit policy form could show a table of reduced paid-up values per $1,000 of actual face amount which would look exactly like the table of such values per $1,000 of face amount which appears in a fixed benefit policy. The formula used to compute such values under a fixed premium variable benefit policy would also be exactly the same as that used in the case of a corresponding fixed benefit policy. For example, consider the calculation of the reduced paid-up value applicable at the end of policy year t under a fixed premium variable benefit whole life policy issued at age x for an initial face amount of $1,000. Let us denote the reduced paid-up value per $1,000 of actual face amount to be shown in the policy by ${}_t\mathrm{RP}_x$. The actual amount of reduced paid-up insurance would therefore be $F_t({}_t\mathrm{RP}_x)$. If ${}_t\mathrm{CV}_x$ is the cash-surrender value per $1,000 of actual face amount under the fixed premium variable benefit whole life policy at the end of policy year t, the actual cash-surrender value would be $F_t({}_t\mathrm{CV}_x)$. It can therefore be seen that ${}_t\mathrm{RP}_x$ should be calculated by the formula

$$F_t({}_t\mathrm{RP}_x) = \frac{F_t({}_t\mathrm{CV}_x)}{A_{x+t}}. \tag{48}$$

This formula reduces to

$$ {}_t\mathrm{RP}_x = \frac{{}_t\mathrm{CV}_x}{A_{x+t}}, \tag{49} $$

which is the regular formula for calculating reduced paid-up values under a fixed benefit whole life policy.

The actual amount of reduced paid-up insurance under the fixed premium variable benefit policy, $F_t({}_t\mathrm{RP}_x)$, could be a guaranteed level amount of insurance. Alternatively, if a company wished to offer variable nonforfeiture benefits in connection with its fixed premium variable benefit policies, $F_t({}_t\mathrm{RP}_x)$ could be the initial amount of variable reduced paid-up insurance. In such case, the actual amount of reduced paid-up insurance as of the end of s policy years after the variable reduced paid-up benefit became effective at the end of policy year t would be given by the expression

$$ F_t({}_t\mathrm{RP}_x) \prod_{j=1}^{s} Z_{t+j} \,. \tag{50} $$

The fixed premium variable benefit policy form could also show a table of extended term periods that would look exactly like the table of such periods appearing in a fixed benefit policy and would be calculated according to the same formula. For example, let us again consider a fixed premium variable benefit whole life policy issued at age x for an initial face amount of \$1,000. The amount of extended term insurance applicable at the end of policy year t will be the actual face amount 1,000 F_t. The extended term period, e, should therefore be computed from the equation

$$ F_t({}_t\mathrm{CV}_x) = 1{,}000\, F_t A^{\;1}_{\overline{x+t}:\overline{e|}} \,. \tag{51} $$

This equation reduces to

$$ {}_t\mathrm{CV}_x = 1{,}000 A^{\;1}_{\overline{x+t}:\overline{e|}} \,, \tag{52} $$

which is the regular equation used in calculating extended term periods under a fixed benefit whole life policy.

The actual amount of extended term insurance under the fixed premium variable benefit policy, 1,000 F_t, could be a guaranteed level amount. Alternatively, if a company wished to offer variable nonforfeiture benefits in connection with its fixed premium variable benefit policies, it could be the initial amount of variable extended term insurance. In such case, the actual amount of extended term insurance as of the end of s policy years after the variable extended term benefit became effective at the end of policy year t would be given by the expression

$$1{,}000\, F_t \prod_{j=1}^{s} Z_{t+j}\,. \tag{53}$$

Thus it can be seen that the implementation in actual policy forms of cash-surrender and nonforfeiture values reflecting the basic concept underlying our proposed method of defining statutory minimum cash-surrender values is quite simple. The fixed premium variable benefit policy form could, under this method, show a table of cash-surrender and nonforfeiture values that would look exactly like the one contained in a fixed benefit policy. The nonforfeiture values shown in the fixed premium variable benefit policy could be computed by using formulas that are exactly the same as those used for a fixed benefit policy. This means that if cash-surrender values per \$1,000 actual face amount under a company's fixed premium variable benefit policies were the same as those per \$1,000 face amount under its corresponding fixed benefit policies, the nonforfeiture values shown in fixed premium variable benefit policies would also be exactly the same as those shown in corresponding fixed benefit policies.

As indicated in Appendix B, there would be serious problems in the illustration in actual policy forms of cash-surrender and nonforfeiture values reflecting the concepts underlying alternative methods of defining statutory minimum cash-surrender values.

V. OTHER STATUTORY CHANGES REQUIRED

In this section, we will briefly outline some other areas in which changes in existing statutory requirements would have to be made in order to accommodate fixed premium variable benefit policies. Also mentioned are several areas in which statutory changes are not required but might be desirable in order to give companies flexibility in designing fixed premium variable benefit policies.

Grace Period

Under present law a premium paid within the grace period must be treated as if it were paid on the due date for the purpose of determining policy benefits and values. Many companies will wish to adhere to this concept under fixed premium variable benefit policies, since they will conclude that the resulting administrative simplifications will outweigh any investment risk which might result from having benefits and values predicated on the assumption that money came into the separate account shortly before the actual premium was paid.

However, some companies may wish to have benefits and values reflect the actual dates on which premium payments are made.

Reinstatement

The basis for reinstatement generally mandated by present law is payment of back premiums with interest. This basis is clearly inappropriate for use under a fixed premium variable benefit contract, because it would permit the policyowner to play the stock market with hindsight. For example, consider the case of a lapsed policy running on extended term insurance. Our testing indicates that, if such a policy were reinstated by payment of back premiums with interest, situations could arise in which the immediate increase in cash-surrender value (from that on the extended term basis to that on the regular premium-paying basis) would be greater than the payment required to reinstate.

Another possibility would be to permit reinstatement on the basis of payment of the increase in cash-surrender value. However, this basis seems to present some significant practical problems. For example, if actual investment performance is better than assumed, the death benefit under the extended term option will increase faster than that on a regular premium-paying basis, because the Y_t factor, which is less than 1 during the premium-paying period whenever the actual face amount exceeds the initial amount, is always 1 on a paid-up basis. Therefore, this alternative basis could result in the anomalous situation in which a policyowner could go on extended term, have bigger death benefits than would be possible if he had continued paying premiums, and yet be able to reinstate by paying an amount less than the premiums that he had skipped.

A third method that appears to be more appropriate for a fixed premium variable benefit policy and that would help resolve these problems is one in which reinstatement would be permitted on the basis of payment of back premiums with interest or the increase in cash-surrender value, if greater. The use of this third method for fixed premium variable benefit policies would require a revision of the reinstatement provisions of the law.

Policy Loans

The present requirement that a policy's loan value be equal to its cash-surrender value appears to be inappropriate for a fixed premium variable benefit policy because the cash-surrender value under such a policy can decrease if actual investment performance is poor. The law would therefore have to be revised to allow a feasible alternative basis (or bases) for loans under fixed premium variable benefit policies.

One approach that appears to be feasible would be to permit policy loans on a basis quite similar to that involved in the present practice of making collateral (or margin) loans secured by common stock. Under this approach, loans would be permitted up to a specified percentage of the actual cash-surrender value. Such a percentage limit would be necessary in order to protect the policyowner from a quick foreclosure (or "margin call" for some repayment) if actual investment performance became unfavorable.

In connection with this approach, procedures could also be developed to make periodic checks of loan status so that the policyowner could be alerted if the relationship between his loan and cash-surrender value were approaching a danger point.

Another alternative would involve the amount of the loan varying to directly reflect actual net investment performance of the separate account. While this alternative is theoretically sound, the concept of variable loans could lead to policyowner confusion and misunderstanding.

Dividend Options

It would be feasible to offer the present types of regular dividend options in connection with fixed premium variable benefit contracts. We also believe that variable analogues of the present paid-up addition and deposit options are feasible and could be quite attractive and that they should therefore be permitted.

A variable paid-up addition option would operate according to the same basic principle that we have previously outlined for the variable reduced paid-up nonforfeiture option; that is, the amount of paid-up coverage provided would vary each year in accordance with the Z_t factor.

Under the variable deposit option, dividends would be applied to purchase units in a separate account, and the value of the dividend deposits would reflect the actual net investment return of the separate account. This option could also be very popular in connection with dividends on fixed benefit policies, as indicated by current experience in Canada, where such an arrangement is legally permissible and has been introduced by several Canadian companies.

Settlement Options

The situation in this area is quite similar to that with respect to dividend options in that (1) it is feasible to use present types of fixed dollar options with a fixed premium variable benefit policy; (2) it is also feasible (and quite attractive, we believe) to have variable options; and (3) variable options would have considerable appeal in connection with regular fixed benefit policies as well as variable policies.

We therefore believe that variable settlement options should be permitted. In fact, their availability appears to be particularly important in connection with one type of situation that could arise under a fixed premium variable benefit policy. This would be where proceeds were at a relatively depressed level due to unfavorable investment performance prior to the time of settlement. Then, a settlement either in cash or under a fixed dollar option would have the effect of "locking in" this unfavorable result, but a variable settlement would give the policyowner (or beneficiary) the opportunity of making some recovery if investment performance subsequently became favorable.

VI. POSSIBLE VARIATIONS IN BASIC CONCEPTS

In this section, we will briefly outline some possible variations in the basic concepts underlying the fixed premium variable benefit policy described in the previous sections of this paper.

Combination of Fixed Benefits and Variable Benefits in Same Policy

In practice, a company might want to permit the policyholder to combine fixed benefits and variable benefits in the same policy. There appears to be no reason why a portion of each premium could not be assigned to fixed benefits and a portion assigned to variable benefits.

A company may also want to permit reserves for fixed benefits under the policy to be transferred from the general account to the separate account to convert fixed benefits into variable benefits and to permit reserves for variable benefits under the policy to be transferred from the separate account to the general account to convert variable benefits into fixed benefits.

Options to Vary Premium within Prescribed Limits

A company might want to give the policyowner the option of varying premiums within certain prescribed limits: for example, it might permit payment of a net premium of $F_{t-1}P_x$ instead of P_x, so that the face amount at the beginning of the tth year (i.e., after payment of the net premium) would be the same as the face amount at the end of the $(t - 1)$st year.

Guarantee of Minimum Benefits for Appropriate Extra Premium

A company might wish to guarantee that the benefits payable under a fixed premium variable benefit policy would in no event be less than a specified minimum, for example, the benefits payable under a corresponding fixed benefit policy, subject to an appropriate extra premium. This would, of course, subject the company to some investment risk.

VII. CONCLUSION

The authors believe that the concepts presented in this paper clearly indicate that it is possible to develop actuarially sound fixed premium variable benefit life insurance policies. These policies would offer the public the opportunity of buying a life insurance product that reflects the investment performance of reserves invested in equities but that has practically all the characteristics of regular fixed benefit life insurance policies.

We have presented this paper in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of fixed premium variable benefit policies along the lines developed in this paper and also of equity-based life insurance products that reflect various alternative approaches. We recognize that the approach described in this paper is merely one of a number of different approaches that may be taken in developing equity-based life insurance products and hope that this paper will stimulate discussion of both the proposed approach and possible alternative approaches to this problem.

APPENDIX A

PROOF OF EQUATION (40)

This equation is

$$A'_{x+t} - P_x\ddot{a}_{x+t} = F_t({}_tV_x)\,, \tag{A1}$$

where it is specified that

$$A'_{x+t} = \frac{1}{D_{x+t}} \sum_{j=0}^{\omega-x-t-1} (C_{x+t+j})(F_{t+j+1}) \tag{A2}$$

with the commutation functions calculated at the assumed interest rate i to reflect the assumption that, in all future years after the end of policy year t, the net investment return on the separate account will be such assumed interest rate i. The series of face amounts $F_{t+1}, F_{t+2}, \ldots$, will therefore be those reflecting this assumption.

We will prove equation (A1) by an inductive process using the following steps:

1. A proof that equation (A1) is true at the end of the next-to-last policy year of the fixed premium variable benefit whole life policy; that is, for $t = \omega - x - 1$.
2. A proof that if equation (A1) is assumed to be true for $t = n$, it is true for $t = n - 1$.

It is clear that, if Steps 1 and 2 are both true, equation (A1) is true for any value of t.

In proving that equation (A1) is true for $t = \omega - x - 1$, we will first examine the right-hand side of this equation, which for $t = \omega - x - 1$ is $F_{\omega-x-1}({}_{\omega-x-1}V_x)$. Let us first recall the general equation of equilibrium for a fixed premium variable benefit whole life policy, previously given as equation (2) in Section II of this paper.

$$[F_{t-1}({}_{t-1}V_x) + P_x](1 + i'_t) = q_{x+t-1}[F_t - F_t({}_tV_x)] + F_t({}_tV_x) \,. \quad \text{(A3)}$$

If we let $t = \omega - x$, we have

$$[F_{\omega-x-1}({}_{\omega-x-1}V_x) + P_x](1 + i'_{\omega-x}) = q_{\omega-1}[F_{\omega-x} - F_{\omega-x}({}_{\omega-x}V_x)] + F_{\omega-x}({}_{\omega-x}V_x) \,. \quad \text{(A4)}$$

We now let $i'_{\omega-x} = i$, to reflect the assumption that the net investment return on the separate account is equal to the assumed interest rate i in all future years. Making this substitution, and also making use of the fact that ${}_{\omega-x}V_x = 1$, we have

$$[F_{\omega-x-1}({}_{\omega-x-1}V_x) + P_x](1 + i) = F_{\omega-x} \,. \quad \text{(A5)}$$

Solving equation (A5) for $F_{\omega-x-1}({}_{\omega-x-1}V_x)$, we have

$$F_{\omega-x-1}({}_{\omega-x-1}V_x) = vF_{\omega-x} - P_x \,. \quad \text{(A6)}$$

Let us now examine the left-hand side of equation (A1), which for $t = \omega - x - 1$ is

$$A'_{\omega-1} - P_x\ddot{a}_{\omega-1} \,. \quad \text{(A7)}$$

By reference to equation (A2) we see that

$$A'_{\omega-1} = \frac{C_{\omega-1}}{D_{\omega-1}} F_{\omega-x} \,. \quad \text{(A8)}$$

Since $C_{\omega-1}/D_{\omega-1} = v$ and $\ddot{a}_{\omega-1} = 1$, it is apparent that

$$A'_{\omega-1} - P_x\ddot{a}_{\omega-1} = vF_{\omega-x} - P_x \,. \quad \text{(A9)}$$

Since the right-hand sides of equations (A6) and (A9) are identical, the left-hand sides of these equations are equal to each other, that is,

$$F_{\omega-x-1}({}_{\omega-x-1}V_x) = A'_{\omega-1} - P_x\ddot{a}_{\omega-1} \,, \quad \text{(A10)}$$

and we have therefore completed Step 1 of the proof, since equation (A10) shows that equation (A1) is true for $t = \omega - x - 1$.

We will now prove that, if equation (A1) is true for $t = n$, it is true for $t = n - 1$, that is, if

$$A'_{x+n} - P_x\ddot{a}_{x+n} = F_n({}_nV_x) \,, \quad \text{(A11)}$$

then

$$A'_{x+n-1} - P_x\ddot{a}_{x+n-1} = F_{n-1}({}_{n-1}V_x)\ .$$

If we let $t = n$ in equation (A3), we have

$$\begin{aligned}[F_{n-1}({}_{n-1}V_x) + P_x](1 + i'_n) = q_{x+n-1}[F_n - F_n({}_nV_x)] \\ + F_n({}_nV_x)\ .\end{aligned} \tag{A12}$$

We now let $i'_n = i$ reflect the assumption that the net investment return on the separate account is equal to the assumed interest rate i, and obtain

$$\begin{aligned}[F_{n-1}({}_{n-1}V_x) + P_x](1 + i) &= q_{x+n-1}[F_n - F_n({}_nV_x)] + F_n({}_nV_x) \\ &= q_{x+n-1}F_n + (1 - q_{x+n-1})F_n({}_nV_x) \\ &= q_{x+n-1}F_n + p_{x+n-1}F_n({}_nV_x)\ .\end{aligned} \tag{A13}$$

We will now make use of the hypothesis that equation (A1) is true for $t = n$ or that

$$A'_{x+n} - P_x\ddot{a}_{x+n} = F_n({}_nV_x)\ . \tag{A14}$$

Substituting for $F_n({}_nV_x)$ in equation (A13), we have

$$\begin{aligned}[F_{n-1}({}_{n-1}V_x) + P_x](1 + i) = q_{x+n-1}F_n \\ + p_{x+n-1}(A'_{x+n} - P_x\ddot{a}_{x+n})\ .\end{aligned} \tag{A15}$$

Solving equation (A15) for $F_{n-1}({}_{n-1}V_x)$, we obtain

$$\begin{aligned}F_{n-1}({}_{n-1}V_x) &= v[q_{x+n-1}F_n + p_{x+n-1}(A'_{x+n} - P_x\ddot{a}_{x+n})] - P_x \\ &= vq_{x+n-1}F_n + vp_{x+n-1}A'_{x+n} - P_x(1 + vp_{x+n-1}\ddot{a}_{x+n})\ .\end{aligned} \tag{A16}$$

If we let $t = n - 1$ in equation (A2), we have

$$\begin{aligned}A'_{x+n-1} &= \frac{1}{D_{x+n-1}} \sum_{j=0}^{\omega-x-n} (C_{x+n+j-1})(F_{n+j}) \\ &= \frac{C_{x+n-1}}{D_{x+n-1}} F_n + \frac{1}{D_{x+n-1}} \sum_{j=1}^{\omega-x-n} (C_{x+n+j-1})(F_{n+j}) \\ &= \frac{C_{x+n-1}}{D_{x+n-1}} F_n + \left(\frac{D_{x+n}}{D_{x+n-1}}\right)\left(\frac{1}{D_{x+n}}\right) \sum_{j=1}^{\omega-x-n} (C_{x+n+j-1})(F_{n+j})\ .\end{aligned} \tag{A17}$$

If we let $t = n$ in equation (A2), we have

$$\begin{aligned}A'_{x+n} &= \frac{1}{D_{x+n}} \sum_{j=0}^{\omega-x-n-1} (C_{x+n+j})(F_{n+j+1}) \\ &= \frac{1}{D_{x+n}} \sum_{j=1}^{\omega-x-n} (C_{x+n+j-1})(F_{n+j})\ .\end{aligned} \tag{A18}$$

Making use of the expression for A'_{x+n} given in equation (A18), we can rewrite equation (A17) as

$$\begin{aligned} A'_{x+n-1} &= \frac{C_{x+n-1}}{D_{x+n-1}} F_n + \frac{D_{x+n}}{D_{x+n-1}} A'_{x+n} \\ &= vq_{x+n-1}F_n + vp_{x+n-1}A'_{x+n} . \end{aligned} \tag{A19}$$

Equation (A19) is seen to be the fixed premium variable benefit analogue of the relationship $A_{x+n-1} = vq_{x+n-1} + vp_{x+n-1}A_{x+n}$.

It is also true that

$$\ddot{a}_{x+n-1} = 1 + vp_{x+n-1}\ddot{a}_{x+n} . \tag{A20}$$

Substituting these expressions for $vq_{x+n-1}F_n + vp_{x+n-1}A'_{x+n}$ and $1 + vp_{x+n-1}\ddot{a}_{x+n}$ in equation (A16), we have

$$F_{n-1}({}_{n-1}V_x) = A'_{x+n-1} - P_x\ddot{a}_{x+n-1} , \tag{A21}$$

and Step 2 of the proof is completed. We have therefore proved that equation (A1) is true for all values of t.

This proof can be generalized in the following ways:

1. To apply to plans generally, rather than just whole life;
2. To apply under any assumption as to future net investment performance of the separate account; and
3. To apply where net annual premiums and reserves are computed according to a reserve method other than the net level premium method (e.g., the commissioners reserve valuation method).

APPENDIX B

ALTERNATIVE APPROACHES TO DEFINITION OF CASH-SURRENDER VALUES

As indicated in Section III, there are several alternative methods of defining statutory minimum cash-surrender values under fixed premium variable benefit policies other than the method proposed in this paper. The two main alternative methods reflect specific concepts as to how I_x (the initial expense deficit specified for purposes of calculating minimum cash-surrender values) should be amortized.

Alternative A reflects the concept that, while the insurance benefits under a fixed premium variable benefit policy reflect investment performance in the separate account, the initial expense deficit set forth for the purpose of computing minimum cash-surrender values is something that is fixed at issue without regard to future investment performance and can properly be amortized at the assumed interest rate without regard to actual investment performance. Therefore, minimum cash-surrender values under this alternative would be equal to the actual reserve less the same unamortized portion of the initial expense deficit involved in the minimum cash-surrender value for a corresponding fixed benefit policy. For example, under a fixed premium variable benefit whole life policy, the minimum cash-surrender value under Alternative A could be expressed as

$$ {}_t(\mathrm{MCV})_x^{\mathrm{A}} = F_t({}_tV_x) - {}_tU_x \,. \tag{B1} $$

Alternative B reflects the concept that a fixed premium variable benefit policy is one under which all financial transactions take place through the separate account and that policy values as well as insurance benefits should therefore reflect actual investment performance in the separate account to the maximum extent possible without subjecting the company to an investment risk. Under this alternative, amortization of the initial expense deficit would directly reflect actual investment performance. For example, again considering a fixed premium variable benefit whole life policy, if we denote the unamortized portion of the initial expense deficit at the end of policy year t under Alternative B as ${}_tU'_x$, it can be shown that, in general

$$ {}_tU'_x = {}_tU_x \prod_{j=1}^{t} Z_j \,. \tag{B2} $$

Minimum cash-surrender values for a fixed premium variable benefit whole life policy under Alternative B can therefore be expressed as

$$ {}_t(\mathrm{MCV})_x^{\mathrm{B}} = F_t({}_tV_x) - {}_tU_x \prod_{j=1}^{t} Z_j \,. \tag{B3} $$

Section IV of this paper indicated that our proposed method would be easy to implement in actual policy forms, since cash-surrender values and reduced paid-up values could be simply shown per $1,000 of actual face amount and extended term periods could also be shown in the same manner as at present. These nonforfeiture benefits would be calculated from the regular equations (49) and (52) previously given in Section IV, that is,

$$ {}_t\mathrm{RP}_x = \frac{{}_t\mathrm{CV}_x}{A_{x+t}} \,; \tag{B4} $$

$$ {}_t\mathrm{CV}_x = 1{,}000 A_{\overset{1}{x+t}:\overline{e}|} \tag{B5} $$

This simple method of illustrating actual cash-surrender values and nonforfeiture benefits could also be used under Alternative A or B at any duration when the cash-surrender value is equal to the full reserve, because such cash-surrender values can be expressed per $1,000 of actual face amount and equations (B4) and (B5) can therefore be used.

It should be clear, however, that these equations cannot be used unless the cash-surrender value is expressed on a per \$1,000 actual face amount basis. This is impossible when computing cash-surrender values according to the basic concept underlying Alternative A or B at any time when such values are less than the full reserve, because the deduction from the reserve under these alternatives is not directly proportional to the actual face amount. Therefore, the following statements can be made regarding illustration of actual cash-surrender values and nonforfeiture benefits, based on Alternative A or B, at durations when cash-surrender values are less than the full reserve.

1. Cash-surrender values would have to be expressed in terms of more than one factor (e.g., 1,000 ${}_tV_x$ per \$1,000 of actual face amount less a deduction per \$1,000 of initial face amount in the case of Alternative A, with the additional complications arising from the existence of the

$$\prod_{j=1}^{t} Z_j$$

factor to be considered under Alternative B).

2. Reduced paid-up values could not be shown per \$1,000 of actual face amount, because equation (B4) for the calculation of such values could not be used. However, it would be feasible to show reduced paid-up values which would be applicable per \$1,000 of actual cash value.

3. There would appear to be no simple way of illustrating extended term periods in the policy form, since equation (B5) could not be used. This, of course, is a serious defect.

We therefore believe that our proposed method of defining cash-surrender values under fixed premium variable benefit policies is, from a practical point of view, far superior to the alternatives discussed above.

We also examined another alternative, which is an attempt to reflect directly the concept of retrospective fund accumulation which underlies the standard nonforfeiture law. This alternative uses an equation of equilibrium for minimum cash-surrender values,

$$\begin{aligned}[F'_{t-1}\cdot {}_{t-1}(\mathrm{MCV})_x + (\mathrm{AP})_x](1+i'_t) = q_{x+t-1}[F'_t - F'_t\cdot {}_t(\mathrm{MCV})_x] \\ + F'_t\cdot {}_t(\mathrm{MCV})_x\,,\end{aligned} \tag{B6}$$

which is a direct analogue of the equation of equilibrium for reserves. The validity of this equation can be proved if it is specified that

$$F'_t = F'_{t-1}Y'_tZ_t \tag{B7}$$

and

$$Y'_t = \frac{{}_{t-1}(\mathrm{MCV})_x + (\mathrm{AP})_x/F'_{t-1}}{{}_{t-1}(\mathrm{MCV})_x + (\mathrm{AP})_x}\,. \tag{B8}$$

This alternative, however, seems to be of no practical significance, because F'_t is different from the actual face amount under the fixed premium variable benefit policy and (because of the fact that ${}_t(\mathrm{MCV})_x$ can be negative in the early policy years) Y'_t can be zero, negative—or infinite, because $[{}_{t-1}(\mathrm{MCV})_x+(\mathrm{AP})_x]$ can be zero!

An offshoot of this alternative was also examined, under which the minimum cash-surrender value would be equal to a fund per survivor, on the assumption that (*a*) the initial fund at issue would be negative to the extent of the initial expense deficit; (*b*) deposits of $(\mathrm{AP})_x$ would be made at issue and for each survivor at the beginning of subsequent years during the premium paying period; and (*c*) death claims, based on the actual face amount, F_t, would be paid out of the fund at the end of each year.

The calculation would reflect interest on the basis of actual investment performance and survivorship according to tabular (i.e., 1958 C.S.O.) mortality rates.

While this approach may seem logical at first glance, it is invalid because the fund per survivor (i.e., the minimum cash-surrender value) will generally not equal the actual terminal reserve at the end of the premium-paying period. The reason for this is that this approach involves "payments" to amortize the initial expense deficit which are calculated at the assumed interest rate but are accumulated at the actual rates.

DISCUSSION OF PRECEDING PAPER

CHARLES B. BAUGHMAN:

Eleven years ago Mr. Fergus McDiarmid presented a paper to the Society entitled "Inflation and Life Insurance." His paper spelled out in detail the need for new life insurance products to enable policyholders to cope better with inflation. Unfortunately, his careful analysis received virtually no support. Today, however, attitudes are greatly different, and the excellent paper that we are discussing should mark the beginning of a new day for our industry and customers.

The first table in Mr. McDiarmid's paper showed that among the most popular plans of insurance the one most vulnerable to inflation was the twenty-payment life policy. It had the longest average elapsed time from the payment of premiums to the receipt of benefits. Because of this and the fact that before inflation became a serious problem the twenty-pay policy was a very popular plan, I am suggesting here a design for a level premium variable twenty-pay policy. The policy should have appeal, because it has no discontinuities in the death benefit and the death benefit at any duration is independent of age at issue.

The policy is a combination of fixed-dollar and variable insurance. The essential feature of the design is that on the issue date and the first nineteen anniversary dates the fixed-dollar portion of the death benefit is reduced by 5 per cent of the initial death benefit and the variable portion is increased by a like amount. Once a portion becomes variable, it varies thereafter according to the Z_t factor in the author's paper. The death benefit therefore remains level over any period that investment results equal the AIR, regardless of prior results. The minimum cash value and reserve at any point in time is the present value of the then existing death benefit less the present value of the adjusted and modified premiums, respectively.

The reserve is partially fixed and partially variable. It will be noted that the major portion of the reserve will be held in the separate account and be subject to the fluctuations of a portfolio invested largely in equities. A very minor portion of the reserve will be invested in a fixed-dollar account. The reserve for the fixed portion will be negative at some durations, but because it is so small it will be netted against the larger positive reserve, as has been frequently done in accident and health insurance.

In order to illustrate that the policy is actuarially sound, let us prove by mathematical induction that the reserve at the end of any policy year is always equal to the present value of future benefits less the present value of future reserve premiums. The proof shows that the retrospective reserve equals the prospective reserve. For the sake of simplicity, let us consider only the terminal net level reserve on an annual basis. Then,

$_{20}P_x = A_x \div \ddot{a}_{x:\overline{20}|}$.
$_nV_x^{(F)}$ = Reserve for fixed portion.
$_nV_x^{(V)}$ = Reserve for variable portion.
i = Assumed investment rate.
j_n = Gross investment rate less margin deduction.
U_n = Value of insurance unit at end of policy year n.
N_n = Number of insurance units of benefit during policy year n.
B_n = Amount of death benefit from variable portion only at end of policy year n.

The total reserve is $_nV_x^{(F)} + {_nV_x^{(V)}}$, where,

$$_nV_x^{(F)} = (1 - 0.05n)A_{x+n} - {_{20}P_x}\ddot{a}_{x+n:\overline{20-n}|}\,; \tag{1}$$

$$_nV_x^{(V)} = B_nA_{x+n}\,. \tag{2}$$

Since at the beginning of the nth year an amount equal to $0.05A_{x+n-1}$ is transferred from the fixed account to the separate account, the reserve formulas are

$$l_{x+n-1}[_{n-1}V_x^{(F)} + {_{20}P_x} - 0.05A_{x+n-1}](1 + i) - (1 - 0.05n)d_{x+n-1} = l_{x+n}\,{_nV_x^{(F)}}\,; \tag{3}$$

and

$$l_{x+n-1}[_{n-1}V_x^{(V)} + 0.05A_{x+n-1}](1 + j_n) - B_nd_{x+n-1} = l_{x+n}\,{_nV_x^{(V)}}\,. \tag{4}$$

Our valuation method and cash-value method are correct if we can show that the values in formulas (1) and (2) are solutions to formulas (3) and (4), respectively.

First we multiply formula (3) by v^{x+n} and substitute the values from formula (1):

$$D_{x+n-1}\{[1 - 0.05(n - 1)]A_{x+n-1} - {_{20}P_x}\ddot{a}_{x+n-1:\overline{21-n}|} + {_{20}P_x} - 0.05A_{x+n-1}\} - (1 - 0.05n)C_{x+n-1}$$
$$= D_{x+n}[(1 - 0.05n)A_{x+n} - {_{20}P_x}\ddot{a}_{x+n:\overline{20-n}|}]\,.$$

In converting to commutation functions,

$$(1 - 0.05n)M_{x+n-1} - {}_{20}P_x(N_{x+n-1} - N_{x+20}) + {}_{20}P_xD_{x+n-1}$$
$$- (1 - 0.05n)C_{x+n-1}$$
$$= (1 - 0.05n)M_{x+n} - {}_{20}P_x(N_{x+n} - N_{x+20}) .$$

This is an identity, since $M_{x+n-1} - C_{x+n-1} = M_{x+n}$ and $N_{x+n-1} - D_{x+n-1} = N_{x+n}$, and we have shown that formula (1) is a solution of formula (3).

In proving formula (4), we use the following formulas, which will be defined in the policy form:

$$U_{n-1}(1 + j_n)v = U_n ; \tag{5}$$

$$B_n = N_nU_n ; \tag{6}$$

$$N_n = N_{n-1} + \frac{0.05}{U_{n-1}} . \tag{7}$$

Multiplying formula (4) by v^{x+n} and substituting the value in formula (2), we have

$$D_{x+n-1}(B_{n-1}A_{x+n-1} + 0.05A_{x+n-1})(1 + j_n)v - B_nC_{x+n-1} = D_{x+n}B_nA_{x+n} .$$

Substituting from formulas (5) and (6), we have

$$D_{x+n-1}(N_{n-1}U_{n-1}A_{x+n-1} + 0.05A_{x+n-1}) \frac{U_n}{U_{n-1}}$$
$$-N_nU_nC_{x+n-1} = D_{x+n}N_nU_nA_{x+n} .$$

Since from formula (7), $0.05 = (N_n - N_{n-1})U_{n-1}$, we get, after adopting commutation functions,

$$[N_{n-1}U_{n-1}M_{x+n-1} + (N_n - N_{n-1})M_{x+n-1}U_{n-1}] \frac{U_n}{U_{n-1}}$$
$$-N_nU_nC_{x+n-1} = N_nU_nM_{x+n} ,$$

which is an identity since

$$M_{x+n-1} - C_{x+n-1} = M_{x+n} .$$

The death benefit based on hypothetical results is a very simple form. If j_n is assumed to be constant at the value j for all years, the death benefit at the end of the year n is $(1 - 0.05n) + 0.05\ddot{s}_{\overline{n}|k}$ during the first twenty years and $0.05\ddot{s}_{\overline{20}|k}(1 + k)^{n-20}$ after twenty years, where $k = [(1 + j)/(1 + i)] - 1$.

The gross premium will be a function of the assumed investment rate. If the AIR for the variable policy is the same as the interest rate used in calculating the premium for a fixed-dollar policy, the gross premiums for both will be equal if all other assumptions are unchanged. In such an instance and if asset share calculations for the fixed policy are satisfactory, asset shares for the variable policy should also be satisfactory, regardless of actual investment results.

By changing the incidence of variability of the benefits, other plans of variable insurance could be developed without incurring any danger of negative reserves.

JOHN K. BOOTH:

Messrs. Fraser, Miller, and Sternhell are to be congratulated on their fine paper, which represents a landmark in the extension of the separate-account concept to life insurance.

The fixed premium variable benefit policy described in their paper is characterized by the fact that both the policy reserve and the net amount at risk vary according to the investment experience of a separate account. One approach which has been suggested overseas is to have the pure insurance portion of the policy remain fixed at each duration during the lifetime of the policy in accordance with a predetermined schedule and to allow only the savings portion of the policy to vary. If we define the pure insurance portion of such a policy as the tabular net amount at risk under a whole life policy, the face amount payable at the end of the year of death is given by $F_t = 1 - {}_tV_x + \mathrm{PS}_t$, where

F_t = face amount at the end of the tth policy year;
${}_tV_x$ = terminal reserve at the end of the tth policy year for a whole life policy;
PS_t = policyholder's share in the separate account at the end of the tth policy year.

This equation may be rewritten and a new symbol X_t defined such that

$$X_t = F_t - 1 = \mathrm{PS}_t - {}_tV_x\,. \tag{1}$$

In other words, the excess, X_t, which may be positive or negative, of the face amount at the end of the tth policy year over the initial face amount of 1 is equal to the excess of the amount of the policyholder's share in the separate account over the tabular reserve for a whole life policy.

If it is assumed that death benefits are paid from the separate account at the end of the year of death, the equation connecting successive policyholder's shares is

$$(\mathrm{PS}_{t-1} + P_x)(1 + i'_t) = q_{x+t-1}(1 - {}_tV_x) + \mathrm{PS}_t\,, \tag{2}$$

where P_x is the net level annual premium for a whole life policy issued at age x and i'_t is the actual net annual investment return on the separate account during the tth policy year, including realized and unrealized appreciation and depreciation.

If we substitute from equation (1),

$$(X_{t-1} + {}_{t-1}V_x + P_x)(1 + i'_t) = q_{x+t-1}(1 - {}_tV_x) + {}_tV_x + X_t \,. \quad (3)$$

We can then substitute the relationship

$$q_{x+t-1}(1 - {}_tV_x) + {}_tV_x = ({}_{t-1}V_x + P_x)(1 + i) \quad (4)$$

and rearrange terms to obtain

$$X_{t-1}(1 + i'_t) + ({}_{t-1}V_x + P_x)(i'_t - i) = X_t \,, \quad (5)$$

where i is the assumed rate of investment return. Equation (5) defines the excess of the face amount over the initial face amount of 1, as of the end of the policy year, as the sum of the excess amount from the end of the previous policy year accumulated at the actual rate of investment return on the separate account during the tth policy year, plus the excess interest on the initial tabular reserve for a whole life policy, where the excess interest factor is based on the excess of the actual rate of investment return for the separate account over the assumed investment rate of return.

The insurer may wish to fund a portion of the death benefit equal to the tabular net amount at risk through its general account. This may be done by deducting the discounted value of the tabular net cost of a benefit equal to the tabular net amount at risk from the net premium as of the beginning of each policy year and transferring this deduction to the general account. In this case the second term of equation (5) becomes $v \cdot {}_tV_x(i'_t - i)$.

It is interesting to note the similarity between the authors' method of defining the variable face amount and the alternative method described above. Under the authors' method $F_t \div \mathrm{PS}_t = 1 \div {}_tV_x$. That is, the variable face amount is defined so that its ratio to the policyholder's share in the separate account is equal to the ratio of the initial benefit to the tabular reserve. Under the alternative method $F_t - \mathrm{PS}_t = 1 - {}_tV_x$. The variable face amount is defined so that the difference between it and the policyholder's share in the separate account is equal to the difference between the initial benefit and the tabular reserve. Therefore, the authors' approach might be appropriately named a "defined ratio method" and the alternative approach could be called a "defined difference method."

A comparison of the face amounts and reserves for a fixed premium variable benefit whole life policy as computed by the defined ratio and by the defined difference methods, under the assumption that the entire death benefit is funded through the separate account, is shown in Table 1. Table 1 shows that the defined difference method places greater emphasis on the investment aspects of the policy than does the defined ratio method. Consequently, when actual investment performance exceeds the assumed interest rate, the defined difference method produces higher reserves. Under these same conditions, the defined difference method produces lower face amounts in the earlier policy years but higher face amounts in the later policy years when the savings element accounts for a larger proportion of the face amount.

TABLE 1

FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED TO MALE AGED 55

(Net Premiums, Reserves, and Tabular Net Amounts at Risk Based on 1958 C.S.O. Table, Net Level Premium Valuation Method, 3 Per Cent Interest, and Traditional Functions)

Net Annual Investment Performance of Separate Account (Per Cent)	End of Policy Year	Defined Ratio Method: Face Amount	Defined Ratio Method: Terminal Reserves	Defined Difference Method: Face Amount	Defined Difference Method: Terminal Reserves
0	5	$ 926	$ 128	$ 986	$ 124
	10	873	239	951	224
	15	825	330	896	296
	20	783	401	824	336
	25	746	458	735	349
3	5	1,000	138	1,000	138
	10	1,000	273	1,000	273
	15	1,000	400	1,000	400
	20	1,000	512	1,000	512
	25	1,000	614	1,000	614
6	5	1,080	149	1,015	154
	10	1,148	314	1,059	333
	15	1,222	488	1,140	540
	20	1,300	666	1,269	782
	25	1,381	848	1,460	1,075
9	5	1,165	161	1,032	170
	10	1,320	361	1,131	404
	15	1,505	601	1,331	730
	20	1,717	880	1,682	1,194
	25	1,963	1,206	2,261	1,875

These comments are strictly my own and should not be construed to

For an endowment of r years with death benefit $F(t)$ and maturity amount

$$\bar{V}(r) = F(r) = e^{\int_0^r (\delta-\delta')ds}$$

equation (2.3) applies, so that

$${}_r p_x e^{-\int_0^r \delta' ds} = \int_0^r [\bar{P}(t) - \mu_{x+t} e^{\int_0^t (\delta-\delta')ds}]_t p_x e^{-\int_0^t \delta ds} dt\,.$$

A solution to this equation is the following:

$$\bar{P}(t) = \bar{P}'_x e^{\int_0^t (\delta-\delta')ds}\,,$$

since

$${}_r p_x e^{-\int_0^r \delta' ds} = \int_0^r (\bar{P}'_x - \mu_{x+t})\,{}_t p_x e^{-\int_0^t \delta' ds} dt\,,$$

which is equation (2.3) for an r-year endowment for unity on the AIR (δ').

Then equation (2.3) becomes

$$\begin{aligned}\bar{V}(r)\cdot{}_r p_x\cdot e^{-\int_0^r \delta ds} &= \int_0^r (\bar{P}'_x - \mu_{x+t}) e^{\int_0^t (\delta-\delta')ds}\cdot{}_t p_x\cdot e^{-\int_0^t \delta ds} dt \\ &= \int_0^r (\bar{P}'_x - \mu_{x+t}) e^{-\int_0^t \delta' ds}\cdot{}_t p_x\cdot dt \\ &= \bar{V}'(r)\,{}_r p_x\cdot e^{-\int_0^r \delta' ds}\,,\end{aligned}$$

so that

$$\bar{V}(r) = \bar{V}'(r)\cdot e^{\int_0^r (\delta-\delta')ds}\,.$$

Since

$$\frac{(au)_{n+t}}{(au)_n} = e^{\int_0^t (\delta-\delta')ds}\,,$$

where n is duration of separate account at issue, it is clear that the completely variable contract follows the standard actuarial formulas utilizing annuity units:

$$\bar{P}(t) = \frac{\bar{P}_x{}'}{(au)_n} \quad \left[\text{annuity units} = \bar{P}'_x \frac{(au)_{n+t}}{(au)_n}\text{ dollars}\right].$$

$$F(t) = \frac{1}{(au)_n} \quad \left[\text{annuity units} = \frac{(au)_{n+t}}{(au)_n}\text{ dollars}\right].$$

$$\bar{V}(t) = \frac{\bar{V}'(t)}{(au)_n} \quad \left[\text{annuity units} = \bar{V}'(t)\frac{(au)_{n+t}}{(au)_n}\text{ dollars}\right].$$

It is notable that the valuation reserve in dollars depends only on the AIR.

The history of our business proves that cash-value life insurance, so valuable to social stability, must be sold by competent, dedicated agents, adequately compensated for their time and skills. Equity-based insurance will become an important part of the insurance portfolio only if appropriate exemptions to the federal security laws permit agent compensation competitive with fixed-dollar contracts, at least for premium levels close to ordinary life.

The complexity of the equity-based contract is another problem. I am inclined to feel that design (4), involving an indefinite period (perhaps with a cost-of-living index), and design (7), involving fixed variable annuity units (which can be tabulated), will best meet the test of simplicity.

STEVEN L. COOPER:

The purpose of this discussion is threefold: first, to express admiration for the authors' elegant approach to variable insurance benefits and to thank them for their lucid exposition of this approach; second, to consider another approach to variable insurance benefits; and, third, to raise a question with regard to policy loan handling as discussed by the authors.

An alternative approach to the fixed premium–variable benefit problem is to split the premium into two parts, investing part of the premium in fixed interest assets—with the rate guaranteed and the investment risk assumed by the insurer on this portion—and the other part of the premium in equities, with the policyowner bearing the investment risk on this part. Such an approach, splitting an endowment premium into the premium for term and the premium for a pure endowment, was proposed by McDiarmid in a paper published in the *Transactions* in 1963. The death benefit was level and guaranteed, while the value of the pure endowment depended entirely upon the appreciation of an equity account.

What is proposed here is a slightly different decomposition of the premium, using the following identity:

$$P = [vq_{x+t-1}(1 - {}_tV)] + (v\cdot{}_tV - {}_{t-1}V) ;$$

$t = 1, 2, \ldots$, maturity year of the contract.

The first term on the right-hand side of this equation is often thought of as the contribution for a one-year term insurance for the net amount at risk, and the second term is an annual contribution to the reserve.

In this instance, the policyowner would be conceptually buying one-year term insurance on the "net amount at risk" and investing the residual amount in equities. Any dividends from the equities purchased are assumed to be reinvested immediately in additional equities. The death benefit at any time would be the net amount at risk plus the entire

YRT case emerges if $\bar{V}(r) = 0$ for r integral and the m-year term insurance case emerges for $\bar{V}(m) = 0$. It should also be noted that, if the particular value of equation (2.3) with $r = m$ is subtracted from the general form of equation (2.3), the prospective reserve formula emerges.

3. Wherever primed functions are used, they will refer to normal fixed-dollar insurance design involving a face amount of one unit, a fixed level premium, and interest on the AIR basis.

4. *Insurance for an indefinite period.*—Integral equation (2.2) indicates that if a term premium for a decreasing face amount of $F(t) - \bar{V}(t)$ is subtracted from a fixed premium of $\bar{P}(t) = \bar{P}$, the remainder accumulates to $\bar{V}(r)$ for all r.

The equation suggests an interesting design: Suppose that $\bar{P}$ is defined as the normal ordinary life premium providing for \$1 of death benefit using some reasonable AIR. A mortality charge equivalent to $\mu_{x+t}[1 - \bar{V}(t)]dt$[1] could be made daily against the equity-based fund $\bar{V}(t)$. The result would be a fixed death benefit policy which might become an endowment prior to the end of the mortality table, or might terminate prior to the end of the mortality table, roughly depending on whether the investment results do exceed the AIR or do not exceed the AIR.

Cost of living variation.—An interesting variation would involve defining $F(t)$ in terms of a cost of living index like the consumer price index. In such a design it would, of course, be desirable to establish the premium at the level of, say, an endowment at 65.

5. *Insurance for a fixed premium, fixed face amount, and positive (or negative) dividend.*—Equation (2.1) applies also to a policy based on an AIR $= \delta'$, providing $F(t) \equiv 1$ at a constant $\bar{P}'$:

$$d\bar{V}'(t) = \{\bar{P}' + \delta'\bar{V}'(t) - \mu_{x+t}[1 - \bar{V}'(t)]\}dt\,. \qquad (2.1)'$$

If $\bar{P}(t)$ and $\bar{V}(t)$ are constrained to equal $\bar{P}'$ and $\bar{V}'(t)$ identically, we obtain the following by subtracting equation (2.1) from equation (2.1)′:

$$F(t) = 1 + \frac{\delta - \delta'}{\mu_{x+t}}\,\bar{V}(t)\,.$$

This is not a marketable design since $F(t)$ can vary widely and might even be negative. However, if this value of $F(t)$ is entered into equation (2.3), we obtain

$$\bar{V}(r)\,{}_rp_x e^{-\int_0^r \delta ds} = \int_0^r [\bar{P}(t) - (\delta - \delta')\,\bar{V}(t) - \mu_{x+t}]\,{}_tp_x e^{-\int_0^t \delta ds} dt\,.$$

Thus, if the premium is adjusted by payment of a positive (or negative)

[1] Here dt is the valuation period (one to three days, for "daily" valuation).

dividend equal to $(\delta - \delta')\ [\bar{V}(t)]$, the premium, face amount, and reserves can be constrained to normal values.

This is the basic concept of the Canadian design in which positive (or negative) dividend additions are used to credit investment gains from the separate account. Usually, 50 per cent of the account is kept in the standard fixed-dollar portfolio, thereby enabling the positive or negative dividend additions to be handled in a conventional manner.

6. *Insurance for fixed premium, fixed period, and variable face amount.*—The New York Life design, presented with such virtuosity in the paper, has the following counterpart in these equations. Equation (2.1) can be taken approximately for dt = daily valuation period. Then constraints are as follows:

$$\begin{aligned} \bar{V}(t) &= F(t)\cdot \bar{V}'(t)\ , \\ \bar{P}(t) &= \bar{P}'_x\ , \end{aligned} \tag{6.1}$$

where the primed functions are for a normal policy with $F(t) \equiv 1$ and premium $\bar{P}'_x$ on the AIR (δ').

Differentiating equation (6.1), we obtain

$$d\bar{V}(t) = F(t)d\bar{V}'(t) + \bar{V}'(t)dF(t)\,. \tag{6.2}$$

Equations (2.1), (2.1)′, and (6.2) can be manipulated as follows:

$$d\bar{V}'(t) = \{\bar{P}'_x + \delta'\bar{V}'(t) - \mu_{x+t}[1 - \bar{V}'(t)]\}\,dt\ ;$$

$$d\bar{V}(t) = \{\bar{P}'_x + \delta\bar{V}(t) - \mu_{x+t}[F(t) - \bar{V}(t)]\}\,dt$$

$$= \{F(t)\cdot\bar{P}'_x + \delta' F(t)\bar{V}'(t) - \mu_{x+t}[F(t) - F(t)\bar{V}'(t)]\}\,dt + \bar{V}'(t)dF(t).$$

Collecting terms

$$\{\bar{P}'_x[1 - F(t)] + (\delta - \delta')\bar{V}(t)\}\,dt = dF(t)\bar{V}'(t)\ ;$$

$$dF(t) = -\frac{\bar{P}'_x[F(t) - 1]}{\bar{V}'(t)}\,dt + (\delta - \delta')F(t)dt\ ;$$

$$\frac{F(t + dt)}{F(t)} = 1 - \frac{\bar{P}'_x[F(t) - 1]}{\bar{V}(t)}\,dt + (\delta - \delta')dt\,.$$

This is the counterpart of the New York Life $Y\cdot Z$.

7. *Insurance for a fixed period, variable premium and variable face amount (fixed premium and fixed face amount in annuity units).*—A completely variable contract can be derived as follows in terms of annuity units. Let

$$F(t) = e^{\int_0^t (\delta - \delta')ds}$$

indicate New York State Insurance Department sanction of any kind of variable life insurance product.

JOHN M. BRAGG:

This paper represents a major achievement; Messrs. Fraser, Miller, and Sternhell deserve the appreciation and thanks of the Society.

The purpose of this discussion is to point out the approximate relationship between the investment performance and the face amount increases that can be expected. Some other comments flowing from this are also included.

Using a constant investment performance (i') of 9 per cent, Table 2 shows a 5.8 per cent increase in face amount in the first year. Thereafter, however, the increases average only to about 2.6 per cent. For the conditions illustrated in the paper, it seems that a very high rate of investment performance is needed, net after expenses and taxes, to achieve modest increases in face amount.

From some sample calculations made by the author of this discussion, it would appear that an annual investment performance in the neighborhood of 12 per cent, net after expenses and taxes, would be needed to achieve face amount increases averaging 4 per cent on the whole life plan.

Some companies might feel a reluctance to rely on very high investment performance; some might be worried about intermediate drops in face amount which can occur if investment performance is not good in a particular year. Such companies might prefer to use a basic product which is not a simple level amount plan but is of an increasing nature—for example, a whole life plan with benefits increasing at the rate of $2\frac{1}{2}$ per cent per annum. In this way, superior investment performance would not be the sole means for protecting the purchasing power of the insured benefits.

DONALD D. CODY:

I presented a simplified general outline of the "Actuarial Mechanics of Variable Annuities" as a discussion of Harry Walker's paper "State Regulation of Individual Variable Annuities" (*TSA*, XX, 456–63). In the following discussion, I am extending that outline to the "Actuarial Mechanics of Equity Based Insurance."

The mathematical development involves the basic differential and integral equations for the most general insurance coverage. By manipulation of these basic equations I have found that all the existing forms of equity-based insurance emerge. Others may find that this technique suggests additional forms of equity-based insurance design.

1. For the basic notation the reader should refer to pages 456–63 of my discussion. In addition, the following notation will be used:

δ' = Force of interest on AIR basis (a constant).

δ = Force of interest on net investment income (a function of t).

$$u_n = u_0 e^{\int_0^n \delta dt} = \text{Investment unit value.}$$

$$(au)_n = (au)_0 e^{\int_0^n (\delta - \delta')dt} = \text{Annuitv unit value.}$$

2. The basic differential and integral equations for the most general insurance coverage are as follows:

$$d\bar{V}(t) = \{\bar{P}(t) + \delta\bar{V}(t) - \mu_{x+t}[F(t) - \bar{V}(t)]\}dt\,. \tag{2.1}$$

If equation (2.1) is expressed as

$$d\bar{V}(t) - \delta\bar{V}(t)dt = \bar{P}(t)dt - \mu_{x+t}[F(t) - \bar{V}(t)]dt$$

and each side is multiplied by

$$e^{-\int_0^t \delta ds}\,,$$

it integrates to

$$\bar{V}(r)e^{-\int_0^r \delta ds} = \int_0^r \{\bar{P}(t) - \mu_{x+t}[F(t) - \bar{V}(t)]\}e^{-\int_0^t \delta ds}dt\,. \tag{2.2}$$

If equation (2.1) is expressed as

$$d\bar{V}(t) - (\delta + \mu_{x+t})\bar{V}(t)dt = [\bar{P}(t) - \mu_{x+t}F(t)]dt$$

and each side is multiplied by

$$e^{-\int_0^t (\mu_{x+s}+\delta)ds}$$

it integrates to

$$\bar{V}(r)_r p_x e^{-\int_0^r \delta ds} = \int_0^r [\bar{P}(t) - F(t)\mu_{x+t}]_t p_x e^{-\int_0^t \delta ds}dt \tag{2.3}$$

where $F(t)$, $\bar{V}(t)$, and $\bar{P}(t)$ are, respectively, face amounts, reserves, and continuous premiums as functions of t. These equations may be made to involve complete generality for the normal modes of premium payment if they are construed in the Lebesgue sense. For instance,

$$\begin{aligned} \bar{P}(t)dt &= P(\bar{A}_x) \quad \text{for} \quad t \text{ integral} \\ &= 0 \quad \text{for} \quad t \text{ nonintegral} \\ \text{for} \quad F(t) &\equiv 1 \quad \text{and} \quad \delta = \text{constant}\,. \end{aligned}$$

If r is the complete duration of the insurance, m, equation (2.3) completely defines the relationship of $F(t)$ and $\bar{P}(t)$. For instance, the trivial value of the policy's equity account. Withdrawal benefits would simply be the value of the policy's equity account at the time of withdrawal. Symbolically, if we let

${}_tP$ = premium for one-year term insurance on the "net amount at risk" in year t,
${}_tR$ = residual contribution to "reserve" in year t,
i'_t = effective yield rate of equity investments in year t, and
${}_t\mathrm{DB}$ = death benefit in year t, payable at end of year,

then we have for the death benefit

$${}_n\mathrm{DB} = (1 - {}_nV) + \sum_{t=1}^{n} {}_tR \prod_{s=t}^{n} (1 + i'_s) \,.$$

If the yield on the "reserve" fund is exactly equal to the interest rate used in calculating the net premium, the death benefit would be the original face amount of the policy. Rates of increase greater than that used in the policy calculations would result in larger death benefits, and lower rates of increase than the rate used in calculations would result in death benefits lower than the original face amount.

To investigate the success of this method, Table 1 has been prepared by use of the proposed method for \$1,000 face amount twenty-year endowment policies issued at three different times to a life aged 35. Market values per share and dividends were taken from annual averages of Moody's common stock averages of 200 stocks. Each of three twenty-year periods was chosen, with the net level premium and reserves calculated on the 1958 C.S.O. Table at 3 per cent. In adjoining columns are the death and endowment benefits which should have been paid that year if a cost-of-living insurance had been issued, with the death benefit and endowment benefit determined by the consumer price index of the Bureau of Labor Statistics.

One could apply the method of splitting the premium into term for the net amount at risk and contributions to reserve to any investment-type life insurance contract. Ordinary whole life would have an added safety factor of reducing the net amount at risk relatively slowly, taking advantage of the long-term tendency of the stock market to rise while reducing the importance of short-term fluctuations.

Such an insurance scheme would probably violate the standard nonforfeiture law currently, but a way around this law would be to issue two contracts for a single policy, one providing term insurance on the "net amount at risk" and another for the equity account. In fact, such an approach or one very similar to it is being used extensively at the present time. It is called "combination sales of life insurance and mutual funds," or "buying decreasing term and investing the difference," or perhaps by other names. The "policyowner" pays for both with one check and really has a protection device similar to that described above. The one difference is that a level premium usually goes into decreasing term life insurance and a level premium goes for mutual funds, rather than the varying split year by year as described above.

To return to the method of the paper, a question arose in my mind as I read the "Policy Loans" section. If loans bearing fixed interest are allowed, this will affect the investment performance of the entire sepa-

TABLE 1

DEATH* AND MATURITY BENEFITS ON A $1,000 TWENTY-YEAR ENDOWMENT POLICY ISSUED TO A LIFE AGED 35

Policy Year	Year Purchased								
	1929			1936			1943		
	Calendar Year	Policy Benefits	CPI Benefits	Calendar Year	Policy Benefits	CPI Benefits	Calendar Year	Policy Benefits	CPI Benefits
1........	1929	991.67	974.87	1936	999.08	1,035.39	1943	1,003.50	1,016.58
2........	1930	967.66	887.77	1937	982.52	1,016.84	1944	1,020.90	1,039.80
3........	1931	930.72	797.32	1938	990.21	1,001.66	1945	1,038.38	1,127.69
4........	1932	954.18	755.44	1939	984.13	1,010.36	1946	1,023.34	1,290.21
5........	1933	970.63	780.57	1940	970.31	1,062.12	1947	1,033.22	1,389.71
6........	1934	990.38	800.67	1941	950.42	1,175.99	1948	1,037.23	1,376.45
7........	1935	1,099.01	809.04	1942	1,030.81	1,248.45	1949	1,113.63	1,389.71
8........	1936	1,091.43	837.52	1943	1,065.79	1,269.16	1950	1,214.21	1,500.82
9........	1937	989.89	822.44	1944	1,164.80	1,298.14	1951	1,279.62	1,533.99
10........	1938	1,024.59	810.72	1945	1,243.05	1,407.87	1952	1,315.63	1,545.60
11........	1939	1,005.38	817.42	1946	1,190.17	1,610.77	1953	1,517.14	1,552.23
12........	1940	966.44	859.30	1947	1,225.73	1,735.00	1954	1,876.88	1,547.26
13........	1941	916.19	951.42	1948	1,241.39	1,718.43	1955	2,079.65	1,570.48
14........	1942	1,105.43	1,010.05	1949	1,454.22	1,735.00	1956	2,064.18	1,625.20
15........	1943	1,183.49	1,026.80	1950	1,718.98	1,873.71	1957	2,209.68	1,669.98
16........	1944	1,393.47	1,050.25	1951	1,887.61	1,915.12	1958	2,725.96	1,683.25
17........	1945	1,554.90	1,139.03	1952	1,981.00	1,929.61	1959	2,662.71	1,709.78
18........	1946	1,451.79	1,303.18	1953	2,463.11	1,937.89	1960	3,214.16	1,728.02
19........	1947	1,523.36	1,403.68	1954	3,300.73	1,931.68	1961	3,151.87	1,747.92
20........	1948	1,555.94	1,390.28	1955	3,771.76	1,960.67	1962	3,657.09	1,796.48
Endowment benefit..	1949	1,555.94	1,390.28	1956	3,771.76	1,960.67	1963	3,657.09	1,796.48

* Death benefits are the net amounts at risk (1958 C.S.O., 3 per cent, net level premium reserves) plus the value of an equity fund purchased with residual net premiums.

rate account. On the other hand, in addition to "policyowner confusion and misunderstanding" caused by relating the amount of the loan to the actual net investment performance, this method takes away the benefits of the fixed premium aspect of the authors' approach, since the loan could fluctuate widely and elude a policyowner's efforts to repay or even to reduce his loan on a systematic basis. Thus we have serious problems either way a loan is handled. One is a problem of equity, since those who take out loans on a fixed interest basis change the investment performance of the whole account. The other problem is the inherent variation of the value of equities and the seeming impropriety of relating debts to such a volatile index, even where the debt is secured by assets which relate in an identical way to the same index. Perhaps since the policyowner has chosen to bear the investment risk on this type of contract, he should be expected to bear the same risk on his loans from the contract and we should expect him to be sophisticated enough to understand this risk. It would certainly be more consistent with the type of contract.

D. FRANK DEAL:

One aspect of such a variable benefit life insurance product should not be overlooked—that is, the probable increase in face amount over an extended period of time (see Tables 3 and 4 in the paper) and its effect on claim experience. With increases as large as the ones shown, the first thought that occurred to me was that there could possibly be some antiselection in these later years. A number of arguments along this line can be made, and I would like to discuss them.

One argument which can be dispensed with immediately is that the company will suffer because of the very large increases in face amount, particularly since there is no way to prevent the amounts on the poorer risks from increasing. The answer to this is that, although the face amount (and the absolute value of the amount at risk) may increase dramatically, it does so for all risks, not just for the impaired ones. The risk relative to the reserve held is no different, as the authors explain, from that on regular fixed benefit coverage. Therefore, unless there is some reason to believe that more of the better risks will lapse their coverage on the variable benefit plan than will on a fixed benefit plan (see below), there is no reason to expect any more adverse claim experience. A reason for the fact that better risks tend to lapse their coverage on traditional forms is the better investment potential of alternative investments. The variable benefit contract quite obviously reduces to a large degree the attractiveness of these alternatives.

Another situation in which the impaired risks would tend to retain their coverage while the better risks would tend to lapse their coverage on the variable contract is when the market is in a temporarily depressed condition. When the market is down, other forms of investment (perhaps fixed-dollar) become more attractive to the better risks, leaving the impaired risks to continue their coverage. In this situation it would seem likely that any effects of antiselection are again minimized as a result of the nature of the variable benefit contract; that is, any claims incurred will be on the basis of the depressed face amount.

If we elaborate to the extreme on this line of thinking, it would seem that the more "ups and downs" of the market, the greater the average mortality rate experienced. Consider a closed block of business. As long as the separate account continues making gains, no abnormal increase in lapses should be expected. At the first major downturn of the market, some of the better risks will lapse (leaving the closed group) and seek alternative investments (perhaps traditional fixed benefit insurance). When the market turns upward, these better risks cannot rejoin the closed group and must, if they wish it, purchase brand-new insurance. On the next downturn of the market, many of those better risks who were not "smart enough" to get out the first time will do so this time. Thus at each major swing of the market some of the better risks will be lost, the poorer risks will remain, and the experience of the remaining group will continue to deteriorate.

The effects of such a process cannot be predicted, and, although they may not be too severe, the possibility should at least be considered, along with other characteristics that are unique to this plan. The effects should not be severe for at least three reasons:

1. People who would tend to purchase the variable benefit plan would probably be familiar enough with the peculiarities of the stock market to expect a few significant downturns over a long period of time and would not panic the first time one occurred.

2. The alternative investments available after the stock market has been down for a significant period are probably not going to be very attractive either. The insured would surely keep his insurance in force rather than lapse it to get a return only slightly better than that his life insurance is providing.

3. The separate account will not consist exclusively of common stocks but a variety of investment forms, although with fairly heavy emphasis on equities. Thus any wide swings in the market would be dampened to some extent, depending on the makeup of the account.

I do not intend in any way whatsoever to take anything away from the attractiveness of the variable benefit product. It seems most desirable, however, to bring out into the open any possible trouble spots and to discuss them in the initial stages of development.

One very desirable attribute of a variable product is that, since the amount of insurance will normally increase, the in force of the company and its investment income will also increase without any additional sales effort or any increase in commissions paid. Accounting systems will have to be modified to accommodate the variable amounts of insurance each year, but this can be built into the over-all system that will necessarily be constructed to handle the other features of this product.

FRANK P. DI PAOLO:

I found this to be one of the most stimulating papers ever published in the *Transactions*. A tremendous amount of spadework has been done by the authors, and, although much digging is yet to be done, they deserve our compliments.

One of the areas in which additional research needs to be done deals with how and to what extent the company's free surplus can and should be used to cover the mortality risk assumed by the company with respect to such variable contracts.

One way to provide a meaningful mortality guarantee would be to set up within the separate account a "mortality stabilization fund" to which insurance costs based on $q_{x+t-1} \cdot F_t(1 - {}_tV_x)$ would be credited and actual death claims less reserves released would be debited. Presumably insurance costs would be calculated according to a safe "valuation" mortality table, but, although significant profits are likely to accrue over the years, it is possible that at some point of time the "mortality stabilization fund" could be depleted. In this case the free surplus of the company must then come to the rescue. Inasmuch as the company assumes the mortality risk, it is only fair that a mechanism be devised whereby a portion of the emerging mortality profits can be released to the company's surplus from time to time. If the variable contracts are nonparticipating, it does not matter greatly what portion of such profits is so released. If they are participating, a problem of equity may arise. Obviously, the portion of the mortality profits to be released to surplus should be commensurate with the risk assumed by the company. But how should this risk be measured? It seems to me that, by means of simulation techniques, reflecting both mortality and stock market fluctuations, it should be possible to obtain a better understanding of the nature of the mortality risk and to find a way to measure it.

I am somewhat disturbed by the results illustrated in Table 3, based on the Standard and Poor's Composite 500. The face amount of the whole life, age 25, drops from 3752 to 782 between 1929 and 1932. It seems to me that additional research needs to be done to find a way in which the volatility of the investment returns generated by a portfolio of common stocks may be harnessed to produce Z factors, which will systematically force the face amount of variable life insurance contracts to follow as closely as possible (and to the extent to which the actual investment returns will permit it) the curve of some economic indicator, such as the consumer price index.

From equation (9) of the paper

$$F_t = F_{t-1}Y_tZ_t \,. \tag{1}$$

If it is required that F_t approach $(1 + C_t)F_{t-1}$, then

$$(1 + C_t)F_{t-1} = F_{t-1}Y_tZ'_t \,, \tag{2}$$

therefore

$$(1 + C_t) = Y_tZ'_t \,, \tag{3}$$

where

C_t = annual rate of change in the consumer price index during the tth policy year

$$\left(C_t = \frac{\mathrm{CPI}_t}{\mathrm{CPI}_{t-1}} - 1\right);$$

$$Z'_t = \frac{1 + j_t}{1 + i};$$

j_t = smallest annual rate of investment return that must be earned by the separate account during the tth policy year in order to support a change in the consumer price index of C_t.

From equation (3),

$$\frac{1 + j_t}{1 + i} = \frac{1 + C_t}{Y_t}; \tag{4}$$

therefore

$$j_t = \frac{1 + C_t}{Y_t}(1 + i) - 1 \,. \tag{5}$$

One of the ways in which the flow of investment income could be stabilized would be to set up, outside the separate account, an "investment stabilization fund" into which excess investment earnings would be deposited and from which deficiencies would be withdrawn. At the end of the tth policy year the equity of a given policyholder in the "investment stabilization fund," payable to him or his beneficiary in the event of surrender, maturity, or death, could be calculated in the following manner:

$$\text{ISF}_x = \sum_{s=1}^{t-1}(i'_s - j_s)[F_{s-1}({}_{s-1}V_x) + P_x]\prod_{r=s+1}^{t}(1 + i''_r) + (i'_t - j_t)[F_{t-1}({}_{t-1}V_x) + P_x]\,, \tag{6}$$

where

i'_s = net annual investment return on the separate account during the sth policy year, including realized and unrealized appreciation and depreciation, and

i''_r = net annual rate of interest earned by the "investment stabilization fund" in the rth policy year.

The reason for suggesting that the "investment stabilization fund" be kept outside the separate account, and possibly invested in fixed income assets, is to avoid its becoming depressed because of a drop in stock prices, precisely at the time when it will be most needed to keep the face amount of variable contracts from nose-diving. The principle here is, in effect, the reverse of that underlying the equity funds created by Canadian companies for investing policy dividends generated by fixed-dollar contracts. The transfer of funds from the separate account into the "investment stabilization fund" may be done by diverting some of the cash flow that would normally go into the separate account rather than by disposing of some of the latter's assets.

Obviously, the "investment stabilization fund" should not be allowed to drop below zero, and the maximum value of C_t that can be recognized in the calculation of j_t should be such that it would not result in a negative balance in the "investment stabilization fund."

Analytically,

$$j_t \ngtr i'_t + \frac{{}_t\text{ISF}_x}{F_{t-1}({}_{t-1}V_x) + P_x}\,; \tag{7}$$

$$\therefore \frac{1 + C_t}{Y_t}(1 + i) - 1 \ngtr i'_t + \frac{{}_t\text{ISF}_x}{F_{t-1}({}_{t-1}V_x) + P_x}\,; \tag{8}$$

$$\therefore C_t \ngtr \frac{Y_t}{(1 + i)}\left[1 + i'_t + \frac{{}_t\text{ISF}_x}{F_{t-1}({}_{t-1}V_x) + P_x}\right] - 1\,. \tag{9}$$

Thus, inequality (9) gives the maximum annual rate of change in the consumer price index that can be recognized in the tth policy year.

The "investment stabilization fund" does not need to be geared strictly and exclusively to changes in the consumer price index. A variable life insurance contract could stipulate, for example, that annual investment returns in excess of, say, 5 per cent would be deposited in the "investment stabilization fund" and that deficiencies below, say, 5 per cent would be withdrawn from it if there is a sufficient balance.

Another purpose of the "investment stabilization fund" would be to act as a vehicle for investing policy dividends. Unless a definite dividend charge is made in the gross premium, dividends would be limited to mortality gains and expense savings, if any. There will not be any excess interest. Thus the small dividends may as well be used to strengthen the "investment stabilization fund."

I am in agreement with the authors that the method proposed by them to specify minimum cash-surrender values is the most practical one, even though it may not be as accurate as the two alternative methods given in Appendix B. It should be noted that the proposed method is likely to produce a small surrender profit to the company. As the premium is fixed, the unamortized expenses (especially acquisition costs) tend to be a function of the initial face amount F_0. Stock prices have historically tended to drift upward, however, at an average annual rate somewhat in excess of the interest rate normally used to value life insurance contracts. Thus it may be reasonable to assume that $F_t \cdot U_x$ is likely to be greater than $F_0 \cdot U_x$. Hence the likelihood of a surrender profit.

With regard to the problem of illustrating cash-surrender values in policy forms, in addition to the table per 1,000 of actual face amount, which in effect would be fully applicable if the net actual rate of investment return is always equal to the valuation rate of interest, I would like to suggest the inclusion of two additional tables, one based on a net rate of return of 0 per cent and another based on a rate equal to twice the valuation rate of interest (e.g., 0, 3, and 6 per cent). This three-way table of cash-surrender values would give a better idea to the policyholder of the range within which his cash value is likely to fluctuate. Other nonforfeiture benefits, however, may well be best illustrated only per 1,000 of face amount.

RALPH E. EDWARDS:

An abbreviated but more generalized derivation of this paper's formula (4), using the same notation, is shown in the following paragraphs.

1. In order that the death benefit in policy year t may be equal to F_t and that the terminal reserve may be $F_t \cdot {}_tV_x$, we require an initial reserve equal to $F_t({}_{t-1}V_x + P_x)$.

2. The initial reserve for policy year t is the sum of (a) the prior year's terminal reserve, equal to $F_{t-1} \cdot {}_{t-1}V_x$; ($b$) the premium actually paid on

the net basis, equal to ${}_t\Pi_x$; and (*c*) any other sum accumulated at the beginning of that policy year.

3. Let us define D'_{x+t} as equal to $l_{x+t} \div (1 + i'_0)(1 + i'_1) \ldots (1 + i'_t)$, where $i'_0 = 0$.

4. Any sum available for distribution to policyholders at the end of policy year $t - r - 1$ (including distribution to those who died in that policy year) may instead be accumulated to distribute to living policyholders at the beginning of policy year t by multiplying by a factor equal to $D'_{x+t-r-2} \div (1 + i'_{t-r-1})(D'_{x+t-1})$.

5. The usual three-factor dividend formula provides an excess interest component, which, for policy year $t - r - 1$ and for face amount F_{t-r-1}, is equal to $F_{t-r-1}({}_{t-r-2}V_x + P_x)(i'_{t-r-1} - i)$.

6. Combining these items, we have the formula

$$F_t({}_{t-1}V + P_x) = F_{t-1} \cdot {}_{t-1}V_x + {}_t\Pi_x$$
$$+ F_{t-r-1}({}_{t-r-2}V_x + P_x)(i'_{t-r-1} - i)D'_{x+t-r-2} \div (1 + i'_{t-r-1})(D'_{x+t-1}) .$$

Formula (4) is obtained by setting ${}_t\Pi_x = P_x$ and $r = -1$.

Immediate payment of death claims and of a pro rata death dividend under the normal dividend scale is accepted practice, but under the paper's proposal it would seem necessary to vary the death benefit daily or else to adopt some procedure that would be inconsistent with the assumptions underlying formula (4). One alternative would be to keep the original face amount unchanged during the first policy year but to use the results of each policy year to calculate the benefit of the subsequent policy year. This has the effect of setting $r = 0$.

Still another possibility would be to determine on December 31 the excess-interest element assumed earned as of policy anniversaries in the preceding calendar year and to accumulate this until the following policy anniversary. That is, set $r = 1$ and have no change in face amount during the first two policy years.

The foregoing assumes that only excess interest is accumulated. If the full dividend ${}_t\Delta_x$ were accumulated, then $F_{t-r-1}({}_{t-r-2}V_x + P_x)(i'_{t-r-1} - i)$ would be replaced by ${}_{t-r-1}\Delta_x$. This suggests that a new dividend option could be offered which would increase the amount of insurance less than that with one-year term additions and more than that with regular paid-up additions.

If legal requirements need to be changed to permit the use of the system proposed by this paper, it might be desirable for the revised requirements to accommodate situations where ${}_t\Pi_x$ is not P_x and r is not necessarily -1.

The authors deserve great credit for devising the only contract that I have seen which seems successfully to join fixed-dollar principles with equity-linked benefits.

GUY L. FAIRBANKS, JR.:

The authors are to be congratulated on writing this scholarly pilot paper on a subject which gives promise of dominating actuarial literature and the insurance trade press for a number of years to come. In this discussion I shall not attempt a comprehensive review of the paper but shall limit my observations to the fundamentals of the proposed policy design.

One point which bothered me considerably when reading the paper was that there appeared at first glance to be a "sawtoothed" effect resulting from the fact that the Y_t factor is applied at the beginning of the year whereas the Z_t factor applies continuously. For example, in the $i'_i = 9$ per cent illustration shown in Table 1, the face amount rises from \$1,000 to \$1,058 during the first year and then falls back to \$1,024 when multiplied by Y_2 $(= 0.9679)$ at the beginning of the second year. The process is repeated in the second and third years, when it rises from \$1,024 to \$1,084, falls back to \$1,049, and then rises to \$1,110. The authors have explained to me that this effect exists only in theory. In actual practice they would regard the net premium as being paid continuously, which will take the kinks out of F_t and make it a continuous function. I trust that the authors will clarify this point further in their response to the discussions of the paper.

Even if given relief from the sawtoothed effect, I still feel that there are some very serious problems associated with this policy design. The stipulation that the face amount must at all times bear the same ratio to the policy reserve as would have existed if the policy had been issued on a fixed basis appears perfectly logical and straightforward on the surface. When used in actual practice, however, I think the results it will produce will prove very difficult to explain.

To illustrate what I have in mind, let us assume that two men buy fixed premium variable life insurance policies following the authors' design. Each policy is issued at age 55 for an initial face amount of \$100,000. Mr. A buys his policy at the beginning of a bull-market swing of three years' duration. This period is followed by return to "normalcy," which I shall define for purposes of illustration as existing when $i'_t = 9$ per cent. Mr. B buys his policy at the beginning of a three-year bear-market swing which, in its departure from normalcy, is the mirror image of Mr. A's bull-market swing. The two policies perform as shown in Table 1.

One can easily imagine the sort of dialogue which might take place between Mr. A and his agent as the agent tries to explain to Mr. A why his face amount did what it did each year in the light of the performance of the equity portfolio. It could be that Mr. B's agent will have an even rougher time, especially at the end of the second year, when Mr. B might say, "Last year you explained to me that your company found it necessary to reduce my face amount 20.4 per cent because you experienced a negative return of 18 per cent on your common stocks against an assumed 3 per cent. Now you tell me that this year things have been not quite so grim, but you have still gone in the hole to the tune of 9 per cent. However, losing 9 per cent is apparently so much more pleasant than losing 18 per cent that you are going to increase the face of my policy by 1.6 per cent. I find all this completely baffling."

TABLE 1

t	i'_t (Per Cent)	Face Amount (End of Year)	Per Cent Change in Face Amount
	Mr. A's $100,000 Policy		
1	36%	$132,000	32.0%
2	27	139,600	5.8
3	18	141,110	1.1
4	9	135,400	− 4.0
5	9	133,500	− 1.4
	Mr. B's $100,000 Policy		
1	−18%	$ 79,600	−20.4%
2	− 9	80,900	1.6
3	0	86,200	6.6
4	9	95,900	11.3
5	9	102,600	7.0

The agent selling variable annuities is frequently confronted with the question, "Suppose your company starts paying me $100 a month when I retire, and you do so well with your common stock portfolio that ten years later I am getting $200 a month. How well do you have to do after that so that you can keep on paying me $200 a month?" If the assumed investment rate is $3\frac{1}{2}$ per cent, the agent is able to say, and it is a really telling sales point, "All we have to do is earn a net rate of $3\frac{1}{2}$ per cent, and your payments will stay at $200 per month for the rest of your life." Clearly, no agent would be able to give such an answer to the corresponding question if it were asked (and it inevitably would be) with regard to a life policy based on the design set forth in this paper. There would be no simple answer. The rate required would vary with type of policy, age at issue, and sex and would not be a constant rate once determined. For example, let us assume that a male aged 55 buys a \$100,000 whole life policy and sees the face amount rise during the first ten years to \$200,000. In order for his face amount to stay at \$200,000, it must follow that $Y_tZ_t = 1$ for the eleventh and subsequent years. This will only be true if $i'_{11} = 9.9$ per cent; $i'_{12} = 9.3$ per cent; $i'_{13} = 8.8$ per cent; $i'_{14} = 8.4$ per cent; and so on. I suspect that the average agent will find it very difficult to explain why this is so.

The fundamental principle that the amount payable under a variable annuity rises when the net investment rate exceeds the assumed investment rate and falls when the reverse is true has been accepted as entirely logical by thousands of variable annuity purchasers. I doubt that the performance of an equity-based variable life policy will make sense to its purchaser unless the same principle obtains. One approach which will accomplish this objective is to define the face amount of the policy as being partially variable and partially fixed at all times during its premium-paying period. The variable portion is the portion which, to borrow a term from pension parlance, may be regarded as "fully funded" by the policy reserve. At the end of t years the fully funded portion of a fixed amount whole life policy is

$$\begin{aligned} {}_tV_x \div A_{x+t} &= (A_{x+t} - P_x\ddot{a}_{x+t}) \div A_{x+t} \\ &= 1 - \frac{P_x}{P_{x+t}}. \end{aligned}$$

The "unfunded" portion is

$$1 - \left(1 - \frac{P_x}{P_{x+t}}\right) = \frac{P_x}{P_{x+t}}.$$

During the tth year the portion which newly achieves fully funded status is

$$\left(1 - \frac{P_x}{P_{x+t}}\right) - \left(1 - \frac{P_x}{P_{x+t-1}}\right) = P_x\left(\frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}}\right).$$

Under the approach which I am proposing this portion of the face amount is converted from a fixed basis to a variable basis at the beginning of the tth year. The number of new variable insurance units created at the beginning of the tth year is

$$P_x\left(\frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}}\right) \div u_{t-1}.$$

The total number of variable insurance units in force at the beginning of the tth year is

$$\sum_{r=1}^{t} P_x\left(\frac{1}{P_{x+r-1}} - \frac{1}{P_{x+r}}\right) \div u_{r-1}\,.$$

The dollar value of these units is

$$u_{t-1}\sum_{r=1}^{t} P_x\left(\frac{1}{P_{x+r-1}} - \frac{1}{P_{x+r}}\right) \div u_{r-1}\,.$$

Adding to this the "unfunded portion," which is still on a fixed basis (and elaborating the authors' notation slightly by introducing superscripts B for "beginning" and E for "ending"), we have, as the face amount at the beginning of the tth year

$$F_t^B = \frac{P_x}{P_{x+t}} + u_{t-1}\sum_{r=1}^{t} P_x\left(\frac{1}{P_{x+r-1}} - \frac{1}{P_{x+r}}\right) \div u_{r-1}\,.$$

To determine the face amount at the end of the tth year, u_{t-1} is simply replaced by u_t, and we have

$$F_t^E = \frac{P_x}{P_{x+t}} + u_t\sum_{r=1}^{t} P_x\left(\frac{1}{P_{x+r-1}} - \frac{1}{P_{x+r}}\right) \div u_{r-1}\,.$$

It can be proved mathematically (but the proof is rather tedious), and it is clearly true by general reasoning, that

$$F_t^B = F_{t-1}^E\,.$$

It can also be proved mathematically and established by general reasoning that

$$F_t^E = \frac{u_t}{u_{t-1}}\left(F_{t-1}^E - \frac{P_x}{P_{x+t}}\right) + \frac{P_x}{P_{x+t}}\,.$$

This expression is the equivalent of

$$F_t^E = F_{t-1}^E Z_t - (Z_t - 1)\,\frac{P_x}{P_{x+t}}\,,$$

which compares directly with the authors' equation (9). The above approach can easily be adapted to any form of policy.

It will be noted that the fact that $F_t^B = F_{t-1}^E$ means that there is no sawtoothed effect either in theory or practice, that is, that the face amount varies continuously throughout the life of the policy even though the premium is regarded as paid annually rather than continuously.

It will also be noted that $F_t^E = F_{t-1}^E$ if $u_t \div u_{t-1} = 1$, that is, if $i_t' = i$. Thus, no matter how high the face amount goes, only a modest return on

TABLE 2

COMPARISON OF YEAR-END FACE AMOUNTS FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000, ISSUED TO MALE AGED 55

(Derived from Table 2 in the Paper Using Values for $i_t' = 9$ Per Cent)

Policy Year	Constant		Simulated		Ratio, Simulated to Constant	
	Policy 1*	Policy 2*	Policy 1	Policy 2	Policy 1	Policy 2
1	$1,058	$1,003	$1,086	$1,004	103%	100%
2	1,084	1,008	1,021	1,003	94	100
3	1,110	1,017	1,152	1,022	104	100
4	1,137	1,028	1,033	1,009	91	98
5	1,165	1,042	1,257	1,061	108	102
6	1,194	1,060	1,264	1,078	106	102
7	1,224	1,081	1,231	1,083	101	100
8	1,255	1,104	1,185	1,080	94	98
9	1,287	1,132	1,136	1,073	88	95
10	1,320	1,163	1,391	1,188	105	102
11	1,355	1,197	1,209	1,127	89	94
12	1,391	1,235	1,511	1,287	109	104
13	1,428	1,277	1,478	1,299	104	102
14	1,466	1,323	1,580	1,384	108	105
15	1,505	1,373	1,536	1,391	102	101
16	1,545	1,427	1,717	1,537	111	108
17	1,586	1,486	1,600	1,499	101	101
18	1,628	1,550	1,597	1,532	98	99
19	1,672	1,618	1,494	1,489	89	92
20	1,717	1,692	1,704	1,681	99	99
21	1,763	1,771	1,741	1,752	99	99
22	1,811	1,856	1,797	1,842	99	99
23	1,860	1,947	1,668	1,774	90	91
24	1,911	2,044	2,042	2,157	107	106
25	1,963	2,148	2,030	2,208	103	103
26	2,016	2,258	2,072	2,311	103	102
27	2,071	2,376	2,260	2,570	109	108
28	2,128	2,501	1,808	2,162	85	86
29	2,186	2,635	1,839	2,245	84	85
30	2,245	2,777	2,322	2,848	103	103
31	2,306	2,928	2,410	3,035	105	104
32	2,369	3,088	2,422	3,140	102	102
33	2,433	3,258	2,519	3,354	104	103
34	2,499	3,439	2,088	2,890	84	84
35	2,567	3,630	2,608	3,663	102	101
36	2,637	3,834	2,857	4,119	108	107
37	2,709	4,051	2,492	3,719	92	92
38	2,783	4,280	2,889	4,413	104	103
39	2,859	4,523	2,993	4,705	105	104
40	2,937	4,781	3,054	4,943	104	103
41	3,018	5,055	2,533	4,237	84	84
42	3,102	5,345	3,068	5,253	99	98
43	3,189	5,654	3,755	6,604	118	117
44	3,279	5,981	3,488	6,326	106	106
45	3,373	6,329	2,973	5,552	88	88

* Policy 1, as proposed in the paper; Policy 2, as proposed in this discussion.

the portfolio equal to the assumed rate is required in order for the face amount to remain at the level which it has reached.

It is interesting to compare the performance of a policy designed as proposed in the paper with a policy following the design proposed in this discussion. Table 2 is derived from the authors' Table 2. In making the comparison of face amounts, I have assumed that the "constant" yield will be 9 per cent and that the "simulated" yield will vibrate back and forth across 9 per cent, as assumed by the authors in constructing the "simulated" column for $i' = 9$ per cent in Table 2. In order to make it

TABLE 3

t	i'_t (Per Cent)	Face Amount at End of Year: Policy 1	Policy 2
		Mr. A's $100,000 Policy	
1	36%	$132,000 (32.0%)	$101,500 (1.5%)
2	27	139,600 (5.8)	104,000 (2.5)
3	18	141,100 (1.1)	106,600 (2.5)
4	9	135,400 (− 4.0)	108,000 (1.3)
5	9	133,500 (− 1.4)	109,700 (1.6)
		Mr. B's $100,000 Policy	
1	−18%	$ 79,600 (−20.4%)	$ 99,000 (−1.0%)
2	− 9	80,900 (1.6)	98,100 (−0.9)
3	0	86,200 (6.6)	97,700 (−0.4)
4	9	95,900 (11.3)	98,600 (0.9)
5	9	102,600 (7.0)	99,800 (1.2)

possible for me to make this comparison, the authors have very generously made available the underlying values which they used in constructing Table 2.

The comparison shows that Policy 1 (the authors' design) provides a more generous face amount during the first twenty years than Policy 2 (my design) but is considerably outdistanced by Policy 2 thereafter. The annual percentage growth rate under "constant" conditions is higher under Policy 1 for the first ten years but much lower thereafter. In the twenty-fifth year, for example, Policy 1's growth rate is 2.7 per cent, whereas Policy 2's is 5.1 per cent. The "ratio, simulated to constant" columns show that Policy 1's face amount is much more volatile than

Policy 2's in the early years and is always at least as sensitive to market fluctuations as Policy 2's face amount. Policy 2 generates more generous cash values at all durations as the result of the more generous death benefits provided by Policy 1 in the early years.

It is also interesting to compare the operation of Policies 1 and 2 in the examples of the aforementioned Mr. A and Mr. B, whose policies perform as shown in Table 3. Mr. A's agent should have little difficulty in explaining the performance of Policy 2. Mr. B's agent should find it much easier to keep Mr. B's policy on the books if he sells him Policy 2.

In general, Policy 2 impresses me as being easier to live with from the standpoint of the purchaser, the agent, and the company. Furthermore, the variable annuity was created as a long-range inflation hedge, and, if this is to be the primary purpose of the equity-based variable life policy, I think Policy 2 will conform to it much better than Policy 1.

DALE R. GUSTAFSON:

If the Society's Program Committee had attempted to set up a symposium for this meeting on "Actuarial Theory, Technical Problems, and Regulatory Considerations in the Development of Equity-Based Life Insurance with Minimum Fixed-Dollar Guarantees," it could not have assembled a broader and more comprehensive set of participants than the authors of the six papers being presented at this meeting. There was no such attempt by the Program Committee, nor did the Committee on Papers plan for this. Nevertheless, I suggest to you that this set of papers presents a broad and very nearly complete foundation for the development of just exactly the type of product envisioned in the hypothetical symposium title used above. Variable ordinary life insurance with minimum fixed-dollar guarantees as to cash values, maturity benefits, and death benefits is just around the corner.

Central to this development is this brilliant paper by Messrs. Fraser, Miller, and Sternhell. One of the authors has been heard to say that, when they first realized the basic nature of the content of this paper, they felt it was so simple that they needed to hurry to complete it and submit it to the Committee on Papers so that someone else would not beat them to it. That apparent simplicity, however, does not detract from the brilliance and importance of the paper. I am quite sure that I am not the only other actuary who has seriously attempted to deal with this matter. Probably most of the rest made the same approach that I did, starting with the assumption that the problem would be extremely complex and technical. As a result, I was never able to see the forest for the trees.

In my opinion this is a most important paper, and it is not my purpose to offer these words of praise as a base from which to launch criticism or disagreement. I simply want to be on record as identifying this paper as a landmark in actuarial literature.

There are some in the Society who, to put it bluntly, are sorry to see the developing interest in equity-based products. They see these new concepts as compromising the very foundation of life insurance. It is obvious that I do not share this traditionalist view, although we may not be as far apart as it may seem. I do not see these developments as leading to a time when we will be dealing principally in products virtually devoid of guarantees. I prefer to start from the base of considering insurance to be a risk-transfer device, or, if you prefer, a risk-sharing device. It is true that the variable annuity, in effect, leaves all the investment risk on the contractholder, but I would suggest that the six papers being presented at this meeting are an eloquent testimony that the variable annuity as we have so far seen it is just the beginning. I believe that in the new emerging scheme of things there will be a definite place for such contracts as the variable annuity that leave the investment risk on the contractholder. I also believe there will continue to be major emphasis on traditional fixed-dollar products. But my point, and the point of this group of papers as a whole, is the importance of developing new forms of risk transfer tuned to the dynamics of a more sophisticated approach to personal security in a more complex social and economic environment.

Messrs. Biggs and Macarchuk deal with definitions and concepts, pinning down more precisely some things that have been only imperfectly understood thus far or that have been mistakenly thought to be the same as similar concepts for traditional products. Messrs. DiPaolo and Turner deal directly with certain aspects of the new risk-transfer concepts, and indirectly Mr. Seal is dealing with this area too.

When considered together, these six papers complement each other and integrate into a package almost as well as if my hypothetical symposium were a reality. It is remarkable in view of the fact that the authors were for the most part totally unaware of each other's efforts.

It would be naïve in the extreme to feel that we now have the problems solved. This is just the beginning, especially when full consideration is given to the regulatory aspects of these new product ideas. I do not think, however, that I am indulging in euphoria if I state it as my opinion that the immediate future of the actuarial profession and the insurance business is going to be exciting as we see accelerating development in this area.

JOHN H. HARDING:

When it was announced that this paper had been written, I looked forward with great interest to reading it. Generally speaking, the technical exposition of the subject is every bit as good as would be expected from these authors. I am very much concerned, however, about the potential practical effects that it may have on our business.

From the standpoint of actuarial neatness, there is no doubt that the suggested method for varying the amount of insurance coverage fits well with the concepts of actuarial mathematics as it had been developed long before I was born. Perhaps much more germane to current problems, however, are the fundamental considerations of sound product design and the practical limitations imposed by the difficulty in getting insurance laws changed in all states.

With regard to sound product design, it is questionable whether the best way to define the total insurance benefit is to equate it with fluctuations in the stock market. There are many possible methods that come to mind as being more desirable, including, but not limited to, varying the face amount with inflationary trends, varying the premium in a parallel manner, incorporating maximum or minimum parameters, and so forth. With the many variations that could potentially be of value to the insured, I find little enthusiasm for placing any special emphasis on a method which would make as much sense to the typical insured as does the standard nonforfeiture law. I am sure that there may be some outstanding salesmen who will be able to paint vivid pictures about why the face amount varies in the manner it does, but few policyholders would retain the concept sufficiently long to have the vaguest notion either of what their coverage is currently or of what it might become.

It was stated that the paper was written "in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of" the type of policy envisioned in the paper. Of course, the particular value of the proposed method is that only minor modifications of existing laws would have to be considered to accommodate it. It is precisely this point that gives me serious concern. Anyone who has been through the experience of trying to get insurance legislation through all the states must be aware of the fact that it requires substantial time and energy. Further, once the laws have been changed, it is very difficult to make significant modifications within a year or two. Therefore, if the minimum changes in existing legislation were made in order to make it possible for this limited style of equity-based product to exist, a substantial delay would automatically be generated before product design incorporating customer-oriented concepts could be introduced.

It is because the standard nonforfeiture and standard valuation laws are so complex as to be virtually incomprehensible to anyone but the expert that it is so hard to change them. We are all aware of the competing sections of these two laws which force nonforfeiture values to be on the same basis as reserves, in spite of the original intent to make them independent. Is it not time for a change? We all recognize that a number of empirical parameters are explicit within them. These parameters were obtained by studying the effects of a financial era and a social structure substantially different from those which we find today. The requirements which forced the spelling-out of lengthy tables of nonforfeiture values and the definitions thereof were set down in an era which precluded any concept of individual tailor-making of policies, variable benefit design, or rapid computation and dissemination of policy values via computer technology.

Some of the areas that should be given substantial reconsideration in addition to the standard valuation and standard nonforfeiture laws are policy loan requirements, dividend option requirements, New York State expense limitations, and investment limitations. While it may be possible to provide for equity-based products in a limited way by introducing minor changes in laws contemplating only fixed-dollar guarantees, the result would be seriously strained. What is a "variable dollar" guarantee? Perhaps an interesting example is the curious treatment of policy loans described in the paper, which would force a life insurance company to provide funds from some unnamed source to place the equity-based policy in a highly leveraged position.

At the end of their paper, the authors have a section describing possible variations in basic concept. In it they hint at some of the possible concepts that could be incorporated in product design. Unfortunately, many of the variations would not be compatible with the minimum suggested changes in the laws.

Many of us have known for a long time that the insurance laws, conceived many years ago to meet the problems of an entirely different era, have seriously hindered the development of consumer-oriented products. It has been hoped that the radically different nature of equity-based products would force a long overdue restructuring of those insurance laws which permit only a few of the many possible approaches currently well within our technological means to be employed to benefit the consumer. It will probably be years before we again have the opportunity to force actuaries, lawyers, and legislators to sit down together and work on the substantial changes that would permit the orderly development of insurance products toward the goal of providing coverage that is appropriate prospectively rather than retrospectively. It will not be of credit to our profession if, instead of accepting the challenge of creative change, we merely make do with what is temporarily convenient.

PAUL M. KAHN:

In his discussion of the stimulating paper by Messrs. Fraser, Miller, and Sternhell of the New York Life, Mr. Harry Walker described an alternative approach to fixed premium variable benefit life insurance.

Of these two, the New York Life method produces face amounts of insurance more closely tied to investment results than the alternative method which, roughly, holds back funds in good years to support benefits in bad years. As a consequence of this, the alternative method, at least a priori, should produce smaller fluctuations in benefits from year to year and should follow more closely a general economic trend line.

That this in fact occurs is borne out by comparing the two methods under the simulated 9 per cent experience illustrated for a whole life policy issued to a male aged 55 in Table 2 of the paper. The alternative method would show decreases in 14 of the 45 years shown, as compared to decreases in 19 years for the New York Life method. The average decrease is 6.3 per cent for the alternative method and 8.1 per cent for the New York Life method, while the average increase is 9.4 and 11.5 per cent under the alternative and New York Life methods, respectively.

The average change from year to year, whether increase or decrease, is 7.9 per cent for the alternative method and 9.8 per cent for the New York Life method; this means that, for this particular simulated experience and this policy, the New York Life method gives face amounts which vary from year to year by 25 per cent more than the alternative method.

GERALD A. LEVY:

The authors are to be congratulated; they have opened a door to a new product in the equity field that could revolutionize our business. Variable life insurance fits our primary roles as insurers and could solve the insurance needs of a policyholder concerned about his loss of insurance dollar purchasing power from inflation.

My discussion points toward a serious problem that could confront a large number of insurers and effectively prevent them from marketing variable life insurance contracts. The solution I offer is "separate account coinsurance." I also discuss the different mortality risks of a variable life insurance policy and how it appears that the risk is either directly or indirectly measured by the accumulation of assets, which suggests that by issuing a variable life insurance policy the insurer is assuming a related investment risk.

The last statement in the authors' summary of their paper is that "the paper was written in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of fixed premium variable benefit policies and of equity-based variable life insurance policies that reflect various alternative approaches." The objective of broad legislation will stimulate a healthy effort to design products that best suit the needs of the policyholder. To that, however, we must add another important, basic objective for enabling legislation—that every life insurer, regardless of size, be able to issue variable life policies. We must permit all companies to share what potentially is a substantial market place for the new equity-based insurance policies. Probably few, if any, of us would disagree with these objectives, and it is my hope that this discussion will provide another point of view to assist members of our Society and those state officials who will be drafting the legislation to permit variable life insurance. It is important to review carefully the intended legislation to see that it really does meet the needs of the industry. I am going to go into this in some detail, to see what the implications are of the existing and proposed legislation as applied to variable life insurance.

Several states currently have capital and/or surplus requirements before approval is given to create a separate account. A separate account is needed to issue equity-funded variable contracts.

While the model bill does not contain such financial requirements, states may include this in their legislation. The basic reasons for requirements to establish a separate account appear to be sound in theory. They seek to protect the company and the policyholder by denying a separate-account vehicle to those companies that may not have the resources to follow it through properly. Of course, we know that meeting requirements does not automatically give a company the resources it needs to effectively invest, and administer, variable life insurance. If we wanted all companies to issue these policies, an alternative solution might be to eliminate all requirements to establish a separate account. On the surface this could allow all companies to enter the business—*if* they have the investment know-how, *if* they can accumulate sufficient funds in their separate accounts to invest efficiently, *if* they can afford the surplus strain from writing new business, *if* they have the administrative capability to follow it through; I am sure that other important *if*s can also be added to the above. Thus, either with, or without, requirements many insurers could be eliminated from this market place.

It appears to me that a large number of insurers will need assistance, and the legislation should leave sufficient doors open that companies with a particular need can satisfy it, whether it be by using another company's separate account, or administrative abilities, or the like. There may be several solutions that could minimize this problem. One such solution, which has conceptual acceptance among insurance department authorities for fixed benefit coverages, is to adapt the reinsurance product of coinsurance to permit a reinsurer to hold all the assets from these variable benefit policies in his separate account. I call this new reinsurance product "separate account coinsurance." This method, a reinsurance solution, can help in those states that decide to maintain requirements to establish a separate account, as the reinsurer will have to qualify his separate account and meet requirements set forth by the insurance department. In those states that chose not to have requirements, the insurance authorities have permitted companies a vehicle to assist them in this complex field, leaving the choice to company management.

What language could be used in legislation to permit separate-account coinsurance? An insertion, such as that illustrated below, to an appropriate section of the enabling legislation discussing the investments allocated to a separate account could be as follows:

> In the case of an insurer which has a separate account with assets represented exclusively by a participation under a reinsurance contract in a separate account of another insurer which is maintained in accordance with the requirements of this section and the separate account of the ceding insurer shall also be considered to be maintained in accordance with the requirements of this section.

Note that a reinsurance contract is specifically included in this language to give the state insurance authority regulatory control over the company holding the invested assets.

What is separate-account coinsurance? It is a logical extension of coinsurance to variable benefit policies. It accomplishes many of the same objectives that coinsurance does for fixed benefit policies. The reinsurer coinsures the variable policy, sharing on either a quota share or excess basis all the traditional risks in the policy—mortality, expense, and lapse; also the reinsurer pays his share of the surplus strain. The entire portion of the net premium is invested in the reinsurer's separate account. This is a major departure from fixed benefit coinsurance, but a necessary one. By contracting to give the reinsurer the investment responsibility, the ceding insurer enjoys a share in a larger investment fund with potentially more sophisticated and more efficient investment means at their command.

Why should an assuming company hold all the assets? Is there a risk that the reinsurer is assuming to support his holding the assets? This question has been asked by an insurance department official. The practical argument for this has already been made; that is, many insurance companies need a vehicle similar to separate-account coinsurance. I believe it can also be justified theoretically: the reinsurer, when he shares in the mortality risk of a variable life policy, accepts a risk which is measured, in part, directly and, in part, indirectly by the asset values and therefore assumes a related investment risk. Let us consider three types of mortality risks which could exist in a variable life insurance policy. The first is created by a minimum benefit guarantee. We expect that many variable life policies will have a minimum death, and maturity, benefit guarantee, probably equal to the initial face amount of insurance. Here the risk is directly related to the assets. A risk payment also occurs from early death, when at the date of death the asset fund is less than the face amount of the policy. In fixed benefit insurance the risk amount decreases as the reserve increases and approaches the face amount. In an equity-linked policy there is an added dimension to this, since, as the investment experience changes, so does the face amount, which affects the risk amount. We can even have the result that reduction in assets could cause an increase in risk amount, especially at early policy durations. The third mortality risk is the annual mortality gain or loss, which in a variable life policy results in a direct payment from the general account to the separate account if a loss has occurred and the reverse if a gain has occurred. The value of the assets is a part of the calculation of the gain or loss and affects the payment that the reinsurer would make as a sharer of the mortality risk.

For the above practical and theoretical reasons, I believe that a vehicle such as the one suggested is necessary. Of course, the actual risk-sharing between a reinsurer and his ceding company will depend on the needs of the ceding company and the decision of its management.

The industry will soon be taking the leap forward into variable life insurance; it may come very quickly. We hope that those individuals with the responsibility will consider very carefully the needs of the industry and will keep the legislative doors open for insurers to seek ways of satisfying their needs.

RUSSELL E. MUNRO AND D'ALTON S. RUDD:

This subject of this paper is very much current in Canada. We are indebted to the authors for their timely and exhaustive paper, which confirms much of our thinking.

We would like to suggest an equity-flavored life insurance contract which is similar to that pioneered by Past President H. R. Lawson, in which variable paid-up insurance provides the main vehicle reflecting the performance of the equity fund. This system can be adapted to any standard form and for any selected equity-based portion of the contract at an annual premium which is the same as that for the corresponding fully guaranteed plans. The basic amount could be guaranteed as a minimum death benefit. Some consideration was given to using a varying basic amount in lieu of positive or negative variable paid-up insurance, but the latter approach appears preferable when "regular" dividends are also involved.

On each policy anniversary (1) any dividend from conventional sources of surplus and net after premium taxes can be applied to purchase variable paid-up insurance, (2) any positive or negative value obtained from the fund's performance on the segregated reserve will be converted to variable paid-up insurance and will increase or decrease the balance of variable paid-up insurance and (3) transfer will be made between the regular fund and the equity fund, so that the specified percentage of the reserve is held in the fund as well as the reserve, positive or negative as the case may be, on the balance of variable paid-up insurance.

On death or surrender during a policy year an interim adjustment is calculated to determine the excess or deficit in the value of the reserve relative to the required level, and the amount is added to or deducted from the proceeds. The authors do not appear to have considered the interim adjustment for terminations within the policy year.

In Canada the popular extended insurance provision is the automatic premium loan. It is possible to continue the insurance coverage in force until the indebtedness including interest equals any guaranteed basic cash value plus some percentage, say, 50 per cent of the basic cash value with respect to the equity portion, and at that time the remaining value could be applied automatically to provide reduced paid-up insurance on a regular guaranteed basis. However, the usual expiry takes place if the net cash value decreases through bad markets to the level of the indebtedness.

Cash loans can be permitted only on the minimum guaranteed cash value. Cash loans would not be available where the equity percentage elected is 100 per cent or at any time any indebtedness exceeds the amount of the minimum guaranteed cash value.

For reduced paid-up insurance policies it does not seem desirable to have both funds involved. Any automatic, reduced, paid-up cases should be fully guaranteed. However, where the option is elected voluntarily and the net cash value is, say, $500 or more, the owner should have the option of electing to have the whole of the reserve invested in the equity fund or guaranteed in the general fund.

Options to change the equity percentage can be made available at, say, quinquennial anniversaries. However, at any time the owner may elect to switch out of the equity fund into a life contract in which all benefits are guaranteed. A positive balance of variable paid-up insurance would be converted to guaranteed paid-up insurance, but, if the balance were negative, it would be canceled and the cash value would be held as indebtedness against the continuing contract. Any interim adjustment, if positive, would reduce this or any indebtedness or provide additional paid-up insurance and, if negative, would become indebtedness under the continuing contract.

The transfer between the regular fund and the equity fund would be as follows:

1. At the end of the first policy year,

$$_1T_x = k\cdot F_0\cdot {_1V_x} + {_1R_x}\cdot A_{x+1}\,,$$

where

$$_1R_x = \frac{(1-t)_1D_x}{A_{x+1}}\,.$$

2. At the end of the policy year n,

$$k\cdot F_0\cdot {_nV_x} + {_{n-1}R_x}A_{x+n} - (1+pi)(kF_0\cdot {_{n-1}V_x} + {_{n-1}R_x}\cdot A_{x+n-1}) + (1-t)_nD_x\,.$$

The variable paid-up insurance after the transfer is

$$_nR_x = {_{n-1}R_x} + [(i'_n - pi)(kF_0\cdot {_{n-1}V_x} + {_{n-1}R_x}\cdot A_{x+n-1}) + (1-t)_nD_x] \div A_{x+n}\,.$$

The interim adjustment for a period j in the policy year $n+1$ is

$$\left(r - \frac{pi}{j}\right)(kF_0\cdot {_nV_x} + {_nR_x}\cdot A_{x+n})\,,$$

where

$$r = \frac{u_{n+j} - u_n}{u_n}\,.$$

In the above formulas, the following notation is used:

$_nT_x$ = Net transfer between funds at end of year n.
F_0 = Basic amount or first-year death benefit.
k = Equity percentage.
i = Interest rate on premiums and reserves.
i'_n = Growth factor of accumulation unit, that is, $\dfrac{u_n - u_{n-1}}{u_{n-1}}$.
pi = Pivotal yield rate.

t = Premium tax rate.
q_{x+n} = Rate of mortality for attained age $x + n$.
A_{x+n} = Single premium for insurance of 1 for age $x + n$.
${}_nV_x$ = Terminal reserve at end of policy year n for policy issued at age x.
${}_nD_x$ = Dividend for year n excluding investment earnings in the fund.
${}_nR_x$ = Variable paid-up insurance at end of year n.

JAMES J. MURPHY:

As presented in the paper, the death benefit for any moment during a given policy year based on fully continuous functions is determined as the product of the previous year-end death benefit—the Y factor and the Z factor. The Y factor is constant for a given policy year, while the Z factor varies throughout the year, depending on the investment results for the fraction of the year that has elapsed.

It will be found that this method of determination will produce marked discontinuities in death benefits between the end of one policy year and the beginning of the next. If the year-end death benefit is greater than the initial face amount, benefits will drop sharply on the anniversary; and, if the year-end death benefit is less than the initial face amount, death benefits will rise sharply on the anniversary. These results seem inconsistent with the assumption of fully continuous functions.

The theory of the fixed premium variable contract assumes equality of reserve per $1,000 of actual face amount between these contracts and traditional fixed-dollar contracts. The fully continuous reserve for any moment during the policy year is found by interpolating the previous and current terminal reserves. Annual premiums are paid on the basis of the present value (at the beginning of the policy year) of the continuous annual premium with interest only at the valuation rate. Because of that fact an additional reserve, the unearned premium reserve, is held. Upon death or surrender (if CV is equal to Reserve) the unearned portion of the annual premium paid is refunded. This reserve can be thought of as a special discounted premium deposit fund. The present value of the year's premiums (payable continuously) is credited to the fund at the beginning of the year, while the continuous premiums are paid from the fund as they "fall due."

The very name of the fixed premium variable benefit contract implies that its unearned premium reserve should be identical to that of a fixed-dollar contract. It does not depend on the variable face amount. Thus, it seems, the face of the FPVB policy should be based only on the actual continuous reserve, as if premiums were credited continuously. The unearned premium reserve would be a separate fund not related to the varying face or reserve. This approach would result in a formula similar to that presented in the paper but with the V factor varying throughout the policy year. The results should show a more continuous pattern of face amounts from policy year to policy year. The method would also be more easily adapted to policies with premium frequencies other than annual.

This discussion leads me to the following questions: (1) How was the formula for fully continuous functions derived? (2) Was the payment of the unearned premium as additional death benefit considered in that derivation?

STEWART G. NAGLER:

Probably no subject has received greater consideration from the actuarial profession in recent years than that of equity-linked contracts. Now that variable annuities are widely accepted, interest is turning to variable life insurance contracts. Messrs. Fraser, Miller, and Sternhell are therefore to be congratulated on their very significant and timely contribution to actuarial literature. The approach to variable life insurance contracts presented in their paper requires that "the reserve per \$1 of face amount at the end of each policy year for the fixed premium variable benefit whole life insurance policy be the same as that for the corresponding fixed premium fixed benefit whole life insurance policy." Consideration of this basic assumption in light of the purpose of variable life insurance leads to a more general approach to the subject, which is presented below.

While maintaining the same underlying assumptions as to the operation of the variable life insurance policy and the calculation of net premiums, we may generalize the authors' results by eliminating the requirement quoted above. That is, in defining the total terminal reserve at duration t, ${}_t(\mathrm{TR})_x$, we may replace the equation

$$ {}_t(\mathrm{TR})_x = F_t \cdot {}_tV_x \tag{1} $$

by

$$ {}_t(\mathrm{TR})_x = {}_tV_x + (F_t - 1) \cdot {}_tR_x \,, \tag{2} $$

where ${}_tR_x$ equals the reserve per \$1 of insurance in excess of the initial face amount of \$1. (Throughout this discussion "excess" will be used to denote both positive and negative amounts.) While a more general definition of the total reserve is possible, this form was chosen to segregate the reserve for the initial face amount of insurance from the reserve generated by the difference between actual and assumed investment earnings. Thus the reserves and benefits under a normal fixed benefit policy will be duplicated if the actual investment performance follows the basic assumptions.

In this generalized case, the equation of equilibrium becomes

$$\begin{aligned}[{}_{t-1}V_x + (F_{t-1} - 1)\cdot{}_{t-1}R_x + P_x](1 + i'_t) \\ = q_{x+t-1}\{F_t - [{}_tV_x + (F_t - 1)\cdot{}_tR_x]\} + {}_tV_x + (F_t - 1){}_tR_x\,.\end{aligned} \tag{3}$$

If the steps outlined in Appendix A of the paper are followed, it can be shown that any function ${}_tR_x$ which satisfies equation (3) at all durations satisfies

$$A'_{x+t} - P_x\ddot{a}_{x+t} = {}_tV_x + (F_t - 1)\cdot{}_tR_x\,, \tag{4}$$

where A'_{x+t} is as defined in the paper.

Solving equation (3) for F_t, we have

$$F_t = 1 + \frac{({}_{t-1}V_x + P_x)(i'_t - i) + (F_{t-1} - 1){}_{t-1}R_x(1 + i'_t)}{q_{x+t-1}(1 - {}_tR_x) + {}_tR_x}\,. \tag{5}$$

Under this formula, the excess earnings in a particular year are used to purchase benefits commencing in that year; current dividend options provide benefits commencing in the following year. This is consistent with the paper and allows the insurance benefit to reflect the investment experience up to the date of the claim. The above assumption is by no means a theoretical necessity and could easily be modified.

In equation (5) the numerator in the second term may be thought of as representing the balance to date of the cumulative excess investment performance over the insurance benefits provided by such excesses, while the reciprocal of the denominator may be thought of as a "multiplier." This "multiplier" translates the excess investment performance into an amount of excess insurance benefit.

It is of interest to note that equation (5) can be rewritten as

$$F_t = 1 + G_t(i'_t - i) + H_t(F_{t-1} - 1)(1 + i'_t)\,, \tag{6}$$

where

$$G_t = \frac{{}_{t-1}V_x + P_x}{q_{x+t-1}(1 - {}_tR_x) + {}_tR_x} \tag{7}$$

and

$$H_t = \frac{{}_{t-1}R_x}{q_{x+t-1}(1 - {}_tR_x) + {}_tR_x}\,. \tag{8}$$

If the values of ${}_tR_x$ are determined at issue, the values of G_t and H_t can be calculated in advance and conveniently incorporated in the policy form. In this way the policy can precisely describe how the insurance benefits will vary in accordance with actual investment performance.

The function ${}_tR_x$ may be viewed as determining the pattern of additional insurance benefits which can be provided by a given level of investment earnings. Therefore, the choice of ${}_tR_x$ values can be considered as an assumption which affects the incidence of benefits and therefore the equity between claimants in various policy years. From a practical point of view, it seems desirable to restrict ${}_tR_x$ so that $0 \leq {}_tR_x \leq 1$. In this way the ${}_tR_x$ function acts to amortize the excess investment performance over the future of the policy. If ${}_tR_x$ were greater than 1, the excess reserve would be greater than the corresponding excess death benefits; if ${}_tR_x$ were less than 0, the excess insurance benefits would move in a direction opposite to the cumulative difference between the actual and assumed investment performance. In either case, the part of equation (3) which represents the cost of the excess insurance would be opposite in sign to the amount of the reserve for such insurance. Therefore, this cost would increase the excess rather than amortize it over the future of the policy.

When ${}_tR_x = 1$ for all t,

$$G_t = {}_{t-1}V_x + P_x \tag{9}$$

and

$$H_t = 1\,. \tag{10}$$

Substituting these values in equation (6), we obtain

$$F_t = 1 + ({}_{t-1}V_x + P_x)(i_t' - i) + (F_{t-1} - 1)(1 + i_t')\,. \tag{11}$$

In effect no additional insurance is purchased with the excess earnings; rather, the excess earnings accumulate at the actual earnings rate and are added to the death benefit. This is analogous to the commonly used "interest only" dividend option. In this case the "multiplier" is 1, which is its minimum value within the limits specified for ${}_tR_x$.

At the other extreme, when ${}_tR_x = 0$ for all t,

$$G_t = \frac{{}_{t-1}V_x + P_x}{q_{x+t-1}} \tag{12}$$

and

$$H_t = 0\,. \tag{13}$$

Substituting in equation (6), we have

$$F_t = 1 + \frac{{}_{t-1}V_x + P_x}{q_{x+t-1}}(i_t' - i)\,. \tag{14}$$

This choice can be likened to the one-year term dividend option, where excess earnings on the initial reserve are used to purchase one-year term insurance. Year-to-year fluctuations in the total insurance benefit are substantial, because, in this case, the "multiplier" has its maximum value. Since negative excess earnings can cause the total death benefit to be less than the total policy reserve or even negative, it would not be practical to set ${}_tR_x = 0$ at all durations.

Within these limits a familiar set of values is produced by setting ${}_tR_x = A_{x+t}$ for all t. Substituting in equation (5), we have

$$F_t = 1 + \frac{({}_{t-1}V_x + P_x)(i'_t - i)}{q_{x+t-1}(1 - A_{x+t}) + A_{x+t}} + \frac{(F_{t-1} - 1)A_{x+t-1}(1 + i'_t)}{q_{x+t-1}(1 - A_{x+t}) + A_{x+t}}. \quad (15)$$

Equation (15) is equivalent to

$$F_t = F_{t-1}\frac{(1 + i'_t)}{(1 + i)} - \frac{P_x \cdot a_{x+t-1}}{(A_{x+t-1})} \cdot \frac{(i'_t - i)}{(1 + i)}. \quad (16)$$

While this equation is not so obviously interpretable as in the previous examples, it can be shown to be analogous to the level paid-up additions dividend option. It is also interesting to note that this assumption is necessary and sufficient to satisfy the condition that $F_t = F_{t-1}$ for any duration where $i'_t = i$. This is similar to the operation of most variable annuity contracts where the annuity payments do not change as long as the actual investment performance equals the assumed investment performance.

To duplicate the results presented in the paper, we could set ${}_tR_x = {}_tV_x$. Substituting in equation (5), we obtain

$$F_t = 1 + \frac{({}_{t-1}V_x + P_x)(i'_t - i)}{q_{x+t-1}(1 - {}_tV_x) + {}_tV_x} + \frac{(F_{t-1} - 1){}_{t-1}V_x(1 + i'_t)}{q_{x+t-1}(1 - {}_tV_x) + {}_tV_x}. \quad (17)$$

This equation may be transformed to

$$F_t = F_{t-1}\frac{(1 + i'_t)}{(1 + i)} \cdot \frac{[{}_{t-1}V_x + (P_x/F_{t-1})]}{({}_{t-1}V_x + P_x)}, \quad (18)$$

which is the result presented in the paper. While this formula is not analogous to any of our present dividend options, it may be viewed as producing paid-up additions which decrease from year to year.

Thus far we have considered values of ${}_tR_x$ which are determined at issue. It is also possible to set the value of ${}_tR_x$ for a particular duration in light of the actual investment performance through the end of that duration to limit the year-to-year fluctuations in the insurance benefit. For example, we could set ${}_tR_x = {}_tV_x$ as long as the change in the face amount of insurance was less than 6 per cent. If this value of ${}_tR_x$ produced a change greater than 6 per cent, a ${}_tR_x$ would be chosen from values between 0 and 1, so that the change in benefits was as close to 6 per cent as possible. From a theoretical point of view, these variable values of ${}_tR_x$ could be

TABLE 1

ILLUSTRATIVE FACE AMOUNTS FOR A FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL FACE AMOUNT OF $1,000 ISSUED IN JULY, 1915, AT AGE 35, WITH SEPARATE ACCOUNT INVESTED IN STANDARD AND POOR'S COMPOSITE 500

(Net Level Premiums and Reserves Based on 1958 C.S.O. Table, 3 Per Cent Interest, and Traditional Functions)

Policy Year Ending in:	${}_tR_x=1$ (Interest Only)	${}_tR_x=0$ (One-Year Term)	${}_tR_x=A_{x+t}$ (Paid-up Additions)	${}_tR_x={}_tV_x$ (Authors' Assumption)	${}_tR_x={}_tV_x$* (Modified)
1916	1,003	2,197	1,008	1,179	1,060
1917	1,003	914	1,007	1,076	1,086
1918	998	− 774	994	937	997
1919	1,016	7,319	1,040	1,247	1,057
1920	1,003	− 2,438	1,007	1,022	1,030
1921	989	− 2,867	973	870	968
1922	1,022	10,765	1,051	1,193	1,026
1923	1,019	282	1,043	1,139	1,087
1924	1,045	5,881	1,099	1,296	1,153
1925	1,100	9,469	1,215	1,594	1,222
1926	1,149	6,423	1,311	1,789	1,295
1927	1,232	8,607	1,474	2,112	1,373
1928	1,373	10,895	1,741	2,622	1,455
1929	1,687	17,362	2,330	3,737	1,674
1930	1,462	− 7,412	1,865	2,641	1,774
1931	1,227	− 9,783	1,406	1,687	1,815
1932	959	−15,615	912	775	952
1933	1,132	21,342	1,219	1,312	1,120
1934	1,079	− 2,624	1,123	1,137	1,187
1935	1,205	9,321	1,329	1,458	1,259
1936	1,460	13,103	1,740	2,063	1,436
1937	1,542	3,157	1,850	2,178	1,522
1938	1,324	− 5,300	1,482	1,606	1,614
1939	1,320	570	1,462	1,554	1,704
1940	1,232	− 1,993	1,316	1,338	1,602
1941	1,269	1,965	1,360	1,382	1,506
1942	1,175	− 1,906	1,213	1,183	1,416
1943	1,450	9,126	1,591	1,647	1,501
1944	1,572	3,141	1,740	1,807	1,591
1945	1,780	4,127	1,997	2,086	1,686
1946	2,124	5,088	2,415	2,539	1,995
1947	1,971	− 766	2,177	2,223	2,114
1948	2,047	1,458	2,236	2,254	2,241
1949	2,046	718	2,196	2,175	2,376
1950	2,461	4,072	2,652	2,629	2,518
1951	3,145	4,811	3,394	3,358	2,838
1952	3,688	3,013	3,935	3,850	3,319
1953	3,769	1,003	3,936	3,779	3,518
1954	4,835	3,999	4,990	4,742	4,324
1955	6,943	5,391	7,073	6,648	6,192

* ${}_tR_x$ varies between 0 and 1 (see text).

TABLE 1—*Continued*

Policy Year Ending in:	${}_tR_x=1$ (Interest Only)	${}_tR_x=0$ (One-Year Term)	${}_tR_x=A_{x+t}$ (Paid-up Additions)	${}_tR_x={}_tV_x$ (Authors' Assumption)	${}_tR_x={}_tV_x^*$ (Modified)
1956	8,125	2,425	8,093	7,471	7,237
1957	8,202	844	7,957	7,197	7,671
1958	8,385	966	7,926	7,029	8,131
1959	10,994	3,318	10,159	8,864	9,643
1960	10,414	395	9,359	8,001	10,222
1961	12,853	2,464	11,266	9,466	11,111
1962	11,599	176	9,883	8,138	11,778
1963	14,148	2,188	11,748	9,510	12,484
1964	17,479	2,200	14,135	11,244	14,624
1965	18,426	1,141	14,493	11,318	15,502
1966	18,614	907	14,236	10,914	16,432
1967	21,771	1,681	16,198	12,202	17,974
1968	23,185	1,165	16,772	12,409	19,136

viewed as merely applying the excess reserves to date partially to term insurance and partially to the "interest only" option.

To appreciate further the impact which ${}_tR_x$ has on the insurance benefits, let us examine the death benefits produced by each of the above values of ${}_tR_x$ for a fixed premium variable benefit whole life policy issued at age 35 under the assumptions used in Table 3 of the paper.

Although the authors' basis for variable insurance shows great promise for combating the effects of inflation on life insurance, it does not appear to be the only practical alternative. No matter what mathematical simplicities a particular method offers, its ultimate acceptability must be determined by how well it provides the security which policyholders expect from life insurance. Considering the unlimited number of benefit patterns which the generalized formula can produce, it is apparent that one must select the ${}_tR_x$ values most carefully in order to achieve results which will be of the greatest value to the policyholder. There is no clear indication that the authors' method will best meet the needs of the public.

As the authors point out, variable insurance products may be designed with many other meaningful variations from the paper's basic assumptions, such as variable premiums or investment guarantees. Any changes in the insurance laws of any state should, therefore, be broad enough to encompass not only the changes necessary to permit the issuance of products based on the assumptions in the paper under discussion but changes which would permit the writing of various other concepts. In this connection, consideration must be given to the requirements of other agencies which may have regulatory authority over the issuance of such contracts. This is particularly pertinent with respect to SEC and the state blue sky legislation.

CECIL J. NESBITT:

It was a great pleasure to read this excellent and timely paper. One thing which appealed to me was that the problem was approached by means of a difference equation (formula [2]) and a solution was developed by use of that equation. In doing so, the authors relied mainly on the recursive character of the equation; I would argue that the equation may be integrated or summed in much the same way that a differential equation is handled to produce a solution. To illustrate this point, I shall outline an alternative for the proof given in Appendix A for the equality of the reserves obtained by a retrospective process and by a prospective process.

For this purpose, I shall rewrite equation (2) of the paper in the form

$$P = v'_h F_h q_{x+h-1} + v'_h p_{x+h-1} F_h ({}_h V) - F_{h-1}({}_{h-1}V) \,, \qquad \text{(a)}$$

where $v'_h = (1 + i_h)^{-1}$ and t has been replaced by h, with the latter to be regarded as a running variable over the domain $1, 2, \ldots, n$. Also, instead of whole life insurance, I have in mind any n-level payment, n-year insurance. If t is the completed duration of the insurance, the values of i'_h, $h = 1, \ldots, t$ are known; for $h = t + 1, \ldots, n$ any reasonable assignment can be made; for example, the authors' assignment $i'_h = i$, $h > t$. The complete assignment of the i'_h will then determine the complete set of benefit amounts F_h, $h = 1, 2, \ldots, n$. Further, one may also define select commutation functions:

$$D'_{x+h} = v'_1 v'_2 \ldots v'_h l_{x+h} \,; \qquad C'^F_{x+h} = D'_{x+h} v'_{h+1} F_{h+1} q_{x+h} \,;$$

$$M'^F_{x+t} = \Sigma_{h=t}^{n-1} C'^F_{x+h} \,; \qquad N'_{x+t} = \Sigma_{h=t}^{n-1} D'_{x+h} \,.$$

Multiplication of formula (a) by D'_{x+h-1} yields

$$P D'_{x+h-1} = C'^F_{x+h-1} + D'_{x+h} F_h({}_h V) - D'_{x+h-1} F_{h-1}({}_{h-1}V) \,. \qquad \text{(b)}$$

On summing formula (b) over $h = 1, 2, \ldots, t$, we obtain

$$P(N'_x - N'_{x+t}) = M'^F_x - M'^F_{x+t} + D'_{x+t} F_t({}_t V) \,, \qquad \text{(c)}$$

and the retrospective formula for the reserve, namely,

$$F_t({}_t V) = [P(N'_x - N'_{x+t}) - (M'^F_x - M'^F_{x+t})]/D'_{x+t} \,. \qquad \text{(d)}$$

Summation of formula (b) over $h = t + 1, t + 2, \ldots, n$ gives the prospective formula

$$F_t({}_tV) = [M'^F_{x+t} + D'_{x+n}F_n({}_nV) - PN'_{x+t}]/D'_{x+t}\,. \qquad \text{(e)}$$

That the retrospective and prospective formulas for $F_t({}_tV)$ are equal follows by the usual argument that such equality is equivalent to the premium benefit equation

$$PN'_x = M'^F_x + D'_{x+n}({}_nV)\,. \qquad \text{(f)}$$

For a whole life insurance, and under the assumption $i'_h = i$ for $h > t$, the right member of (e) is equal to the authors' formula (38), so that their equation (40) is now proved.

It was also my pleasure to read D. D. Cody's extremely interesting discussion of the paper, and my other comment will be related to that discussion.

Mr. Cody indicated a number of designs for equity-based insurances. His design 4 is based on the principle of deducting from the fixed premium a mortality charge for the net amount of risk and of investing the balance of the premium in a savings fund that would be available in case of death or survival. If the fund were to grow to equal the sum insured, the policy would mature as an endowment; if the fund were exhausted, the policy would terminate. Such a free-form policy design might be used in relation to a company-operated mutual fund. An alternative, also free-form, design could be based on the principle of deducting from the fixed premium a mortality charge for the full sum insured (which could be varied relative to some index) and placing the balance of the premium into a fund accumulating under both investment income and survivorship, such fund to be available only in case of survival. If ${}_tW$ denotes the insured's reserve at the end of t years under these conditions, one has the relation

$${}_tW = [(P - v'_tF_tq_{x+t-1}) + {}_{t-1}W](1 + i'_t)l_{x+t-1}/l_{x+t}\,. \qquad \text{(g)}$$

As for design 4, the term of the insurance would be indefinite. Further, $(P - v'_tF_tq_{x+t-1})$ might well be negative at the higher ages and imply a charge against rather than a deposit to the survivorship fund. Both the insurance benefit and the survival benefit components of the premium would here be applied in ways involving mortality risk, and from that point of view the alternative design might be more appropriate for an insurance company than design 4. In both designs P is fixed, F_t may be determined by some external index and not depend on the reserve, and consequently the reserve itself is the adjusting item.

Whether either of these two designs is workable in practice and whether they would differ in their results, I leave to others to consider.

ROBERT J. RANDALL:

A key assumption of this paper is that the reserve per \$1 of face amount at the end of each policy year is the same as that for a corresponding fixed benefit life insurance policy. The reasons for this somewhat arbitrary assumption are not explicitly stated by the authors. The implication is that the assumption is well justified by the fact that the resulting nonforfeiture and minimum reserve formulas work out so neatly. Under this approach, increases in the face amount reflecting investment performance in excess of the assumed rate are reduced not only to allow the premium to remain level but also to meet this reserve assumption. The result is that the face amount changes each year not only in relation to investment results but also by a factor varying by plan, issue age, and duration.

An alternative, and perhaps more natural, approach would be to allow face amounts to vary directly in proportion to investment results, that is, by the ratio of $(1 + i_t)$ to $(1 + i)$, and to absorb the other effects in the change in cash values and reserves. Time and lack of ability have prevented me from developing the formulas for this approach comparable to the formulas presented in the papers. It seems to me, however, that both reserves and cash values satisfying the standard laws could be defined by insurance cost factors and an accumulation formula (set forth in the contract for cash values); this formula would produce conventional fixed benefit values if actual investment results follow exactly the assumed rate.

The accumulation formula would be

$$V_t = (V_{t-1} + P)(1 + i_t)f_t - g_tF_t ,$$

where F_t would be taken as $F_{t-1}(1 + i_t)/(1 + i)$; i_t is the actual earned rate; i is the assumed rate; $f_t = 1/p_{x+t-1}$, and $g_t = q_{x+t-1}/p_{x+t-1}$.

Cash values satisfying the standard nonforfeiture laws would be accumulated in a comparable manner, except that P would be replaced by (AP) as defined in the paper, and an additional deduction equal to I, the initial expense deficit, would enter into the accumulation process in the first year. The resulting cash values would be, I believe, comparable to cash values developed by the authors' formula (B3) in Appendix B so far as amortization of the initial expense deficit is concerned.

The advantage, if any, of this alternate approach is that the face-amount changes are keyed directly to investment performance. Under either approach, increases (or decreases) in premiums might be permitted, as the authors suggest, in order to keep changes in both face amounts and cash values in proportion to investment results. The approach suggested here seems to accommodate premium changes more naturally.

The obvious disadvantage of this approach is that reserves and cash values would eventually become negative if the actual investment returns exceeded the assumed interest rate for substantial periods. I believe that, for an ordinary life policy, for example, this would not happen for a great many years; nevertheless, some practical contractual treatment would have to be devised before this approach could be considered feasible. One possible solution might be a provision providing for either increase in the premium or lapse to paid-up variable insurance in the event the cash value fell below some prescribed minimum.

The paper ties the relationship between the initial face amount and the premium to the traditional forms of fixed benefit insurance, ordinary life, twenty-payment life, and so forth. This relationship tends to vanish quickly with duration, and the actual nature of the policy may depend more on investment performance than on the original assumptions. This suggests to me that it might be more natural to offer policies where the premium could be selected from a range of percentages of the initial face amount and where the duration of premium payments would not be specified at issue. Cash values could be calculated on the assumption that premiums would be continued for the maximum duration permitted by the contract, which might be either for life or to age 65. Where the selected percentage was high enough, something comparable to the old retirement income form of policy would result.

Messrs. Fraser, Miller, and Sternhell are to be congratulated on a brilliant and timely solution to a difficult problem.

MEL STEIN:

Messrs. Fraser, Miller, and Sternhell are to be congratulated for presenting the basic actuarial theory for the fixed premium variable benefit policy in such a concise, logical, and straightforward manner. This paper meets the standards that one would expect from such an august team of coauthors. This discussion will touch upon one very critical area not covered in this paper—pricing.

Utilizing the tables of death benefits shown in Table 2, I made ordinary life gross premium calculations for a whole life policy issued at age 55 under four conditions: (1) 6 per cent constant yield fixed premium fixed benefit plan, (2) 6 per cent constant yield fixed premium variable benefit plan, (3) 9 per cent constant yield fixed premium variable benefit plan, and (4) 9 per cent simulated yield fixed premium variable benefit plan.

The first question that came to mind was "What amount must be added to the gross premium for a fixed premium fixed benefit policy to obtain a gross premium of equal profitability for a comparable fixed premium variable benefit plan?" A 6 per cent constant yield and the break-even approach produced the following results:

Break-Even Period	Gross Premium Increase
5	$2.46
10	3.67
15	4.60
20	5.27
25	5.66
30	5.87

The fact that the additional amount of required gross premium increases as the break-even period increases is not at all surprising, particularly when the additional costs of the variable benefit policy are for higher cash-surrender values, death benefits, and reserves. The relative additional cost of each of these three variables is, of course, a function of parameters, such as plan, issue age, reserve basis, and cash-value basis.

Another question that immediately came to mind was "What happens if fixed premium variable benefit gross premiums are based on a 6 per cent constant yield and a 9 per cent yield is earned?" The answer, or perhaps I should say answers, proved to be quite interesting.

First, when a break-even premium approach was used, the 6 per cent constant yield break-even premiums were lower during the first eighteen years, while the 9 per cent constant yield break-even premiums were slightly lower thereafter. On the surface, this does not seem to be very conclusive. Next, when a gross premium of $39.10 for the fixed premium variable benefit plan was used, the following seemingly contradictory results were obtained: (1) If a 9 per cent yield was earned, the accumulated book earnings at the end of thirty years were roughly twice what they were if a 6 per cent yield was earned. (2) The value at issue of book profits less losses during the thirty-year gross premium calculation period was 22 per cent greater when a 6 per cent yield was earned.

This apparent contradiction may be explained as follows: (1) The accumulated book earnings comparison is invalid, since it compares apples and oranges. The cause for this is that the book profits are accumulated at different interest rates. This, in turn, will very likely result in a different rate of inflation and different dollars. (2) The value at issue of future book profits less losses compares apples and apples and ties in with the results obtained under the break-even premium analysis. Under each yield basis, the book profits were discounted to issue by the yield, which represents the value of money to the company. If the value at issue of book profits less losses were the same under both 6 and 9 per cent yields, the accumulated book earnings at the end of the thirty-year gross premium calculation period under the 9 per cent yield basis would be equal to the thirty-year accumulated book earnings under the 6 per cent yield basis times $(1.09/1.06)^{30}$.

It must be emphasized that these results represent only calculations done for a whole life policy with net level premium reserves and minimum cash values issued at age 55. Much more extensive calculations will be necessary to show a reasonably complete or general picture of the peculiar problems of calculating premiums for fixed premium variable benefit policies and the resulting patterns of premiums and profitability levels. It will be interesting to see how the results of such calculations vary by plan, issue age, reserve basis, cash-value basis, and yield pattern (which does not have to be level).

The gross premium calculations for the 9 per cent *simulated* yield fixed premium variable benefit plan showed very large erratic fluctuations in book profits. This raises the question whether it would be desirable for the insurance industry to issue fixed premium variable benefit policies. The fixed premium variable benefit policy should result in life insurance companies' becoming increasingly vulnerable to fluctuations in annual statement earnings and resulting stockholder dividends, sudden large drains on surplus, and, in the case of small companies, insolvency.

It should be noted that the financial dangers of fixed premium variable benefit policies are much greater for small companies than for giants, such as N.Y.L.I.C.

A closing thought is that the fixed premium variable benefit policy seems to offer some intriguing problems in the area of setting and adjusting dividend scales.

SAMUEL H. TURNER:

The authors are quite deserving of the accolades which will certainly be bestowed upon them for their excellent paper. The theory is simple, yet profound. The practical treatment is thorough and informative.

I am concerned with the fact that the approach proposed by the authors, that is, using an equation of equilibrium for reserves, inextricably links the level of cash values for a particular policy to the level of reserves maintained for that policy. This is true, of course, since the cash value for a fixed premium variable benefit policy is equal to the cash value per $1,000 face amount times the adjusted face amount, which is a function of the reserve maintained for the policy. The alternative approach of using an equation of equilibrium for cash values is considered by the authors in Appendix B but is discarded as being of no practical significance. I am not convinced that this alternative is of no practical significance.

Consider two fixed premium variable benefit policies—both of which provide "minimum" cash-surrender values per $1,000 face amount with the face amount adjusted in accordance with the equation of equilibrium for reserves (as proposed by the authors)—reserves being determined in accordance with the net level premium valuation method for one policy and the commissioners reserve valuation method for the other policy. Consider, further, the same two policies, except with the face amount being adjusted in accordance with the equation of equilibrium for cash values (formulas [B6]–[B8]) *stated in the policy*; that is, negative cash values are taken as zero. Adjusted face amounts and cash values under the latter two policies would, of course, be identical, since they are independent of the reserves maintained.

Adjusted face amounts and "minimum" cash-surrender values computed for the cases outlined above are shown in Table 1. Based on the data illustrated in Table 1, the following observations may be noted:

1. It is apparent that, *where the equation of equilibrium for reserves is applied*, adjusted face amounts and cash values can be significantly greater under a policy with net level reserves than under an otherwise similar policy with CRVM reserves. There is, therefore, no unique scale of "minimum" cash-surrender values for a fixed premium variable benefit policy, if the equation of equilibrium for reserves is applied, such values likely varying from company to company and even within a particular company based on the reserve actually maintained.

2. It is apparent that, *celeris paribus*, adjusted face amounts and cash values for a particular fixed premium variable benefit policy can be significantly greater where derived from an equation of equilibrium for reserves than those derived from an equation of equilibrium for cash values.[1]

Theoretically, the fund underlying a fixed premium variable benefit policy, on which are based adjustments to reflect actual investment performance, should be reasonably representative of the *actual* fund ac-

[1] Although not illustrated in the table, the same relationship would exist for adjusted reserves since the reserve under the policy is equal to the reserve per $1,000 times the adjusted face amount.

cumulated under the policy. The policy cash value is a more realistic approximation to the *actual* fund accumulated under the policy than is the statutory reserve and is, therefore, the theoretically preferred basis for applying the equation of equilibrium.

In conclusion, there would appear to be both practical and theoretical justifications for utilizing an equation of equilibrium for cash values

TABLE 1

ACTUAL FACE AMOUNTS AND MINIMUM CASH-SURRENDER VALUES FOR FIXED PREMIUM VARIABLE BENEFIT WHOLE LIFE POLICY WITH INITIAL AMOUNT OF $1,000 ISSUED TO MALE AGED 55

(1958 C.S.O. 3 Per Cent and Net Annual Investment Rate of Separate Account, 9 Per Cent)

End of Policy Year	Equation of Equilibrium for Reserves, Using:				Equation of Equilibrium for (Minimum) Cash Values*	
	Net Level Reserves		CRVM Reserves			
	Face Amount	"Minimum Cash Value"	Face Amount	"Minimum Cash Value"	Face Amount	"Minimum Cash Value"
1	$1,058	$ 0	$1,058	$ 0	$1,058	$ 0
2	1,084	12.82	1,058	12.52	1,058	12.52
3	1,110	45.25	1,083	44.15	1,072	43.71
4	1,137	79.21	1,109	77.26	1,096	76.36
5	1,165	114.74	1,136	111.88	1,122	110.51
6	1,194	151.85	1,164	148.04	1,149	146.13
7	1,224	190.60	1,193	185.77	1,177	183.28
8	1,255	230.97	1,223	225.08	1,206	221.95
9	1,287	273.00	1,254	266.00	1,236	262.18
10	1,320	316.68	1,286	308.52	1,267	303.97
15	1,505	559.78	1,461	543.42	1,437	534.49
20	1,717	841.24	1,661	813.81	1,632	799.60

* Negative cash-surrender values taken as zero.

rather than for statutory reserves. Under this alternative approach, the face amount would be adjusted on the basis of an equation of equilibrium for cash values, the adjusted cash value and reserve being equal to the adjusted face amount times the respective cash value and reserve per $1,000, respectively. The advantages of this alternative approach would appear to include the following:

1. Cash values are independent of the particular statutory reserve maintained.
2. A unique scale of "minimum" cash values is preserved.

3. Adjustments are based on the most realistic representation available of *actual* funds underlying the policy.
4. The approach is practical in application, if cash values *shown in the policy* are utilized.

HARRY WALKER:

We are indebted to Messrs. Fraser, Miller, and Sternhell for their paper which presents the basic actuarial theory for a form of variable life insurance with fixed premiums. As one who has been involved in helping to draft model state legislation and model regulations to cover the field of variable life insurance, I can assure you that this paper has proved to be most valuable in helping us to visualize the variety of forms of variable life insurance which should be covered by such model legislation and regulations.

There is one feature of this approach to fixed premium variable life insurance that may be disturbing. It may be difficult to explain a reduction in the face amount of insurance at a time when the actual net investment return on the separate account exceeds the interest rate assumed in the calculation of the net annual premium and reserves. Nevertheless, this is precisely what could happen under the contract described in the paper if, after several years of unusually good investment performance, there is a reduction in the net investment return but the company still has earned somewhat more than the assumed interest rate. Under these circumstances the Z factor will be greater than 1, but the Y factor could be so much smaller than 1 that the product of the two factors would be less than 1.

In Table 2 of the paper, for issue age 55, the face amount is shown assuming a constant annual return of 9 per cent. At the end of the fifth policy year the face amount has increased to $1,165. If the net investment return for the sixth year falls below 6.3 per cent, there will be a reduction in the death benefit from the fifth to the sixth year. I have had a calculation made, assuming a 25 per cent net investment return for each of the first five years, in which event the death benefit in the fifth year would have risen to $1,732—and a net return of 10.3 per cent would be necessary in the sixth year to maintain the death benefit at that level without any reduction. It will be difficult to explain to the policyholder why the death benefit is being reduced in a year in which the net investment return is appreciably greater than the "interest rate assumed in the calculation of the net annual premium and reserves."

This suggests consideration of a fixed premium variable life insurance policy under which the death benefit and reserves at any time are so determined that the death benefit would remain level thereafter if the net investment return thereafter should be exactly equal to the assumed interest rate. This objective can be achieved if the excess of the net investment return over the assumed interest rate is used to buy paid-up insurance, the reserves for which remain in the separate account and the benefits under which will vary with the investment results of the separate account. It is evident that the face amount of the base policy, as well as the amount of the paid-up insurance, will remain constant if the net investment return thereafter is exactly equal to the assumed interest rate.

Under this approach to a fixed premium variable life insurance policy, if the net investment return is less than the assumed interest rate, the death benefit would be reduced in effect by buying a negative amount of paid-up insurance. Anticipating that this might suggest the likelihood of the reserve disappearing entirely with bad investment performance, I have had a calculation made indicating that for issue age 35 the company would have to suffer a net investment return of worse than -85 per cent during the first five years to wipe out the reserve in any of these years. At issue age 55, the company would have to suffer a net investment return worse than -63 per cent to wipe out the reserve in any of the first five years. The probability of this happening is obviously very small.

If the policy is participating with respect to mortality and loading gains, the surplus apportioned annually to reflect these gains could be used to augment the paid-up variable insurance purchased by the excess of the net investment return over the assumed interest rate. In the case of a declining market, this would serve to cushion the decrease in death benefit.

It may be noted that the approach suggested in this discussion is similar to the technique used for variable annuity contracts. Under a variable annuity the excess of the net investment return over the "assumed interest rate" is effectively used to buy an additional paid-up annuity, and in that way there is no reduction in annuity payments at any time when the net investment return is at least equal to the assumed interest rate.

In comparison with the fixed premium variable life insurance policy described in the paper, the alternative approach suggested in this discussion will, in the case of a rising market, produce substantially lower increases in death benefits and correspondingly higher cash values in the early policy years. A point will be reached, however, beyond which the death benefits under this alternative approach would exceed those described in the paper. If we look at the figures for a 9 per cent constant return in Table 2 of the paper, the death benefits shown in the table would be higher than those under this alternative approach for the first nineteen years. In the twentieth year, the death benefit under the approach I am suggesting would be \$1,745, in comparison with the \$1,717 figure in the paper. By the thirtieth year, the death benefit would equal \$2,894, in comparison with the figure of \$2,245 shown in the table. In the forty-fifth year, the alternative approach produces a death benefit about twice the \$3,373 figure shown in the paper.

I wish to acknowledge the help I have had from Harold Wiebke and Paul Kahn in preparing this discussion.

(AUTHORS' REVIEW OF DISCUSSION)

JOHN C. FRASER, WALTER N. MILLER, AND CHARLES M. STERNHELL:

The authors wish to express their sincere appreciation to the many persons who have submitted discussions of the paper. As we had hoped, the paper stimulated considerable discussion of both our proposed approach and possible alternative approaches.

To us, it is extremely significant that the central theme of the discussions is "What's the best way to design a sound, salable variable life insurance policy?" rather than "Is variable life insurance a good idea?" While we believe that our proposed design embodies a number of important plus factors, we recognize that many of the alternative designs presented are worthy of serious consideration.

We will begin these remarks by summarizing the actuarial aspects of the alternative designs, using a generalized approach. This generalized approach was suggested by Mr. Nagler's very interesting formulation of what might be termed a family of variable life insurance policies. We have extended this formulation to encompass designs other than those suggested by Mr. Nagler, including approaches where part of the benefits are funded on a fixed-dollar basis and part on a variable basis.

General Equation of Equilibrium

Let us begin with the general equation of equilibrium for the separate account. We will use traditional assumptions and define the following:

F_t = Total face amount payable at end of policy year t.
P_t = Total net premium payable at beginning of policy year t.
F_t^G = Portion of F_t payable from general account.
P_t^G = Portion of P_t payable to general account. It is assumed that the F_t^G's and P_t^G's are in actuarial balance in the general account.
${}_tJ$ = Separate account reserve at end of policy year t.
i_t' = Actual net annual investment return on the separate account during the tth policy year.

All other actuarial symbols have the usual meaning.

The general equation of equilibrium for the separate account in the tth policy year for a policy issued at age x is

$$({}_{t-1}J + P_t - P_t^G)(1 + i_t') = q_{x+t-1}(F_t - F_t^G) + p_{x+t-1}({}_tJ) . \quad (1)$$

From this general equation of equilibrium it is possible to derive various benefit designs for variable life insurance.

Designs Using Only Separate Account

Let us first consider the situation in which the general account is not used. In this case both F_t^G and P_t^G are zero, and the separate account equation of equilibrium becomes

$$({}_{t-1}J + P_t)(1 + i_t') = q_{x+t-1}F_t + p_{x+t-1}({}_tJ) . \quad (2)$$

This is the counterpart of Mr. Cody's basic equation, but traditional assumptions are used instead of integrals. In the special case where the premium P_t is fixed, this is also the "free form" design described by Dr. Nesbitt.

This equation of equilibrium will hold, irrespective of the investment experience, i_t', of the separate account, provided the quantities are determined in such a way that there is a balancing item. For example, in Mr. Cody's design that provides insurance for an indefinite period, the face amount is either level or follows a cost-of-living index, the premium is fixed, and the reserve ${}_tJ$ is the balancing item. In this case the policy may either terminate with no value when the reserve reaches zero or mature as an endowment when the reserve equals the face amount, depending on the actual investment experience of the separate account.

Let us now consider those designs in which the face amount is the balancing item and the reserve is defined as a function of the face amount. It is possible, of course, for the reserve to be any function of the face amount and for the function to (1) be predetermined at issue or (2) depend on the investment performance of the separate account as it emerges.

We will begin by discussing what we call the "single ratio" methods. If we introduce a term, ${}_tW$, which represents the reserve per \$1 of actual face amount F_t, so that ${}_tJ = F_t({}_tW)$, we get the "single ratio" methods, for which the equation of equilibrium is

$$[F_{t-1}({}_{t-1}W) + P_t](1 + i_t') = q_{x+t-1}F_t + p_{x+t-1}F_t({}_tW) . \quad (3)$$

The left-hand side of the equation is the initial reserve brought up to the end of the policy year at rate i_t'. This is the amount available for allocation at the end of the year between those dying during the year and those surviving to the end of the year.

The right-hand side of the equation shows how this amount is allocated. Each person dying during the year is paid F_t at the end of the year, and each person surviving to the end of the year is credited with a reserve of $F_t({}_tW)$. Thus, if each person dying is considered to have received a "full share," worth F_t, each person surviving receives a portion, ${}_tW$, of a full share.

If the survivor's portion, ${}_tW$, of a full share is less than zero, negative reserves will result, which means that the separate account has become insolvent, at least with respect to the specific policy. If the survivor's portion, ${}_tW$, of a full share exceeds 1, each of the survivors is credited with a larger share than each of those dying, which is not a desirable result.

Thus it seems appropriate to restrict the value of ${}_tW$ in these "single ratio" methods to the range 0 to 1. If ${}_tW = 0$, all terminal reserves are zero and the premiums paid by the entire group at the beginning of the year, brought up to the end of the policy year at rate i'_t, are distributed entirely to those dying during the year, with nothing to the survivors. If ${}_tW = 1$, both those dying during the year and the survivors receive or are credited with a full share, and we have simply a deposit fund without any life contingency element.

There are many possible approaches using values of ${}_tW$ that lie between these two extremes of 0 and 1. For example, using a whole life policy for illustrative purposes, if we let ${}_tW = {}_tV_x$ (where ${}_tV_x$ is the regular traditional net level terminal reserve) and $P_t = P_x$ (where P_x is the regular traditional net level annual premium), we get the New York Life design. If ${}_tW = {}_tV_x$ and $P_t = F_{t-1}P_x$, we get the fully variable Dutch-type design, where the premiums as well as the benefits vary to reflect the investment performance of the separate account.

Although many other approaches using values of ${}_tW$ that lie between 0 and 1 are theoretically possible, it should be noted that another important criterion from a practical point of view is maintaining actuarial balance between the net premiums P_t and the reserve factors ${}_tW$. In other words, one should use values of P_t and ${}_tW$ that are in actuarial balance on the basis of an assumed mortality table and an assumed interest rate i. This criterion is necessary in order to satisfy the condition that the actual face amounts will always remain equal to the initial face amount if the actual investment performance of the separate account i'_t is always equal to the assumed interest rate i.

The importance of this criterion can be illustrated by analyzing the situation where, for a whole life policy issued at age x, $P_t = P_x$ and

${}_tW = \frac{1}{2}\,({}_tV_x)$. It is true that the reserve will never become negative and that the survivors at the end of any policy year will never receive a larger share than persons dying during the year. It is apparent, however, that, if $i'_t = i$ for all policy years, then the death benefit will not be level but will start out higher than the initial face amount and decrease each year, ending up much lower than the initial face amount.

The New York Life design in the fixed premium case and the fully variable Dutch-type design in the variable premium case are the only "single ratio" designs where the face amount will always remain level if the actual investment performance of the separate account i'_t is always equal to the assumed interest rate i.

Let us now explore what we call "double ratio" methods. We will introduce two new terms, F^W_t and ${}_tR$, which satisfy the equation ${}_tJ = F^W_t({}_tW) + (F_t - F^W_t){}_tR$. In these "double ratio" methods the reserve factor ${}_tW$ is applied to only a portion F^W_t of the face amount and the factor ${}_tR$ is applied to the remainder $F_t - F^W_t$ of the face amount. The equation of equilibrium for these "double ratio" methods is

$$
\begin{aligned}
&[F^W_{t-1}({}_{t-1}W) + (F_{t-1} - F^W_{t-1})_{t-1}R + P_t](1 + i'_t) \\
&\qquad = q_{x+t-1}F_t + p_{x+t-1}[F^W_t({}_tW) + (F_t - F^W_t){}_tR]\,.
\end{aligned}
\tag{4}
$$

There are many interesting combinations possible with these "double ratio" methods, and we will illustrate only a few of the possible combinations below:

1. Whole life combined with paid-up life insurance (i.e., ${}_tW = {}_tV_x$ and ${}_tR = A_{x+t}$).
2. Whole life combined with a deposit fund (i.e., ${}_tW = {}_tV_x$ and ${}_tR = 1$).
3. Whole life combined with one-year term insurance (i.e., ${}_tW = {}_tV_x$ and ${}_tR = 0$).
4. Paid-up life insurance combined with one-year term insurance (i.e., ${}_tW = A_{x+t}$ and ${}_tR = 0$).

It should be noted that, for each of the possible combinations, there is also a wide choice of possible values of F^W_t. However, if two of the criteria used for "single ratio" methods (namely, that reserves should not be negative and that a person dying should not receive less than a person surviving) are to be satisfied, the following requirement must be met:

$$0 \le F^W_t({}_tW) + (F_t - F^W_t){}_tR \le F_t\,. \tag{5}$$

This may be rewritten as

$$0 \le F^W_t({}_tW - {}_tR) + F_t({}_tR) \le F_t\,. \tag{6}$$

Since F_t must be positive in any design meeting these requirements, we can divide equation (6) through by F_t without changing the direction of the inequalities:

$$0 \leq \frac{F_t^W}{F_t} ({}_tW - {}_tR) + {}_tR \leq 1 \,. \tag{7}$$

This requirement will always be met if ${}_tR = {}_tW$ and $0 \leq {}_tW \leq 1$, but this simply changes the "double ratio" design to a "single ratio" design meeting our criteria that the reserve factor lie between 0 and 1. The requirement can also be met if F_t^W is defined as a predetermined percentage, k_t, of F_t in the tth policy year, provided

$$0 \leq k_t({}_tW - {}_tR) + {}_tR \leq 1 \,. \tag{8}$$

However, any "double ratio" design involving a percentage breakdown of the face amount F_t is simply a "single ratio" design, since the reserve

$$k_tF_t({}_tW) + (1 - k_t)F_t({}_tR) = F_t[k_t({}_tW) + (1 - k_t)({}_tR)] \tag{9}$$

can be obtained by applying a single composite reserve factor $[k_t({}_tW) + (1 - k_t)({}_tR)]$ to the face amount F_t.

If the ratio of F_t^W to F_t in the tth policy year is not predetermined (e.g., if F_t^W is fixed at the beginning of the year before F_t is known), it appears that there will always be the possibility of not meeting the requirements in formula (7), unless we have a design where ${}_tR = {}_tW$ and $0 \leq {}_tW \leq 1$, which, as previously mentioned, is simply a "single ratio" design.

In theory, at least, F_t can decrease to zero or can increase enormously during the year as a result of the year's investment performance. There are four possible situations where F_t^W is not a predetermined percentage of F_t in the tth policy year and where ${}_tW$ and ${}_tR$ are not equal:

1. ${}_tW > {}_tR$ and F_t^W/F_t increases as F_t increases. In this case it may be impossible to find a design where F_t cannot become large enough to cause the upper bound in formula (7) to be broken.
2. ${}_tW < {}_tR$ and F_t^W/F_t increases as F_t increases. In this case it may be impossible to find a design where F_t cannot become large enough to cause the lower bound in formula (7) to be broken.
3. ${}_tW > {}_tR$ and F_t^W/F_t increases as F_t decreases. In this case it may be impossible to find a design where F_t cannot become small enough to cause the upper bound in formula (7) to be broken.
4. ${}_tW < {}_tR$ and F_t^W/F_t increases as F_t decreases. In this case it may be impossible to find a design where F_t cannot become small enough to cause the lower bound in formula (7) to be broken.

Consequently, it is entirely possible that there may be no "double ratio" methods using only the separate account (aside from those that reduce to "single ratio" methods) where there can be absolute assurance that reserves will never be negative and that a person dying will never receive less than a person surviving. The chances of this happening in practice may be extremely small, but it is a factor to be considered by a company evaluating a "double ratio" design that uses only the separate account.

One interesting type of "double ratio" design is what we will call the "excess insurance" type of design where $F_t^W = 1$, the initial face amount. These methods break the reserve into two pieces by applying the reserve factor ${}_tW$ to the initial face amount of \$1 and the other reserve factor ${}_tR$ to the difference, $F_t - 1$, between the actual face amount of F_t and the initial face amount. The equation of equilibrium for these "excess insurance" methods is

$$\begin{aligned}&[{}_{t-1}W + (F_{t-1} - 1){}_{t-1}R + P_t](1 + i_t') \\ &\qquad = q_{x+t-1}F_t + p_{x+t-1}[{}_tW + (F_t - 1){}_tR] \,. \end{aligned} \tag{10}$$

If we use a whole life policy for which $P_t = P_x$, ${}_tW = {}_tV_x$ and ${}_tR = {}_tR_x$, we get Mr. Nagler's family of policies. The general equation of equilibrium for his family of policies is

$$\begin{aligned}&[{}_{t-1}V_x + (F_{t-1} - 1){}_{t-1}R_x + P_x](1 + i_t') \\ &\qquad = q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1){}_tR_x] \,, \end{aligned} \tag{11}$$

which may be rewritten as

$$\begin{aligned}&({}_{t-1}V_x + P_x)(1 + i) + ({}_{t-1}V_x + P_x)(i_t' - i) \\ &\qquad + (F_{t-1} - 1)({}_{t-1}R_x)(1 + i_t') = [q_{x+t-1} + p_{x+t-1}({}_tV_x)] \\ &\qquad + q_{x+t-1}(F_t - 1) + p_{x+t-1}(F_t - 1){}_tR_x \,. \end{aligned} \tag{12}$$

Since the first terms on each side are equal, this becomes

$$\begin{aligned}&({}_{t-1}V_x + P_x)(i_t' - i) + (F_{t-1} - 1)({}_{t-1}R_x)(1 + i_t') \\ &\qquad = q_{x+t-1}(F_t - 1) + p_{x+t-1}(F_t - 1){}_tR_x \,. \end{aligned} \tag{13}$$

The left-hand side of this equation gives the extra earnings (positive or negative) available as a result of the separate-account returns being more or less than the assumed rate i. The first term on the left-hand side represents the extra earnings for the current year on the tabular initial reserve for the initial face amount. The second term represents the undistributed portion of the extra earnings for prior years brought up to the end of the current year at rate i_t'.

The right-hand side of the equation shows how these extra earnings are allocated. Each person dying during the year is paid $F_t - 1$ (positive or negative) at the end of the year in addition to the initial \$1 face amount, and each person surviving to the end of the year is credited with an extra reserve (positive or negative) of $(F_t - 1)_tR_x$ in addition to the regular reserve of ${}_tV_x$ on the initial \$1 face amount. Thus, if each person dying is considered to have received a full "extra share" worth $F_t - 1$, each person surviving receives a portion, ${}_tR_x$, of a full extra share.

Mr. Nagler limits ${}_tR_x$ to the range from 0 to 1. When ${}_tR_x = 0$, we get an unsatisfactory design, mentioned by Mr. Cody, that gives the entire excess (positive or negative) of the actual investment performance over the assumed interest rate i to those dying during the year and keeps none for the survivors. As Mr. Nagler shows, this produces wildly fluctuating death benefits, including negative amounts.

When ${}_tR_x = 1$, we get what Mr. Nagler refers to as the analogy to the interest-only deposit option, under which the cumulative excess investment performance is shared equally by everybody, those dying during the year as well as those surviving at the end of the year. This specific design was also described by Mr. Booth as a "defined difference" method.

In between Mr. Nagler's two limits of 0 and 1 for ${}_tR_x$, we obtain the New York Life design when ${}_tR_x = {}_tV_x$, under which the cumulative excess investment performance is distributed on the basis of a full extra share to those dying during the year and a portion, ${}_tV_x$, of a full extra share to those surviving at the end of the year. A variation on this is Mr. Walker's design under which ${}_tR_x = A_{x+t}$. This is referred to by Mr. Nagler as the analogy to the level paid-up addition option, under which the cumulative excess investment performance is distributed on the basis of a full extra share to those dying and a portion, A_{x+t}, of a full extra share to those surviving.

Mr. Nagler indicated that it would be desirable to limit ${}_tR_x$ to the range from 0 to 1 on the theory that if ${}_tR_x$ were negative, the deads would receive more than 100 per cent of the cumulative extra investment performance, and, if ${}_tR_x$ were to exceed 1, the survivors would receive a larger share of the cumulative extra investment performance than the deads. We feel that this is not a sufficiently restrictive limitation on the range of ${}_tR_x$, since it is possible for the total reserve to be negative, which means that the separate account can become insolvent, or for the total amount paid to each of the deads to be less than the reserve held for each of the survivors.

The total amount paid to each dead under Mr. Nagler's family of policies is, of course, F_t, and the total terminal reserve held for each survivor is ${}_tV_x + (F_t - 1){}_tR_x$. Thus we believe that it is desirable to impose additional restrictions on ${}_tR_x$ so that

$$0 \leq {}_tV_x + (F_t - 1){}_tR_x \leq F_t \,. \tag{14}$$

We will now show that the only predetermined values of ${}_tR_x$ (i.e., the only values independent of the actual investment results to date) that necessarily meet these additional restrictions are ${}_tV_x$. Expression (14) can be rewritten as

$$0 \leq ({}_tV_x - {}_tR_x) + F_t({}_tR_x) \leq F_t \,. \tag{15}$$

It is clear that the condition always holds if ${}_tR_x = {}_tV_x$, since this reduces to the requirement that $0 \leq {}_tV_x \leq 1$, which is true. If ${}_tR_x$ exceeds ${}_tV_x$, it is clear that, even though F_t is a function of ${}_tR_x$, unfavorable enough investment performance can cause the value of F_t to be so small that $({}_tV_x - {}_tR_x) + F_t({}_tR_x)$ would be less than zero. If ${}_tR_x$ is less than ${}_tV_x$, but not negative, it is clear that a small enough value of F_t can cause F_t to be less than $({}_tV_x - {}_tR_x) + F_t({}_tR_x)$. If ${}_tR_x$ is negative, there can still be a value of F_t which is large enough so as to cause $({}_tV_x - {}_tR_x) + F_t({}_tR_x)$ to be less than zero.

Thus we have shown that the only predetermined values of ${}_tR_x$ in Mr. Nagler's family of policies that will *necessarily* avoid negative separate-account reserves and at the same time avoid paying more to those surviving than to those dying are ${}_tV_x$, which are the values for the New York Life policy design. This result is consistent with our previous discussion, where we expressed doubt that any "double ratio" method using only the separate account, which could not be reduced to a "single ratio" method, would necessarily meet these conditions. The New York Life design meets these conditions because it also belongs to the acceptable family of "single ratio" methods and represents the intersection of Mr. Nagler's family of "excess insurance" methods with the family of "single ratio" methods.

This does not mean that Mr. Nagler's "excess insurance" methods are necessarily unsound, although some (such as for ${}_tR_x = 0$) undoubtedly are. It simply means that, if ${}_tR_x$ is to be defined in advance with values other than ${}_tV_x$, there is always the possibility either of (1) the face amount being less than the reserve or (2) negative face amounts and/or negative reserves. As indicated by Mr. Walker, there may be little chance of possibility 2 in practice under some of these designs, but the possibility does exist and should be considered by a company evaluating such a design.

This concludes our discussion of "double ratio" methods, and, although "triple ratio" and higher ratio methods are theoretically possible, we will not attempt to explore them herein.

Designs Using Both the General Account and the Separate Account

Our discussion of designs using both the general account and the separate account will be limited to a discussion of the "double ratio" methods; there can be no "single ratio" methods when both accounts are being used. For the "double ratio" methods using both accounts, we set ${}_tJ = (F_t - F_t^G)_tW$ in the general equation of equilibrium, equation (1). Note that, although this is a "double ratio" method, only one ratio ${}_tW$ appears in the definition of the separate-account reserve. The other ratio, with which we are not explicitly concerned (since the general account is assumed to be in actuarial balance), is applied to F_t^G in order to obtain the general-account reserve. For the "double ratio" methods using both accounts, the general equation of equilibrium for the separate account, equation (1), becomes, with this substitution,

$$[(F_{t-1} - F_{t-1}^G)_{t-1}W + P_t - P_t^G](1 + i_t') \\ = q_{x+t-1}(F_t - F_t^G) + p_{x+t-1}(F_t - F_t^G)_tW\,. \tag{16}$$

As is true in the case of the "double ratio" methods using only the separate account, many interesting combinations are possible. Combined with various types of general-account coverage, the separate account can be used for whole life (i.e., ${}_tW = {}_tV_x$), paid-up life (i.e., ${}_tW = A_{x+t}$), one-year term (i.e., ${}_tW = 0$), a deposit fund (i.e., ${}_tW = 1$), and so forth, all with different portions of premiums and benefits paid to and paid from each account.

Let us now consider the "double ratio" methods that use the general account only for one-year term insurance, that is, the case where $P_t^G = vq_{x+t-1}F_t^G$. Under these conditions the equation of equilibrium for the separate account, equation (16), becomes

$$[(F_{t-1} - F_{t-1}^G)_{t-1}W + P_t - vq_{x+t-1}F_t^G](1 + i_t') \\ = q_{x+t-1}(F_t - F_t^G) + p_{x+t-1}(F_t - F_t^G)_tW\,. \tag{17}$$

If we use the separate account only as a deposit fund (i.e., ${}_tW = 1$) and purchase one-year term insurance in the general account for the amount at risk on a corresponding fixed-dollar policy (i.e., $F_t^G = 1 - {}_tV$), we get Mr. Cooper's design. This design was also mentioned by Mr. Booth. Using whole life as an illustration and substituting $F_t^G = 1 - {}_tV_x$,

$P_t = P_x$, and ${}_tW = 1$ in equation (17), we get the separate-account equation of equilibrium for Mr. Cooper's benefit design:

$$[F_{t-1} - (1 - {}_{t-1}V_x) + P_x - vq_{x+t-1}(1 - {}_tV_x)](1 + i'_t)$$
$$= q_{x+t-1}[F_t - (1 - {}_tV_x)] + p_{x+t-1}[F_t - (1 - {}_tV_x)] \quad (18)$$
$$= F_t - (1 - {}_tV_x) \,.$$

This clearly indicates the fact that the separate-account activity under this benefit design involves only an investment accumulation at interest rate i'_t and does not include any life contingency element. Note that Mr. Cooper's design can lead to negative reserves, since the premium $vq_{x+t-1}(1 - {}_tV_x)$ paid for the one-year term insurance in the general account will eventually exceed the total premium P_x, thus requiring funds to be withdrawn from the separate-account deposit fund, which, because of poor investment performance, may not be able to stand the strain.

We will next demonstrate the relationship between (1) Mr. Cooper's benefit design, under which $vq_{x+t-1}(1 - {}_tV_x)$ is used to purchase one-year term insurance for an amount $1 - {}_tV_x$ in the general account and the balance of P_x is accumulated in a deposit fund in the separate account, and (2) the benefit design referred to earlier as Mr. Booth's "defined difference" method and also as Mr. Nagler's analogy to the interest-only deposit option, under which variable one-year term insurance for an initial amount of $1 - {}_tV_x$ is purchased in the separate account and the balance of P_x is accumulated in a deposit fund in the separate account.

Equation (18) can be rewritten as follows:

$$[{}_{t-1}V_x + (F_{t-1} - 1) + P_x](1 + i'_t) - vq_{x+t-1}(1 - {}_tV_x)(1 + i'_t)$$
$$= q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1)] - q_{x+t-1}(1 - {}_tV_x) \,. \quad (19)$$

If we take the one-year term premium $vq_{x+t-1}(1 - {}_tV_x)$, which was placed in the general account under Mr. Cooper's method, with interest at the assumed interest rate i, and add the result $vq_{x+t-1}(1 - {}_tV_x)(1 + i)$ to both sides of equation (19), we get

$$[{}_{t-1}V_x + (F_{t-1} - 1) + P_x](1 + i'_t) - vq_{x+t-1}(1 - {}_tV_x)(1 + i'_t)$$
$$+ vq_{x+t-1}(1 - {}_tV_x)(1 + i) = q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1)] \quad (20)$$
$$- q_{x+t-1}(1 - {}_tV_x) + vq_{x+t-1}(1 - {}_tV_x)(1 + i) \,.$$

Equation (20) can be rewritten as follows:

$$[{}_{t-1}V_x + (F_{t-1} - 1) + P_x](1 + i'_t) - vq_{x+t-1}(1 - {}_tV_x)(i'_t - i)$$
$$= q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1)] \,. \quad (21)$$

This can be seen to be very similar to Mr. Booth's "defined difference" design by setting ${}_tR_x = 1$ in our equation (11), which is Mr. Nagler's general equation of equilibrium for "excess insurance" whole life policies using only the separate account. The equation of equilibrium for Mr. Booth's design and for Mr. Nagler's analogy to the interest-only dividend deposit option is

$$[{}_{t-1}V_x + (F_{t-1} - 1) + P_x](1 + i_t') = q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1)] \,. \tag{22}$$

Note that equation (22) and Mr. Cooper's equation (21) are identical except for the additional term $[-vq_{x+t-1}(1 - {}_tV_x)(i_t' - i)]$ on the left-hand side of Mr. Cooper's equation (21). This term is usually fairly small and reflects the fact that one-year term benefits were purchased in the general account rather than in the separate account. In general, the use of the separate account rather than the general account for one-year term insurance will have little effect on the total face amounts, although it does, of course, shift the mortality risk for the one-year term insurance from the general account to the separate account.

Let us turn now to Mr. Fairbanks' very interesting whole life design, which uses the general account for one-year term insurance of $F_t^G = 1 - ({}_tV_x/A_{x+t}) = P_x/P_{x+t}$ (i.e., the excess of a level fixed-dollar whole life face amount of \$1 over the amount of reduced paid-up insurance that can be purchased by the terminal reserve of ${}_tV_x$ at the end of the tth policy year) and uses the separate account for variable paid-up insurance (i.e., ${}_tW = A_{x+t}$). The separate-account equation of equilibrium for Mr. Fairbanks' benefit design can be obtained from our equation (17) by substituting $F_t^G = P_x/P_{x+t}$, $P_t = P_x$ and ${}_tW = A_{x+t}$:

$$\left[\left(F_{t-1} - \frac{P_x}{P_{x+t-1}}\right) A_{x+t-1} + P_x - vq_{x+t-1}\frac{P_x}{P_{x+t}}\right](1 + i_t') = q_{x+t-1}\left(F_t - \frac{P_x}{P_{x+t}}\right) + p_{x+t-1}\left(F_t - \frac{P_x}{P_{x+t}}\right) A_{x+t} \,. \tag{23}$$

At the beginning of each year Mr. Fairbanks purchases additional variable paid-up insurance in the separate account with an initial face amount at the time of purchase, using the tth policy year as an illustration, of $P_x[(1/P_{x+t-1}) - (1/P_{x+t})]$. He also purchases one-year term insurance in the general account in the amount of P_x/P_{x+t}. The total premium for these two benefits combined is

$$P_x\left(\frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}}\right) A_{x+t-1} + \frac{P_x}{P_{x+t}}\, vq_{x+t-1} \,. \tag{24}$$

This can be rewritten as

$$P_x\ddot{a}_{x+t-1} - \frac{P_x}{P_{x+t}}(A_{x+t-1} - vq_{x+t-1})\,, \tag{25}$$

which becomes

$$P_x\ddot{a}_{x+t-1} - \frac{P_x}{P_{x+t}}(vp_{x+t-1}A_{x+t}) = P_x(\ddot{a}_{x+t-1} - vp_{x+t-1}\ddot{a}_{x+t}) = P_x\,. \tag{26}$$

We have shown that

$$P_x = P_x\left(\frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}}\right)A_{x+t-1} + \frac{P_x}{P_{x+t}}vq_{x+t-1}\,, \tag{27}$$

which indicates that when each premium P_x is paid under Mr. Fairbanks' design, it is exactly sufficient to purchase the new benefits provided in the general account and the separate account without any need to draw on the separate-account reserve. Substituting equation (27) into equation (23), we obtain

$$\begin{aligned}\Bigg[\left(F_{t-1} - \frac{P_x}{P_{x+t-1}}\right)A_{x+t-1} &+ P_x\left(\frac{1}{P_{x+t-1}} - \frac{1}{P_{x+t}}\right)A_{x+t-1}\\ &+ \frac{P_x}{P_{x+t}}vq_{x+t-1} - vq_{x+t-1}\frac{P_x}{P_{x+t}}\Bigg](1+i'_t)\\ &= q_{x+t-1}\left(F_t - \frac{P_x}{P_{x+t}}\right) + p_{x+t-1}\left(F_t - \frac{P_x}{P_{x+t}}\right)A_{x+t}\,,\end{aligned} \tag{28}$$

which becomes

$$\begin{aligned}\left[\left(F_{t-1} - \frac{P_x}{P_{x+t}}\right)A_{x+t-1}\right](1+i'_t)&\\ &= [q_{x+t-1} + p_{x+t-1}A_{x+t}]\left(F_t - \frac{P_x}{P_{x+t}}\right)\\ &= (1+i)A_{x+t-1}\left(F_t - \frac{P_x}{P_{x+t}}\right).\end{aligned} \tag{29}$$

Thus,

$$F_t = \frac{P_x}{P_{x+t}} + \left(F_{t-1} - \frac{P_x}{P_{x+t}}\right)\left(\frac{1+i'_t}{1+i}\right), \tag{30}$$

which says that the total face amount, F_t, at the end of the tth year is the sum of (1) the fixed-dollar one-year term insurance of P_x/P_{x+t} and (2) the variable paid-up life insurance at the end of the tth year, which is equal to the variable paid-up life insurance at the beginning of the year,

$[F_{t-1} - (P_x/P_{x+t})]$, multiplied by $(1 + i'_t)/(1 + i)$, that is, by Z_t, as defined in the paper.

Since Mr. Fairbanks' design exactly uses up each premium as it is paid and since the amount of variable paid-up insurance in the separate account at the beginning of any policy year changes only according to the Z_t factors, there is no possibility of the separate-account reserve being wiped out. In addition, it is evident that Mr. Fairbanks' design meets all our other criteria for a sound design, because (1) there is no possibility of a person dying receiving less than a person surviving and (2) the face amount will remain level if the actual investment performance of the separate account is always equal to the assumed interest rate i.

We will now show that Mr. Walker's design, referred to by Mr. Nagler as the analogy to the paid-up addition dividend option, differs from Mr. Fairbanks' design only to the extent that the separate account rather than the general account is used for the one-year term insurance of P_x/P_{x+t}.

The equation of equilibrium for Mr. Walker's design is found by setting ${}_tR_x = A_{x+t}$ in our equation (11), which is Mr. Nagler's general equation of equilibrium for "excess insurance" whole life policies using only the separate account. We obtain

$$\begin{aligned}&[{}_{t-1}V_x + (F_{t-1} - 1)A_{x+t-1} + P_x](1 + i'_t) \\ &\qquad = q_{x+t-1}F_t + p_{x+t-1}[{}_tV_x + (F_t - 1)A_{x+t}] \,.\end{aligned} \tag{31}$$

Substituting ${}_{t-1}V_x = A_{x+t-1} - P_x\ddot{a}_{x+t-1}$ and ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$, we obtain

$$\begin{aligned}&(A_{x+t-1} - P_x\ddot{a}_{x+t-1} + F_{t-1}A_{x+t-1} - A_{x+t-1} + P_x)(1 + i'_t) \\ &\qquad = q_{x+t-1}F_t + p_{x+t-1}(A_{x+t} - P_x\ddot{a}_{x+t} + F_tA_{x+t} - A_{x+t}) \,.\end{aligned} \tag{32}$$

Canceling and substituting $\ddot{a}_{x+t-1} = A_{x+t-1}/P_{x+t-1}$ and $\ddot{a}_{x+t} = A_{x+t}/P_{x+t}$, we obtain

$$\begin{aligned}&\left[\left(F_{t-1} - \frac{P_x}{P_{x+t-1}}\right) A_{x+t-1} + P_x\right](1 + i'_t) \\ &\qquad = q_{x+t-1}F_t + p_{x+t-1}\left[\left(F_t - \frac{P_x}{P_{x+t}}\right) A_{x+t}\right] \,.\end{aligned} \tag{33}$$

The similarity of Mr. Walker's equation (33) to Mr. Fairbanks' equation (23) can be seen if we take the one-year term premium $vq_{x+t-1}(P_x/P_{x+t})$ that is placed in the general account under Mr. Fairbanks' method, with interest at the assumed interest rate i, and add the result vq_{x+t-1} $(P_x/P_{x+t})(1+i)$ to both sides of equation (23). This gives

$$\left[\left(F_{t-1}-\frac{P_x}{P_{x+t-1}}\right)A_{x+t-1}+P_x\right](1+i'_t)-vq_{x+t-1}\frac{P_x}{P_{x+t}}(1+i'_t)$$

$$+\,vq_{x+t-1}\frac{P_x}{P_{x+t}}(1+i)=q_{x+t-1}F_t+p_{x+t-1}\left(F_t-\frac{P_x}{P_{x+t}}\right)A_{x+t} \quad (34)$$

$$-\,q_{x+t-1}\frac{P_x}{P_{x+t}}+vq_{x+t-1}\frac{P_x}{P_{x+t}}(1+i)\,,$$

which becomes

$$\left[\left(F_{t-1}-\frac{P_x}{P_{x+t-1}}\right)A_{x+t-1}+P_x\right](1+i'_t)-vq_{x+t-1}\frac{P_x}{P_{x+t}}(i'_t-i)$$
$$=q_{x+t-1}F_t+p_{x+t-1}\left[F_t-\frac{P_x}{P_{x+t}}\right]A_{x+t}\,. \quad (35)$$

Note that Mr. Walker's equation (33) and Mr. Fairbanks' equation (35) are identical except for the additional term $[-vq_{x+t-1}(P_x/P_{x+t})$ $(i'_t-i)]$ on the left-hand side of Mr. Fairbanks' equation. Mr. Walker's equation does not have this term since he is using the separate account rather than the general account to purchase the one-year term insurance of P_x/P_{x+t}. In any event, the differences between the face amounts produced by Mr. Walker's design and Mr. Fairbanks' design are not very large.

It is interesting, however, to note that Mr. Walker's design can produce negative reserves in extreme circumstances whereas Mr. Fairbanks' cannot. To understand why this is so, we will compare the equation for obtaining face amounts under Mr. Walker's design with the corresponding equation under Mr. Fairbanks' design.

From Mr. Walker's equation (33), it is possible to derive the following equation for obtaining his face amounts:

$$F_t=\frac{P_x}{P_{x+t}}+\left(F_{t-1}-\frac{P_x}{P_{x+t}}\right)\left(\frac{1+i'_t}{1+i}\right)+\frac{vq_{x+t-1}(P_x/P_{x+t})(i'_t-i)}{q_{x+t-1}+p_{x+t-1}A_{x+t}}\,. \quad (36)$$

This equation is the same as Mr. Fairbanks' equation (30) except for the additional term

$$\left[\frac{vq_{x+t-1}(P_x/P_{x+t})(i'_t-i)}{q_{x+t-1}+p_{x+t-1}A_{x+t}}\right]$$

The numerator of this additional term represents the extra interest earned by Mr. Walker on the one-year term premium of $vq_{x+t-1}(P_x/$

P_{x+t}) because he is investing it in the separate account at rate i'_t instead of in the general account at the assumed rate i, as in Mr. Fairbanks' design. The denominator of this additional term $(q_{x+t-1} + p_{x+t-1}A_{x+t})$ shows how Mr. Walker is distributing the extra interest, a full share to those dying and a portion, A_{x+t}, of a full share to those surviving. However, Mr. Walker's extra interest becomes a charge when the investment performance of the separate account is less than i, and, with poor enough investment performance, the separate-account reserve can be wiped out.

We turn now to some "double ratio" methods that use both the general account and the separate account but use the general account for more than just one-year term insurance.

On page 90 of Jordan's life contingency textbook, it is shown that the n-year endowment insurance premium, $P_{x:\overline{n}|}$, can be separated into (1) its pure investment element, $(1/\ddot{s}_{\overline{n}|})$, and (2) its pure insurance element, $P_{x:\overline{n}|} - (1/\ddot{s}_{\overline{n}|})$, which provides decreasing term insurance of $1 - (\ddot{s}_{\overline{t}|}/\ddot{s}_{\overline{n}|})$ in each year t from 1 to $n - 1$ to make up the balance of the \$1 face amount. This suggests the possibility of using the separate account as a deposit fund for the pure investment element of the premium and using the general account to provide the decreasing term insurance of $1 - (\ddot{s}_{\overline{t}|}/\ddot{s}_{\overline{n}|})$. If we let $F_t^G = 1 - (\ddot{s}_{\overline{t}|}/\ddot{s}_{\overline{n}|})$, ${}_tW = 1$, $P_t = P_{x:\overline{n}|}$, and $P_t^G = P_{x:\overline{n}|} - (1/\ddot{s}_{\overline{n}|})$, equation (16) becomes

$$\left(F_{t-1} - 1 + \frac{\ddot{s}_{\overline{t-1}|}}{\ddot{s}_{\overline{n}|}} + \frac{1}{\ddot{s}_{\overline{n}|}}\right)(1 + i'_t) = q_{x+t-1}\left(F_t - 1 + \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}\right) + p_{x+t-1}\left(F_t - 1 + \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}\right), \tag{37}$$

which becomes

$$(F_{t-1} - 1)(1 + i'_t) + Z_t \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}} = F_t - 1 + \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}, \tag{38}$$

so that

$$F_t = F_{t-1} + (F_{t-1} - 1)i'_t + \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}(Z_t - 1) . \tag{39}$$

Although this design can never have negative reserves in the separate account, it can have negative reserves in the general account.

Mr. Baughman's interesting design uses the separate account for variable paid-up insurance and the general account for decreasing term insurance. He starts with a regular twenty-pay life policy, which we will generalize to an n-pay life policy. Then at the beginning of each year,

$1/n$th of the original amount of fixed insurance is converted to paid-up variable life insurance. If for $t \leq n$, we let $F_t^g = (n - t)/n$, ${}_tW = A_{x+t}$, and $(P_t - P_t^g) = A_{x+t-1}/n$, equation (16) becomes

$$\left[\left(F_{t-1} - 1 + \frac{t-1}{n}\right) A_{x+t-1} + \frac{A_{x+t-1}}{n}\right](1 + i_t') = q_{x+t-1}\left(F_t - 1 + \frac{t}{n}\right) + p_{x+t-1}\left(F_t - 1 + \frac{t}{n}\right) A_{x+t}\,, \tag{40}$$

which becomes

$$\left(F_{t-1} - 1 + \frac{t}{n}\right) A_{x+t-1}(1 + i_t') = \left(F_t - 1 + \frac{t}{n}\right)(1 + i) A_{x+t-1}\,, \tag{41}$$

so that

$$F_t = \left[F_{t-1} - \left(1 - \frac{t}{n}\right)\right] Z_t + \left(1 - \frac{t}{n}\right), \tag{42}$$

which is Mr. Baughman's design.

This design cannot produce negative separate-account reserves but can produce negative general-account reserves, because it can involve actual transfers of general account assets to the separate account. This is apparent from the fact that the amount deposited in the separate account, $(P_t - P_t^g)$, is equal to A_{x+t-1}/n. Under an n-pay life policy, P_t would be equal to ${}_nP_x$, and it is clear that there are many situations in the later policy years where A_{x+t-1}/n $(t < n)$ is larger than ${}_nP_x$. Any such situation would require the withdrawal of $[(A_{x+t-1}/n) - {}_nP_x]$ from the general account in order to make the required deposit of A_{x+t-1}/n into the separate account.

This concludes our analysis of various benefit designs for variable life insurance that are derivable from the general equation of equilibrium.

Questions Raised concerning New York Life Design

Let us now turn to some of the other points raised in the discussions. Several discussions were critical of one aspect or another of the design proposed in the paper. One point which was made by Messrs. Fairbanks and Walker is that, when $i_t' = i$ in a particular policy year, the face amount under our design will not remain the same as that at the end of the prior policy year but will change due to the effect of the Y_t factor (except, of course, in the special case when the face amount at the end of the prior policy year is equal to the initial face amount). Furthermore, they pointed out that under our design it is possible to have the face amount decrease from one year to the next even if i_t' for the year in question is greater than i.

The alternative designs suggested by Messrs. Fairbanks and Walker (which, as we have previously pointed out, are quite similar) do not have these characteristics. In examining alternative designs, however, it is important to consider not only changes in face amounts from one year to the next but also the level and general pattern of face amounts. For example, let us suggest that Mr. Fairbanks might have difficulty explaining to his Mr. A why the face amount of his Fairbanks' design policy increased only from \$100,000 to \$101,500 during the first policy year when the actual investment return in the separate account was 36 per cent.

As for the general pattern of face amounts, Messrs. Fairbanks and Walker each pointed out that, given the same favorable investment performance, face amounts under their designs would be lower than those under our design for a considerable number of years and then become higher.

Dr. Kahn has characterized the difference between results under these designs by stating that "the New York Life method produces face amounts of insurance more closely tied to investment results than the alternative method which roughly holds back funds in good years to support benefits in bad years." Actually, we believe it is better to examine this difference in light of the fact that, as we have previously indicated, our design distributes relatively more of each year's "excess" investment performance to the deads, and relatively less to the survivors, than the designs of Messrs. Fairbanks and Walker. This, we believe, is the basic reason why our design produces face amounts which are more responsive to current investment performance in the early policy years and (assuming favorable investment performance) are higher for a number of years than those under the Fairbanks and Walker designs.

To sum up this point, while we recognize that Mr. Fairbanks' and Mr. Walker's designs have some favorable features, we wonder whether it is advantageous to have the face amounts as unresponsive to current investment performance in the early policy years as they are under these designs. We do, however, recognize the fact that our design may involve some problems in explaining to policyowners how the death benefit is calculated and how it varies from year to year.

Another aspect of our proposed design was questioned by Messrs. Murphy and Fairbanks. These questions stemmed from the fact that, for simplicity, almost all the derivations and formulas presented in the paper were based on traditional functions, that is, assuming annual premiums payable at the beginning of the policy year and death benefits payable at the end of the policy year of death. Under these assumptions, the only relevant death benefits are those payable as of anniversaries, that is, values of F_t for integral values of t.

In practice, of course, companies will calculate and pay death benefits as of the moment of death. This led Messrs. Fairbanks and Murphy to examine the progress of F_t throughout a policy year and to comment on a "sawtoothed" effect occurring at each point where an annual premium is paid. Thus, if the actual face amount at the end of a policy year is greater than the initial face amount, it can be said that the actual face amount at the beginning of the next policy year will decrease because the Y_t factor is less than 1 under such circumstances. Similarly, if the actual face amount at the end of a policy year is less than the initial face amount, it can be said that the actual face amount at the beginning of the next policy year will increase because the Y_t factor is greater than 1.

While this result is to be expected theoretically, it is true that it may be difficult to explain to the policyowner. There are several ways to eliminate or mitigate this effect:

1. The premiums could be credited to the separate account more frequently than annually, that is, on a monthly or even daily basis. The Y_t factor would then change accordingly on a monthly or daily basis. Where premiums were payable to the company less frequently than they were credited to the separate account (e.g., annual premium payments but with premiums credited monthly to the separate account), the general account could be used as a "holding account" for portions of premiums not yet credited to the separate account.
2. As noted in section VI of the paper, one variation in policy design is to permit payment of a net premium of $F_{t-1}P_x$ so that the face amount at the beginning of one policy year is the same as the face amount at the end of the prior year.
3. A continuous functions approach could be used. Mr. Murphy's discussion concentrates on this approach, and he raised several questions regarding our formulas involving continuous functions.

In considering these questions, let us first review the implicit assumptions underlying the "fully continuous" basis commonly used for fixed benefit policies, where the discounted annual premium for a whole life policy is $(d/\delta)\bar{P}(\bar{A}_x)$. This basis assumes that premiums are payable annually in advance, that death benefits are payable at the moment of death, and that a refund of premium is payable at death equal to $\ddot{a}_{\overline{1-f}|}$ times the annual premium, where f is the portion of a year elapsed from the preceding policy anniversary to the moment of death. Under this type of refund benefit, the annual premium of $(d/\delta)\bar{P}(\bar{A}_x)$ provides exactly (under any assumption for the distribution of deaths during the policy year) for immediate payment of claims and partial refund of itself in the year of death on the basis described.

In deriving the "fully continuous" formula for fixed premium variable benefit policies shown in the paper, we have assumed that the entire annual premium $(d/\delta)\bar{P}(\bar{A}_x)$ is placed in the separate account each year. Under this assumption, the entire reserve, including the reserve for the refund of premium feature, is invested in the separate account. At first glance, it might seem that the portion of the reserve which provides for the premium refund benefit is zero on each policy anniversary. Actually, however, there is a reserve build-up for the premium refund benefit, since the "premium" for this benefit (i.e., the portion of the total premium $(d/\delta)\bar{P}(\bar{A}_x)$ which is required to fund the premium refund benefit) is level while the risk involved is always increasing as long as q_{x+t} increases.

Thus, under our formulation the "unearned premium reserve" referred to by Mr. Murphy is not a separate fund unrelated to the varying face amount or reserve. In fact, the premium refund benefit is subject to the same Y_t and Z_t factors as the basic face amount. Thus, the premium refund benefit is equal to the corresponding benefit for a fixed premium fixed benefit policy times the face amount of the fixed premium variable benefit policy payable at the moment of death as given by equation (35) in the paper.

Because this type of result may be difficult to explain in practice, various alternatives may be considered. One possibility is to carry in the separate account only the reserve for the basic face amount with immediate payment of claims (i.e., the "semi-continuous" reserve) but not the reserve for refund of premium. The basic death benefit would then vary according to the Y_t and Z_t factors shown by equations (30) and (31) in the paper, but the refund of premium benefit would be the same as that under a fixed premium fixed benefit policy, since the reserve for it would be carried in the general account. Another alternative, suggested by Mr. Murphy, would be to credit the separate account with premiums on a continuous basis (which, in practice, would probably be on a daily basis) and to determine the proper Y_t factors varying throughout the policy year.

Other Comments regarding Discussions

Mr. Bragg makes the interesting suggestion that, in order to minimize the possibility of reductions in face amount due to unfavorable investment performance, companies consider a design under which the face amount increases if actual investment performance is at the assumed rate. Similar reasoning might also suggest consideration of a design involving level "basic" benefits but a very low assumed interest rate. Consideration of approaches like these, however, must take into account the fact that an additional element of cost to the policyowner is involved. This can be illustrated by the traditional net level annual premiums for whole life insurance, computed using the 1958 C.S.O. Table for a male aged 55 at issue (see accompanying tabulation).

Assumed Interest Rate (Per Cent)	Benefit	Net Annual Premium
$2\frac{1}{2}$%	\$1,000 level	\$40.57
0	\$1,000 level	49.47
$2\frac{1}{2}$	\$1,000 first year, increasing \$25 each subsequent year	57.73
$2\frac{1}{2}$	\$1,000 $(1.025)^{n-1}$ in policy year n	63.38

Mr. Cody's discussion is a valuable contribution to the actuarial theory underlying variable life insurance. His formulations are based on a general differential equation of equilibrium. In similar fashion, these remarks have made use of general equations of equilibrium covering a one-year period under the "traditional functions" approach.

In substance, then, our development, as indicated in these remarks, has followed a path parallel to Mr. Cody's. We believe it is apparent that the technique of establishing a general equation of equilibrium which will cover all the possible variable life insurance designs is a very powerful one.

We agree with Mr. Cody that, of the specific designs he enumerated, the two simplest are his design involving a level benefit for an indefinite period and the Dutch-type design where both premiums and benefits vary. While there is no definitive answer as to what will be the "best" variable life insurance design, we believe that these discussions indicate that simplicity is only one of a number of criteria which must be considered.

Mr. Cooper commented on problems involved in connection with policy loans under a variable life insurance policy either (*a*) on a variable basis with the amount of the outstanding loan reflecting actual net investment performance of the separate account or (*b*) on a fixed interest basis through the separate account, thus causing separate-account investment performance to reflect the existence of such loans.

We did not intend to imply that the fixed interest basis could be handled only through the separate account; in other words, policy loans could also be made on a fixed interest basis through the general account. Under this alternative, it would be essential for the interest rate to be comparable to the "new money" level of interest rates being earned on new general-account investments at the time the loan was made, so that the company and owners of fixed benefit contracts would not be adversely affected by an influx of policy loans on variable benefit policies.

Because of the problems involved in connection with policy loans on variable insurance policies and the fact that there are several possible alternative bases for such loans, we believe that insurance laws should be revised to provide flexibility with respect to (*a*) whether a variable life insurance policy should or should not include a policy loan provision and (*b*) the type of provision which might be included. In this connection, let us note that the model legislation which was largely developed by an industry committee, and endorsed by the N.A.I.C. at its December, 1969, meeting, does not contain any requirement that a variable life insurance policy contain a loan provision.

Mr. Deal has some interesting comments as to the possibility of antiselection arising on variable life insurance because of the variable nature of the benefits provided. We agree with his conclusion that this should not be a serious problem.

Mr. DiPaolo correctly points out that an important question to be considered in connection with variable life insurance concerns the company's mortality risk and the handling of mortality profits as they emerge. This, of course, is also a key question in traditional fixed-dollar insurance. We agree with his conclusion that techniques to handle this question can be developed in connection with variable life insurance, and some work that we have done indicates that such techniques will be logical extensions of those used today.

Mr. DiPaolo's suggestion of an "investment stabilization fund" is an interesting one which, like many of the alternative designs presented in these discussions, deserves serious consideration. While his formulation of how such a fund might operate is linked to the design presented in the paper, his concept can also be considered in connection with many of the alternative designs which were proposed.

Mr. Edwards presented a method under which a given year's "excess" investment performance could be withheld from both deads and survivors for one or two years beyond the end of the year when such excess performance arose. While this method may be feasible, it would have the effect of making face amounts less responsive to current investment performance than would otherwise be the case. It would also appear that under this method excess investment performance in the last one or two policy years of a particular contract would never be credited to the policyowner. This point might particularly be considered in connection with any possible application of Mr. Edwards' method to endowment or term plans.

We are sure that the authors of the other papers relating to equity products which were presented at this meeting will join us in thanking Mr. Gustafson for his kind words regarding our paper and theirs. Mr. Gustafson makes some interesting observations concerning the future of life insurance equity products, and we certainly endorse his general theme that the advent of such products represents a development for the insurance industry which is logical and beneficial rather than dangerous. Like Mr. Gustafson, we expect that this subject will be an important one in actuarial and industry circles for many years to come.

Mr. Harding claims that "it was stated that the paper was written 'in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of' the type of policy envisioned in the paper." We are sorry that Mr. Harding did not choose to give a complete quotation of the sentence in question, which is as follows (italics supplied): "We have presented this paper in order to stimulate the enactment of appropriate legislation that would be sufficiently broad to permit the introduction of fixed premium variable benefit policies along the lines developed in this paper *and also the introduction of equity-based life insurance products that reflect various alternative approaches*." It should therefore be very clear that we have never suggested that any legislative changes be limited to those which would permit only the design proposed in the paper.

As of the time when these remarks are being written, the outlook for broad legislative changes is a bright one. We have previously mentioned the fact that at its December, 1969, meeting the N.A.I.C. endorsed model legislation and regulations which were largely developed by an industry committee. This committee functioned under the able chairmanship of Mr. Walker, and the model legislation would permit almost all the alternative designs proposed in these discussions, as well as the variations mentioned in section VI of the paper.

Mr. Levy's concept of "separate account coinsurance" is an interesting example of how present techniques can be extended and adapted to variable life insurance. We certainly agree with his feeling that, to the greatest extent possible, company size should not be a factor in determining whether a company is able to offer variable life insurance.

Messrs. Munro and Rudd have presented a design which reflects the same basic concept as that underlying Mr. Walker's design, namely, that "excess" investment performance (which may be positive or negative) in the separate account is used to purchase positive or negative amounts of variable paid-up insurance. Their discussion is particularly interesting because it covers suggested approaches in a number of areas which relate to practical problems rather than to pure theory. These areas include flexibility as to the portion of the policy which is funded on a variable basis, an approach to the use of dividends, bases for nonforfeiture benefits and policy loans, and the use of a "pivotal yield rate" instead of the interest rate assumed for calculating net premiums and reserves.

The Munro-Rudd treatment of policy loans is interesting in that it requires any cash loans to be fully secured by the guaranteed cash value of the policy's fixed-dollar element but permits premium loans up to such guaranteed cash value plus a percentage of the cash value of the policy's variable element. On the assumption that all loans are made through the general account, this approach assures that there will never be a cash drain on general-account assets because of loans secured by assets in the separate account. It appears, however, that the practice of allowing premium loans to be made from the general account, for total amounts in excess of guaranteed cash values, could have some adverse effect on "book" general-account earnings rates. Of course, limiting this situation to premium loans will tend to minimize any such adverse effect.

We wish to thank Dr. Nesbitt for his sound, short, alternative proof of equation (40) in the paper.

Mr. Randall mentioned a possible design involving fixed premiums but with face amounts varying to reflect only the Z_t factors. Therefore, there is no adjustment to reflect the fact that fixed premiums are payable. He mentioned that such a design will produce negative reserves "if the actual investment returns exceeded the assumed rate for substantial periods." Actually, reserves under this design will always become negative at some point if cumulative actual investment performance exceeds that according to the assumed interest rate, and the better the actual investment performance, the sooner the time when reserves will become negative.

Mr. Stein makes some interesting observations concerning the matter of pricing variable life insurance policies. His discussion covers three main points, based on calculations which (Mr. Stein has told us) are on a nonparticipating basis.

First, he presents some figures which he believes may be indicative of the amount by which a gross premium for a fixed premium variable benefit policy must exceed that for a comparable fixed benefit policy in order to produce "equal profitability." He states that increases in gross premium are to be expected in view of "the additional costs of the variable benefit policy . . . for higher cash-surrender values, death benefits, and reserves."

It seems to us that such additional costs are not the basic reason for the premium increases indicated by his calculations, since higher death benefits, cash-surrender values, and reserves are implicitly provided for by the "excess" earnings available when separate-account investment performance exceeds the assumed interest rate. Instead, we believe that results like Mr. Stein's basically stem from the fact that nonparticipating gross premiums for fixed benefit policies calculated according to his methods reflect interest margins which are largely absent under a variable life insurance policy. This is true because, under a variable life insurance policy, all the "excess" earnings on the net premiums are used—sooner or later, depending on the particular design—to support higher benefits and reserves. An offsetting factor is that, when separate-account investment performance is favorable and actual mortality is more favorable than that assumed in the premium calculation, a variable policy will produce larger mortality profits than a corresponding fixed-dollar policy.

Another important pricing element to be considered is the possibility of making a charge against separate-account assets, as has become common practice in connection with variable annuities. Such a charge could be used to provide an interest margin.

All things considered, we do not believe that gross premiums for variable insurance policies must, as Mr. Stein seems to imply, necessarily be higher than those for corresponding fixed benefit policies.

Mr. Stein's second main point involves a worthwhile warning to be very careful when you enter the fascinating world of valuing income, outgo, and profits according to several different interest rates. Here is an area where approaches and philosophies may differ widely, and seemingly contradictory results must be evaluated with care.

Finally, Mr. Stein suggests that issuance of variable insurance policies may be undesirable because of the possible magnitude of fluctuations in book profits and raises the specter of possible insolvency. On the basis of our research, we see absolutely nothing to justify this viewpoint, assuming, of course, that the particular variable life insurance design offered meets the appropriate criteria for a sound design. Book profits under variable life insurance policies will certainly fluctuate, but fluctuations need not lead to insolvency. Just as has been the case with fixed benefit insurance, we believe that issuance of variable life insurance is perfectly feasible on the basis of sound product design and pricing, and maintenance of adequate reserves and surplus. In contrast to Mr. Stein's gloomy outlook, we believe that variable life insurance will prove to be a most beneficial development for the life insurance industry.

Mr. Turner suggests an alternative approach for minimum cash values, based on adjustments designed to make an equation of equilibrium for minimum cash values workable in practice, namely, (*a*) having the actual face amount be that derived from the equation of equilibrium for minimum cash values and (*b*) assuring positive Y'_t factors by imposing the requirement that any negative minimum cash value be taken as zero. He states that this alternative is desirable because it will produce minimum cash values that are independent of the reserve method actually adopted.

It seems to us that there is no need to have such independence. The key point is that benefits under a variable life insurance policy necessarily reflect the level and incidence of funds (i.e., actuarial net premiums) deposited in the separate account. This was illustrated in the paper with respect to our proposed design but also applies with respect to almost any alternative design involving fixed gross premiums. We therefore believe that it is perfectly natural for minimum cash values to reflect such differences.

In closing, we would like to state that our preparation of these remarks, including our analysis of the various discussions, was most stimulating and rewarding—as much so as our preparation of the paper itself. We hope that the paper, the discussions, and these remarks—taken as a whole—will be viewed as a valuable basic reference regarding the actuarial theory of variable life insurance. We look forward with interest and anticipation to future developments in this area, which will encompass not only extensions of the theory involved but also all the aspects of translating theory into practice.