ASSET VALUE GUARANTEES UNDER
EQUITY-BASED PRODUCTS

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ABSTRACT

Under current equity-based products offered by life insurance companies in the United States, the contractholder generally assumes the full investment risk and has no guarantee as to the asset value of his contract at any point in time. It is both reasonable and appropriate for a life insurance company to offer an additional assurance under such products whereby the investment risk is assumed by the company. This assumption of investment risk would mean a guarantee of asset value under the contract but would not mean that such a guarantee need be made at every point in time.

Accordingly, this paper considers only the case in which a minimum asset value is guaranteed at the end of a specified investment period under an equity-based contract. The intention of this paper is to present the bases and methodology for evaluating the risk inherent in such a guarantee and to indicate the relative significance of several underlying factors. Therefore, this paper considers the following three areas:

1. Analysis of the nature of the probability density function of rates of return on common stocks, to include descriptive characteristics of several sources of data by various historical periods; correlation between such sources of data; and consideration of the hypothesis that the statistic, rate of return on common stocks, is an independent random variable over time.

2. Presentation of a general simulation model based on the Monte Carlo "rejection technique" for evaluating the net risk premium for an asset value guarantee at the end of a specified investment period under an equity-based product. (It may be noted in this regard that the model probability density function of the random variable, rate of return on common stocks, is intended only to represent the *nature* of such function, the long-term mean rate of return being treated as a controlled parameter.)

3. Determination of the net risk premium, utilizing a set of "basic" assumptions, and analysis of the sensitivity of the net risk premium to changes in several underlying parameters, to include investment period; total charge deducted from the equity investment account rate of return; tax on equity investment account rate of return; long-term, mean rate of return on equity investment account; total charge deducted from each periodic payment; and decrements of mortality and withdrawal.

For example, consider an equity-based contract maturing at the end of twenty years and providing an asset value guarantee at maturity equal to the total payments (including the premiums charged for such guarantee) made under the contract. If it can be reasonably anticipated that the underlying equity investment account will realize a total annual rate of return of 10.0 per cent, over the long term, the *net* risk premium for such a guarantee (payable in addition to the regular annual payment) would not appear to exceed 1 per cent of the annual payment otherwise made under the contract. However, significant variations in the net risk premium were noted with respect to the length of the investment period, and, to a lesser extent, variations were also noted with respect to other underlying factors. It would therefore appear that an array of risk premiums would be appropriate for an asset value guarantee, such premiums varying in much the same manner as do life insurance premiums—by age at issue, benefit (or investment) period, and so forth.

I. INTRODUCTION

VARIABLE," "cost-of-living," "equity-based," "equity-linked"—the life insurance industry is in the embryonic stage of a revolution in product design. Traditional insurance products, by providing fixed-dollar guarantees, are essentially appropriate within a static economy. Because of the economic loss of purchasing power suffered under fixed-dollar guarantees in an inflationary economy, however, traditional insurance products are not "in step" with an economic system which, historically, has been characterized by persistent inflation.

Three major forces permeate the current revolution in insurance product design: assertion of life insurance industry leadership in meeting consumer demands for financial products which provide for participation in economic growth and/or which, either explicitly or implicitly, provide some assurance against the economic loss of purchasing power; increased technology and methodology in evaluating the risks inherent in such products; and permissive change in the regulatory environment within which the life insurance industry must operate.

Under traditional life insurance products the contractholder does not assume any investment risk and has full guarantee as to the asset value of his contract at every point in time (except to the extent that a nominal risk is assumed and, hence, asset value is not guaranteed, under a traditional participating life insurance product). Under current equity-based products[1] offered in the United States, the contractholder assumes the entire investment risk and has no guarantee as to the asset value of his contract at any point in time, except to the extent that such products may

[1] As used in this paper, "equity-based" products shall mean products the underlying assets of which are invested substantially in equities. While the term "equities" includes forms of investment other than common stocks (e.g., real estate), it will relate only to common stocks in this paper.

provide for allocation of some portion of payments to a nonequity, general investment account which guarantees a minimum rate of return.

It would appear reasonable and appropriate for a life insurance company to offer an additional assurance under its equity-based products whereby the investment risk is assumed by the company. This assumption of investment risk would mean a guarantee of asset value under the contract but would not mean that such a guarantee need be made at every point in time. This paper will consider the case in which an asset value guarantee is made only at the end of a specified investment period, for example, maturity of the contract. A guarantee which provides that the asset value of the contract at maturity will not be less than some stipulated amount (typically the sum of gross periodic payments made under the contract) is currently being offered by many life insurance companies outside the United States. It should be noted that a guarantee equal to the sum of gross periodic payments made under the contract not only guarantees preservation of principal but also guarantees an unstated rate of return since the value guaranteed is a function of gross payments and the asset value of the contract is a function of net periodic payments credited to the underlying equity investment account.

The purpose of this paper is threefold: (1) to analyze the nature of the probability density function of rates of return on common stocks; (2) to develop a simulation model for evaluating the net risk premium for an asset value guarantee at the end of a specified investment period under an equity-based product; and (3) to analyze the sensitivity of the net risk premium to changes in several underlying parameters. Accordingly, the remainder of this paper is arranged into the following sections: Nature of Probability Density Function, Development of Simulation Model, and Sensitivity Analysis of Net Risk Premium.

II. NATURE OF PROBABILITY DENSITY FUNCTION

This section will consider descriptive characteristics of historical probability density functions of the statistic, annual rate of return on common stocks, and will examine the hypothesis that this statistic is an independent random variable.

Quantitative Characteristics

Several sources of historical data regarding rates of return on common stocks are available. The following sources will be considered initially to present an overview of the relative characteristics of such sources of data:

1. Standard and Poor's Composite Index (500 common stocks)
2. Standard and Poor's Industrial Index (425 common stocks)
3. Dow-Jones Industrial Index (30 common stocks)

4. "Rates of Return on Investments in Common Stock: The Year-by-Year Record, 1926–1965," *Journal of Business*, Vol. XL, No. 3 (July, 1968).[2]

For each of the above sources, annual rates of return (from December to December) were calculated, in total and separately for the portions arising from dividends (i.e., "yield") and from changes in market price (i.e., "capital gain or loss"), for the three periods, 1926–65, 1936–65, and 1946–65. Table 1 presents the mean and standard deviation of these annual rates of return. It is apparent that the mean total rate of return has

TABLE 1

MEAN AND STANDARD DEVIATION OF ANNUAL RATES OF RETURN

(December to December)

INDEX: PERIOD	MEAN RATE OF RETURN			STANDARD DEVIATION		
	Price	Dividend	Total	Price	Dividend	Total
S & P Composite:						
1926–65.........	7.42%	4.84%	12.26%	21.25%	1.46%	20.43%
1936–65.........	8.22	4.83	13.05	17.45	1.46	16.84
1946–65.........	9.73	4.53	14.26	15.13	1.33	15.05
S & P Industrial:						
1926–65.........	8.62	4.80	13.41	23.74	1.56	22.88
1936–65.........	8.69	4.77	13.46	17.85	1.51	17.25
1946–65.........	10.37	4.49	14.85	16.05	1.41	15.98
D-J Industrial:						
1929–65.........	6.53	4.88	11.40	22.72	1.34	22.13
1936–65.........	7.81	4.66	12.47	15.84	1.08	15.53
1946–65.........	9.16	4.57	13.73	13.56	1.18	13.48
NYSE, U. Chicago:						
1926–65.........	10.57	3.43	14.00	31.49	1.52	31.72
1936–65.........	11.18	3.84	15.01	25.27	1.44	25.71
1946–65.........	8.77	3.85	12.62	18.99	1.56	19.41

generally increased over the period considered and that such increase has been accompanied by a decrease in the corresponding standard deviation as earlier, more volatile, market periods are eliminated. For example, the mean total rate of return for the Standard and Poor's Composite Index increased from 12.26 per cent for the period 1926–65 to 14.26 per cent for the period 1946–65, and the corresponding standard deviation decreased from 20.43 to 15.15 per cent. The "coefficient of variation" (ratio of standard deviation to the mean) decreased from 167 per cent for the

[2] This article presents results for portfolios of all common stocks listed on the New York Stock Exchange for the period 1926–65, assuming equal initial investments in the stock of each corporation. This source will be referred to hereinafter as "NYSE, U. Chicago."

period 1926–65 to 106 per cent for the period 1946–65. It is also apparent that an increase in mean rate of return for that portion attributable to changes in market price has occurred simultaneously with a decrease in mean rate of return for that portion attributable to dividends.

Comparing the standard deviation with the corresponding mean for each portion of the total rate of return, it is apparent that the standard deviation, or "volatility," for the portion of return attributable to changes in market price is much greater than that for the portion attributable to dividends, the standard deviation for the former being roughly 3 times

TABLE 2

COEFFICIENTS OF CORRELATION OF
ANNUAL RATES OF RETURN

(December to December)

INDEX: PERIOD	CORRELATION TO S & P COMPOSITE		
	Price	Dividend	Total
S & P Industrial:			
1926–65.........	0.98	0.99	0.98
1936–65.........	0.99	1.00	0.99
1946–65.........	1.00	1.00	1.00
D-J Industrial:			
1929*–65........	0.95	0.74	0.95
1936–65.........	0.97	0.92	0.97
1946–65.........	0.96	0.98	0.96
NYSE, U. Chicago:			
1926–65.........	0.92	0.32	0.93
1936–65.........	0.91	0.53	0.93
1946–65.........	0.92	0.74	0.95

* S & P Composite recalculated for same period.

the mean and the standard deviation for the latter being roughly one-third of the mean.

Table 2 presents, for the same periods shown in Table 1, the coefficients of correlation between the Standard and Poor's Composite Index and the other sources considered for the annual rate of return in total and separately for each portion. Except for the portion of the rate of return attributable to dividends for the NYSE, U. Chicago index, a high degree of correlation is apparent.

Due to the high degree of correlation indicated between the Standard and Poor's Composite Index and the other sources considered and the availability of reliable data for this Index for years prior to 1926, it was used for further analysis of the nature of the probability density function.

Using the Standard and Poor's Composite Index, annual rates of return, in total and separately for each portion, were calculated for each calendar month (e.g., March, 1928, to March, 1929) for the period 1880–1967, inclusive.[3]

Table 3 presents the mean, standard deviation, and third standardized moment calculated for various historical periods. The third standardized moment about the mean, a_3, may provide an indication of symmetry about the mean, that is, "skewness." The graphical interpretation generally placed on the value of a_3 is as follows: If values of r (annual rates of return) are distributed symmetrically about the mean, then $a_3 = 0$. If the distribution has a longer tail out to the right than to the left, then $a_3 > 0$, and the distribution is said to have positive skewness. If the distribution has a longer tail to the left, then $a_3 < 0$, and the distribution is said to have negative skewness. Such graphical interpretations must, however, be made with care since $a_3 = 0$ is neither a necessary nor a sufficient condition for symmetry.

As is true of Table 1, Table 3 also indicates that the mean total rate of return has generally increased throughout the total period considered. The decrease in the corresponding standard deviation, however, occurs only in periods beginning after 1926 (the periods considered in Table 1), and, in fact, the standard deviation increases as the mean rate of return increases for periods beginning prior to 1926. The probability density functions would appear to have positive skewness (i.e., a longer tail on the right than on the left), although this interpretation cannot be made conclusively at this point.

Rates of Return as Independent Random Variables

A basic assumption underlying the simulation model developed in this paper is that the rate of return on common stocks can be treated as an independent random variable. The very notion of this assumption is at first repelling, if not downright frustrating, to adherents of such classical financial theories as the Dow-Jones theory and the "point and figure" school.

Jackson [1] makes the following statement:

The last few years have seen a revival of interest in the study of price movements in the Stock Markets by means of mathematical models. Most of the

[3] Data for years prior to 1918 converted from Cowles Commission Stock Price Indexes. The Cowles Commission indexes are an extension of the Standard and Poor's indexes, the same method of construction being used and, as far as possible, the same companies. The annual "yield" (i.e., dividend portion of total rate) was not available prior to 1926 by calendar month, but by calendar year only. The annual "yield" for each calendar month within a calendar year prior to 1926 was, therefore, assumed to be equal to the corresponding average annual "yield" for such year.

studies have been empirical in nature and concerned primarily with investigating the stochastic process underlying the movement of prices. . . . Many theories, facile and sophisticated, have been put forward to aid prediction of the prices of stocks and shares. They crumble when statistical analysis reveals that price variations conform very closely to something as simple as the flip of a coin. . . . The major differences are that the "coin" is slightly biased in favor of a head and the price change is not limited to unit steps.

TABLE 3

STANDARD AND POOR'S COMPOSITE INDEX:
STATISTICAL CHARACTERISTICS

(Percentage Annual Rate of Return, Calculated
for Each Calendar Month)

Historical Period	Mean Rate of Return	Standard Deviation	Third Standardized Moment (α_3)
Total Rate of Return			
1880–1967	10.18	19.53	0.01
1896–1967	11.13	20.48	.03
1906–67	10.96	21.01	.04
1916–67	12.13	21.83	.10
1926–67	12.55	22.92	.09
1936–67	13.19	17.84	.14
1946–67	13.89	14.63	0.10
Change in Market Price Portion Only			
1880–1967	5.57	20.03	0.10
1896–1967	6.47	21.05	.00
1906–67	6.14	21.59	.02
1916–67	7.26	22.48	.06
1926–67	7.95	23.58	.03
1936–67	8.60	18.30	.10
1946–67	9.55	14.73	0.11
Dividend Portion Only			
1880–1967	4.61	1.17	0.94
1896–1967	4.66	1.24	.83
1906–67	4.82	1.26	.66
1916–67	4.86	1.35	.56
1926–67	4.61	1.34	.94
1936–67	4.58	1.32	.64
1946–67	4.34	1.23	0.58

The major postulates of the theory that stock price movements are random have been verified independently by many investigators and for several markets, including the American, French, Belgian, and British. In England, Kendall [2] applied the techniques of spectral analysis to several British price series, verifying the independent increments (changes in stock price) assumption and noting negative first serial correlation coefficients. Studies by Granger, Godrey, and Morgenstern [4, 5] independently confirmed the theory for the New York market.

Jackson [6] summarized the basic premises underlying the theory that stock price movements are random as follows:

> The basic element of randomness in a stock's price from day to day is caused by the input of random information—information concerning mergers, stock splits, earnings, dividends, new products, international crises, etc. This information arrives in a continuous stream via ticker tape, radio, newspaper, and even in the instantaneous reactions of the market itself, to a myriad of traders all across the country who immediately translate each bit of information into a re-evaluation of each stock's value, hence into an immediate stock purchase or sale to reflect the latest information input.
>
> The independence of day to day changes in the variate (stock price) follows a similar line of reasoning and is based on the premise that "favorable" and "non-favorable" bits of information are equally likely.

A revolution in attitudes and approach to understanding the stock market phenomenon is taking place, led "not by the financial community, nor by the classical economists, but by a new breed of men trained in scientific business management, operational research, and statistical methods. . . . One by one, the myriad of myths (which have enshrouded understanding of the stock market phenomenon) are quietly being revealed for what they are, and the new theorists are strengthening their position" [Jackson (1)]. In this regard, it is interesting to note the following, somewhat colorful, statement by Karl Borch, one of the new theorists:

> With the present state of our knowledge it seems quite fair to consider security analysts, at least those who belong to the "technical school," as astrologers of our century. The horoscopes they prepare for individual stocks are based on price behaviour in the past, but they may equally be based on the constellations of the stars when the companies were founded. Security analysts are respected members of our society, like astrologers once were, and they are amply rewarded when their predictions prove right—as they are bound to be in about half of the cases" [7].

In addition to the above, two rather basic nonparametric tests of randomness were made using the total annual rates of return (December to December only) for the Standard and Poor's Composite Index for

the period 1880–1967, inclusive. The first test was based on the number of runs. For the 88 rate of return values, the total number of runs was 41—21 runs above the median and 20 runs below the median. Using a table of the frequency function of the statistic total number of runs with the number of runs above and below the median fixed, tests against the following two hypotheses were made: one, there are too many runs; two, there are too few runs. The hypothesis of randomness was *not* rejected at the $2\frac{1}{2}$ per cent level. The second test was based on serial correlation. If a set of observations is ordered with respect to time and if time is irrelevant to the variable (e.g., total annual rate of return) being considered, no correlation would be expected to exist, for example, between successive pairs of values. If the sequence tested yields a large positive or negative value of the serial correlation coefficient, its randomness would be questioned. The serial correlation coefficient between successive pairs of values of total annual rates of return, using the same data as those used in the first test, was -0.0045, and therefore the hypothesis of randomness was *not* rejected. It is interesting to note that the "slightly negative serial correlation" evident in the investigations of Kendall [2] and Moore [3] does actually exist for the data being considered.

It should be noted that some scholars, or investigators, holding the theory that stock price movements are random qualify the theory by observing that random movement takes place within the framework of a long-term "drift"—that is, over long periods of time, stock prices move higher. The general increase in the mean total rate of return noted earlier in this paper would seem to support this observation of long-term "drift."

In summary, many investigators support the theory that the rate of return on common stocks is an independent, random variable. Some of these investigators qualify the theory by adding that random movement takes place within the framework of a long-term, upward "drift" in rates of return. The analyses that I have made, while admittedly basic, support the hypotheses of randomness and long-term "drift."

Before proceeding, it seems appropriate to note that the model probability density function developed in the next section of this paper is intended only to represent the *nature* of the probability density function of the random variable, rate of return on common stocks, the long-term mean rate of return being treated as a controlled parameter, either fixed or variable over time.

III. DEVELOPMENT OF SIMULATION MODEL

Generally, a model should be a sufficiently precise representation of the essential features of a system so that observations and conclusions ob-

tained from the model are valid for the "real" system. In this respect the proper criterion for judging the validity of a model is whether or not it predicts relative effects of alternative courses of action with sufficient accuracy to permit a sound decision. This section will summarize the development of the model probability density function of the total annual rate of return on common stocks and of other elements of the simulation model.

Probability Density Function of Total Annual Rate of Return

As previously stated, the model probability density function developed is intended only to represent the *nature* of the probability density function of total annual rates of return, the mean of the function being controlled as a parameter.[4]

The model probability density function is based on the Standard and Poor's Composite Index for two reasons: (1) this index provided the greatest volume of reliable data, and (2) it indicated a high degree of correlation with several other possible sources of data. With the use of this index, total annual rates of return were calculated for each calendar month in the period 1880–1967, inclusive, yielding 1,044 rate of return values. The discrete probability density function and the corresponding cumulative function were then constructed by use of 1 per cent class intervals, the range being $(-56.0\%$ to $+127.0\%)$. Both functions were graphically plotted. Because of the very smooth series of class marks exhibited for the cumulative function, this function was used to obtain pivotal values, truncating the high-order rates above $+82.0$ per cent (where there were only three frequencies) and establishing the low order limit at -58.0 per cent. The pivotal values were then graduated and interpolated at $\frac{1}{10}$th per cent intervals to yield five-digit values of the cumulative function by using Jenkin's fifth-difference modified osculatory interpolation formula. The resulting cumulative function represented 1,401 discrete rate of return values in the range $(-58.0\%$ to $+82.0\%)$. This function was then differenced to obtain the corresponding probability density function. Both these functions are illustrated in Figure 1, *a* and *b*. The mean, standard deviation, and third standardized moment about the mean, calculated for the graduated data, compared satisfactorily with similar characteristics for the crude historical data.

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[4] It will be assumed in this paper that the long-term, mean rate of return is fixed over time. The alternative assumption that such mean rate increases over time in accordance with the hypothesis of long-term "drift" was not considered necessary for the purposes of this paper. In this respect, the observations and conclusions obtained from the simulation model may tend to be conservative.
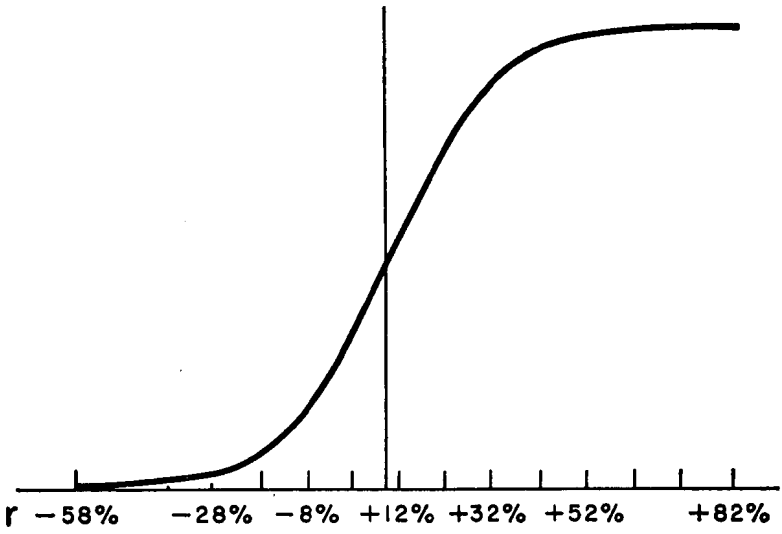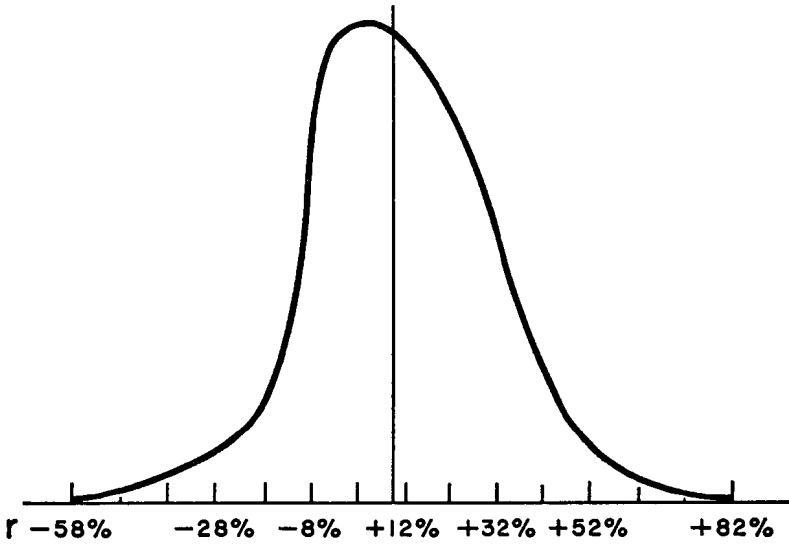
FIG. 1.—Illustration of model density functions. (a) Cumulative density function; (b) probability density function (Standard and Poor's Composite Index; total annual rate of return, r, by calendar month, 1880–1967).

Random Number Generation

Implementing a simulation model requires random numbers in the range (0, 1) in order to obtain random observations from the defined probability density function. While various methods are available, a computerized multiplicative congruential method was utilized to generate the required random numbers. This method is one of the most popular and well-tested methods and obtains the $(n + 1)$st random number from the nth random number by using a recurrence relationship. The first random number (not used to obtain a random observation) is obtained by selecting any large integer, subject to certain constraints. While, strictly speaking, the numbers generated are not random numbers but pseudo-random numbers, they do satisfactorily play the role of random numbers. The procedure was defined such that a ten-digit number was generated and then truncated, retaining the high-order five-digit number. By adding a decimal in front of the first position, the number retained was then utilized to obtain a random observation from the defined probability density function (the values of which were also expressed as a five-digit number).

Sampling Technique

In the interest of minimizing computer storage requirements and minimizing the time required to obtain a sample observation, the Monte Carlo "rejection technique" was utilized. This technique is based on the probability density function rather than the corresponding cumulative function, which is the basis for more common sampling techniques, and may be summarized as follows:

1. Transform the probability density function, $f(r)$, by a scale factor E, such that
$$E \cdot f(r) \leq 1 \qquad (a \leq r \leq b).$$

2. Define r as a linear function of the random number n:
$$r = a + (b - a) \cdot n .$$

3. Generate pairs of random numbers (n_1, n_2).

4. If $n_2 \leq E \cdot f[a + (b - a) \cdot n_1]$, the pair of random numbers is "accepted" and the random observation generated is taken as $r = a + (b - a) \cdot n_1$.

5. If $n_2 > E \cdot f[a + (b - a)n_1]$, the pair of random numbers is "rejected" and another pair of random numbers is generated.

The theory behind this technique is that $P[n \leq E \cdot f(r)] = E \cdot f(r)$. Consequently, if r is chosen at random from the range (a, b) according to the

equation in item 2 and then rejected if $n > E \cdot f(r)$, the probability density function of the accepted r's will be exactly $f(r)$. If no rejection procedure had been used, then r would be uniformly distributed in the range (a, b). The expected number of trials before a pair is accepted is equal to $1/E$, and the probability of accepting a pair on the first trial is equal to E, the "efficiency" of the technique.[5]

Tests of Model Probability Density Function and Sampling Technique

Based on the assumption that an error in the mean of the sampled rates of return equal to approximately 0.25 per cent would be acceptable, a sample of 10,000 trials was generated. The mean, standard deviation, and third standardized moment about the mean, calculated for this sample, compared satisfactorily with the corresponding model values. In addition a *chi-square test* for goodness of fit was made between the frequency distribution for the observed sample and the corresponding model distribution by utilizing nine classes. The value of χ_s^2 obtained was 4.81 with 8 degrees of freedom. Since the probability of a value of χ_s^2 at least as great as this is about 78 per cent, the fit of the observed sample was regarded as satisfactory.

Determination of Net Risk Premium

Having developed the probability density function of total annual rates of return and defined the technique for obtaining sample observations from this function, we will now consider the remaining elements of the simulation model necessary to determine the net risk premium for an asset value guarantee under an equity-based contract. As noted in the Introduction, this paper will consider only the case in which such a guarantee is made at the end of a specified investment period, for example, maturity of the contract. The remaining elements of the simulation model may be summarized as follows:

1. Calculate trial values of p (a percentage increment or decrement in the periodic payment allocable to the equity investment account and otherwise made under the contract) such that the total asset value of the contract at maturity will exactly equal the asset value guaranteed.

2. Determine the discrete probability density function for the calculated values of p based on a sufficient number of trials and calculate the net risk premium by evaluating the right-hand tail of this function where $p > 0$.

[5] The "rejection technique" is considered in both the following sources: Herman Kahn, "Applications of Monte Carlo" (Research Memorandum RM-1237-AEC; Santa Monica: The Rand Corporation, 1956), pp. 12–17; and J. L. Balintfy, D. S. Burdick, Kong Chu, and T. H. Nalyor, *Computer Simulation Techniques* (New York: Wiley & Sons, 1968), pp. 73–75.

Calculation of trial values of p.—The procedure used assumes reinvestment of all dividends and that the following two amounts will be available at maturity to meet then guaranteed values with respect to contracts then in force: (1) the asset value accumulated in the equity investment account to which has been credited net periodic payments (i.e., net after any deductions applicable to such payments); (2) a risk fund accumulated in the general investment account to which has been credited net risk premium payments.

As noted under item 1 above, it has also been assumed that the net risk premium will be expressed as a percentage of the periodic payment otherwise made under the contract and allocable to the equity investment account. An alternative approach could be taken such that the net risk charge is expressed as a deduction from the equity investment account rate of return. Further consideration of this alternative, however, was not considered necessary for the purposes of this paper.

Definitions and formulas applicable to the calculation of trial values of p are presented below:

m = Mean total annual rate of return of the model probability density function developed.

μ = Assumed mean, long-term total annual rate of return for the equity investment account (i.e., the assumed mean of the model probability density function of total annual rates of return, as distinguished from the actual mean of the function developed, m).

c_t = Total deduction (e.g., investment management fee, mortality guarantee, if applicable) in the tth contract year from the equity investment account rate of return.

r_t = Sample value of total annual rate of return for tth year obtained from probability density function.

y_t = Net total annual rate of return on equity investment account for tth contract year (after a shift in mean from m to μ).[6]

$= [r_t + (r - m)] - c_t.$

j_t = Net total annual rate of return on general investment account for tth contract year.

[6] The following approximate adjustment to y_t was made to estimate the corresponding net "after-tax" rate of return for purposes of analyzing the sensitivity of the net risk premium to this item: $y'_t = (y_t - d)(1 - T^a) + d \cdot (1 - T^p)$, where, d is assumed mean rate of return attributable to dividend income, T^a is effective tax rate on capital gains or losses, and T^p is effective tax rate on dividend income.

While a more sophisticated approach could be developed, the approximate adjustment indicated was considered sufficient for the purposes of this paper, since other assumptions are involved, for example, appropriate tax rates, distribution of capital gains and losses into realized and unrealized portions, and so forth.

$q_{[x]+t-1}$ = Rate of mortality for individual age x at issue, $[x]$, in tth contract year.

$w_{[x],t}$ = Rate of voluntary withdrawal for $[x]$ in tth contract year.

$P_{[x],t}$ = Annual periodic payment allocable to equity investment account in tth contract year under contract issued to $[x]$.

$L_{[x],t}$ = Total deduction from $P_{[x],t}$, expressed as a percentage of $P_{[x],t}$.

n = Investment period in years.

${}_tAV^y_{[x],n}$ = Contract asset value in equity investment account at end of tth contract year.

$\quad = [{}_{t-1}AV^y_{[x],n} + P_{[x],t}(1 - L_{[x],t})]\cdot(1 + y_t)$.

${}_tS^j_{[x]:\overline{n}|}$ = Value at end of tth contract year of a fund accumulated at the general investment account rate of return based on annual payments of $P_{[x],t}$.

$$= \frac{[{}_{t-1}S^j_{[x]:\overline{n}|} + P_{[x],t}(1 - L_{[x],t})]\cdot(1 + j_t)}{1 - q_{[x]+t-1} - w_{[x],t}}.$$

$AVG_{[x],n}$ = Asset value guaranteed at end of investment period with respect to contractholders then alive and persisting.

$\quad$ = Either (a): \$X

$\qquad$ or (b):

$$\sum_{t=1}^{n}(1 + p_{[x],n})\cdot P_{[x],t}\cdot(1 + i)^{n-t+1}.[7]$$

${}_kp_{[x],n}$ = kth trial value of p (percentage increment or decrement to annual periodic payment) for contract issued to $[x]$.

$$= \frac{AVG_{[x],n} - {}_nAV^y_{[x],n}}{{}_nS^i_{[x]:\overline{n}|}}.$$

Negative values of ${}_kp_{[x],n}$ represent those cases in which the asset value in the equity investment account is greater than that guaranteed and therefore no "benefit" would be payable. Positive values of ${}_kp_{[x],n}$ represent those cases where the asset value in the equity investment account is less than that guaranteed and therefore a "benefit" equal to the difference would be payable.

It may be noted that the method used to calculate net risk premiums

[7] If $i = 0$, the asset value guaranteed would be total periodic payments made under contract, including net risk premium for the asset value guarantee. If $i > 0$, the asset value guaranteed would be the accumulated value of total periodic payments, including net risk premiums, at a stated rate of interest i. As noted in the Introduction, even when $i = 0$, an unstated rate of interest on the total periodic payments is implicitly guaranteed, if, as is the usual case, $L_{[x],t} > 0$.

assumes that a risk reserve will be accumulated, which, together with the asset value of the contracts persisting to the end of the investment period, will be sufficient to provide the asset value guaranteed with respect to all contracts then in force. Such risk reserve would be equal to the net risk premiums accumulated with interest (at the rate applicable to the general investment account), mortality, and withdrawal.

Calculation of net risk premium, $\pi_{[x],n}$.—Given a sufficient number of trial values of $_k p_{[x],n}$, the discrete probability density function $f(p_{[x],n})$ can be determined. The net risk premium, $\pi_{[x],n}$, can then be obtained by evaluating the critical region of the probability density function, $p_{[x],n} > 0$ (see Fig. 2).
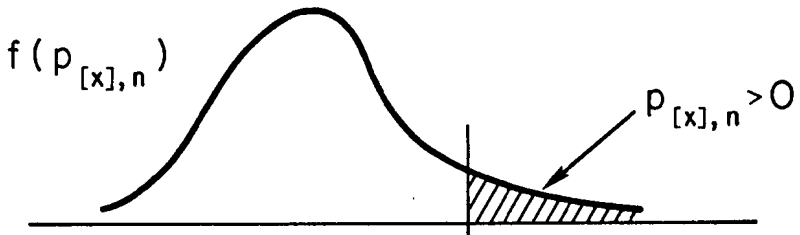
$$\pi_{[x],n} = E\left\{ p_{[x],n} > 0 \right\}$$

$$= \sum_{p_{[x],n} > 0} f(p_{[x],n}) \cdot p_{[x],n}$$

FIG. 2

IV. ANALYSIS OF SENSITIVITY OF NET RISK PREMIUM

The purpose of this section is to indicate the relative level, or sensitivity, of the net risk premium with respect to variations in the following parameters: investment period; total charge deducted from equity investment account rate of return; tax on equity investment account rate of return; long-range, mean total rate of return for the equity investment account; total charge deducted from the periodic payment; and decrements of mortality and withdrawal. With respect to these parameters, the following "basic" assumptions apply to all sensitivity tests performed, *unless* otherwise indicated:

1. Investment period: twenty years
2. Long-term total rate of return for the equity investment account: 10.00 per cent

3. Total charge (investment management fee; mortality guarantee, if applicable) deducted from the equity investment account rate of return: 0.5 per cent
4. Tax applicable to the equity investment account rate of return: none
5. Total percentage charge ("sales load"; premium tax, if applicable) deducted from each periodic payment: 8.5 per cent
6. Decrements of mortality and withdrawal: 1955–60 Select and Ultimate Basic Mortality Tables and Moorhead "S" Withdrawal Rates

For all sensitivity tests, net risk premiums were calculated for issue ages 25, 35, and 45, and the following assumptions were applied: (1) the net annual rate of return for the general investment account is 4.50 per cent; (2) the maturity value guaranteed is equal to the total periodic payments made, including the net risk premiums; (3) the annual periodic payment is equal to $100; and (4) the entire amount of each periodic payment is allocated to the equity investment account. For all tests made, the number of trial values of $p_{[x],n}$ was equal to 1,000 but not less than the number of trial values (up to a maximum of 5,000 such trial values) required to generate twenty-five trial values of $p_{[x],n} > 0$.

A "basic" set of net risk premiums was calculated by utilizing the "basic" assumptions set forth above for the six parameters to be considered. Then, for each parameter, hypothetical variations were made from the "basic" assumption and a set of net risk premiums was calculated, by utilizing "basic" assumptions for all other parameters. In this manner it is possible to analyze the sensitivity of the net risk premium to specific changes in each of the parameters considered. Table 4 presents a summary of the results of all calculations made. It is apparent from Table 4 that the net risk premium is particularly sensitive to variations in the investment period, long-term total rate of return for the equity investment account, and decrements of mortality and withdrawal.

One method of determining a security loading for the net risk premium considers the standard deviation of the critical values, that is, positive values of $p_{[x],n}$. It may be of interest, therefore, to note that the standard deviations of positive values of $p_{[x],n}$ for the "basic" set of net risk premiums were as follows: age 25, 1.09 per cent; age 35, 1.00 per cent; and age 45, 0.82 per cent. Expressed as a percentage of the corresponding net risk premium, these values of the standard deviation are as follows: age 25, 156 per cent; age 35, 167 per cent; and age 45, 155 per cent.

TABLE 4

Net Risk Premiums as Percentage of Periodic Payment

Issue Age on Initial Contract Date (1)	"Basic" Assumptions (2)	Investment Period		Mean Equity Rate of Return		Total Charge from Equity Rate of Return		Tax on Equity Rate of Return		Total Charge from Periodic Payment		Mortality and Withdrawal	
		10 Years (3)	30 Years (4)	8.0% (5)	12.0% (6)	0.0% (7)	1.0% (8)	See Notes (9)	See Notes (10)	7.5% (11)	9.5% (12)	None (13)	200% Withdrawal (14)
25.........	0.70%	5.30%	0.24%	2.00%	0.25%	0.52%	0.94%	0.23%	1.10%	0.65%	0.76%	1.56%	0.38%
35.........	.64	5.17	.20	1.83	.23	.48	.86	.21	1.00	.59	.69	1.56	.35
45.........	0.53	4.83	0.13	1.51	0.19	0.39	0.71	0.17	0.82	0.49	0.57	1.56	0.30

Notes.—Column 9: Effective tax rate of 25 per cent applicable to capital gains *and losses;* no tax applicable to assumed dividend "yield" of 4.5 per cent. Column 10: Effective tax rate of 25 per cent applicable to capital gains *and losses;* effective tax rate of 50 per cent applicable to assumed dividend "yield" of 4.5 per cent. Column 14: Withdrawal rates equal to 200 per cent of Moorhead "S" rates, the "basic" assumption; mortality rates in accordance with "basic" assumption.

necessary computer systems, without which this paper would have been impossible.

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DISCUSSION OF PRECEDING PAPER

SIDNEY BENJAMIN:

I have had the pleasure of reading an advance copy of this paper, and I commend it as being a most valuable piece of work, carried out in a very thorough fashion.

Before the paper is published, it would be useful to have tables appended of the basic eighty-eight values of the Standard and Poor's Composite Index, which the author used, and, for future reference and scrutiny, the 1,044 annual rates of return commencing on each calendar month, which were used to develop the probability density function. It would be interesting to compare these with the values used in the preparation of the memorandum dated September 5, 1969, on "Guarantees in Variable Life Insurance and Annuity Contracts," prepared by the Canadian Life Insurance Association for the Association of Superintendents of Insurance of the Provinces of Canada.

I have some difficulty in being satisfied with the author's tests of randomness, and I will admit that this is largely because he appears to me to have demonstrated by implication that the business cycle does not exist. I find this hard to accept. In a private letter he has given me the distribution of lengths of run referred to on page 467 of his paper, and I think this could be published. I would add at this point that my own investigations indicated that the length of cycle time, if cycles existed, did not appear to influence the result. This is another set of results which I find hard to accept.

We have been faced with the problems of equity-linked products in the United Kingdom for a long while now, and the first well-publicized maturity guarantees were given about ten years ago, as far as I remember. They have become increasingly popular for obvious reasons, but there has been strong pressure against them from the actuarial profession. Most of the preliminary calculations which I have seen have been based on historic tests which merely investigate what would have happened to various policies if they had been taken out at different times in the past. It is not true that the author's assumption of randomness is necessarily a more stringent test.

I would, however, like to draw a very clear distinction between the author's work and the problem of reserves. It was the latter problem which occupied my own work on the subject. The author has produced a premium rate, and on page 474 he suggests that an appropriate reserve

would be obtained by accumulating the premiums. My own feeling is that, if a basis is correct for premiums, it must *ipso facto* be too weak for valuation and vice versa—if it is correct for valuation, it is too strong for a commercial premium. Furthermore, a retrospective reserve is dangerous.

I myself tried to produce prospective reserves on a basis which I assumed would be stronger than market experience. In fact, I assumed that the running yield for new money at any point of time would be uniformly distributed between 2 per cent per annum and 6 per cent per annum before tax. This assumption was commented upon by A. C. Stalker in a note "Frequency Distributions of Investment Index Yields."[1]

It is interesting and surprising to note that this assumption of a uniform distribution for a running yield based on a range of $1\frac{1}{4}$–$3\frac{3}{4}$ per cent (i.e., 2–6 per cent net of United Kingdom tax at $37\frac{1}{2}$ per cent) produces a distribution of annual return similar in shape to that produced by the author, but with a mean of almost 10 per cent per annum growth rate and a standard deviation of almost 50 per cent.

I do not agree that the accumulation of a commercial premium is a suitable actuarial reserve. A prospective reserve on a cautious basis should be set up, and the interest and profit return required on this capital should be incorporated in the commercial premium charged. My own arithmetic trebled the commercial premium chargeable when I inserted a profit return on the reserves into the calculation.

There are cases of companies issuing policies for one or two years and then ceasing issue. The result could be a lack of spread of maturity dates, and hence I think there is a very real problem of evaluating reserves at the commencement of this type of business.

The author is to be congratulated on the careful work he has done and the way in which he has presented it. In spite of this, he is likely to have his own results quoted back at him out of context for a long time to come.

FRANCIS H. GEORGE:

Mr. Turner's timely paper was especially valuable to our company, since our proposed variable annuity contracts provide a minimum surrender value at every point in time. This surrender value is determined by applying a percentage to the sum of the net premiums paid. This percentage increases from 50 per cent the first year to 100 per cent after ten years. We were able to calculate a net risk premium for this benefit by using the cumulative distribution function of monthly percentage changes in stock price indexes (with dividends reinvested) determined by

[1] *Journal of the Institute of Actuaries*, XCIII, 113.

Mr. DiPaolo and contained in his paper "An Application of Simulated Stock Market Trends to Investigate a Ruin Problem" and an approach similar to Mr. Turner's.

The guaranteed asset value and the asset value in the equity investment account were calculated at the end of every month for thirty years. These two values were compared and a net risk premium calculated each month which, if accumulated in the general account of the company, would equal the difference between the two values.

The guaranteed value is defined by the contract. The asset value of the equity account is the accumulation of premiums using monthly index changes determined by the cumulative distribution function mentioned above and a Monte Carlo sampling technique. The net risk premium is expressed as a per cent of gross premium. If this percentage is negative, it is taken as zero. Each sample consists of one set of three hundred and sixty months of simulated experience. Two net risk premiums are calculated for each sample—the minimum, which is the first positive premium obtained (assuming that everyone withdraws as soon as the guarantee is greater than the asset value of the equity account), and the maximum, which is the largest positive premium obtained in the three hundred and sixty simulated months (assuming that everyone withdraws when the positive difference between the guaranteed account and the equity account is the largest).

Four hundred samples were taken in this way for issue age 35, and an expected premium was determined. The expected premium under the first assumption was 0.5 per cent of the gross premium. The expected premium under the second assumption was 2.4 per cent of the gross premium. About 70 per cent of the net risk premiums came up negative and were taken as zero.

Our company also guarantees a death benefit equal to the return of gross premiums or the asset value of the equity account, whichever is greater, and we plan to use the same type of approach in pricing this benefit.

<center>RICHARD G. HORN:</center>

Mr. Turner's paper develops an excellent method for evaluating the cost of an asset guarantee in an equity-based product. Some questions come to mind, however, and I wonder to what extent the following problems were considered?

1. Is it practical to assume that the rate of investment yield on common stocks is an independent variable from year to year?

As Mr. Turner has suggested, this concept is rather difficult to digest.

For example, the yields on the Dow-Jones Industrials (including estimated cash dividends) for the depression years 1930–32 were −30, −45, and −18 per cent, respectively; the fact that the 1931 yield was a negative 48 per cent would seem to be in some way related to the fact that the 1930 yield was a negative 30 per cent, and the negative yield for 1932 would hardly seem to be completely independent of the 1930 and 1931 yields. Perhaps the concept of randomness of yield rates is practical for evaluating the cost of an asset guarantee over a long period, such as twenty years, but not for relatively short periods.

2. Is it possible to evaluate the risk that future common stock yields will form quite a different probability density function from that formed by historical yields?

It seems quite possible that the social, economic, and political problems of today might produce an investment pattern in the future that is not well related to what has happened historically. In any event, it would seem that the use of yield rates obtained by sampling from a historical population tacitly ignores a significant risk.

3. What about the statutory surplus strain prior to maturity that an asset guarantee could generate because of statutory policy reserve requirements?

Under present law, an asset guarantee at maturity would require that statutory fixed-dollar reserves be established each year in support of the fixed-dollar obligation at maturity. Such a requirement could expose statutory surplus to nearly the same risk as would the provision for an asset guarantee at every point in time. The only significant difference would likely be that, with an asset guarantee at every point in time, the probable cost-of-surrender benefits paid would be substantially increased because of antiselection during severe market declines. An asset guarantee premium, as determined by Mr. Turner's method, measures the long-term actual cost of an asset guarantee, but it does not consider the problem of intermediate surplus drain. Any company contemplating the introduction of an asset guarantee into an equity-based product would be well advised to run model office studies of asset share accumulations (cycling through suitable series of common stock yield rates) in order to observe the effect on statutory surplus.

Mr. Turner has explored a subject that is nearly certain to become increasingly important to the actuarial profession. His paper is a welcome addition to actuarial literature.

HAROLD G. INGRAHAM, JR.:

A basic assumption underlying the simulation model developed in this paper is that stock price movements are random.

A recent best-seller entitled *The Money Game*, by an individual with the nom de plume of Adam Smith, contains an entertaining chapter with the colorful title "What the Hell Is a Random Walk?" The chapter states that the first premise of random walk theory is that the market—say, the New York Stock Exchange—is an "efficient" market, containing numbers of rational, profit-maximizing investors who are competing, with roughly equal access to information, in trying to predict the future course of prices. A second premise is that stocks do have an intrinsic value and that, at any point in time, the price of a stock will be a good estimate of its intrinsic value, the intrinsic value depending on the earning power of the stock. But, since no one is exactly sure what the intrinsic value is, the actions of the many competing participants should cause the actual price of a security to wander randomly about its intrinsic value.

This chapter also points out, however, that stock price movements are not entirely random, because the market strays from the "efficient." This is a result of a herd psychology that occasionally infects even cold, austere professional money managers. Thus it concludes that, while over an extended period of time price movements are random, in the short run the dominant factor may be the temper of the crowd.

PAUL H. JACKSON:

In the first section of Mr. Turner's very interesting paper he analyzes the nature of the probability density function of rates of return on common stock. Mr. Turner states the basic assumption underlying the simulation model: "The rate of return on common stocks can be treated as an independent random variable." Mr. Turner then proceeds to present many facts and authoritative references supporting the general reasonableness of this assumption. Perhaps the most impressive array of facts supporting the randomness of common stock price movements is contained in Eugene F. Fama's "The Behavior of Stock Market Prices."[2] And yet all the authoritative references do not assuage the sense of uneasiness with this assumption but rather bring to mind the melancholy Dane and "the lady doth protest too much, methinks." If the assumption were at all a reasonable one, it should not be necessary to develop so great a body of authority to support it.

[2] *Journal of Business* (January, 1965).

While the author admits that "the very notion of this assumption is at first repelling," the facts and references listed are all from a common school of thought—supporters of the random walk theory. Among the facts which are not included are the following:

1. The price which will be received in the next transaction is dependent upon the number of shares sold. The differential in selling price between a 100-share block and a 100,000-share block is significant and imposes a bias on the end results.

2. Some of the statistical distributions assembled on market price changes have developed a clear trimodal pattern, whereas the assumption of randomness implies a normal distribution which is unimodal. Benoit Mandelbrot, in "The Variation of Certain Speculative Prices,"[3] and A. C. Stalker, in "Frequency Distributions of Investment Index Yields,"[4] both report this triple peak.

3. Common stock prices are normally reported on the trading-day basis, and normality is assumed to apply to the change in price from one trading day to the next. The trading-day concept implies, however, that in some instances the period of time covered may vary from one to four or five calendar days. The assumption that the price changes are independent of the calendar period spanned from one trading day to the next seems unreasonable.

4. There are specialists who are supposed to provide an orderly market through the judicious purchase and sale of specific securities from time to time. If these specialists have any impact whatever on the market, the price changes will not be random. And yet, if they have none, why are they considered necessary?

5. Most indices and most common stock reports are based on the final sale of the day—the closing price. In the course of a day many thousands of shares can be traded, but it is the final trade, even though it is only 100 shares, that sets the stock price quotations for the financial pages and various common stock indices. It is obviously all too easy for a number of traders to get together and slip in the last trade of the day in order to buttress prices. Any such final trade activity clearly imposes a bias.

6. There are too many occasions when the prices of almost all securities are moving in a common direction—from financial panics to the Kennedy market. And there are significantly more large price changes than can be supported by an assumption of randomness.

7. While the chartists' methods may resemble the black arts, the existence of a large number of followers does impose on the market a pattern of results that must more or less follow the particular theory, simply because so many people expect such results.

8. Fama and others have dropped the normal curve as a reasonable representative in favor of a symmetric stable distribution—one lying somewhere be-

[3] *Journal of Business* (October, 1963).

[4] *Journal of the Institute of Actuaries*, Vol. XCIII.

tween the normal curve and the Cauchy distribution. One characteristic of such distributions is that a second moment does not exist, and the "standard deviation" of annual rates of return for an increasing sample size would never approach a single, finite limit, no matter how long data are collected. Clearly, if a second moment does not exist, the third standardized moment would similarly not exist, so that the development of such numbers in Mr. Turner's paper must be questioned.

It is not my intention to detract from the value of Mr. Turner's paper by grousing about a simple assumption. It is my feeling, however, that the presentation would have been far more convincing if Mr. Turner had been able to show that, even if the underlying assumption of randomness were dropped in favor of, say, a stable symmetric distribution or the Cauchy distribution, there would have been less variation in the net risk premium than that due to the other variable factors studied, that is, the assumed mean rate of return, total equity charge, tax on equities, and mortality and withdrawal rates. It may well be, as a practical matter, that the assumption of randomness can be justified simply because the differences likely to result from other assumptions would not warrant the added complexity.

GORDON D. SHELLARD:

This excellent and lucid paper on the determination of net investment risk premiums for equity-based products is logically divided into sections clearly labeled by the author. My discussion is limited to the section of the paper with the title "Nature of the Probability Density Function," which analyzes historical market data to summarize and describe them in terms of a mathematical model.

Table 1 of the paper presents the mean and standard deviation of certain annual rates of return from December to December of successive years. Table 3 similarly presents the mean and standard deviation, and third standardized moment about the mean (or coefficient of skewness), of certain annual rates of return, but these rates are for every possible set of 12 consecutive months within the period (e.g., March, 1928, to March, 1929; April, 1928, to April, 1929; etc.). Thus for the period 1946–67, involving 22 complete years or 264 months, a total of 253 annual rates of return were calculated for each series.

This calculation of annual rates for each set of 12 consecutive months is somewhat unusual, in that the annual rates for two periods beginning on contiguous months are by no means independent, the rate of return for 11 of the 12 months covered by each of the annual rates being identical. This leads to a considerable duplication of monthly rates implicitly

covered in the table. Thus for the period 1946–67, while rates for January, 1946, and December, 1967, enter only once, those for February, 1946, and November, 1967, enter twice, and those for each of the months December, 1946, through January, 1967, enter 12 times.

Despite this nonindependence and duplication, the characteristics of the distributions presented in Table 3 (the estimated means, standard deviations, and third standardized moments) differ not at all or only inconsequentially from those that could have been calculated from independent annual rates for periods with no overlap. On the other hand, the distribution characteristic estimates shown in Table 3 are no more accurate or precise than those that could have been calculated from independent annual rates for periods with no overlap. Generally speaking, the accuracy of the statistical estimates computed increases with the number of observations included in the calculations. Thus the variance (square of the standard deviation) of the mean, calculated from a sample of size n, is

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n(n-1)} ,$$

where $\bar{x}$ is the sample mean. This is the formula if each of the observations x_i is independent. But if the "number of observations" is artificially increased, as by including each value of x_i twelve times in the summation, and thus apparently increasing n by a factor of 12, the calculation formula changes to yield a value for variance exactly the same as would have been obtained from the uninflated number of observations.

The practical meaning of all this is that the values in Table 3 are, to all intents and purposes, based on the number of years indicated. Thus the values for the period 1946–67 are practically the same as, and just about as accurate as, those that could have been calculated from the twenty-two independent annual rates covering the period.

The monthly rates themselves, which according to the model are independent of each other and of which the annual rates are functions, may be of interest. Using Standard and Poor's Composite Index for 1946–67, we substantially duplicated the figures shown in Table 3 for this period. We then computed for the corresponding independent monthly rates the mean, standard deviation, and third standardized moment (or coefficient of skewness). The resulting figures were 1.08, 3.64, and −0.337. It may be noted that the figures for the mean and standard deviation are reasonably consistent with the corresponding ones for an-

nual rates shown in Table 3, one-twelfth of 13.89 being 1.16, and 14.63 ÷ $\sqrt{12}$ = 4.22. (Note particularly that the standard deviation of the monthly rate is not the standard deviation of one-twelfth the annual rate.) It is less easy to compare skewness, but by the central limit theorem the skewness of the annual rates should be less than that of the monthly, though of the same sign, and from this it follows that, if the distribution of the monthly rates were truly symmetrical (0 skewness), the distribution of annual rates would also be truly symmetrical. Computed estimates of the coefficients of skewness are naturally subject to chance fluctuation.

If the interest rates for successive periods, whether months or years, may be considered a stochastic variable i, then the amount X to which a unit of principal will increase in n periods is also a stochastic variable, $X = \prod_{j=1}^{n} (1 + i_j)$. As the number of factors in the product increases, the distribution of X approaches the logarithmic normal, given by formula (1).

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp \left[- (\ln x - \mu)^2/2\sigma^2\right],$$

where

$$\mu = n \cdot \ln \left[\frac{(1 + \mu_i)}{\sqrt{1 + \left(\frac{\sigma_i}{1 + \mu_i}\right)^2}}\right];$$

$$\sigma^2 = n \cdot \ln \left[1 + \left(\frac{\sigma_i}{1 + \mu_i}\right)^2\right];$$

μ_i is the mean interest rate for one period; and σ_i is the standard deviation of interest rate for one period.

It should be noted that the mean of this distribution of X is at $(1 + \mu_i)^n$, but its mode is at $\exp (\mu - \sigma^2)$. For example, if $\mu_i = 1.08$ per cent and $\sigma_i = 3.64$ per cent, as found above for monthly interest, and if $n = 12$, the mean value of 1 invested twelve months earlier would be 1.1376, but its modal value would be only 1.1114. This illustrates positive skewness, which is characteristic of the logarithmic-normal distribution.

Everything to this point in this discussion has been based on an assumption that the interest rate for one month is uncorrelated with that for any other month preceding or succeeding, and this is the position taken in the paper. Indeed, the section on the nature of the probability density function ends with quotations from a number of authorities and studies to the effect that "the rate of return on common stocks can be treated as an independent random variable," yet this assumption can be critical in the analysis of market behavior and not all authorities or studies are agreed on

the independent random hypothesis. This hypothesis is treated and examined in a collection of papers edited by Paul H. Cootner under the title of *The Random Character of Stock Market Prices* [1]. The editor writes of one of the papers as follows:

> There are two separate issues involved in the study of stock prices. The one he chooses to study is whether, in point of fact, stock prices are a random walk, and his conclusion is that they certainly are not. A quite different question is whether or not stock prices are a sufficiently close approximation of a random walk, where the standard of "closeness" is an economic one—that market participants cannot improve their performance by acting on the regularities.

A somewhat similar pair of questions may be before us in the determination of net investment risk premiums. Answers may be sought through further analyses of actual market histories and through comparison of figures resulting from trial of various models.

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(AUTHOR'S REVIEW OF DISCUSSION)

SAMUEL H. TURNER:

I wish to express my personal appreciation for the valuable discussions prepared by Messrs. Benjamin, George, Horn, Ingraham, Jackson, and Shellard. Because of the limitations of time and space, it was not possible to comment on all points made by the discussants, and an apology is made for this. I have arranged my comments under three headings, although the topics treated are interrelated in many respects.

Rates of Return as Independent Random Variables

Messrs. Benjamin, Horn, Ingraham, Jackson, and Shellard offer valuable comments regarding the hypothesis that the rate of return on common stocks can be treated as an independent random variable. Obviously, acceptance of this hypothesis is not unanimous. Although this topic has not been previously considered in United States actuarial literature, a considerable volume of literature has developed since the late 1950's in nonactuarial publications. It would therefore appear that the greatest value which could be served here is to provide a source of refer-

ences on this topic (sometimes treated under the heading of "random walk theory"). Eleven additional references are included in a supplemental bibliography at the end of this review, four of which are cited in the discussion by Mr. Jackson and Mr. Shellard. The situation may be accurately described in Mr. Jackson's comment: "If the assumption [of randomness] were at all a reasonable one, it should not be necessary to develop so great a body of authority to support it."

Messrs. Ingraham, Jackson, and Shellard offer comments related to the sufficiency of the assumption of randomness from a practical viewpoint. As noted in the paper, the proper criterion for judging a model is whether or not it provides a sufficiently precise representation of the essential features of the system so that observations and conclusions obtained from the model are valid for the "real" system. Within this context, the simulation model developed is intended solely to represent long-term investment performance as one parameter underlying the valuation of risk for an asset value guarantee. It does not represent, or imply an ability to represent, short-term investment performance.

Probability Density Function of Rates of Return

Mr. Horn indicates a possible risk in utilizing a probability density function (p.d.f.) developed from historical investment data to simulate future investment experience and states that the model developed tacitly ignores this risk.

In general, there is always a risk that actual future experience will not conform to that expected or assumed. This is, of course, true for several parameters underlying traditional insurance products—mortality, lapse, interest, expense, etc.—where historical experience is a significant, if not a necessary, consideration in developing assumptions as to expected experience. The return for the assumption of this risk is frequently accounted for in a contingency loading which is determined, where feasible, by evaluating alternative assumptions as to possible future experience.

Specifically, to what extent is the model dependent upon historical investment experience and to what extent is the risk that actual investment experience will not conform to that assumed ignored? The model provides that the mean of the p.d.f. (i.e., the expected long-term total rate of return) is a controlled parameter and, as such, provides that various assumptions as to the expected long-term rate of return can be evaluated. Therefore, reliance upon historical investment experience in the model p.d.f. is with respect to the shape of this p.d.f. only. In this regard, the period of experience underlying the p.d.f. (1880–1967) covers the longest period for which reliable data were available; reflects a variety of social,

economic, and political conditions; and has the greatest "coefficient of variation" of any of the periods indicated in Table 3 of the paper. If a particular investigator feels that the shape of the p.d.f. developed from historical experience may not be representative of that expected, then one approach would be to determine a contingency loading to be included in the price structure. Such a loading could be based solely on the investigator's judgment or, where feasible, by evaluating possible alternative assumptions. There are several possible alternative assumptions regarding the p.d.f. of investment rates of return. Mr. Jackson suggests the use of a stable symmetric distribution or the Cauchy distribution as possible alternative assumptions. Mr. Benjamin's investigations, for example, assume a uniform distribution of a running yield for new money and consider various lengths of cycle time. Mr. Benjamin indicates in his discussion that the length of cycle time does not appear to influence the result. In addition, Mr. Benjamin indicated in a private letter that the pure risk premiums generated in his investigations appear to be of the same order of magnitude as those indicated in the paper.

Reserves for Asset Value Guarantees

Messrs. Benjamin and Horn offer comments related to reserves for asset value guarantees. Although this topic was not an area of specific consideration in the paper, at least within the context of valuation, the following interpretation of the method used to calculate net risk premiums was presented on page 474 of the paper:

> The method . . . assumes that a risk reserve will be accumulated, which together with the asset value of the contracts persisting to the end of the investment period, will be sufficient to provide the asset value guaranteed with respect to all contracts then in force. Such risk reserve would be equal to the net risk premiums accumulated with interest . . . , mortality and withdrawal.

Mr. Benjamin apparently misinterpreted the above explanation of the procedure used to calculate net risk premiums as a suggestion for an "appropriate" reserve for valuation purposes. While the risk reserve defined should be sufficient, if actual experience conforms to that assumed to provide the expected amounts payable under an asset value guarantee, I have neither stated nor hopefully implied that such a risk reserve is an "appropriate" reserve for valuation of asset value guarantees.

Mr. Benjamin states that "a retrospective reserve is dangerous . . . [and that] a prospective reserve on a cautious basis should be set up." I cannot agree that a reserve calculated prospectively is, *ceteris paribus*, necessarily a more conservative representation of liability than a reserve

calculated retrospectively, except where actual experience differs appreciably from that assumed in calculating the reserve and such actual experience is ignored. Furthermore, a prospective reserve of the type referred to by Mr. Benjamin, where the simulation process is carried out each year, would appear to be impractical in its application as a general valuation procedure. Finally, I am not convinced that it is necessary to consider reserves in determining gross premiums for an asset value guarantee, unless the net valuation premium exceeds the net risk premium and applicable contingency loading.

Mr. Horn has assumed that the inclusion of an asset value guarantee in an equity-based contract would require that traditional fixed-dollar reserves be maintained each year for the contract in its entirety. Justification for such an assumption would appear to exist only if an asset value guarantee was made with respect to the amounts available under the contract upon withdrawal at *every* point in time. I do not believe, however, that it would be economically feasible to offer this type of guarantee, since *every* contract in force would be exposed to demand for payment at guaranteed minimum values at *every* point in time. Consequently, I do not believe that the reserve requirement assumed by Mr. Horn should, or will, be imposed.

Because of the nature of asset value guarantees and the different benefit forms in which such guarantees can be applied (e.g., benefits payable at death or maturity, annuity benefits, etc.), an "appropriate" reserve for valuation will depend to a great extent on the actuary's judgment in dealing with each particular situation. Therefore, regulations pertaining to reserves for asset value guarantees might include the following:

1. That the reserve liability for asset value guarantees be established in accordance with actuarial procedures that recognize the nature of benefits provided and, to the extent applicable, the requirements of the standard valuation law.
2. That a statement of valuation standard be filed setting forth the bases, methods, and assumptions used in determining the reserves for asset value guarantees. Subsequently, a company could indicate its continued compliance with such statement of valuation standard.
3. That an opinion by a qualified actuary be filed stating that the reserves established for an asset value guarantee place a sound value on the liability with respect to such guarantees as of the valuation date.

As an example, the following valuation procedure for an asset value guarantee at maturity of an equity-based contract would appear to be consistent with the above and practical in its application. Valuation premiums would be determined for the guarantee using 1958 C.S.O. mortality, $3\frac{1}{2}$ per cent interest, and such actuarial procedures for evaluat-

ing the risk as are deemed sufficient and appropriate by a qualified actuary. Average or composite net valuation premiums based on representative groupings would be acceptable. Such net valuation premiums would be accumulated in the general account of the company in a manner consistent with the reserve bases used. Additional contingency reserves could be maintained as deemed necessary or desirable by the actuary. Prospective gross premium valuations would be made after several years of experience became available and at periodic intervals thereafter (say, every five years) to test the adequacy of the reserves maintained.

In conclusion, the life insurance industry does not adequately respond to the needs of individuals by providing equity-based products which force such individuals to assume the full investment risk under such products. This is particularly true where the assumption of investment risk by the individuals insured substantially reduces or eliminates the security such individuals seek. As stated in the paper, it is both reasonable and appropriate for life insurance companies to offer an additional assurance under its equity-based products whereby the investment risk is assumed by the company. Such assurance, whether related to benefits payable at death or maturity or to annuity benefits, can be provided through asset value guarantees. Because asset value guarantees under equity-based products is a new topic in United States actuarial literature and will be a significant product characteristic of the new generation of insurance products, the discussions of this paper are of considerable value in molding a nucleus of information from which a continuing treatise of actuarial research on the topic of asset value guarantees will hopefully develop.

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