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ACTUARIAL FUNCTIONS AS EXPECTED VALUES

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ABSTRACT

Actuarial calculations involving life contingencies are traditionally made on a present value basis, in which the functions used are discounted for interest and survivorship. This paper shows that some interesting results can be obtained by using expected values rather than present values.

The basic principle introduced in this paper is that the expected value of receipts on any insurance policy is equal to the expected value of payouts. Several identities are derived which illustrate the application of this basic principle in a variety of situations.

The expected value approach not only offers an interesting conceptual framework together with a group of new identities but hopefully might provide a possible starting point in risk theory to consider statistical variation about these expected values.

THE BASIC CONCEPT

ONE of the fundamental principles of life contingencies is that the present value of receipts on a life insurance contract is equal to the present value of the payments to be made on the contract. For the standard ordinary life policy with a death benefit of 1 payable at the end of the policy year of death, this result produces the familiar

$$P_x \cdot \ddot{a}_x = A_x . \qquad (1)$$

An analogous principle using expected values would state that the expected value of receipts is equal to the expected value of the payments to be made. The equation corresponding to formula (1) for the ordinary life policy would be

$$P_x \sum_{t=0}^{\infty} {}_tp_x + i \sum_{t=0}^{\infty} ({}_tV_x + P_x)\, {}_tp_x = 1 . \qquad (2)$$

This equation states that the expected value of premium income at the beginning of each policy year plus the expected value of interest income on the initial reserve for each policy year is equal to the expected (and in this case certain) value of the payment to be made.

Equation (2) can be derived by starting with a formula connecting successive reserves,

$$({}_tV_x + P_x)(1 + i) = q_{x+t} + p_{x+t}\cdot {}_{t+1}V_x \,, \tag{3}$$

and then multiplying by ${}_tp_x$ and rearranging terms, to give

$${}_tp_x({}_tV_x + P_x)i + {}_tp_x\cdot P_x + {}_tp_x\cdot {}_tV_x - {}_{t+1}p_x\cdot {}_{t+1}V_x = {}_{t|}q_x \,. \tag{4}$$

We now sum equation (4) from $t = 0$ to $t = \infty$, and note that all the terms ${}_tp_x\cdot {}_tV_x - {}_{t+1}p_x\cdot {}_{t+1}V_x$ drop out, which gives

$$i\sum_{t=0}^{\infty}{}_tp_x({}_tV_x + P_x) + P_x\sum_{t=0}^{\infty}{}_tp_x = \sum_{t=0}^{\infty}{}_{t|}q_x \,,$$

or

$$P_x\sum_{t=0}^{\infty}{}_tp_x + i\sum_{t=0}^{\infty}({}_tV_x + P_x)\,{}_tp_x = 1 \,. \tag{2}$$

The above result can be generalized somewhat in considering forms of insurance other than ordinary life, summing from $t = 0$ to $t = n - 1$, to give

$$P\sum_{t=0}^{n-1}{}_tp_x + i\sum_{t=0}^{n-1}({}_tV + P)\,{}_tp_x = {}_np_x\cdot {}_nV + {}_nq_x \,. \tag{5}$$

For an n-year term policy, equation (5) becomes

$$P^{\,1}_{x:\overline{n|}}\sum_{t=0}^{n-1}{}_tp_x + i\sum_{t=0}^{n-1}({}_tV^{\,1}_{x:\overline{n|}} + P^{\,1}_{x:\overline{n|}})\,{}_tp_x = {}_nq_x \,.$$

For an n-year endowment policy, equation (5) becomes

$$P_{x:\overline{n|}}\sum_{t=0}^{n-1}{}_tp_x + i\sum_{t=0}^{n-1}({}_tV_{x:\overline{n|}} + P_{x:\overline{n|}})\,{}_tp_x = {}_np_x + {}_nq_x = 1 \,.$$

In both cases the expected value of premium income plus the expected value of interest income is equal to the expected value of the payment to be made.

APPLICATION TO MODIFIED RESERVE SYSTEMS

The approach using expected values rather than present values produces a new insight into the nature of modified reserve systems.

Consider the same ordinary life policy with a first-year net premium

α and renewal net premiums β. Instead of formula (3), we have the following two equations:

$$\alpha(1+i) = q_x + p_x \cdot {}_1V_x^{\text{Mod}} \tag{6}$$

and

$$({}_tV_x^{\text{Mod}} + \beta)(1+i) = q_{x+t} + p_{x+t} \cdot {}_{t+1}V_x^{\text{Mod}} \qquad \text{for} \quad t \geq 1\,. \tag{7}$$

If we multiply both sides of equation (7) by ${}_tp_x$, sum the result from $t=1$ to $t=\infty$, and add equation (6), we obtain

$$\left(\alpha + \beta\sum_{t=1}^{\infty}{}_tp_x\right) + i\left[\alpha + \sum_{t=1}^{\infty}({}_tV_x^{\text{Mod}} + \beta)\,{}_tp_x\right] = 1\,, \tag{8}$$

which is analogous to formula (2).

If equation (8) is subtracted from equation (2), we obtain

$$\left[(P_x - \alpha) + (P_x - \beta)\sum_{t=1}^{\infty}{}_tp_x\right]$$

$$+\, i\left\{(P_x - \alpha) + \sum_{t=1}^{\infty}[({}_tV_x + P_x) - ({}_tV_x^{\text{Mod}} + \beta)]\,{}_tp_x\right\} = 0\,,$$

and, rearranging,

$$\begin{aligned}(\beta - P_x)\sum_{t=1}^{\infty}{}_tp_x &= (P_x - \alpha) \\ &\quad + i\left\{(P_x - \alpha) + \sum_{t=1}^{\infty}[({}_tV_x + P_x) - ({}_tV_x^{\text{Mod}} + \beta)]\,{}_tp_x\right\}.\end{aligned} \tag{9}$$

Formula (9) shows that the expected value of the excesses of modified renewal net premiums over net level premiums is equal to the extra expense allowance in the first policy year plus the expected value of the loss of interest incurred because of the lower reserves held under the modified reserve system.

SOME GENERALIZATIONS

Single Premium Insurance

The preceding results have assumed annual premium contracts. However, similar results will hold for single premium plans. If we take the familiar identity

$$A_{x+t}(1+i) = q_{x+t} + p_{x+t} \cdot A_{x+t+1}\,, \tag{10}$$

multiply by ${}_tp_x$, and sum from $t=0$ to $t=\infty$, we obtain

$$A_x + i\sum_{t=0}^{\infty}A_{x+t} \cdot {}_tp_x = 1\,. \tag{11}$$

Hence the net single premium for whole life insurance of 1 plus the expected value of interest over the life of the contract is equal to the expected (and certain) benefit of 1.

Annuities

The above results for insurance can be extended to annuities as well. As an example, consider a single premium annuity immediate. We know that

$$a_{x+t}(1 + i) = p_{x+t} + p_{x+t} \cdot a_{x+t+1} \, . \tag{12}$$

Again, if we multiply by ${}_tp_x$ and sum from $t = 0$ to $t = \infty$, we obtain

$$a_x + i\sum_{t=0}^{\infty} a_{x+t} \cdot {}_tp_x = \sum_{t=0}^{\infty} {}_{t+1}p_x = e_x \, . \tag{13}$$

Hence the net single premium plus the expected value of the interest is equal to the expected value of the annuity payments to be made, which, in turn, is equal to the curtate expectation of life.

Continuous Functions

The above results have assumed curtate functions throughout. However, analogous results hold for continuous functions.

From life contingencies we have the following differential equation, which describes the instantaneous rate of change of the aggregate reserve fund in terms of premium income, interest income, and benefit payments:

$$\frac{d}{dt}\left({}_t\bar{V} \cdot l_{x+t}\right) = \bar{P} \cdot l_{x+t} + \delta \cdot {}_t\bar{V} \cdot l_{x+t} - l_{x+t} \cdot \mu_{x+t} \, . \tag{14}$$

If we divide by l_x and integrate from $t = 0$ to $t = \infty$ (again assuming a whole life plan), we have

$$\begin{aligned} \int_0^{\infty} d_t\bar{V} \cdot {}_tp_x &= {}_t\bar{V} \cdot {}_tp_x \big|_0^{\infty} = 0 \\ &= \bar{P}\int_0^{\infty} {}_tp_x dt + \delta \int_0^{\infty} {}_t\bar{V} \cdot {}_tp_x dt - \int_0^{\infty} {}_tp_x \cdot \mu_{x+t} dt \, . \end{aligned}$$

Since the last integral is equal to 1, this produces the following formula analogous to formula (2):

$$\bar{P}\int_0^{\infty} {}_tp_x dt + \delta \int_0^{\infty} {}_t\bar{V} \cdot {}_tp_x dt = 1 \, . \tag{15}$$

Compound Interest

In compound interest all payments are assumed certain to be made, and probabilities of survivorship do not enter the calculations. Thus, expected values are also certain values. Nevertheless, some interesting analogous results can be derived which are not normally encountered in the study of compound interest.

As one example, consider an n-year certain annuity immediate. It is simple to derive the following formula, which is analogous to formula (13):

$$a_{\overline{n}|} + i\sum_{t=0}^{n-1} a_{\overline{n-t}|} = n\,. \tag{16}$$

As a second example, it can be shown that the price of a bond plus the value of interest (at the yield rate) on the amortized book values over the life of the bond is equal to the redemption price of the bond plus the sum of the coupons.

Other Generalizations

The above examples illustrate the fact that the concept of using expected values has wide applicability in actuarial theory. Other illustrations easily suggest themselves.

As one example, the above approach can be extended to multilife functions. This would lead to many new identities; for example, in analogy to formula (11) we would have

$$A_{\overset{1}{x}y} + i\sum_{t=0}^{\infty} A_{\overset{1}{\overline{x+t}}:y+t}\cdot {}_tp_{xy} = {}_{\infty}q_{\overset{1}{x}y}\,. \tag{17}$$

As a final example, in setting gross premiums for nonparticipating insurance using a double-decrement table, it is clear that the expected value of gross premiums plus the expected value of interest on the natural reserve (asset share) would have to be equal to the expected value of death benefits to those who die plus the expected value of nonforfeiture benefits to those who lapse plus the expected value of expenses and profit.

SUMMARY

This actuarial note has introduced a very familiar concept, expected value, into actuarial theory in a new fashion. This approach not only offers an interesting conceptual framework together with a group of new identities but hopefully might provide a possible starting point in risk theory to consider statistical variation about these expected values.