TRANSACTIONS OF SOCIETY OF ACTUARIES
1971 VOL. 23 PT. 1 NO. 65

A NEW LOOK AT GAIN AND LOSS ANALYSIS

ARTHUR W. ANDERSON

ABSTRACT

This paper addresses a problem which haunts pension actuaries: how to explain yearly fluctuations and computed pension cost in terms of deviations of actual from expected experience. It presents a method for deriving gain-and-loss-analysis formulas directly from the valuation formulas and shows that the analysis is implicitly defined by the valuation formulas, rather than something which can be considered apart from them.

The paper shows how the method can be used to attack both individual and aggregate cost methods (the resultant analysis formulas for aggregate methods are not merely approximations but exact expressions). It also shows that the development of exact analysis formulas provides not only the basis for computations but also valuable insight into the mechanisms by which cost methods adjust computed costs to reflect actual experience.

THIS paper is written primarily for those actuaries who are involved with work on pension plans, and it presumes on the reader's part a working knowledge of the mechanics of pension valuations. Its purpose is to suggest a new approach to the old problem of gain and loss analysis, based on deductive reasoning.

The quantities computed in a pension valuation, such as normal cost, accrued liability, and the like, have precise mathematical definitions. That is, we can describe to a clerk (or to a computer) how to perform the computations in simple, objective terms: "multiply these quantities by this type of factor and sum the products," and so on. Each of the elemental quantities involved in the valuation this year can be related to its counterpart in next year's valuation by some sort of equation. Therefore, since the computation of the actuarial gain involves only the valuation results for two successive years, if such a thing as a gain and loss analysis exists, it too must be capable of exact mathematical description.

Such an exact formulation of a gain and loss analysis, if it could be found, would not only tell us how to compute each component gain, such as the gain from mortality, but would also yield a complete list of the possible *sources* of gain or loss. Hence we could approach gain and loss analysis from a new, more scientific direction. The traditional approach, described by Dreher,[1] is first to list the possible sources of gain and then to use actuarial intuition in arriving at the formulas to be used to compute the component gains; our approach would be first to acknowledge that the definitions of unfunded accrued liability and normal cost implicitly define the gain and loss analysis and then to derive by direct reasoning (as opposed to "general" reasoning) the sources of gain and the definition of each component.

The more rigorous approach would have practical advantages: First, it would afford 100 per cent accuracy in the computation and thereby make it possible to routinize not only the valuation itself but also the validation thereof by means of analysis of the gain. The traditional method requires the direct attention of the actuary or at least of a highly trained person with considerable powers of judgment, because the analysis is only approximate, and one must decide (*a*) what sources of gain are pertinent and (*b*) what tolerance between actual and analyzed gains is acceptable (the new method would always involve a tolerance of zero, since the algebraic equivalence of the computed gain and the sum of the components is established beforehand). Second, the actuary would be able to sleep more soundly, knowing that he has indeed substituted demonstrations for impressions rather than the other way around as with the traditional approach.

An additional benefit is derived from the deduction of the gain analysis from first principles, in that the actuary gains greater insight into the internal mechanisms of his mathematical model and can make more informed actuarial judgments. For example, it is occasionally desirable to allow for the effect of salary increases by a more conservative interest assumption in place of a salary scale. While the assumption that gains from interest offset losses from salary increases may hold true in the long run, in the short run (especially if there is a low fund balance) it may not. An examination of precise formulas for these two components of the gain would enable the actuary to decide whether the size of probable losses in early years is tolerable.

Since each actuarial valuation is a unique model and the gain and loss analysis is therefore also unique, although there will be a noticeable similarity between analysis formulas from case to case (just as there is similarity in the models themselves), it is difficult to make generalizations powerful enough to be applied effectively in all valuation situations. Therefore, the aim of this paper will be to demonstrate a method of

[1] William A. Dreher, "Gain and Loss Analysis for Pension Fund Valuations," *TSA*, XI, 588.

deriving formulas to suit a particular case rather than to set forth general theories and formulas. Such derivations involve a great many terms and symbols, and it is therefore essential to go about them methodically to avoid utter confusion.

The general method will be first illustrated and then described in Section I, with a simple unit-credit valuation as an example. Section II shows how the method works in a more complicated unit-credit situation. Section III looks at entry age normal in a simple case, and Section IV adds the complication of a salary scale. Section V deals with aggregate funding methods and shows that, contrary to popular misconception, it is possible to analyze "gains" and "losses" under these methods in a completely accurate and straightforward way.

I. THE GENERAL APPROACH: A UNIT-CREDIT EXAMPLE

Consider, as an introductory example, a simple unit-benefit plan providing for retirement benefits in the form of a monthly life annuity (no optional forms) and no ancillary benefits such as for death or severance. Assume no vesting or disability benefits—or at least assume that they will be ignored for funding purposes—so that the valuation is concerned only with active and retired employees. For purposes of the valuation in year zero, define the following three sets of persons included in the valuation:

A_0: Active employees whose respective ages (nearest birthday) on the valuation date are strictly less than their corresponding assumed retirement ages (an assumed retirement age is to be used in the valuation, rather than a retirement-probability table)

B_0: Active employees whose attained ages are equal to their corresponding assumed retirement ages; members of B_0 are assumed to retire immediately

R_0: Retired employees as of the valuation date ("time 0")

The primary actuarial computations are those of the normal cost (NC_0) and the accrued liability (AL_0), which we have decided to compute according to the unit-credit method as follows:

$$NC_0 = \sum_{A_0} \widetilde{UB}_j \frac{N_{y_j}^{(12)}}{D_{x_j}}, \tag{1}$$

$$AL_0 = \sum_{A_0} {}_0AB_j \frac{N_{y_j}^{(12)}}{D_{x_j}} + \sum_{B_0} {}_0AB_j \frac{N_{x_j}^{(12)}}{D_{x_j}} + \sum_{R_0} B_j \frac{N_{x_j}^{(12)}}{D_{x_j}}, \tag{2}$$

where the subscript j refers to each individual in the set over which the summation is being taken and where

x_j = Attained age nearest birthday of j at time 0;

y_j = Assumed retirement age of j (member of A_0);

$\widetilde{UB}_j$ = Unit annual benefit expected to be credited to j during the year following the valuation date; note that no such quantity is defined for members of set B_0, since they are expected to retire immediately;

${}_0AB_j$ = Accrued annual benefit of j at time 0;

B_j = Annual pension of retired life j;

$N_x^{(12)} = N_x - \frac{11}{24}D_x$.

If we let F_0 denote the assets of the pension fund at time 0, we can define the unfunded accrued liability (UAL) as

$$UAL_0 = AL_0 - F_0 . \tag{3}$$

At time 1, a year later, we compute a new unfunded accrued liability as

$$UAL_1 = \sum_{A_1} {}_1AB_j \frac{N_{y_j}^{(12)}}{D_{x_j+1}} + \sum_{B_1} {}_1AB_j \frac{N_{x_j+1}^{(12)}}{D_{x_j+1}} + \sum_{R_1} B_j \frac{N_{x_j+1}^{(12)}}{D_{x_j+1}} - F_1 , \tag{4}$$

where A_1, B_1, and R_1 are sets of employees defined in the same manner as A_0, B_0, and R_0, ${}_1AB_j$ is the accrued benefit of j at time 1, and F_1 is the value of pension-fund assets at time 1.

Also at time 1, according to custom, we compute the gain (G) for the prior year as

$$G = (UAL_0 + NC_0)(1 + i) - K - I_K - UAL_1 , \tag{5}$$

where i is the valuation interest rate, K is the amount of (employer) contributions accrued for the prior year, and I_K is the amount of interest on K at rate i from the date of deposit to the end of the year.

The objective of gain and loss analysis is to break G up into several components, where each component is associated with exactly one of the forces influencing the computation of UAL, so that if the actuarial assumption regarding that force were exactly realized—regardless of whether any other assumptions are realized—the value of the component would be identically zero. Since G is an algebraic expression, we should be able to accomplish this purely by algebraic manipulation of equation (5). The product of such manipulation will be an equation for G which will be the sum of several components of the sort described above, i.e., an exact algebraic expression of the gain and loss analysis.

The general approach to such manipulation is to establish recursion relationships between the various sets and quantities with subscripts 0 and 1; then substitute these relationships into equation (4) and gather terms; and then substitute the resultant expression for UAL_1 into equation (5).

First, let us establish how the sets A_1, B_1, and R_1 are related to A_0, B_0, and R_0. Define the following subsets of $A_0 + B_0 + R_0$ (the union of the three sets):

T: Employees terminating participation in the plan, for whatever reason, during the year

R: Employees retiring during the year; to avoid unnecessary complication at this stage, assume that all employees who retire do so on a monthly pension

Let N be the set of new entrants at time 1. Then it is obvious that

$$A_1 = A_0 - T \cap A_0 - R \cap A_0 - A_0 \cap B_1 + N \cap A_1 , \tag{6}$$

$$B_1 = B_0 + A_0 \cap B_1 - T \cap B_0 - R \cap B_0 + N \cap B_1 , \tag{7}$$

$$R_1 = R_0 - T \cap R_0 + R \tag{8}$$

(assuming that no one enters the retired status directly, without first having appeared as an active participant). Assume that F_1 is related to F_0 as follows:

$$F_1 = F_0 + K + I - B , \tag{9}$$

where I is the interest earnings (implicitly defined by the asset-valuation method used to determine F_0 and F_1) for the year and B is the amount of benefits payable for the year (in this case consisting entirely of monthly pensions).

Before substituting equations (6)–(9) into equation (4), we must define ${}_1AB_j$ on T and R, since this quantity is not normally defined on those sets; let

$${}_1AB_j = {}_0AB_j + \widetilde{UB}_j \qquad \text{for} \quad j \in T \quad \text{or} \quad R . \tag{10}$$

That is, set the accrued benefit at time 1 for those who retire, terminate, die, and so on, equal to the accrued benefit at the end of the year which would be anticipated at time 0.

Now substitute equations (6)–(9) into equation (4):

$$\begin{aligned} UAL_1 = {} & \sum_{A_0} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} - \sum_{T\cap A_0} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} - \sum_{R\cap A_0} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} \\ & - \sum_{A_0\cap B_1} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} + \sum_{N\cap A_1} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} + \sum_{B_0} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} \\ & + \sum_{A_0\cap B_1} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - \sum_{T\cap B_0} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - \sum_{R\cap B_0} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} \qquad (11) \\ & + \sum_{N\cap B_1} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} + \sum_{R_0} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - \sum_{T\cap R_0} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} \\ & + \sum_{R} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - F_0 - K - I + B\,. \end{aligned}$$

Note that the fourth and seventh terms cancel each other.

Define UB_j as the actual unit benefit credited for the year in question (as opposed to $\widetilde{UB}_j$, the benefit expected to be credited in the valuation at time 0); i.e.,

$$ {}_1AB_j = {}_0AB_j + UB_j \qquad \text{for} \quad j \in A_0 \quad \text{or} \quad j \in B_0\,, \qquad (12)$$

where $UB_j = \widetilde{UB}_j$ for $j \in T$ or $j \in R$, according to equation (10). Then the first term of equation (11) can be rewritten as

$$\begin{aligned} \sum_{A_0} {}_1AB_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} = {} & \sum_{A_0} ({}_0AB_j + \widetilde{UB}_j) \frac{N^{(12)}_{v_j}}{D_{x_j+1}} \\ & + \sum_{A_0\cap (A_1+B_1)} (UB_j - \widetilde{UB}_j) \frac{N^{(12)}_{v_j}}{D_{x_j+1}}\,, \end{aligned} \qquad (13)$$

and the sixth term can be rewritten

$$\sum_{B_0} {}_1AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} = \sum_{B_0} {}_0AB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} + \sum_{B_0 \cap B_1} UB_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} . \tag{14}$$

We can now restate equation (11) as follows, where the subscripts j within summation signs are understood:

$$\begin{aligned} UAL_1 = {} & \sum_{A_0} ({}_0AB + \widetilde{UB}) \frac{N^{(12)}_y}{D_{x+1}} + \sum_{A \cap (A_1 + B_1)} (UB - \widetilde{UB}) \frac{N^{(12)}_y}{D_{x+1}} \\ & - \sum_{T \cap A_0} {}_1AB \frac{N^{(12)}_y}{D_{x+1}} - \sum_{R \cap A_0} {}_1AB \frac{N^{(12)}_y}{D_{x+1}} + \sum_{N \cap A_1} {}_1AB \frac{N^{(12)}_y}{D_{x+1}} \\ & + \sum_{B_0} {}_0AB \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{B_0 \cap B_1} UB \frac{N^{(12)}_{x+1}}{D_{x+1}} - \sum_{T \cap B_0} {}_1AB \frac{N^{(12)}_{x+1}}{D_{x+1}} \\ & - \sum_{R \cap B_0} {}_1AB \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{N \cap B_1} {}_1AB \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} \\ & - \sum_{T \cap R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R} B \frac{N^{(12)}_{x+1}}{D_{x+1}} - F_0 - K - I + B . \end{aligned} \tag{15}$$

Now, as a final step in our manipulation of the expression for UAL_1, note the following two identities:

$$\frac{N^{(12)}_y}{D_{x+1}} = \frac{N^{(12)}_y}{D_x} (1 + i) + q_x \frac{N^{(12)}_y}{D_{x+1}} \quad \text{for} \quad x < y \tag{16}$$

and

$$\frac{N^{(12)}_{x+1}}{D_{x+1}} = \frac{N^{(12)}_x}{D_x} (1 + i) - 1 - \tfrac{13}{24} i + q_x \left(\frac{N^{(12)}_{x+1}}{D_{x+1}} + \tfrac{11}{24} \right) , \tag{17}$$

where q_x comprises the total probability of withdrawal from the plan, from all causes.

Now rewrite equation (15), substituting expression (16) into the first term and expression (17) into the sixth and eleventh terms:

$$
\begin{aligned}
UAL_1 = {} & \sum_{A_0} ({}_0AB + \widetilde{UB}) \left[\frac{N_v^{(12)}}{D_x} (1+i) + q_x \frac{N_v^{(12)}}{D_{x+1}} \right] \\
& + \sum_{A_0 \cap (A_1 + B_1)} (UB - \widetilde{UB}) \frac{N_v^{(12)}}{D_{x+1}} - \sum_{T \cap A_0} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} \\
& - \sum_{R \cap A_0} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} + \sum_{N \cap A_1} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} \\
& + \sum_{B_0} {}_0AB \left[\frac{N_x^{(12)}}{D_x} (1+i) - 1 - \tfrac{13}{24} i + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\
& + \sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& + \sum_{N \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R_0} B \left[\frac{N_x^{(12)}}{D_x} (1+i) - 1 - \tfrac{13}{24} i \right. \\
& \qquad\qquad \left. + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\
& - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - F_0 - K - I + B \,.
\end{aligned} \tag{18}
$$

Recalling equations (1), (2), and (10), we can now rewrite equation (18) as

$$
\begin{aligned}
UAL_1 = {} & (UAL_0 + NC_0)(1+i) + iF_0 - K - I + B \\
& + \sum_{A_0} q_x \, {}_1\widetilde{AB} \frac{N_v^{(12)}}{D_{x+1}} + \sum_{A_0 \cap (A_1 + B_1)} (UB - \widetilde{UB}) \frac{N_v^{(12)}}{D_{x+1}} \\
& - \sum_{T \cap A_0} {}_1\widetilde{AB} \frac{N_v^{(12)}}{D_{x+1}} - \sum_{R \cap A_0} {}_1\widetilde{AB} \frac{N_v^{(12)}}{D_{x+1}} + \sum_{N \cap A_1} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} \\
& - \sum_{B_0} {}_1\widetilde{AB} (1 + \tfrac{13}{24} i) + \sum_{B_0} q_x \, {}_1\widetilde{AB} \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\
& + \sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_1\widetilde{AB} \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_1\widetilde{AB} \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& + \sum_{N \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R_0} B (1 + \tfrac{13}{24} i) \\
& + \sum_{R_0} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} \,,
\end{aligned} \tag{19}
$$

where ${}_1\widetilde{AB}$ denotes ${}_0AB + \widetilde{UB}$.

Finally, substitute expression (19) into equation (5):

$$
\begin{aligned}
G = I - iF_0 - I_K - B - \sum_{A_0} q_x\, {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} \\
- \sum_{A_0 \cap (A_1+B_1)} (UB - \widetilde{UB})\, \frac{N_y^{(12)}}{D_{x+1}} + \sum_{T \cap A_0} {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} \\
+ \sum_{R \cap A_0} {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} - \sum_{N \cap A_1} {}_1AB\, \frac{N_y^{(12)}}{D_{x+1}} + \sum_{B_0} {}_1\widetilde{AB} + \tfrac{13}{24} i \sum_{B_0} {}_1\widetilde{AB} \\
- \sum_{B_0} q_x\, {}_1\widetilde{AB} \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) - \sum_{B_0 \cap B_1} UB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{T \cap B_0} {}_1\widetilde{AB}\, \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
+ \sum_{R \cap B_0} {}_1\widetilde{AB}\, \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{N \cap B_1} {}_1AB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R_0} B + \tfrac{13}{24} i \sum_{R_0} B \\
- \sum_{R_0} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) + \sum_{T \cap R_0} B\, \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R} B\, \frac{N_{x+1}^{(12)}}{D_{x+1}} .
\end{aligned} \tag{20}
$$

Now we can improve the appearance of equation (20) by rearranging terms, grouping all terms having to do with interest; all having to do with withdrawal (i.e., all summations over T and all those involving q_x); all having to do with retirement; and all others. Then, remembering that ${}_1\widetilde{AB} = {}_0AB$ for members of B_0, we write:

$$
\begin{aligned}
G = \left[I - iF_0 - I_K + \tfrac{13}{24} i \left(\sum_{B_0} {}_0AB + \sum_{R_0} B \right) \right] \\
+ \left[\sum_{T \cap A_0} {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} + \sum_{T \cap B_0} {}_0AB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{T \cap R_0} B\, \frac{N_{x+1}^{(12)}}{D_{x+1}} \right. \\
- \sum_{A_0} q_x\, {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} - \sum_{B_0} q_x\, {}_0AB \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\
\left. - \sum_{R_0} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\
+ \left[\sum_{R \cap A_0} {}_1\widetilde{AB}\, \frac{N_y^{(12)}}{D_{x+1}} + \sum_{R \cap B_0} {}_0AB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R} B\, \frac{N_{x+1}^{(12)}}{D_{x+1}} \right. \\
\left. - \sum_{B_0 \cap B_1} UB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{B_0} {}_0AB + \sum_{R_0} B - B \right] \\
- \left[\sum_{A_0 \cap (A_1+B_1)} (UB - \widetilde{UB})\, \frac{N_y^{(12)}}{D_{x+1}} \right] \\
- \left[\sum_{N \cap A_1} {}_1AB\, \frac{N_y^{(12)}}{D_{x+1}} + \sum_{N \cap B_1} {}_1AB\, \frac{N_{x+1}^{(12)}}{D_{x+1}} \right] .
\end{aligned} \tag{21}
$$

The bracketed terms in equation (21) represent, respectively, the following.

1. The gain from interest, which can be interpreted as the interest actually earned (I), less the interest expected to be earned on the fund.
2. The gain from mortality, termination, and so on, expressed as the actual accrued liability released, less that expected to be released. This term could be made more specific by defining T and q_x in more detail, e.g., $T = W + D$ and $q_x = q_x^{(T)} = q_x^{(w)} + q_x^{(m)}$, where T consists of withdrawals and deaths.
3. The gain from retirements, expressed in two parts: (a) the accrued liability expected for those who retired, less that actually held at time 1, plus (b) the excess of benefits expected to be paid over those actually paid. Note that benefits expected to be paid to members of B_0 are included here, since B_0 is treated in every respect as R_0.
4. The loss from benefit increases (or salary increases), i.e., from unit benefits granted which exceed those anticipated at the beginning of the year. Note, in this connection, that the term

$$-\sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

   was included in the gain from retirements.
5. The loss from new entrants, consisting of the present value of accrued benefits for new entrants. For plans in which employees enter the plan on the valuation date, this term would be zero, unless credit were given for service prior to inclusion in the valuation.

Situations as simple as the case just examined are relatively rare, so that equation (21) will not have general applicability. It is, however, an exact expression of the gain G in terms of what appear to be reasonably defined components according to the criteria established at the beginning of the discussion; thus it assures us that the whole will exactly equal the sum of the parts, making gain and loss analysis according to this formula particularly valuable, since the "tolerance" between the actual gain and the analyzed gain is zero (neglecting possible rounding discrepancies).

Another advantage of analysis by exact formula is that one knows exactly what data should be computed in each year's valuation in anticipation of the next year's gain and loss analysis. In the example just discussed, one should compute (in addition to the usual quantities of normal cost and accrued liability) the following.

1. $\sum_{B_0} {}_0AB$ and $\sum_{R_0} B$

2. Accrued liability one year hence for each active and retired member included in the valuation, to allow such terms as

$$\sum_{T \cap A_0} {}_1\widetilde{A}B \frac{N_y^{(12)}}{D_{x+1}}$$

to be computed next year, when the members of set T will first become known
3. Expected termination releases—last three terms in the second set of brackets of equation (21)

These quantities could be difficult to ferret out if the valuation included many members and one had to reconstruct a prior year's computations. If a formula for the gain and loss analysis has been derived, the analysis can be programmed into the regular valuation procedure; then one should be able to arrive at exactly the same gain figure by two independent methods.

A third advantage of the formula method is that, although the derivation of the formula is somewhat lengthy, one will probably spend less time—with more certain results—deriving and applying the formula than by using the usual intuitive approach to gain and loss analysis. Once one has done for himself a few derivations like the one above, he finds new situations substantially easier to work out.

Let us now review the steps by which formula (21) was derived, which steps, as we shall show, hold the key to solution of any particular gain and loss problem.

1. Write down the formulas for the basic valuation results (normal cost, accrued liability, and so on) in terms of summations over various classifications (sets) of plan participants (cf. formulas [1]–[4]) for each of two successive years.
2. Write the formula for actuarial gain (except for aggregate funding methods, where these steps are modified somewhat, as discussed later), i.e., the quantity to be analyzed into components (formula [5]).
3. Express the recursion relations between each set used in step 1, defining necessary subsets (formulas [6]–[8]), as well as recursion relations between terms not expressed as summations over member sets (e.g., between F_0 and F_1; see formula [9]).
4. Rewrite the expressions for the second-year valuation results of step 1, using the recursion relations of step 3; make sure to extend definitions of terms within summation signs to sets on which they may not have been defined originally (formulas [10]–[15]).
5. Using recursion relationships among the various actuarial functions

used for the second year (formulas [16]–[17], for example), rewrite step 4 (formulas [18]–[19]), and substitute this result into the formula for aggregate gain (formula [20]).

6. Rearrange terms so as to express the components of the total gain (formula [21]), and assign verbal descriptions to the computation of each component.

The presentation of the gain and loss analysis to the client has been well covered by Dreher[2] and need not be discussed further, although this step is no less important than the actual computation.

II. APPLICATION OF THE METHOD TO A MORE COMPLEX UNIT-CREDIT SITUATION

Consider a unit-benefit, contributory plan with a modified-cash-refund postretirement death benefit. For active lives, rather than make a refined calculation of the present value of this benefit, we have decided merely to add 2 per cent to the straight life-annuity value. For retired lives we shall use a more refined approach, as discussed below. According to the terms of our hypothetical plan, employee contributions are refundable with interest upon death or termination before retirement. However, the rate at which such interest is computed is less than the valuation rate, and we have decided to simplify our formulas by assuming that contributions are paid at the beginning of each year and are credited with the valuation interest rate for future years.

Thus we have decided to use the following formulas, which will have a familiar look:

$$NC_0 = \sum_{A_0}\left[\widetilde{UB}_j \frac{N_{y_j}^{(12)}}{D_{x_j}}(1.02) + \widetilde{C}_j \frac{l_{x_j} - l_{y_j}}{l_{x_j}}\right], \tag{22}$$

$$\begin{aligned} UAL_0 = {} & \sum_{A_0}\left[{}_0AB_j \frac{N_{y_j}^{(12)}}{D_{x_j}}(1.02) + {}_0AC_j \frac{l_{x_j} - l_{y_j}}{l_{x_j}}\right] \\ & + \sum_{R_0} {}_0AB_j \frac{N_{x_j}^{(12)}}{D_{x_j}}(1.02) \\ & + \sum_{R_0} B_j \frac{N_{x_j}^{(12)} + n_j^0 M_{x_j} - R_{x_j}^{(12)} + R_{x_j+n_j^0}^{(12)}}{D_{x_j}} \\ & + \sum_{R_0'} B_j \frac{N_{x_j}^{(12)}}{D_{x_j}} - F_0, \end{aligned} \tag{23}$$

[2] *Op. cit.*

where the following new symbols have been introduced:

$\widetilde{C}_j$ = Contribution anticipated during ensuing year by employee j;
${}_0AC_j$ = Accumulated contributions of employee j at time 0, at the refund interest rate;
n_j^0 = Number of years (rounded to the nearest integer) remaining, at time 0, in the death-benefit period;
$R_x^{(12)} = R_x - \frac{11}{24}M_x$;
R_0' = Set of all retired lives whose death-benefit period has expired;
R_0 = Set of retired employees still in death-benefit period.

The following year we have

$$
\begin{aligned}
UAL_1 = {} & \sum_{A_1}\left[{}_1AB_j \frac{N_{y_j}^{(12)}}{D_{x_j+}}(1.02) + {}_1AC_j \frac{l_{x_j+1} - l_{y_j}}{l_{x_j+1}}\right] \\
& + \sum_{B_1} {}_1AB_j \frac{N_{x_j+1}^{(12)}}{D_{x_j+1}}(1.02) \\
& + \sum_{R_1} B_j \frac{N_{x_j+1}^{(12)} + n_j^1 M_{x_j+1} - R_{x_j+1}^{(12)} + R_{x_j+n_j^1+1}^{(12)}}{D_{x_j+1}} \\
& \qquad + \sum_{R_1'} B_j \frac{N_{x_j+1}^{(12)}}{D_{x_j+1}} - F_1 ,
\end{aligned}
\tag{24}
$$

where n_j^1 is the (integral) number of years remaining in the death-benefit period at time 1, and symbols with the subscript 1 are defined as their counterparts at time 0. Note that NC_0 is actually the total normal cost to be borne by employees as well as employer in this case. The "normal cost" which would be quoted the employer would probably be

$$
NC_0 - \sum_{A_0} \widetilde{C}_j = \sum_{A_0}\left[\widetilde{UB}_j \frac{N_{y_j}^{(12)}}{D_{x_j}}(1.02) - \widetilde{C}_j \frac{l_{y_j}}{l_{x_j}}\right].
$$

We have now completed step 1 of the general plan of attack outlined in Section I. Step 2 consists of writing the formula by which the aggregate gain, G, is computed:

$$
G = (UAL_0 + NC_0)(1 + i) - K - I_K - UAL_1 , \tag{25}
$$

where K is the total amount of employer and employee contributions paid into the fund for the year, and I_K is the interest thereon at rate i from the date of payment into the fund to the end of the year (time 1). Contributions made after the year end applicable to the prior year are generally not discounted back to the year end but are simply disregarded in the computation of I_K.

Step 3 consists of stating the relationships between similar sets of employees at times 0 and 1:

$$A_1 = A_0 - T \cap A_0 - R \cap A_0 - A_0 \cap B_1 + N \cap A_1 , \tag{26}$$

$$B_1 = B_0 + A_0 \cap B_1 - T \cap B_0 - R \cap B_0 + N \cap B_1 , \tag{27}$$

$$R_1 = R_0 - R_0 \cap R_1' + R \cap R_1 - T \cap R_0 , \tag{28}$$

$$R_1' = R_0' + R_0 \cap R_1' + R \cap R_1' - T \cap R_0' , \tag{29}$$

and stating the relationship between F_0 and F_1:

$$F_1 = F_0 + K + I - B^{(t)} - B^{(r)} , \tag{30}$$

where $B^{(t)}$ represents termination and death benefits payable for the year (i.e., refunds of employee contributions) and $B^{(r)}$ is retirement benefits.

In step 4 expressions (26)–(30) are substituted into equation (24), after the definitions of certain terms are extended so that the summations after the substitutions will be well defined; let

$$_1AB_j = {}_1\widetilde{AB}_j = {}_0AB_j + \widetilde{UB}_j , \tag{31}$$

$$_1AC_j = {}_1\widetilde{AC}_j = {}_0AC_j(1 + i') + \widetilde{C}_j , \tag{32}$$

for $j \in T$ or $j \in R$, where i' is the rate of interest credited to employee contributions.

Now we can rewrite equation (24) as follows (where the subscript j will be understood):

$$\begin{aligned} UAL_1 = {} & \sum_{A_0} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} (1.02) + \sum_{A_0} {}_1AC \frac{l_{x+1} - l_v}{l_{x+1}} \\ & - \sum_{T \cap A_0} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} (1.02) - \sum_{T \cap A_0} {}_1AC \frac{l_{x+1} - l_v}{l_{x+1}} \\ & - \sum_{R \cap A_0} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} (1.02) - \sum_{R \cap A_0} {}_1AC \frac{l_{x+1} - l_v}{l_{x+1}} \\ & - \sum_{A_0 \cap B_1} {}_1AB \frac{N_v^{(12)}}{D_{x+1}} (1.02) - \sum_{A_0 \cap B_1} {}_1AC \frac{l_{x+1} - l_v}{l_{x+1}} \end{aligned} \tag{33}$$

$$+ \sum_{N \cap A_1} {}_1AB \frac{N_y^{(12)}}{D_{x+1}} (1.02) + \sum_{N \cap A_1} {}_1AC \frac{l_{x+1} - l_y}{l_{x+1}} \qquad (33\ cont.)$$

$$+ \sum_{B_0} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) + \sum_{A_0 \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02)$$

$$- \sum_{T \cap B_0} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) - \sum_{R \cap B_0} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02)$$

$$+ \sum_{N \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) + \sum_{R_0} Bf(x+1, n^1)$$

$$- \sum_{R_0 \cap R_1'} Bf(x+1, n^1) + \sum_{R \cap R_1} Bf(x+1, n^1)$$

$$- \sum_{T \cap R_0} Bf(x+1, n^1) + \sum_{R_0'} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R_0 \cap R_1'} B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$+ \sum_{R \cap R_1'} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0'} B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- F_0 - K - I + B^{(t)} + B^{(r)} ,$$

where

$$f(x, n) = \begin{cases} \dfrac{N_x^{(12)} + nM_x - R_x^{(12)} + R_{x+n}^{(12)}}{D_x} & \text{for} \quad n > 0 , \\[2ex] \dfrac{N_x^{(12)}}{D_x} & \text{for} \quad n = 0 . \end{cases}$$

Look over the terms of equation (33), and note the following: (1) the seventh and twelfth terms cancel; (2) the eighth term is identically zero, because $x_j + 1 = y_j$ for $j \in B_1$; (3) the seventeenth and twenty-first terms cancel, since $n_j^1 = 0$ for $j \in R_1'$, and $f(x+1, 0) = N_{x+1}^{(12)}/D_{x+1}$ by definition.

For step 5, note the following recursion relationships.

$${}_1AB_j = {}_0AB_j + UB_j = {}_0AB_j + \widetilde{UB}_j + (UB_j - \widetilde{UB}_j) ,$$

$$j \in A_0 \cap (A_1 + B_1) ;$$

$${}_1AC_j = {}_0AC_j(1 + i') + \widetilde{C}_j + (C_j - \widetilde{C}_j) , \qquad j \in A_0 \cap (A_1 + B_1) .$$

Equation (33) can now be restated thus:

$$
\begin{aligned}
UAL_1 = {} & \sum_{A_0}({}_0AB + \widetilde{UB})\,\frac{N_y^{(12)}}{D_{x+1}}\,(1.02) \\
& + \sum_{A_0 \cap (A_1+B_1)}(UB - \widetilde{UB})\,\frac{N^{(12)}}{D_{x+1}}\,(1.02) \\
& + \sum_{A_0}[{}_0AC(1+i') + \widetilde{C}]\,\frac{l_{x+1}-l_y}{l_{x+1}} \\
& + \sum_{A_0 \cap (A_1+B_1)}(C - \widetilde{C})\,\frac{l_{x+1}-l_y}{l_{x+1}} \\
& - \sum_{T \cap A_0}({}_0AB + \widetilde{UB})\,\frac{N_y^{(12)}}{D_{x+1}}\,(1.02) \\
& - \sum_{T \cap A_0}[{}_0AC(1+i') + \widetilde{C}]\,\frac{l_{x+1}-l_y}{l_{x+1}} \\
& - \sum_{R \cap A_0}({}_0AB + \widetilde{UB})\,\frac{N^{(12)}}{D_{x+1}}\,(1.02) \\
& - \sum_{R \cap A_0}[{}_0AC(1+i') + \widetilde{C}]\,\frac{l_{x+1}-l_y}{l_{x+1}} \\
& + \sum_{N \cap A_1}{}_1AB\,\frac{N_y^{(12)}}{D_{x+1}}\,(1.02) + \sum_{N \cap A_1}{}_1AC\,\frac{l_{x+1}-l_y}{l_{x+1}} \\
& + \sum_{B_0}{}_0AB\,\frac{N_{x+1}^{(12)}}{D_{x+1}}\,(1.02) - \sum_{R \cap B_0}({}_0AB + UB)\,\frac{N_{x+1}^{(12)}}{D_{x+1}}\,(1.02) \\
& + \sum_{N \cap B_1}{}_1AB\,\frac{N_{x+1}^{(12)}}{D_{x+1}}\,(1.02) + \sum_{R_0}Bf(x+1, n-1) \\
& + \sum_{R \cap R_1}Bf(x+1, n-1) - \sum_{T \cap R_0}Bf(x+1, n-1) \\
& + \sum_{R_0'}B\,\frac{N_{x+1}^{(12}}{D_{x+1}} + \sum_{R \cap R_1'}B\,\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0'}B\,\frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& \qquad - F_0 - K - I + B^{(t)} + B^{(r)}\,,
\end{aligned}
\tag{34}
$$

where n denotes n^0.

The reader will easily verify that

$$\frac{l_{x+1}-l_y}{l_{x+1}} = \frac{l_x - l_y}{l_x} - q_x\,\frac{l_y}{l_{x+1}} \qquad \text{for} \quad y > x$$

and

$$
\begin{aligned}
f(x+1, n-1) = {} & f(x, n)(1+i) - 1 - \tfrac{13}{24}i \\
& + q_x[f(x+1, n-1) - (n-1)] \qquad \text{for} \quad n > 0\,.
\end{aligned}
\tag{35}
$$

Then, noting equations (16) and (17), we can substitute recursion relationships into those terms of equation (34) that are likely to yield expressions like those of equations (22) and (23), as follows:

$$
\begin{aligned}
UAL_1 = {} & \sum_{A_0} ({}_0AB + \widetilde{UB})(1.02)\left[\frac{N_y^{(12)}}{D_x}(1+i) + q_x \frac{N_y^{(12)}}{D_{x+1}}\right] \\
& + \sum_{A_0 \cap (A_1 + B_1)} (UB - \widetilde{UB}) \frac{N_y^{(12)}}{D_{x+1}} (1.02) \\
& + \sum_{A_0} [{}_0AC(1+i') + \widetilde{C}] \left(\frac{l_x - l_y}{l_x} - q_x \frac{l_y}{l_{x+1}}\right) \\
& + \sum_{A_0 \cap (A_1 + B_1)} (C - \widetilde{C}) \frac{l_{x+1} - l_y}{l_{x+1}} \\
& - \sum_{T \cap A_0} \left\{ ({}_0AB + \widetilde{UB}) \frac{N_y^{(12)}}{D_{x+1}} (1.02) \right. \\
& \qquad \left. + [{}_0AC(1+i') + \widetilde{C}] \frac{l_{x+1} - l_y}{l_{x+1}} \right\} \\
& - \sum_{R \cap A_0} \left\{ ({}_0AB + \widetilde{UB}) \frac{N_y^{(12)}}{D_{x+1}} (1.02) \right. \\
& \qquad \left. + [{}_0AC(1+i') + \widetilde{C}] \frac{l_{x+1} - l_y}{l_{x+1}} \right\} \\
& + \sum_{N \cap A_1} \left[{}_1AB \frac{N^{(12)}}{D_{x+1}} (1.02) + {}_1AC \frac{l_{x+1} - l_y}{l_{x+1}} \right] \\
& + \sum_{B_0} {}_0AB \left[\frac{N_x^{(12)}}{D_x} (1+i) - 1 - \tfrac{13}{24} i \right. \\
& \qquad \left. + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] (1.02) \\
& + \sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) - \sum_{T \cap B_0} {}_0AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) \\
& - \sum_{R \cap B_0} ({}_0AB + UB) \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) + \sum_{N \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) \\
& + \sum_{R_0} B \{ f(x, n)(1+i) - 1 - \tfrac{13}{24} i \\
& \qquad + q_x [f(x+1, n-1) - (n-1)] \} \\
& + \sum_{R \cap R_1} Bf(x+1, n-1) - \sum_{T \cap R_0} Bf(x+1, n-1) \\
& + \sum_{R_0'} B \left[\frac{N_x^{(12)}}{D_x} (1+i) - 1 - \tfrac{13}{24} i + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\
& + \sum_{R \cap R_1'} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{T \cap R_0'} B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& \qquad - F_0 - K - I + B^{(i)} + B^{(r)} \,.
\end{aligned} \tag{36}
$$

If we now utilize the notation of equation (10), namely,

$${}_1\widetilde{AB}_j = {}_0AB_j + \widetilde{UB}_j$$

for members of A_0 and B_0, and

$${}_1\widetilde{AC}_j = {}_0AC_j(1 + i') + \widetilde{C}_j\,, \tag{37}$$

we can rewrite equation (36) as follows:

$$\begin{aligned}
UAL_1 = {} & (UAL_0 + NC_0)(1 + i) + \sum_{A_0} q_x \, {}_1\widetilde{AB}(1.02) \frac{N_y^{(12)}}{D_{x+1}} \\
& + \sum_{A_0 \cap (A_1 + B_1)} (UB - \widetilde{UB}) \frac{N_y^{(12)}}{D_{x+1}} (1.02) \\
& - i \sum_{A_0} \widetilde{C} \frac{l_x - l_y}{l_x} - (i - i') \sum_{A_0} {}_0AC \frac{l_x - l_y}{l_x} \\
& - \sum_{A_0} q_x \, {}_1\widetilde{AC} \frac{l_y}{l_{x+1}} + \sum_{A_0 \cap (A_1 + B_1)} (C - \widetilde{C}) \frac{l_{x+1} - l_y}{l_{x+1}} \\
& - \sum_{T \cap A_0} \left[{}_1\widetilde{AB} \frac{N_y^{(12)}}{D_{x+1}} (1.02) + {}_1\widetilde{AC} \frac{l_{x+1} - l_y}{l_{x+1}} \right] \\
& - \sum_{R \cap A_0} \left[{}_1\widetilde{AB} \frac{N_y^{(12)}}{D_{x+1}} (1.02) + {}_1\widetilde{AC} \frac{l_{x+1} - l_y}{l_{x+1}} \right] \\
& + \sum_{N \cap A_1} \left[{}_1AB \frac{N_y^{(12)}}{D_{x+1}} (1.02) + {}_1AC \frac{l_{x+1} - l_y}{l_{x+1}} \right] \\
& + \sum_{B_0} {}_0AB[-1 - \tfrac{13}{24} i](1.02) \\
& + \sum_{B_0} q_x \, {}_0AB(1.02) \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\
& + \sum_{B_0} q_x \, {}_0AB(1.02) \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\
& + \sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) - \sum_{T \cap B_0} {}_0AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) \\
& - \sum_{R \cap B_0} {}_0AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) - \sum_{R \cap B_0} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02)
\end{aligned} \tag{38}$$

$$+ \sum_{N\cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02) - \sum_{R_0} B(1 + \tfrac{13}{24}i) \qquad \text{(38 cont.)}$$

$$+ \sum_{R_0} q_x[f(x+1, n-1) - n - 1]$$

$$+ \sum_{R\cap R_1} Bf(x+1, n-1) - \sum_{T\cap R_0} Bf(x+1, n-1)$$

$$- \sum_{R_0'} B(1 + \tfrac{13}{24}i) + \sum_{R_0'} q_x B\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right)$$

$$+ \sum_{R\cap R_1'} B\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T\cap R_0'} B\frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$+ iF_0 - K - I + B^{(t)} + B^{(r)} .$$

Let us now substitute expression (38) into equation (25):

$$G = -I_K + I - iF_0 - B^{(t)} - B^{(r)} - \sum_{A_0} q_x\, {}_1\widetilde{AB}(1.02)\frac{N_y^{(12)}}{D_{x+1}}$$

$$- \sum_{A_0\cap(A_1+B_1)} (UB - \widetilde{UB})\frac{N_y^{(12)}}{D_{x+1}}(1.02)$$

$$+ i\sum_{A_0}\widetilde{C}\frac{l_x - l_y}{l_x} + (i - i')\sum_{A_0} {}_0AC\frac{l_x - l_y}{l_x}$$

$$+ \sum_{A_0} q_x\, {}_1\widetilde{AC}\frac{l_y}{l_{x+1}} - \sum_{A_0\cap(A_1+B_1)} (C - \widetilde{C})\frac{l_{x+1} - l_y}{l_{x+1}}$$

$$+ \sum_{T\cap A_0}\left[{}_1\widetilde{AB}\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1\widetilde{AC}\frac{l_{x+1} - l_y}{l_{x+1}}\right] \qquad (39)$$

$$+ \sum_{R\cap A_0}\left[{}_1\widetilde{AB}\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1\widetilde{AC}\frac{l_{x+1} - l_y}{l_{x+1}}\right]$$

$$- \sum_{N\cap A_1}\left[{}_1AB\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1AC\frac{l_{x+1} - l_y}{l_{x+1}}\right]$$

$$+ 1.02\sum_{B_0} {}_0AB + \tfrac{13}{24}i(1.02)\sum_{B_0} {}_0AB$$

$$- \sum_{B_0} q_x\, {}_0AB(1.02)\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right) - \sum_{B_0\cap B_1} UB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)$$

$$+\sum_{T\cap B_0} {}_0AB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)+\sum_{R\cap B_0} {}_0AB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02) \qquad (39\ cont.)$$

$$-\sum_{N\cap B_1} {}_1AB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)+\sum_{R_0}B+\tfrac{13}{24}i\sum_{R_0}B$$

$$-\sum_{R_0}q_xB[f(x+1,n-1)-(n-1)]$$

$$-\sum_{R\cap R_1}Bf(x+1,n-1)+\sum_{T\cap R_0}Bf(x+1,n-1)$$

$$+\sum_{R_0'}B+\tfrac{13}{24}i\sum_{R_0'}B-\sum_{R_0'}q_xB\left(\frac{N_{x+1}^{(12)}}{D_{x+1}}+\tfrac{11}{24}\right)$$

$$-\sum_{R\cap R_1'}B\frac{N_{x+1}^{(12)}}{D_{x+1}}+\sum_{T\cap R_0'}B\frac{N_{x+1}^{(12)}}{D_{x+1}}.$$

This completes step 5 of the plan outlined in Section I and marks the end of the scientific phase of the derivation. Step 6 consists of rearranging the terms in equation (39) and attaching a verbal interpretation to the results, a task which calls upon the imaginative resources of the actuary. It is important to realize that the designation of some portion of equation (39) as "gain from mortality" or the like is fairly arbitrary, but certain designations will have an intuitive appeal. As in the example of Section I, let us group together those terms involving interest, those involving mortality (or termination from all causes), and so on:

$$G=\Big\{I-iF_0-I_K+i\sum_{A_0}\widetilde{C}\frac{l_x-l_y}{l_x}+(i-i')\sum_{A_0}{}_0AC\frac{l_x-l_y}{l_x}$$

$$+\tfrac{13}{24}i\Big(\sum_{R_0}B+\sum_{R_0'}B+1.02\sum_{B_0}{}_0AB\Big)\Big\}$$

$$+\Big\{\sum_{T\cap A_0}\Big[{}_1\widetilde{AB}\frac{N_y^{(12)}}{D_{x+1}}(1.02)+{}_1\widetilde{AC}\frac{l_{x+1}-l_y}{l_{x+1}}\Big] \qquad (40)$$

$$+\sum_{T\cap B_0}{}_0AB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)+\sum_{T\cap R_0}Bf(x+1,n-1)$$

$$+\sum_{T\cap R_0'}B\frac{N_{x+1}^{(12)}}{D_{x+1}}-\sum_{A_0}q_x\Big[{}_1\widetilde{AB}(1.02)\frac{N_y^{(12)}}{D_{x+1}}$$

$$+{}_1\widetilde{AC}\frac{l_{x+1}-l_y}{l_{x+1}}-{}_1\widetilde{AC}\Big]$$

$$- \sum_{B_0} q_x \, {}_0AB \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) (1.02) \qquad (40 \textit{ cont.})$$

$$- \sum_{R_0} q_x B [f(x+1, n-1) - (n-1)]$$

$$- \sum_{R_0'} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) + 0.02 \sum_{B_0} {}_0AB - B^{(t)} \Big\}$$

$$+ \Big\{ \sum_{R \cap A_0} \left[{}_1\widetilde{AB} \frac{N_y^{(12)}}{D_{x+1}} (1.02) + {}_1\widetilde{AC} \frac{l_{x+1} - l_y}{l_{x+1}} \right]$$

$$+ \sum_{R \cap B_0} {}_0AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) - \sum_{B_0 \cap B_1} UB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02)$$

$$- \sum_{R \cap R_1} B f(x+1, n-1) - \sum_{R \cap R_1'} B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$+ \sum_{B_0} {}_0AB + \sum_{R_0} B + \sum_{R_0'} B - B^{(r)} \Big\}$$

$$- \Big\{ \sum_{N \cap A_1} \left[{}_1AB \frac{N_y^{(12)}}{D_{x+1}} (1.02) + {}_1AC \frac{l_{x+1} - l_y}{l_{x+1}} \right]$$

$$+ \sum_{N \cap B_1} {}_1AB \frac{N_{x+1}^{(12)}}{D_{x+1}} (1.02) \Big\}$$

$$- \Big\{ \sum_{A_0 \cap (A_1 + B_1)} \Big[(UB - \widetilde{UB}) \frac{N_y^{(12)}}{D_{x+1}} (1.02) + (C - \widetilde{C}) \frac{l_{x+1} - l_y}{l_{x+1}} \Big] \Big\} .$$

Equation (40) may be interpreted as follows:

I. Gain from interest equals

(1) actual interest income, I,

less (2) required interest, which equals

(*a*) interest on initial fund balance, iF_0,

plus (*b*) interest on contributions (employer plus employee) from receipt, I_K,

less (*c*) interest on expected benefit payments,

$$\tfrac{13}{24} i \left(\sum_{R_0} B + \sum_{R_0'} B + 1.02 \sum_{B_0} {}_0AB \right),$$

less (*d*) "systematic" interest gain due to lower rate of interest credited to employee contributions,

$$i\sum_{A_0}\widetilde{C}\,\frac{l_x - l_y}{l_x} + (i - i')\sum_{A_0}{}_0AC\,\frac{l_x - l_y}{l_x}\,.$$

II. Gain from terminations and deaths equals

(1) liability actually released,

$$\sum_{T\cap A_0}\left[{}_1\widetilde{AB}\,\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1\widetilde{AC}\,\frac{l_{x+1} - l_y}{l_{x+1}}\right]$$

$$+ \sum_{T\cap B_0}{}_0AB\,\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02) + \sum_{T\cap R_0}Bf(x+1, n-1)$$

$$+ \sum_{T\cap R_0'}B\,\frac{N_{x+1}^{(12)}}{D_{x+1}} - B^{(t)}\,,$$

less (2) liability expected to be released,

$$\sum_{A_0}q_x\left[{}_1\widetilde{AB}\,\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1\widetilde{AC}\,\frac{l_{x+1} - l_y}{l_{x+1}} - {}_1\widetilde{AC}\right]$$

$$+ \sum_{B_0}q_x\,{}_0AB\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right)(1.02)$$

$$+ \sum_{R_0}q_xB[f(x+1, n-1) - (n-1)]$$

$$+ \sum_{R_0'}q_xB\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right) - 0.02\sum_{B_0}{}_0AB\,.$$

III. Gain from retirements equals

(1) liability held for employees retired during the year,

$$\sum_{R\cap A_0}\left[{}_1\widetilde{AB}\,\frac{N_y^{(12)}}{D_{x+1}}(1.02) + {}_1\widetilde{AC}\,\frac{l_{x+1} - l_y}{l_{x+1}}\right]$$

$$+ \sum_{R\cap B_0}{}_0AB\,\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)\,,$$

less (2) liability actually incurred,

$$\sum_{R\cap R_1}Bf(x+1, n-1) + \sum_{R\cap R_1'}B\,\frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{B_0\cap B_1}UB\,\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)\,,$$

plus (3) expected retirement benefits for year,

$$\sum_{R_0}B + \sum_{R_0'}B + \sum_{B_0}{}_0AB\,,$$

less (4) actual retirement benefits, $B^{(r)}$.

IV. Loss (negative gain) from new entrants:

$$\sum_{N\cap A_1}\left[{}_1AB\frac{N_y^{(12)}}{D_{x+1}}(1.02)+{}_1AC\frac{l_{x+1}-l_y}{l_{x+1}}\right]+\sum_{N\cap B_1}{}_1AB\frac{N_{x+1}^{(12)}}{D_{x+1}}(1.02)\,.$$

V. Loss from salary increases:

$$\sum_{A_0\cap(A_1+B_1)}\left[(UB-\widetilde{UB})\frac{N_y^{(12)}}{D_{x+1}}(1.02)+(C-\widetilde{C})\frac{l_{x+1}-l_y}{l_{x+1}}\right]\,.$$

Again, the analysis could be further refined by splitting Part II into the gain from mortality and gain from terminations other than deaths: define $q_x = q_x^{(m)} + q_x^{(w)}$ and $B^{(t)} = B^{(m)} + B^{(w)}$, where the superscript m denotes mortality and the superscript w withdrawal for reasons other than death.

Note that the term

$$0.02\sum_{B_0}{}_0AB$$

in Part II(2) is included in the category of terminations and deaths, since the 2 per cent loading was added to account for the postretirement death benefit.

When an approximation is employed in the valuation computations, a *systematic* gain or loss may result. For example, the assumption that employee contributions will receive interest at the valuation rate and that the contributions for each year are made in a lump sum at the beginning of each year substantially simplified the valuation calculations in the preceding example; but the approximation resulted in a systematic gain (Part I[2][*d*] above). A systematic gain is one which is not zero when actual experience is in complete accord with expected. Note that another, but smaller, systematic gain results from the approximation $\ddot{a}_x^{(12)} \doteq \ddot{a}_x - \frac{11}{24}$, which causes the expected interest on benefit payments (see Part I[2][*c*] above) to be expressed in an approximate way. Thus the ideal set forth in Section I of having each component gain be zero when the assumption associated with that gain is exactly realized is seldom fulfilled in actual practice. This is just another way of saying that the valuation is just a *model* of actual events and that the gain and loss analysis must be consistent *with that model* and cannot be accurately defined without reference to the peculiarities of each individual valuation.

III. A SIMPLE ENTRY-AGE-NORMAL EXAMPLE

Although application of the principles discussed so far to the entry-age-normal cost method is relatively straightforward, there are enough small differences in technique to make it worthwhile to examine a simple case.

First, we set down the valuation formulas for this year and next:

$$NC = \sum_{A_0} {}_0NC_j = \sum_{A_0} {}_0B'_j \frac{N^{(12)}_{v_j}}{N_{w_j} - N_{v_j}}, \tag{41}$$

where ${}_0B'_j$ is a hypothetical projected benefit at retirement, assuming that employee j entered the plan at age w_j at his current salary and that the current future-service benefit formula was always in effect. We define ${}_1NC_j$ in a similar manner. Letting ${}_kB_j$ be the actual projected benefit of j at time k, we write

$$\begin{aligned} UAL_0 = {} & \sum_{A_0}\left[{}_0B_j \frac{N^{(12)}_{v_j}}{D_{x_j}} - {}_0NC_j \frac{N_{x_j} - N_{v_j}}{D_{x_j}}\right] \\ & + \sum_{B_0} {}_0B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} + \sum_{R_0} B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} - F_0 , \end{aligned} \tag{42}$$

$$\begin{aligned} UAL_1 = {} & \sum_{A_1}\left[{}_1B_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} - {}_1NC_j \frac{N_{x_j+1} - N_{v_j}}{D_{x_j+1}}\right] \\ & + \sum_{B_1} {}_1B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} + \sum_{R_1} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - F_1 . \end{aligned} \tag{43}$$

If we assume that no new entrants appear in sets B_1 and/or R_1, we can write our set relationships (cf. eqs. [6]–[8]) as

$$\begin{aligned} A_1 &= A_0 - T \cap A_0 - R \cap A_0 - A_0 \cap B_1 + N , \\ B_1 &= B_0 + A_0 \cap B_1 - T \cap B_0 - R \cap B_0 , \\ R_1 &= R_0 + R - T \cap R_0 . \end{aligned} \tag{44}$$

Now rewrite equation (43), omitting the subscript j for clarity:

$$\begin{aligned} UAL_1 = {} & \sum_{A_0}\left({}_1B \frac{N^{(12)}_{v}}{D_{x+1}} - {}_1NC \frac{N_{x+1} - N_{v}}{D_{x+1}}\right) \\ & - \sum_{T\cap A_0}\left({}_1B \frac{N^{(12)}_{v}}{D_{x+1}} - {}_1NC \frac{N_{x+1} - N_{v}}{D_{x+1}}\right) \\ & - \sum_{R\cap A_0}\left({}_1B \frac{N^{(12)}_{v}}{D_{x+1}} - {}_1NC \frac{N_{x+1} - N_{v}}{D_{x+1}}\right) \\ & - \sum_{A_0\cap B_1}\left({}_1B \frac{N^{(12)}}{D_{x+1}} - {}_1NC \frac{N_{x+1} - N_{v}}{D_{x+1}}\right) \end{aligned} \tag{45}$$

$$+ \sum_{N} \left({}_1B \frac{N_y^{(12)}}{D_{x+1}} - {}_1NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) \tag{45 cont.}$$

$$+ \sum_{B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{A_0 \cap B_1} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- \sum_{R \cap B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - F_1 ,$$

where we note that ${}_1NC_j = 0$ for $j \in B_1$, and we have extended the definitions of ${}_1B_j$ and ${}_1NC_j$ as follows:

$${}_1B_j = {}_0B_j \quad \text{and} \quad {}_1NC_j = {}_0NC_j \qquad \text{for} \quad j \in T, R .$$

Let $\Delta B_j = {}_1B_j - {}_0B_j$ and $\Delta NC_j = {}_1NC_j - {}_0NC_j$. Then equation (45) becomes, when we note that the fourth and seventh terms cancel and that the fifth term may be taken as zero,

$$UAL_1 = \sum_{A_0} \left[({}_0B + \Delta B) \frac{N_y^{(12)}}{D_{x+1}} - ({}_0NC + \Delta NC) \frac{N_{x+1} - N_y}{D_{x+1}} \right]$$

$$- \sum_{T \cap A_0} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right)$$

$$- \sum_{R \cap A_0} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) \tag{46}$$

$$+ \sum_{B_0} ({}_0B + \Delta B) \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$+ \sum_{R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - F_1 .$$

Now verify that

$$\frac{N_{x+1} - N_y}{D_{x+1}} = \left(\frac{N_x - N_y}{D_x} - 1 \right) (1 + i) + q_x \frac{N_{x+1} - N_y}{D_{x+1}} \qquad \text{for} \quad x < y , \tag{47a}$$

and reintroduce the equation of the fund,

$$F_1 = F_0 + K + I - B . \tag{47b}$$

Then equations (16), (17), and (47) allow us to rewrite equation (46) as

$$
\begin{aligned}
UAL_1 = {} & \sum_{A_0} \Big\{ {}_0B\Big[\frac{N_y^{(12)}}{D_x}(1+i) + q_x\frac{N_y^{(12)}}{D_{x+1}}\Big] \\
& - {}_0NC\Big[\Big(\frac{N_x - N_y}{D_x} - 1\Big)(1+i) + q_x\frac{N_{x+1} - N_y}{D_{x+1}}\Big]\Big\} \\
& + \sum_{A_0 \cap (A_1+B_1)} \Big(\Delta B\frac{N^{(12)}}{D_{x+1}} - \Delta NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big) \\
& - \sum_{T \cap A_0} \Big({}_0B\frac{N_y^{(12)}}{D_{x+1}} - {}_0NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big) \\
& - \sum_{R \cap A_0} \Big({}_0B\frac{N_y^{(12)}}{D_{x+1}} - {}_0NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big) \qquad (48)\\
& + \sum_{B_0} {}_0B\Big[\frac{N_x^{(12)}}{D_x}(1+i) - 1 - \tfrac{13}{24}i + q_x\Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\Big)\Big] \\
& + \sum_{B_0 \cap B_1} \Delta B\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B\frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& + \sum_{R_0} B\Big[\frac{N_x^{(12)}}{D_x}(1+i) - 1 - \tfrac{13}{24}i + q_x\Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\Big)\Big] \\
& \quad + \sum_{R} B\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B\frac{N_{x+1}^{(12)}}{D_{x+1}} - F_0 - K - I + B\,,
\end{aligned}
$$

where the second and sixth summations make use of the fact that ΔB and ΔNC are zero on sets T and R.

A suitable rearrangement of terms in equation (48) allows us to write, remembering equations (41) and (42),

$$
\begin{aligned}
UAL_1 = {} & (UAL_0 + NC)(1+i) \\
& + \sum_{A_0} q_x\Big({}_0B\frac{N_y^{(12)}}{D_{x+1}} - {}_0NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big) \\
& + \sum_{A_0 \cap (A_1+B_1)} \Big(\Delta B\frac{N_y^{(12)}}{D_{x+1}} - \Delta NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big) \qquad (49)\\
& - \sum_{T \cap A_0} \Big({}_0B\frac{N_y^{(12)}}{D_{x+1}} - {}_0NC\frac{N_{x+1} - N_y}{D_{x+1}}\Big)
\end{aligned}
$$

$$- \sum_{R \cap A_0} \left({}_0B \frac{N^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) \tag{49 cont.}$$

$$- \sum_{B_0} {}_0B(1 + \tfrac{13}{24}i) + \sum_{B_0} q_x \, {}_0B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right)$$

$$+ \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- \sum_{R_0} B(1 + \tfrac{13}{24}i) + \sum_{R_0} q_x \, B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right)$$

$$+ \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + iF_0 - K - I + B \, .$$

As before, we assume that the total actuarial gain G is defined by equation (5). If we substitute equation (49) into equation (5) and rearrange terms, we get our formula for the gain and loss analysis:

$$\begin{aligned}
G = {} & \left[I - iF_0 - I_K + \tfrac{13}{24}i \left(\sum_{B_0} {}_0B + \sum_{R_0} B \right) \right] \\
& + \left[\sum_{T \cap A_0} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) + \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \right. \\
& \quad + \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{A_0} q_x \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) \\
& \quad \left. - \sum_{B_0} q_x \, {}_0B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) - \sum_{R_0} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\
& + \left[\sum_{R \cap A_0} \left({}_0B \frac{N^{(12)}}{D_{x+1}} - {}_0NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) + \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \right. \\
& \quad \left. - \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{B_0} {}_0B + \sum_{R_0} B - B \right] \\
& - \left[\sum_{A_0 \cap (A_1 + B_1)} \left(\Delta B \frac{N_y^{(12)}}{D_{x+1}} - \Delta NC \frac{N_{x+1} - N_y}{D_{x+1}} \right) \right. \\
& \qquad \left. + \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} \right] .
\end{aligned} \tag{50}$$

Although the components differ in the manner of computation, equation (50) can be given a verbal interpretation similar to that of equation (21).

We now go on to see how equation (50) would have looked if we had used a salary scale in our valuation.

IV. ENTRY AGE NORMAL WITH SALARY SCALE

When a salary-increase assumption is used in an entry-age-normal valuation, it is usual to define the normal cost not as a level annual premium in number of dollars, but level as a percentage of salary:

$$NC = \sum_{A_0} {}_0NC_j = \sum_{A_0} {}_0B'_j \frac{N^{(12)}_{v_j}}{D_{x_j}} \frac{{}^sD_{x_j}}{{}^sN_{w_j} - {}^sN_{v_j}},$$

where s_x is the salary index at age x, and

$${}^sD_x = s_x D_x, \quad {}^sN_x = \sum_{z=x}^{\infty} {}^sD_z .$$

The unfunded accrued liability is then defined as

$$UAL_0 = \sum_{A_0} \left({}_0B_j \frac{N^{(12)}_{v_j}}{D_{x_j}} - {}_0NC_j \frac{{}^sN_{x_j} - {}^sN_{v_j}}{{}^sD_{x_j}} \right)$$

$$+ \sum_{B_0} {}_0B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} + \sum_{R_0} B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} - F_0 ,$$

$$UAL_1 = \sum_{A_1} \left({}_1B_j \frac{N^{(12)}_{v_j}}{D_{x_j+1}} - {}_1NC_j \frac{{}^sN_{x_j+1} - {}^sN_{v_j}}{{}^sD_{x_j+1}} \right)$$

$$+ \sum_{B_1} {}_1B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} + \sum_{R_1} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - F_1 .$$

Using the set relationships of Section III, we can go directly to

$$\begin{aligned} UAL_1 = & \sum_{A_0} \left({}_1B \frac{N^{(12)}_v}{D_x} - {}_1NC \frac{{}^sN_{x+1} - {}^sN_v}{{}^sD_{x+1}} \right) \\ & - \sum_{T \cap A_0} \left({}_1B \frac{N^{(12)}_v}{D_{x+1}} - {}_1NC \frac{{}^sN_{x+1} - {}^sN_v}{{}^sD_{x+1}} \right) \\ & - \sum_{R \cap A_0} \left({}_1B \frac{N^{(12)}_v}{D_{x+1}} - {}_1NC \frac{{}^sN_{x+1} - {}^sN_v}{{}^sD_{x+1}} \right) \\ & - \sum_{A_0 \cap B_1} \left({}_1B \frac{N^{(12)}_v}{D_{x+1}} - {}_1NC \frac{{}^sN_{x+1} - {}^sN_v}{{}^sD_{x+1}} \right) \end{aligned} \quad (51)$$

$$+ \sum_{B_0} {}_1B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{A_0 \cap B_1} {}_1B \frac{N^{(12)}_{x+1}}{D_{x+1}} - \sum_{T \cap B_0} {}_1B \frac{N^{(12)}_{x+1}}{D_{x+1}} \qquad (51\ \textit{cont.})$$

$$- \sum_{R \cap B_0} {}_1B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R} B \frac{N^{(12)}_{x+1}}{D_{x+1}}$$

$$- \sum_{T \cap R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} - F_1 .$$

Not surprisingly, the fourth and sixth terms of expression (51) cancel, and we have implied the definitions of ${}_1B_j$ and ${}_1NC_j$ on sets T and R. Since we are about to follow the procedure of Section III by defining ΔB_j and ΔNC_j, and, since we want both of these to be zero on sets T and R, we exercise our freedom to extend the definitions of ${}_1B_j$ and ${}_1NC_j$ to T and R as follows: let ${}_1B_j = {}_0B_j$ and ${}_1NC_j = {}_0NC_j$ for $j \in T, R$. We do not define ${}_1NC$ to be $(s_{x+1}/s_x){}_0NC$ as some might expect, because this would make it impossible for ΔNC to be zero on T and R as well as consistent in definition with ΔNC on $A_0 \cap (A_1 + B_1)$. Define

$$\Delta B_j = {}_1B_j - {}_0B_j , \quad \Delta NC_j = {}_1NC_j - {}_0NC_j ,$$

and rewrite equation (51) as:

$$UAL_1 = \sum_{A_0} \left({}_0B \frac{N^{(12)}_y}{D_{x+1}} - {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right)$$

$$+ \sum_{A_0 \cap (A_1+B_1)} \left(\Delta B \frac{N^{(12)}_y}{D_{x+1}} - \Delta NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right)$$

$$- \sum_{T \cap A_0} \left({}_0B \frac{N^{(12)}_y}{D_{x+1}} - {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right)$$

$$- \sum_{R \cap A_0} \left({}_0B \frac{N^{(12)}_y}{D_{x+1}} - {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right) \qquad (52)$$

$$+ \sum_{B_0} {}_0B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{B_0 \cap B_1} \Delta B \frac{N^{(12)}_{x+1}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N^{(12)}_{x+1}}{D_{x+1}}$$

$$- \sum_{R \cap B_0} {}_0B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} + \sum_{R} B \frac{N^{(12)}_{x+1}}{D_{x+1}}$$

$$- \sum_{T \cap R_0} B \frac{N^{(12)}_{x+1}}{D_{x+1}} - F_0 - K - I + B .$$

We now require the following recursion relation, which the reader can readily verify:

$$\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} = \left(\frac{{}^sN_x - {}^sN_y}{{}^sD_x} - 1\right)(1 + i) + q_x \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} - r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}, \tag{53}$$

where

$$r_x = \frac{s_{x+1} - s_x}{s_x}.$$

Now we can write

$$\begin{aligned}
UAL_1 = \sum_{A_0} \Big\{ {}_0B\Big[\frac{N_y^{(12)}}{D_x}(1 + i) + q_x \frac{N_y^{(12)}}{D_{x+1}}\Big] \\
- {}_0NC\Big[\frac{{}^sN_x - {}^sN_y}{{}^sD_x}(1 + i) - (1 + i) \\
+ q_x \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} - r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\Big]\Big\} \\
+ \sum_{A_0 \cap (A_1 + B_1)} \left(\Delta B \frac{N_y^{(12)}}{D_{x+1}} - \Delta NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \\
- \sum_{T \cap A_0} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \\
- \sum_{R \cap A_0} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \\
+ \sum_{B_0} {}_0B\Big[\frac{N_x^{(12)}}{D_x}(1 + i) - 1 - \tfrac{13}{24}i + q_x\Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\Big)\Big] \\
+ \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
+ \sum_{R_0} B\Big[\frac{N_x^{(12)}}{D_x}(1 + i) - 1 - \tfrac{13}{24}i + q_x\Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\Big)\Big] \\
+ \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - F_0 - K - I + B.
\end{aligned} \tag{54}$$

Note that $A_0 = A_0 \cap (A_1 + B_1) + A_0 \cap T + A_0 \cap R$, so that

$$\sum_{A_0} r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} = \sum_{A_0 \cap (A_1+B_1)} r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} + \sum_{T \cap A_0} r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} + \sum_{R \cap A_0} r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}. \tag{54a}$$

Therefore,

$$\begin{aligned} UAL_1 = {} & (UAL_0 + NC)(1+i) + iF_0 - K - I + B \\ & + \sum_{A_0} q_x \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - \frac{s_{x+1}}{s_x} {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right) \\ & + \sum_{A_0 \cap (A_1+B_1)} \left[\Delta B \frac{N^{(12)}}{D_{x+1}} - (\Delta NC - r_x\, {}_0NC) \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right] \\ & - \sum_{T \cap A_0} \left[{}_0B \frac{N_y^{(12)}}{D_{x+1}} - (1 + r_x)_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right] \\ & - \sum_{R \cap A_0} \left[{}_0B \frac{N_y^{(12)}}{D_{x+1}} - (1 + r_x)_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \right] \\ & - \sum_{B_0} {}_0B(1 + \tfrac{13}{24}i) + \sum_{B_0} q_x\, {}_0B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\ & + \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\ & - \sum_{R_0} B(1 + \tfrac{13}{24}i) + \sum_{R_0} q_x B \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \\ & + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}}. \end{aligned} \tag{55}$$

Now, if we note that $1 + r_x = s_{x+1}/s_x$ and that $\Delta NC - r_x\, {}_0NC = {}_1NC - (s_{x+1}/s_x)_0NC$, we can substitute expression (55) into equation (5), rearrange terms, and get our gain and loss analysis:

$$G = \Big\{ I - iF_0 - I_K + \tfrac{13}{24} i \Big(\sum_{B_0} {}_0B + \sum_{R_0} B \Big) \Big\}$$

$$+ \Big\{ \sum_{T \cap A_0} \Big({}_0B \frac{N_y^{(12)}}{D_{x+1}} - \frac{s_{x+1}}{s_x} {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big)$$

$$+ \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- \sum_{A_0} q_x \Big({}_0B \frac{N_y^{(12)}}{D_{x+1}} - \frac{s_{x+1}}{s_x} {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big)$$

$$- \sum_{B_0} q_x \, {}_0B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) - \sum_{R_0} q_x B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) \Big\} \tag{56}$$

$$+ \Big\{ \sum_{R \cap A_0} \Big({}_0B \frac{N_y^{(12)}}{D_{x+1}} - \frac{s_{x+1}}{s_x} {}_0NC \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big) + \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}}$$

$$- \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{B_0} {}_0B + \sum_{R_0} B - B \Big\}$$

$$- \Big\{ \sum_{A_0 \cap (A_1 + B_1)} \Big[\Delta B \frac{N_y^{(12)}}{D_{x+1}} - \Big({}_1NC - \frac{s_{x+1}}{s_x} {}_0NC \Big) \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big]$$

$$+ \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big\} .$$

Again, the grouped terms (in braces) represent, respectively, the gains from interest, mortality, retirement, and salary increases. The reader will find it instructive to compare equation (56) with equation (50), and note that the latter is just a special case of equation (56) with $s_x = s_{x+1}$.

V. AGGREGATE FUNDING METHODS

For the aggregate funding methods the term "gain" is undefined, and therefore no gain can be analyzed. The objective of "gain and loss analysis" with respect to the aggregate methods should be to analyze the net change in unit normal cost (normal cost per dollar of covered payroll or per employee) from one year to the next. Because both numerator and denominator of the expression of unit normal cost (see below) contain life-contingent functions, however, the aggregate methods have seemed most resistant to analysis. Dreher[3] suggests an approximate approach based on analysis of the numerator only. Others have worked out analogue

[3] *Op. cit.*

methods based on a simultaneous valuation on an individual-level-premium method. These approaches are nevertheless only approximate and are subject to distortion in cases in which the unit normal cost has changed radically. What one would like to see is a directly deducible method which shows the various components of the change in unit normal cost adding up exactly to the gross change. One would expect this method to be particularly revealing of the inner workings of aggregate methods insofar as they adjust the unit normal cost to reflect actual experience. For example, it is often heard that with aggregate cost methods "gains are spread into the future." What gains? Over what period are they spread? We shall show that such an analysis of the unit normal cost is indeed possible, and that it not only enables us to explain cost fluctuations to clients, but yields valuable insight into the mechanics of this class of cost methods.

For purposes of illustration, let us again assume that we have a simple salary-based plan with a life-annuity normal form and no optional forms, and that our assumptions include the use of a salary scale. We compute the unit normal costs for this year and next as follows:

$$U_0 = \frac{\sum_{A_0} {}_0B_j \frac{N^{(12)}_{y_j}}{D_{x_j}} + \sum_{B_0} {}_0B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} + \sum_{R_0} B_j \frac{N^{(12)}_{x_j}}{D_{x_j}} - UL_0 - F_0}{\sum_{A_0} {}_0S_j \frac{{}^sN_{x_j} - {}^sN_{y_j}}{{}^sD_{x_j}}},$$

$$U_1 = \frac{\sum_{A_1} {}_1B_j \frac{N^{(12)}_{y_j}}{D_{x_j+1}} + \sum_{B_1} {}_1B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} + \sum_{R_1} B_j \frac{N^{(12)}_{x_j+1}}{D_{x_j+1}} - UL_1 - F_1}{\sum_{A_1} {}_1S_j \frac{{}^sN_{x_j+1} - {}^sN_{y_j}}{{}^sD_{x_j+1}}},$$

where U_0 and U_1 are the unit normal costs, UL_k is the unfunded prior service liability or, better, unfunded supplemental liability, at time k, and ${}_kS_j$ is the salary of employee j at time k ($k = 0, 1$). The other symbols have been defined in preceding sections.

Our plan of attack is first to break down the numerator of U_1, using our recursion and set relationships in the now customary manner; second, to break down the denominator in a similar fashion; and, finally, to combine analogous components from numerator and denominator into a linear relationship.

Let the numerator for year k be denoted by N_k and the denominator by D_k ($k = 0, 1$). Also, note that the unfunded supplemental liability is carried forward from year to year in the following way:

$$UL_1 = (UL_0 + NC)(1 + i) - K - I_K\,, \tag{57}$$

where

$$NC = U_0 \sum_{A_0} {}_0S_j\,,^4$$

and the other symbols are as defined earlier. With the aid of equations (44) and (47b), we can write (omitting the subscript j)

$$\begin{aligned} N_1 = {} & \sum_{A_0} {}_1B \frac{N_y^{(12)}}{D_{x+1}} - \sum_{T \cap A_0} {}_1B \frac{N_y^{(12)}}{D_{x+1}} - \sum_{R \cap A_0} {}_1B \frac{N_y^{(12)}}{D_{x+1}} - \sum_{A_0 \cap B_1} {}_1B \frac{N_y^{(12)}}{D_{x+1}} \\ & + \sum_{N} {}_1B \frac{N_y^{(12)}}{D_{x+1}} + \sum_{B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{A_0 \cap B_1} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\ & - \sum_{R \cap B_0} {}_1B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\ & - (UL_0 + NC)(1 + i) + K + I_K - F_0 - K - I + B\,, \end{aligned} \tag{58}$$

as long as we define ${}_1B_j = {}_0B_j$ for $j \in T, R$. Noting that the fourth and seventh terms of expression (58) cancel and recalling equations (16) and (17),

$$\begin{aligned} N_1 = {} & \sum_{A_0} {}_0B \left[\frac{N_y^{(12)}}{D_x} (1 + i) + q_x \frac{N_y^{(12)}}{D_{x+1}} \right] + \sum_{A_0} \Delta B \frac{N_y^{(12)}}{D_{x+1}} \\ & - \sum_{T \cap A_0} {}_0B \frac{N_y^{(12)}}{D_{x+1}} - \sum_{R \cap A_0} {}_0B \frac{N_y^{(12)}}{D_{x+1}} + \sum_{N} {}_1B \frac{N_y^{(12)}}{D_{x+1}} \\ & + \sum_{B_0} {}_0B \left[\frac{N_x^{(12)}}{D_x} (1 + i) - 1 - \tfrac{13}{24} i + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \\ & + \sum_{B_0} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\ & + \sum_{R_0} B \left[\frac{N_x^{(12)}}{D_x} (1 + i) - 1 - \tfrac{13}{24} i + q_x \left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \right) \right] \end{aligned} \tag{59}$$

[4] Some actuaries would use

$$NC = U_0 \left(\sum_{A_0} {}_0S_j + \sum_{B_0} {}_0S_j \right).$$

$$+ \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - (UL_0 + NC)(1 + i) \qquad (59\ cont.)$$
$$+ I_K - F_0 - I + B\,,$$

where $\Delta B = {}_1B_j - {}_0B_j$; or

$$\begin{aligned}
N_1 = {} & N_0 + iN_0 + iF_0 - iNC + I_K - I - NC \\
& - \tfrac{13}{24} i \Big(\sum_{B_0} {}_0B + \sum_{R_0} B \Big) \\
& + \Big[\sum_{A_0} q_x\, {}_0B \frac{N_y^{(12)}}{D_{x+1}} + \sum_{B_0} q_x\, {}_0B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) \\
& \quad + \sum_{R_0} q_x B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) - \sum_{T \cap A_0} {}_0B \frac{N_y^{(12)}}{D_{x+1}} \\
& \quad - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big] \\
& + \Big(\sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap A_0} {}_0B \frac{N^{(12)}}{D_{x+1}} - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} \\
& \quad - \sum_{B_0} {}_0B - \sum_{R_0} B + B \Big) \\
& + \Big(\sum_{A_0 \cap (A_1 + B_1)} \Delta B \frac{N^{(12)}}{D_{x+1}} + \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big) + \sum_{N} {}_1B \frac{N_y^{(12)}}{D_{x+1}}\,.
\end{aligned} \tag{60}$$

Now we turn our attention to D_1 (the denominator of U_1):

$$\begin{aligned}
D_1 = \sum_{A_1} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} = {} & \sum_{A_0} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
& - \sum_{T \cap A_0} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} - \sum_{R \cap A_0} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
& - \sum_{A_0 \cap B_1} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} + \sum_{N} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\,,
\end{aligned} \tag{61}$$

where we consider ${}_1S_j = {}_0S_j$ for $j \in T, R$. The fourth term of expression (61) is identically zero (since $x_j + 1 = y_j$ for $j \in A_0 \cap B_1$). Now, with reference to equation (53), we can write

$$
\begin{aligned}
D_1 = \sum_{A_0} {}_0S\Big[\Big(\frac{{}^sN_x - {}^sN_y}{{}^sD_x} - 1\Big)(1+i) + q_x \frac{s_{x+1}}{s_x}\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
- r_x \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\Big] + \sum_{A_0} \Delta S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
- \sum_{T\cap A_0} {}_0S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} - \sum_{R\cap A_0} {}_0S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
+ \sum_{N} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}},
\end{aligned}
\tag{62}
$$

where $\Delta S_j = {}_1S_j - {}_0S_j$. Note that $\Delta S - r_x\, {}_0S = {}_1S - (s_{x+1}/s_x)_0S$, and then rearrange expression (62) as follows, recalling equation (54a):

$$
\begin{aligned}
D_1 = D_0 - \sum_{A_0} {}_0S + i\Big(D_0 - \sum_{A_0} {}_0S\Big) \\
+ \sum_{A_0\cap(A_1+B_1)} \Big({}_1S - \frac{s_{x+1}}{s_x}\, {}_0S\Big)\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \\
+ \Big(\sum_{A_0} q_x\, {}_0S \frac{s_{x+1}}{s_x}\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} - \sum_{T\cap A_0} {}_0S \frac{s_{x+1}}{s_x}\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\Big) \\
- \sum_{R\cap A_0} {}_0S \frac{s_{x+1}}{s_x}\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} + \sum_{N} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}.
\end{aligned}
\tag{63}
$$

Now note that equation (60) expresses N_1 in the form

$$N_1 = N_0 + \Delta N$$

and that equation (63) expresses D_1 in the form

$$D_1 = D_0 + \Delta D\,,$$

so that

$$U_1 = \frac{N_1}{D_1} = \frac{N_0 + \Delta N}{D_0 + \Delta D}. \tag{64}$$

We are, however, looking for an expression of the form

$$U_1 = U_0 + \Delta U\,.$$

We can transform equation (64) into this form by noting the following algebraic identity:

$$\frac{N_0 + \Delta N}{D_0 + \Delta D} = \frac{N_0 + \Delta N - \dfrac{N_0 + \Delta N}{D_0 + \Delta D}\Delta D}{D_0}. \tag{65}$$

In other words,

$$U_1 = U_0 + \frac{1}{D_0}(\Delta N - U_1 \Delta D),$$

or

$$\Delta U = \frac{1}{D_0}(\Delta N - U_1 \Delta D). \tag{66}$$

Taking the terms of ΔN from equation (60) and the terms of ΔD from equation (63), and remembering that

$$NC = U_0 \sum_{A_0} {}_0S ,$$

we can write

$$\begin{aligned}
\Delta U = & - \frac{1}{D_0} U_0 \sum_{A_0} {}_0S + \frac{1}{D_0} U_1 \sum_{A_0} {}_0S \\
& + \frac{1}{D_0} \Big\{ iN_0 + iF_0 - iU_0 \sum_{A_0} {}_0S + I_K - I - \tfrac{13}{24} i \Big(\sum_{B_0} {}_0B + \sum_{R_0} B \Big) \\
& \qquad\qquad - iU_1 D_0 + iU_1 \sum_{A_0} {}_0S \Big\} \\
& + \frac{1}{D_0} \Big\{ \sum_{A_0} q_x \Big[{}_0B \frac{N_y^{(12)}}{D_{x+1}} - U_1\, {}_0S \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big] \\
& \qquad + \sum_{B_0} q_x\, {}_0B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) + \sum_{R_0} q_x B \Big(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24} \Big) \\
& \qquad - \sum_{T \cap A_0} \Big({}_0B \frac{N_y^{(12)}}{D_{x+1}} - U_1\, {}_0S \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big) \\
& \qquad\qquad - \sum_{T \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{T \cap R_0} B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big\} \\
& + \frac{1}{D_0} \Big\{ \sum_{R} B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{R \cap A_0} \Big({}_0B \frac{N_y^{(12)}}{D_{x+1}} - U_1\, {}_0S \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big) \\
& \qquad\qquad - \sum_{R \cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{B_0} {}_0B - \sum_{R_0} B + B \Big\} \\
& + \frac{1}{D_0} \Big\{ \sum_{A_0 \cap (A_1 + B_1)} \Big[\Delta B \frac{N_y^{(12)}}{D_{x+1}} - U_1 \Big({}_1S - \frac{s_{x+1}}{s_x}\, {}_0S \Big) \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big] \\
& \qquad + \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big\} + \frac{1}{D_0} \sum_{N} \Big({}_1B \frac{N_y^{(12)}}{D_{x+1}} - U_1\, {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big).
\end{aligned} \tag{67}$$

Equation (67) can be rewritten in the form

$$\Delta U = \frac{\Delta U}{D_0} \sum_{A_0} {}_0S + iU_0 - iU_1 + i\frac{\Delta U}{D_0} \sum_{A_0} {}_0S$$
$$+ \frac{1}{D_0}\left\{iF_0 + I_K - I - \tfrac{13}{24}i\left(\sum_{B_0} {}_0B + \sum_{R_0} B\right)\right\} + \ldots \tag{68}$$

or

$$\Delta U = \Delta U\left(\frac{1+i}{D_0} \sum_{A_0} {}_0S - i\right) + \frac{1}{D_0}\{\ldots\} ;$$

that is,

$$\Delta U\left(1 - \frac{1+i}{D_0} \sum_{A_0} {}_0S + i\right) = \frac{1}{D_0}\{\ldots\} ,$$

$$\Delta U(1+i)\left(1 - \frac{\sum_{A_0} {}_0S}{D_0}\right) = \frac{1}{D_0}\{\ldots\} ,$$

$$\Delta U = \frac{1}{(1+i)\left(D_0 - \sum_{A_0} {}_0S\right)}\{\ldots\} .$$

Therefore, let us define a "spread factor" as follows:

$$SF = \frac{1}{(1+i)\sum_{A_0} {}_0S_j\left(\dfrac{{}^sN_{x_j} - {}^sN_{y_j}}{{}^sD_{x_j}} - 1\right)} \tag{69}$$

and write our analysis formula in final form:

$$\Delta U = U_1 - U_0 = -SF\left\{I - iF_0 - I_K + \tfrac{13}{24}i\left(\sum_{B_0} {}_0B + \sum_{R_0} B\right)\right\}$$

$$- SF\left\{\sum_{T\cap A_0}\left({}_0B\,\frac{N_y^{(12)}}{D_{x+1}} - U_1\,{}_0S\,\frac{s_{x+1}}{s_x}\,\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) + \sum_{T\cap B_0} {}_0B\,\frac{N_{x+1}^{(12)}}{D_{x+1}}\right.$$

$$+ \sum_{T\cap R_0} B\,\frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_{A_0} q_x\left({}_0B\,\frac{N_y^{(12)}}{D_{x+1}} - U_1\,{}_0S\,\frac{s_{x+1}}{s_x}\,\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \tag{70}$$

$$\left. - \sum_{B_0} q_x\,{}_0B\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right) - \sum_{R_0} q_x B\left(\frac{N_{x+1}^{(12)}}{D_{x+1}} + \tfrac{11}{24}\right)\right\}$$

$$- SF\left\{\sum_{R\cap A_0}\left({}_0B\,\frac{N_y^{(12)}}{D_{x+1}} - U_1\,{}_0S\,\frac{s_{x+1}}{s_x}\,\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right)\right.$$

$$+ \sum_{R\cap B_0} {}_0B \frac{N_{x+1}^{(12)}}{D_{x+1}} - \sum_R B \frac{N_{x+1}^{(12)}}{D_{x+1}} + \sum_{B_0} {}_0B + \sum_{R_0} B - B \Big\} \quad (70\ cont.)$$

$$+ SF \Big\{ \sum_{A_0 \cap (A_1+B_1)} \Big[\Delta B \frac{N_y^{(12)}}{D_{x+1}} - U_1 \Big({}_1S - \frac{s_{x+1}}{s_x} {}_0S \Big) \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big] + \sum_{B_0 \cap B_1} \Delta B \frac{N_{x+1}^{(12)}}{D_{x+1}} \Big\}$$

$$+ SF \Big\{ \sum_N \Big({}_1B \frac{N_y^{(12)}}{D_{x+1}} - U_1 \, {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} \Big) \Big\} ,$$

which bears a startling resemblance to equation (56)! Note that the "gains" (in braces) are spread by application of the spread factor and that the effect of a "gain" is to reduce the unit normal cost. The individual "accrued liabilities" are computed using the aggregate unit normal cost applied to individual salaries (see the first term in the second pair of braces); otherwise the braced amounts are computed in much the same way as are gains under the individual entry-age-normal method.

It is now possible not only to say with complete authority that under aggregate funding methods "gains are spread into the future" but also to say exactly what one means by "gains" and to say that the number of years over which these "gains" are spread is exactly

$$\frac{(1+i) \sum_{A_0} {}_0S_j \Big(\frac{{}^sN_{x_j} - {}^sN_{y_j}}{{}^sD_{x_j}} - 1 \Big)}{\sum_{A_0} {}_0S_j}$$

in the case under consideration. The ability to identify separately the "gain" and the spread factor offers interesting possibilities for presentation of the analysis of change in cost to the client.

Note also that, unlike the situation with individual funding methods, the "loss" from new entrants under aggregate methods is not normally zero. In order for the addition of new entrants not to disturb the unit normal cost, all other factors being equal, the following must hold, according to equation (70):

$$\sum_N {}_1B \frac{N_y^{(12)}}{D_{x+1}} = U_1 \sum_N {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}$$

or

$$U_1 = \frac{\sum_N {}_1B \frac{N_y^{(12)}}{D_{x+1}}}{\sum_N {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}} .$$

That is, the unit normal cost computed for the new entrants as a separate group must be identical with the unit normal cost for the entire group at time 1. This result will not surprise the reader, but it will strengthen his confidence in the analysis.

Examination of equation (70) will reveal that not only terminal values of present value of future benefits need to be computed for each participant, but also terminal values of present value of future salary. These figures will be required separately, since U_1 is not known until the following year.

It is well to repeat here our earlier cautioning remarks. The foregoing derivation should be used only as a guide in determining the analysis formula for each case; a change in any one of the underlying assumptions could invalidate formulas (69) and (70).[5] However, a similar derivation for a different situation should be fairly straightforward if the reader has familiarized himself with the basic approach; the twist for aggregate methods is noticing formula (65), which is the key to the whole derivation.

VI. SOME CONCLUDING REMARKS

So much for our new look at gain and loss analysis. The reader who has followed the derivations carefully will have noted that each of them was accomplished with essentially the same bag of tricks. He may now find it helpful to review some fine points that, for reasons of clarity, were not emphasized earlier.

First, the employee sets, by which the valuation computations are defined, should be as few as possible, since each additional set generates another set relationship and thus several additional terms in each step of of the derivation. In general, a separate set must be defined for each different actuarial factor to be applied: "different" factors are those which have different Fackler-type recursion relationships. Thus the set of active participants was broken down into two subsets, A and B, in our examples, because the recursion relation of the factor $N_y^{(12)}/D_x$ for $y > x$ is not the same as for $y = x$. On the other hand, it proved unnecessary to break down set T into deaths, terminations, withdrawals, and so on, until after the derivation was completed. The set of retired lives may need to be subdivided as in the example of Section II, but even that subdivision might have been eliminated by a more generalized recursion relationship.

Note that we consistently let x represent the age on the *first* valuation

[5] In particular, if the "pure" aggregate method, with no supplemental liability, were used the contribution to the fund for the year would no longer be immaterial to ΔU, and equation (70) would contain a term expressing the "gain" from contributions in excess of the normal cost.

date, even for new entrants (who were not in the first valuation). This device makes the substitution of recursion formulas much more obvious and natural.

It is important to be careful about extending the definitions of certain symbols to sets on which they were not defined originally. Theoretically, it should not make any difference in the final result how these definitions are extended, but it will be found practical to extend definitions in the same ways as in the examples. The ambitious reader may wish to try the derivation of Section IV letting ${}_1S_j = (s_{x+1}/s_x){}_0S_j$ for $j \in T, R$. The end result will be the same, but the derivation will take a slightly different course. How these definitions are extended is less important than the fact that they are explicitly extended: if this is not done, considerable confusion could result.

Finally, it is clear that one should have some idea of the sort of formula he wants to end up with, since this enables him, near the end of the derivation, to gather terms in a reasonably efficient way—there is much groping on one's first try..

The writer hopes that this paper has dispelled some of the mystery which has enshrouded the subject of gain and loss analysis, by showing that such problems can be successfully attacked algebraically. It is further hoped that at least a few pension actuaries have been convinced of the superior practicality of a rigorous approach—how satisfying when rigor is practical!

The author is gratefully indebted to Henry Winslow for his patient review and thoughtful criticism of the original manuscript.