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SOME OBSERVATIONS ON THE ACTUARIAL ASPECTS OF THE INSURED VARIABLE ANNUITY

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ABSTRACT

This paper describes some of the more important actuarial characteristics of the insured variable annuity of the type currently offered by a number of life insurance companies in the United States. Some commentary on these characteristics is included in the paper.

The concept of a unit value is developed, and its significant characteristics are discussed. Various ramifications of these characteristics are mentioned. In addition, the unit value concept is contrasted with the concept of a share value.

The concept of a separate account is also discussed. Emphasis is placed on the very general nature of a separate account which is a vehicle for funding insured variable annuities. The discussion points up the technical utility of the unit value as a measure of insured benefits in a separate account. Some comments on several actuarial aspects of funding insured variable benefits and the unusual nature of separate account surplus are included.

INTRODUCTION

The author's recent experience in the insured variable annuity field has brought to his attention a number of misunderstandings of the actuarial nature of the insured variable annuity. These misunderstandings result in many practical difficulties among and between members of the insurance industry, members of regulatory bodies, and members of the general public.

The purpose of this paper is to describe some of the more important characteristics of the insured variable annuity and to provide some commentary thereon from the actuarial point of view. It is hoped that the paper will stimulate discussion of the subject among members of the Society. It is also hoped that the paper will serve as a catalyst to the dissemination beyond the membership of the Society of a broad understanding of the actuarial nature of the insured variable annuity.

The paper will deal only with the insured variable annuity of the form which is currently offered by a number of life insurance companies in the United States. The benefits provided by such a variable annuity are funded by a block of equity investments. The principal actuarial conditions underlying the contract are that investment performance will be credited at the net rate actually experienced by the block of investments used to fund contract benefits and that the mortality and expense guarantees normally contained in insured annuities will apply. Stated in different terms, the insured variable annuity being considered in this paper is fundamentally identical to its insured fixed amount benefit counterpart, with the sole exception that, in the case of the variable annuity, investment performance is credited at the net rate actually experienced by the underlying fund of investments rather than at a rate guaranteed by the insurance company. Typically, the net rate credited reflects a percentage reduction in the gross investment performance to cover the cost of mortality and expense guarantees, taxes, and investment management fees for the variable operation. This percentage reduction should reflect the particular mortality and expense basis established for the variable annuity operation and the absence of the interest margin normally present in fixed amount annuity pricing.

THE UNIT VALUE CONCEPT

Many of the misunderstandings of the insured variable annuity can probably be attributed to a lack of understanding of the essence of the unit value concept. In order to stress basic principles, it is helpful to develop the unit value concept within the confines of the traditional fixed dollar annuity concept. This approach makes it possible to identify clearly the essence of the unit value concept before one grapples with the additional concept of an annuity whose benefits vary.

During the period prior to commencement of income payments, the benefits provided by the garden variety form of individual annual premium retirement annuity are generally related to a table of cash values per \$100 of annual premium payable under the contract. This table of cash values is constructed from a series of net premiums, N_t, and the guaranteed interest rate, i. (Usually, cash values are equal to the accumulation of net premiums less a surrender charge. For simplicity's sake, surrender charges are ignored in this development.) Mathematically, the cash value table can be represented by an expression of the following type:

$$\langle\{[N_1(1+i)+N_2](1+i)\}+N_3\rangle(1+i)\ldots.$$

Rearranging this expression, we have the following formula for the tth cash value, $(\mathrm{CV})_t$:

$$(\mathrm{CV})_t=N_1(1+i)^t+N_2(1+i)^{t-1}+\ldots+N_t(1+i)\,.$$

Let us now define a series of accumulation unit values $v_1, v_2, \ldots, v_t$ such that $v_{k+1} = v_k(1+i)$. By successively applying this relationship, we find that

$$v_{k+1} = v_k(1+i) = v_{k-1}(1+i)^2 = \ldots = v_1(1+i)^k .$$

If we now express each net premium in terms of accumulation units, we find that the number of accumulation units which can be used to represent N_k is u_k, where

$$u_k = \frac{N_k}{v_k} .$$

This expression yields the formula $N_k = u_k v_k$, which, when used in the cash value formula, yields

$$(\mathrm{CV})_t = u_1 \cdot v_1(1+i)^t + u_2 \cdot v_2(1+i)^{t-1} + \ldots + u_t \cdot v_t(1+i) .$$

However, since $v_{k+1} = v_1(1+i)^k$, we have further that

$$(\mathrm{CV})_t = u_1 \cdot v_{t+1} + u_2 \cdot v_{t+1} + \ldots + u_t \cdot v_{t+1}$$

or

$$(\mathrm{CV})_t = (u_1 + u_2 + \ldots + u_t) v_{t+1} .$$

The foregoing elementary demonstration has significant value. It clearly demonstrates that we can express traditional fixed benefit annuity cash values in terms of a number of accumulation units and a chain of unit values which reflect the interest rate guarantee implicit in the traditional contractual table of cash values. It also demonstrates that the essence of the accumulation unit concept is a chain of unit values which are interrelated by an investment rate of return, which in the case of the fixed amount annuity is the guaranteed interest rate.

After commencement of annuity payments, benefits under the traditional fixed amount annuity are measured in terms of a dollar amount of annuity payment for each specified time interval of the annuity payment period, \$100 per month, for example. This dollar amount of annuity payment is also the basis of the actuarial value for the benefit payment commitment under the annuity agreement.

From a pedagogical point of view, the dollar amount of annuity payment under the traditional fixed amount annuity can profitably be viewed as the amount which will actuarially amortize the funds which constitute the actuarial value of the annuity and investment earnings thereon. Let us call the rate of interest upon which this amortization is based j. In order to explore the nature of the annuity unit value concept, let us define the annuity unit value, v_t^a, so that

$$v_t^a = v_{t-1}^a \cdot \left(\frac{1+j'}{1+j}\right) .$$

If we let P_i be the amount of the initial annuity payment and u^a be the number of annuity units which is used to represent P_i, then

$$u^a = \frac{P_i}{v_i^a} \quad \text{or} \quad P_i = u^a \cdot v_i^a .$$

Since $v_{i+1}^a = v_i^a \cdot [(1+j')/(1+j)]$, we find that $P_{i+1} = u^a \cdot v_i^a \cdot [(1+j')/(1+j)]$. Hence, we observe that, if we wish to design a fixed amount annuity with level payments, we can do so by setting $j' = j$. Here, j' is probably best interpreted as the rate of interest credited to annuity benefits. In the insured situation, it is, of course, guaranteed.

It should be noted that we can construct a fixed amount annuity with monotonically increasing benefits by imposing the conditions that both j and j' are constants and that $j < j'$. Alternatively, an increasing annuity which becomes level after k periods can be constructed by adding to the foregoing conditions the additional condition that $j = j'$ after k periods. Many other forms of benefit are obviously possible.

The actuarial value, V_{i+r}, of the benefit payment commitment can be determined from the equation $V_{i+r} = m \cdot P_{i+r} \cdot \ddot{a}_{y+r}^{(m)}$, which in unit value form is

$$V_{i+r} = m \cdot u^a \cdot v_{i+r}^a \cdot \ddot{a}_{y+r}^{(m)} .$$

Since $v_{i+r+1}^a = [(1+j')/(1+j)] \cdot v_{i+r}^a$, we note that

$$V_{i+r+1} = \left(\frac{1+j'}{1+j}\right) \cdot m \cdot u^a \cdot v_{i+r}^a \cdot \ddot{a}_{y+r+1}^{(m)} .$$

Again, if we let j' be the rate of interest credited, we note that the annuity unit, u^a, becomes in effect a base for the equitable distribution of interest earnings. It should also be noted that the distribution is a distribution of the interest earnings of a given period over the entire future benefit payment period.

The foregoing demonstrations of the unit value concept in the realm of fixed amount annuities suggest some further conclusions. It should be noted, for example, that the accumulation unit value concept frees us from the rigidity of the traditional form of cash value tables and permits considerable flexibility in the amount of premium acceptable under the contract. It also permits considerable flexibility in the interest rates guaranteed over the life of the contract. All of this is accomplished in a very simple, practical manner. In the process, no damage is done to such fundamental actuarial concepts as the fact that insurance reserves are not generally equal to the cash values provided under the contract. Furthermore, reserve-strengthening procedures will not interfere with cash value determinations under the unit value concept, just as they do not interfere with such cash value determinations under the traditional forms of doing business.

In this connection, it is well to note that it is often difficult for people who are not actuarially oriented to recognize the life contingency base of the reserve for a traditional annual premium retirement annuity during the accumulation period. In order to overcome this difficulty, it is probably desirable to stress the fact that such annuities are actually deferred annuities whose actuarial value can be represented prospectively as $v^n \cdot \ddot{a}^{(m)}_{x+t+n}$. It may also be helpful to stress the fact that the reason for the extremely simple mathematical form of the retrospective reserve formula during the accumulation period is the fact that death benefits are substantially equal to cash values prior to the commencement of annuity payments and hence can, for reserve computation purposes, be assumed to be equal to cash values.

It is interesting to note that the accumulation unit and annuity unit described above have identical fundamental characteristics. Both describe contract benefits in terms of a chain of unit values which are interrelated by an investment rate of return. As one might suspect under such circumstances, reduction of the two unit values to a single unit value is possible. One way in which this can be accomplished is to define contract values during the so-called accumulation period, n, in terms of the chain of annuity unit values v^a_i defined previously. If this is done, each net premium, N_t, is applied in the deferred annuity form, and the number of annuity units credited to the contract as a result of the payment of N_t is

$$u^a_t = \frac{N_t}{m\cdot\ddot{a}^{(m)}_y\cdot v^a_i\cdot[(1+j')/(1+j)]^{-(n-t+1)}},$$

if we establish the condition that j and j' are constants. Under this kind of arrangement, contract values during the accumulation period may be expressed in the form

$$(u^a_1+u^a_2+\ldots+u^a_k)\cdot m\cdot\ddot{a}^{(m)}_y\cdot v^a_i\cdot\left(\frac{1+j'}{1+j}\right)^{-(n-k+1)},$$

where the number of annuity units credited under the contract for the first k premium payments is $u^a_1+u^a_2+\ldots+u^a_k$.

From a purely technical standpoint, perhaps the most intriguing aspect of the unit value concept is the distributive characteristic of the annuity unit value. The annuity unit value concept coupled with the rate of investment performance effortlessly distributes such investment performance over the future annuity payment period. Pedagogically, the process can be viewed as a series of annuity purchases to adjust annuity payments to the level necessary to reflect the constantly changing investment performance. Perhaps this approach can be used to help explain away the mysteries of the variable annuity unit.

VARIABLE ANNUITY UNITS

The preceding section illustrates the fact that the fixed dollar annuity business can be conducted on a unit value basis if one so desires. In the variable annuity business, the unit value comes close to being a practical necessity. For example, the option of using contractual cash value tables for determination of cash values is not available to the designer of a variable annuity contract.

Theoretically, it would be possible to maintain records of individual contract values in terms of dollar amounts and constantly to update such dollar amounts by applying the investment rate of return for each valuation period to each individual record. (Many companies have already achieved a considerable degree of flexibility in their fixed amount annuity operations without the use of unit values.) From a practical standpoint, however, such a record-keeping system is very undesirable for variable benefit operations. Application of the unit value concept to the variable annuity business makes possible relatively simple contract value administration. It also makes provision for variations in the rate of investment performance credited to contract values, an essential requirement of variable annuities. At the same time, the unit value concept significantly increases the flexibility available in the area of product design. It accomplishes this without changing any of the fundamental concepts of the underlying insurance business.

The unit value concepts developed with respect to fixed amount annuities are directly applicable to variable annuities. During the annuity payment period, j becomes the rate of investment earnings assumed by the designer of the product. It should be noted that j is purely and simply a product design feature. In this context, j' becomes a variable from period to period and is the actual net investment earnings rate for the block of assets which fund contract benefits.

For simplicity's sake, the accumulation unit is commonly used for the period prior to the commencement of annuity payments, and the unit values are interrelated by the series of rates j'. For convenience sake, the investment earnings rate assumed during the accumulation period is zero.

The unit value, coupled with the rate of investment performance of the assets used to fund variable benefits, provides complete flexibility with respect to the choice of j, the assumed rate. Benefits based upon several assumed rates can be funded by one block of assets without impairing equity among annuitants. Rate bases for contract benefits which vary from group to group are likewise as equitable as they are for comparable fixed amount benefits and can be funded by one block of assets.

Reserve bases need not be identical with premium rate bases. Both will be changed from time to time to accommodate changing conditions. Reserves must, of course, take into account the number of units credited (i.e., the amount of the initial variable annuity payment) and the assumed investment performance coupled with the number of units credited. In the United States, reserves must also comply with the minimum standards specified by the various state laws and regulatory authorities.

SHARE VALUES AND UNIT VALUES

For a number of reasons there is in the United States a strong tendency to relate the variable annuity business to the mutual fund business and vice versa. One result of this tendency is the confusion of the unit value concept with the share value concept. Technically, the two concepts are distinct and very different. As has been demonstrated in this paper, the essence of the unit value concept is a unit value coupled with a rate of investment return. Among other things, this concept does not require any restriction on the amount of assets used to fund the variable annuity benefits. The share value concept, on the other hand, is irrevocably committed to a specified amount of assets. This commitment arises from the fact that share value is defined as the amount of assets divided by the number of shares.

The share value concept appears to be quite appropriate to the conduct of the mutual fund business. The objective of the mutual fund's share value is, in effect, to "divide up" a given block of net assets among the owners of such assets. The share value concept accomplishes this division of assets in an equitable and direct manner.

The unit value, on the other hand, is very appropriate to the insured variable annuity business. The objective of the insured variable annuity's unit value is to distribute investment earnings, including capital gains and losses, and the charges for mortality and expense guarantees, taxes, and investment management fees over contract benefits within the confines of the mortality and expense guarantees provided by the variable annuity contract. The unit value concept accomplishes this distribution in an equitable and direct manner. It does so by basing such distribution on the net rate of return, including capital gains and losses, experienced by the block of assets used to fund variable annuity contract benefits.

The unit value concept intentionally avoids dependence on a specific amount of assets. It specifically avoids this dependence in order to make feasible the insurance company's fulfillment of the mortality and expense guarantees which it makes under its contracts. The company is obligated by contract to provide the funds to pay the benefits guaranteed under its contracts. The unit value concept makes it possible for the company to provide funds or take profits without upsetting the contractowner's equity under his contract.

The share value concept is not appropriate to such a situation. Attempts to employ such a concept involve the risk that share values and hence contract benefits may be affected by the insurance company's approach to funding the benefits under its contracts. This is obviously undesirable, for it may lead to default by the company under its contract guarantees. The possibility of such a default arises from the fact that share values distribute any change in the amount of assets regardless of the source of such a change. For example, if a reduction of, say, 10 per cent were to occur in reserves, the share value would distribute such a reduction among shareholders just as it distributes a 10 per cent loss in investment performance. The unit value concept coupled with a rate of return which reflects only the net investment experience of the assets used to fund contract benefits, regardless of how large or small such a block of assets is, avoids such problems and provides an ideal technical solution to the problem.

THE SEPARATE ACCOUNT

The separate account concept has been used by life insurance companies in the United States as a method of determining contract values prior to retirement under group deposit administration-type contracts with respect to the equity funding portion of such contracts. The share value approach to contract values appears to have been widely applied, probably because of the desire to compete with institutions providing uninsured equity funding in elementary terms which were familiar to the clients of such institutions and in common use in the market place.

The shortcoming of a separate account based on the share value concept becomes evident when one attempts to use the separate account as a funding device for more general forms of the life insurance product, such as the insured variable annuity. At this point, one finds that the most desirable definition is that the separate account is merely a segregated fund of equity assets used to fund the variable benefits provided by the contracts. The amount of this segregated fund is not precisely specified. Furthermore, since the contract guarantees that variable benefits will reflect the net investment performance of such assets, the separate account must also serve as a device for measuring the investment performance of such assets.

When the separate account definition becomes broad enough to encompass the funding concept inherent in all long-term insurance contracts, the amount of assets required to be maintained in the account becomes essentially and necessarily indeterminate. This indeterminate nature of the amount of required assets arises from the essentially indeterminate character of the life insurance reserve liability. While this essentially indeterminate character of the life insurance reserve is as old as the life insurance business itself, it is sometimes not readily apparent in its new variable benefits cloak. As actuaries, we probably need to stress this fact when we deal with the variable benefit contract. We need to emphasize the fact that the amount of life insurance reserve established in an insurance operation at any point in time is essentially an informed, professional opinion of the amount of liability and nothing more.

The unit value concept, coupled with a rate of return formula for the separate account, appears to be the ideal solution to the problems presented by a broad definition of separate account. Technically, such a concept provides for perfect equity among contractowners and complete funding flexibility by the insurance company, all within the conditions guaranteed by the insurance contract.

When one recognizes the need for a broad form of separate account, one also recognizes that the separate account concept must embrace the concept of separate account surplus. This same conclusion can also be reached from other considerations. For example, since the performance of the separate account is merely a measure of the company's benefit liability, its assets might at some point in time be less than, equal to or greater than the benefit liability.

Recognition of separate account surplus leads one to the question of what level of assets should be maintained in the separate account. Since the insurance company's management is responsible for ensuring the payment of contractual benefits, it needs to be free to fund such benefits on the basis of its best judgment. Certainly, prudent management will generally insist on maintaining fund assets at a level substantially equal to the company actuary's current estimate of the liability for contractual variable benefits. Generally, management will normally want to maintain some surplus in the separate account as additional protection against unforeseen contingencies. There may, however, be occasions when prudent management may dictate funding some portion of variable contract liabilities, at least temporarily, by means of general account assets. A well-conceived separate account will make possible all desirable funding and general management objectives.

The advent of the variable annuity changes the character of the surplus requirements of a life insurance company. For example, maintenance of significant amounts of surplus in the separate account during periods when mortality losses on annuity risks are on the upswing is very desirable when management feels that future equity investment experience is likely to be favorable. On the other hand, during periods when favorable mortality is anticipated and unfavorable investment experience is likely, the actuary may recommend maintenance of a lower separate account surplus and a higher general account surplus.

Where preretirement death benefits under the variable annuity are equal to the greater of premiums paid and current surrender value, the actuary may recommend the maintenance of reserves or surplus for this risk in the company's general account, where it can be funded by debt-type securities. This action will minimize losses during periods when the investment performance of the separate account is poor. If funding for this liability were to be restricted to the separate account during such periods, assets which fund this type of liability would shrink at precisely the time when the liabilities which they fund increase. This is obviously an undesirable result and should be avoided.

Increased attention to these matters will obviously promote sounder companies and lower costs to contractowners. The result will be a furthering of the public interest as well as the interest of the companies.

CONCLUSION

The author believes that there is a real need for actuaries to participate in the equity contract evolution which is currently occurring in the life insurance business. Hopefully, this paper will help to stimulate such participation.

Much of the commentary in this paper is in terms of the individual product rather than the group product. This has been done for simplicity's sake only. The concepts and basic fundamentals apply to both forms of products.

This paper does not deal with the actuarial aspects of equity-funded variable annuities which do not guarantee mortality and expenses. As a result, the comments made herein may not be applicable to variable annuities issued without such guarantees.

The opinions expressed in this paper are, of course, those of its author. They are not necessarily the official views of the company which the author represents.

DISCUSSION OF PRECEDING PAPER

PHILIP J. FEUER:

Mr. Macarchuk expresses concern about practical difficulties which result from "a number of misunderstandings of the actuarial nature of the insured variable annuity." Because he believes that many of the misunderstandings can probably be attributed to a lack of appreciation of the unit value concept, he shows how unit values could be used in conducting traditional fixed-dollar annuity business. After identifying the essence of the unit value concept in this manner, he introduces the more complicated variable annuity as an extension of the ideas previously presented, showing how the same unit value concepts are directly applicable.

In facing similar misunderstandings and practical difficulties, I have also found it helpful to use a two-stage approach to explain the nature of insured variable annuities. My approach, however, works in the reverse order of the one used by Mr. Macarchuk. First, the fundamental concept of insured variable annuities is examined without any reference to unit values. Then administration is considered, and the unit value is seen as a logical means of accomplishing the necessary work simplification. In the discussion following, the first stage of this approach is outlined in the hope that others may find it a useful tool.

The basic ideas underlying insured variable annuities can be brought into focus by examining the components of an annuity premium in order to answer the following questions:

1. Where do investment gains and losses go under fixed-dollar annuities and under variable annuities?
2. Why is a benefit payment of an insured variable annuity obtained by "charging" the prior payment $(1 + i)$ and "crediting" it with $(1 + i')$? (i is the assumed investment rate and i' the net rate earned during the period between the two payments.)

Consider a group of 100,000 lives aged 65, of whom 98,000 survive to 66; 95,000 to 67; and so on. Assuming a future investment return of 4 per cent, the total net single premium collected from the group for an annuity due paying \$1 a year to each member of the group is $\$[100{,}000 + 98{,}000/1.04 + 95{,}000/(1.04)^2 + \ldots]$. If all assumptions are realized, the funds represented by each term of this expression will grow to an amount exactly sufficient to pay the benefits as they fall due in the future. For example, the funds represented by the third term will amount to [$95,000/$(1.04)^2$] (1.04) (1.04), or $95,000, at the end of two years. This is exactly sufficient to pay $1 to each of the 95,000 survivors at that time. The exact sufficiency is not affected if emerging experience differs from that assumed, because excess earnings are paid as dividends or transferred to company surplus and deficiencies are made up from surplus. So the answer to question 1 for fixed-dollar annuities is that investment gains and losses are absorbed by surplus.

Now consider the same group of lives paying the same premium but sharing in the investment experience, that is, buying variable annuities. If mortality and expense assumptions are exactly realized, surplus does not enter the picture. Annuity payments increase in years when more than 4 per cent is earned and decrease in years when less than 4 per cent is earned. Suppose, for example, that the first year's rate of return is +10 per cent and the second year's is −10 per cent. Then the second term in the premium expression will accumulate to ($98,000/1.04) (1.10) at the end of one year; that is, the payment to each of the 98,000 survivors at age 66 will be (1.10/1.04) what it was at age 65. The third term in the premium expression will accumulate after two years to [$95,000/$(1.04)^2$] (1.10) (0.90), or $95,000 (1.10/1.04) (0.90/1.04), so each of the 95,000 survivors at age 67 will receive a payment equal to (0.90/1.04) times what he received at age 66.

This answers question 2. Benefit payments in both the fixed and the variable annuity are "charged" 4 per cent because the premium discounts future benefit payments at a 4 per cent assumed rate of investment return; the credit $(1 + i')$ reflects actual investment earnings (taken as equal to assumed earnings in fixed-dollar annuities). The relationship $(\text{Payment}_{t+1}) = [(1 + i')/(1 + i)]\ (\text{Payment}_t)$ is usually presented without any rationale.

After this explanation, most people see intuitively what happens to investment gains and losses under variable annuities; that is, the factors (1.10/1.04), (0.90/1.04), and so on, are the result of using investment gains and losses to adjust benefit payments. Furthermore, since (1.10/1.04) appears in the amounts payable at ages 66, 67, and so on, it is easy to point out that any year's investment gains are distributed over the entire future-benefit period.

To demonstrate what happens to investment gains and losses, reserves may be introduced as follows:

1. Total net single premium collected from the group at 65 for either a fixed-dollar annuity or a variable annuity.	$\left[100{,}000+\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$.
2. Fixed-dollar annuity reserve and variable annuity reserve at 65 after benefit payment of $100,000.	$\left[\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$.
3. Fixed-dollar annuity reserve at 66 before payment of benefit = (2) accumulated at 4 per cent.	$\left[98{,}000+\frac{95{,}000}{(1.04)}+\ldots\right]$.
4. Variable annuity reserve at 66 before payment of benefit = (2) accumulated at 10 per cent.	$(1.10)\left[\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$.
5. [(4) − (2)] − [(3) − (2)].	$(1.10-1)\left[\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$ $-(1.04-1)\left[\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$ $=0.06\left[\frac{98{,}000}{1.04}+\frac{95{,}000}{(1.04)^2}+\ldots\right]$.

Thus the excess of the increase in variable annuity reserve over the increase in fixed-dollar annuity reserve is seen to be equal to the fixed-dollar annuity's investment gain, 6 per cent of the beginning of year reserve. This sets question 1 regarding variable annuities to rest—investment gains and losses are channeled into the variable annuity reserve.

By rewriting the variable annuity reserve at age 66 in the form (1.10/1.04) [98,000 + 95,000/(1.04) + . . .], where the expression multiplied by (1.10/1.04) is the fixed-dollar annuity reserve at age 66, this example provides a convenient starting place for introducing unit values.

DONALD D. CODY:

Mr. Macarchuk has made a timely and worthwhile contribution to the actuarial aspects of variable annuities. The reader will also find a similar formulation of variable annuity actuarial mechanics, more detailed in some respects, in my discussion of Harry Walker's paper "State Regulation of Individual Variable Annuities" (*TSA*, XX, 456–63). Mr. Macarchuk has filled out the background of the theory in a very helpful manner.

I am extending below the formulations from my discussion, in abbreviated form, to the more generalized actuarial mathematics of variable annuities during the payout period in the hope that it will be of interest to students of the subject. I have found it to be the easiest approach to an understanding of variable annuities.

The following notation is needed in addition to that already set forth on pages 456–63 of Volume XX of the *Transactions:*

δ' = force of interest on AIR basis (a constant),

δ = force of interest on net investment income (a function of t),

$$\bar{u}_m = \bar{u}_0 e^{\int_0^m \delta dt} = \text{investment unit value,}$$

$$(\bar{a}\bar{u})_m = (\bar{a}\bar{u})_0 e^{\int_0^m (\delta-\delta')dt} = \text{annuity unit value,}$$

where m measures from the date of establishment of the unit values; $\bar{u}_0$ and $(\bar{a}\bar{u})_0$ commonly equal \$1.

The general annuity differential equation is the following:

$$\frac{d\bar{a}(t)}{dt} = (\mu + \delta)\bar{a}(t) - \mu I(t) - \bar{p}(t) , \tag{1}$$

where $\bar{a}(t)$ is the annuity reserve at time t, μ is the force of mortality at time t, $I(t)$ is the death benefit at time t, and $\bar{p}(t)$ is the rate of annuity payment at time t. This equation is fully representative of all common annuities, provided $\bar{a}(t)$, $\bar{p}(t)$, and $I(t)$ are taken in the Lebesgue sense. For instance, $\bar{a}(t) = \ddot{a}(t)$ if

$$\bar{p}(t)dt = \begin{cases} 1 & \text{for} \quad \text{integral } t , \\ 0 & \text{for} \quad \text{nonintegral } t . \end{cases}$$

$I(t)$ has discontinuous values and derivatives at t integral. If $\mu \equiv 0$, interest-only annuities emerge.

If integrating factors of

$$e^{-\int_0^t (\mu+\delta)ds}$$

and

$$e^{-\int_0^t \delta ds}$$

are applied, the following integral equations, respectively, result:

$$\bar{a}(r)_r p_x e^{-\int_0^r \delta ds} = \bar{a}(0) - \int_0^r [\bar{p}(t) + \mu I(t)]_t p_x e^{-\int_0^t \delta ds} dt . \tag{2}$$

$$\bar{a}(r) e^{-\int_0^r \delta ds} = \bar{a}(0) - \int_0^r \{\bar{p}(t) - \mu[\bar{a}(t) - I(t)]\} e^{-\int_0^t \delta ds} dt . \tag{3}$$

If $r = \infty$, equations (2) and (3) define $\bar{a}(0)$ prospectively, since $\bar{a}(\infty) = 0$. The fundamental definition of the variable annuity is

$$\bar{p}(t) = e^{\int_0^t (\delta-\delta')ds} \tag{4}$$

for one annuity unit. Introducing equation (4) into equation (2), we obtain the variable annuity reserve at duration r:

$$\bar{a}(r)_r p_x e^{-\int_0^r \delta ds} = \bar{a}(0) - \bar{a}'_{x:\overline{r|}} - \int_0^r {}_t p_x \mu_{x+t} I(t) e^{-\int_0^t \delta ds} dt , \tag{5}$$

where the primed function is based on the AIR (δ').

It can also be shown that, for an n-year certain life annuity,

$$I(t)e^{-\int_0^t \delta ds} = V'^t \bar{a}'_{\overline{n-t|}} , \tag{6}$$

so that

$$\bar{a}(r)_r p_x e^{-\int_0^r \delta ds} = \bar{a}(0) - \bar{a}'_{x:\overline{r|}} - \int_0^r V'^t {}_t p_x \mu_{x+t} \bar{a}'_{\overline{n-t|}} dt . \tag{7}$$

The right-hand side is equal to

$$\bar{a}'(r)_r p_x e^{-\int_0^r \delta' ds} ,$$

so that

$$\bar{a}(r) = \bar{a}'(r) e^{\int_0^r (\delta-\delta')ds} = \bar{p}(r)\bar{a}'(r) . \tag{8}$$

This demonstrates that the variable annuity reserve at duration r is equal to the annuity reserve on the AIR for annuity payments at the level existing at duration r. It is notable that relationship (6) is a requirement for this to be true; a corollary to this is that the refund period in a refund annuity must be defined in annuity units so that the refund period is constant.

Equation (3) can be reduced to the following for the variable annuity by using equation (4):

$$\bar{a}(r)e^{-\int_0^r \delta ds} = \bar{a}(0) - \bar{a}'_{\overline{r|}} + \int_0^r \mu_{x+t} V'^t [\bar{a}'(t) - \bar{a}'_{\overline{n-t|}}] dt . \tag{9}$$

One form of variable annuity on the market leaves the assets in a trust fund (earning δ) and moves the risk element to an insurance company, subject to an annual charge for annuity payments during the ensuing year and subject to payment of any balance of the trust fund to the insurance company at the time the annuitant dies. The mechanics of this is based on equation (9).

By differencing equation (9) and rearranging terms, we obtain the following:

$$\bar{a}(r)e^{\int_r^{r+1} \delta ds} - \bar{a}(r+1) \tag{10}$$
$$= e^{\int_0^r (\delta-\delta')ds} \{ \bar{a}'_{\overline{1|}} - \int_0^1 V'^t \mu_{x+t} [\bar{a}'(t) - \bar{a}'_{\overline{n-t|}}] dt \} .$$

The right-hand side is the annual charge made for year r to $r+1$. The insurance company pays annuity payments at the rate of

$$\bar{p}(t) = e^{\int_0^t (\delta-\delta')ds} .$$

The final charge at date of death at $r = u$ is $\bar{a}(u)$, the balance of the fund in the case of a no-refund life annuity. If the annuity certain period has not terminated, the insurance company receives

$$(\bar{a}'(u) - \bar{a}'_{\overline{n-u}|})e^{\int_0^u (\delta-\delta')ds}$$

at death. The trust fund is then reduced to

$$\bar{a}'_{\overline{n-u}|}e^{\int_0^u (\delta-\delta')ds} = \bar{p}(u)\bar{a}'_{\overline{n-u}|} ,$$

and equation (10) becomes

$$\bar{a}(r)e^{\int_0^{r+1} \delta ds} - \bar{a}(r+1) = e^{\int_0^r (\delta-\delta')ds}\bar{a}'_{\overline{1}|} = \bar{p}(r)\bar{a}'_{\overline{1}|} , \tag{11}$$

the right side being the annual charge thereafter.

Provision for expenses is introduced by determining annuity income by applying a suitably loaded annuity rate to the consideration and thus holding margin in the trust fund to produce expense charges for the insurance company over the lifetime of the annuity.

An insurance company conceivably could use this approach to move all risk into its general portfolio, using its separate account only for the pure investment function.

I would also comment that, since daily computerized valuations of variable annuities and equity-based life insurances are normal, the actuarial approach on the basis of calculus (where dt is the valuation period) tends to be natural.

PHILIP A. RABENAU:

Mr. Macarchuk's paper should be helpful and stimulating to actuaries who have not been directly involved in close study of the underlying concepts of variable annuities. I found the idea of developing the unit value concept for the fixed-dollar annuity an interesting one, although I suspect that most of us will continue to feel more at home with the traditional approach.

In the section on "The Separate Account," however, the author offers the thought that "the advent of the variable annuity changes the character of the surplus requirements of a life insurance company." I am sure that many of us see no such sweeping implication to the problems of developing and managing surplus. Extending our regular annuity business to include variable annuity contracts hardly seems to dictate a fundamental change in surplus considerations.

The author seems also to favor maintaining surplus funds within any separate account. I believe it well to make clear in this connection that the holding of surplus within the account is not an inherent or necessary arrangement. It may well be a valid decision for any company management to take in the light of its own separate account operations and procedures and its surplus philosophy. On the other hand, it seems clear that to deal with the surplus arising from separate account transaction as part of general company surplus to be held in its regular account is also a valid decision. In a particular company's operations, this approach might offer simpler, less expensive, and clearer accounting. Where numerous separate accounts were established in a single company, a considerable degree of fragmented accounting for surplus items would be avoided.

But the choice remains one for company management.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN MACARCHUK:

Mr. Feuer's development of the annuity unit value is a useful demonstration of the concept of variable annuity unit values. It should help further to clarify the concept.

Mr. Feuer refers to the actuarial value which is used as a base for distributing actual net investment earnings as the reserve. Generally, this will not be the case. For example, when a company strengthens variable annuity reserves to a basis more stringent than that used in determining the number of annuity units credited to an annuitant, it will not generally want to credit the annuitant with the investment earnings on company surplus invested in the reserve strengthening. To credit earnings on such surplus would constitute the payment of a special kind of dividend to the annuitant.

This complication was avoided in my paper by using the generic term "actuarial value." This broad definition of value base should cover all possible situations.

It is interesting to observe that, since the annuity unit faithfully and automatically preserves the mortality and expense guarantees implicit in the insured variable annuity, dividend declarations such as those mentioned above require specific distributive actions. For example, such dividends could be distributed by the crediting of additional annuity units or by the payment of lump-sum cash dividends from time to time.

Mr. Cody's discussion adds some generalized actuarial mathematics to the information available on the mathematics of variable annuities. His mathematical formulations should be useful to actuaries and to students in particular.

Toward the end of his discussion, Mr. Cody comments that an insurance company could move all risk into its general portfolio, using its separate account only for the pure investment function. This comment contradicts several fundamental aspects of the insured variable annuity. The company which offers a variable annuity that guarantees mortality and expenses assumes liabilities for the variable benefits guaranteed. The liabilities are liabilities of the company as a whole and not of a portion of its assets. The liabilities are derived from the benefits provided by the company's variable annuity contracts, not from the assets which fund these liabilities.

Since a separate account is a group of assets which belong to the company and since the amount of such assets may or may not be adequate to provide the benefits guaranteed by company contracts, separate account assets must be included in company assets in any test of company solvency. While the separate account's investment performance is vital to the determination of variable annuity benefits, the assets maintained in such an account at any point in time are purely and simply a portion of the company's total asset pool.

An arrangement between an insurance company and a trust fund such as that mentioned by Mr. Cody apparently does remove the risk element from the trust entity. It should be noted, however, that this is accomplished by means of additional contractual stipulations, including the requirement to pay special annuity premiums throughout the annuity payment period. Trust assets are not legally a part of the insurance company. As a result, the commitment between the two legal entities is consummated in a manner considerably different from the manner in which the more common variable annuity contract commitment is consummated.

Mr. Rabenau stresses the fact that surplus management is a decision for company management. I tried to stress this point in my paper.

My remarks were also aimed at stressing the point that the assets which fund the surplus of a variable benefits operation should be invested with the variability of the benefits clearly in mind. It is certainly correct to fund such surplus with debt securities if certain conditions exist. It is likewise correct to fund such surplus with securities identical to those used to fund variable benefits, if contrary operating conditions exist.

Mr. Rabenau comments that maintaining the bulk of surplus in the general account rather than the separate account would offer simpler, less expensive, and clearer accounting. Surplus is merely the excess of assets over liabilities. Accounting for variable annuity surplus is not, therefore, a complex, expensive, or unclear matter any more than it is for fixed-dollar benefits. Hence, it would appear that one will find such a process complex, expensive, and unclear only if one makes his accounting complex, expensive, and unclear.