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DISCUSSION OF PAPERS PRESENTED AT EARLIER REGIONAL MEETING

ACTUARIAL FUNCTIONS AS EXPECTED VALUES

JOHN A. FIBIGER AND STEPHEN G. KELLISON

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JOHN E. MORRILL* AND CECIL J. NESBITT:

In a discussion of this paper in the actuarial seminar at the University of Michigan, the question came up as to whether the apparently basic formulas in the paper had not been expressed previously in actuarial literature. The answer we found is that the basic formulas have appeared before, but in a different form and a different context. The previous form is as the "equation of maturity," namely,

$$C + i(F + C) = B\,, \tag{1}$$

which appears, for instance, on page 271 of Charles L. Trowbridge's paper "Funding of Group Life Insurance" (*TSA*, VII [1955], 270). To show the connection, one may multiply through formula (2) of the paper under discussion by l_x, to obtain

$$P_x\sum_{t=0}^{\infty} l_{x+t} + i\sum_{t=0}^{\infty}({}_tV_x + P_x)l_{x+t} = l_x\,, \tag{2}$$

which agrees with a formula on page 274 of Trowbridge's paper if one assumes that there are no withdrawals and makes appropriate changes in notation.

Similarly, by multiplying through formula (15) of the paper by l_x, we have

$$\bar{P}T_x + \delta\int_0^{\infty} {}_t\bar{V}\cdot l_{x+t}dt = l_x\,, \tag{3}$$

or

$$\int_0^{\infty} {}_t\bar{V}\cdot l_{x+t}dt = (l_x - \bar{P}T_x)/\delta\,. \tag{4}$$

Here formula (3) is the equation of maturity for an ordinary life insurance covering all persons aged x or greater in the stationary population repre-

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sented by a mortality table, with level annual premiums $\bar{P}$ payable continuously from age x and with unit benefits payable momently, so that the annual rate of benefits is l_x. The remaining term in equation (3) represents the annual flow of interest income. Equation (4) gives expressions for the total reserve.

It should be stated that, while the equation of maturity is mathematically equivalent to the basic formula of the paper, the two differ in the sense of probability. In fact, the equation of maturity is usually thought of on the basis of a deterministic model, while the paper's basic formula is a probability statement.

As a second remark, we note for the discrete case of an insurance with annual premiums P payable at the beginning of each insurance year, and with unit benefit payable at the end of the year of death, what might be called the equation at immaturity, namely,

$$P\sum_{t=0}^{r-1} l_{x+t} + i\sum_{t=0}^{r-1}({}_tV + P)l_{x+t} = (l_x - l_{x+r}) + {}_rV \cdot l_{x+r}\,. \tag{5}$$

This indicates that the premiums for those lives below age $x + r$, plus the interest on the reserves for those below age $x + r$, provide the claims for insureds dying between ages x and $x + r$ and also provide the reserves for the l_{x+r} survivors to age $x + r$. This could be used as a recursive relation to compute or check reserve values. Similarly, the equation at immaturity for the continuous model is

$$\bar{P}(T_x - T_{x+r}) + \delta\int_0^r {}_t\bar{V} \cdot l_{x+t}dt = (l_x - l_{x+r}) + {}_r\bar{V} \cdot l_{x+r}\,. \tag{6}$$

Our last remark is that many actuarial problems, and the mathematical models for their solutions, are best approached by means of a differential equation or a difference equation (linear recurrence relation) for the reserve function. In the discrete case for a standard insurance, the difference equation is

$$({}_tV + P)(1 + i) = q_{x+t} + p_{x+t} \cdot {}_{t+1}V\,, \tag{7}$$

which may be solved by use of various integrating factors. One possibility is to rearrange the formula as

$$P + i({}_tV + P) - q_{x+t}(1 - {}_{t+1}V) = {}_{t+1}V - {}_tV \tag{8}$$

and sum from $t = 0$ to $t = r - 1$ to obtain

$$r \cdot P + i\sum_{t=0}^{r-1}({}_tV + P) - \sum_{t=0}^{r-1} q_{x+t}(1 - {}_{t+1}V) = {}_rV\,. \tag{9}$$

A second possibility is to use an interest function such as v^t as an integrating factor, and this leads to formulas such as (5.18) on page 107 of Jordan's *Life Contingencies*. A third possibility, and one that seems to have been largely overlooked, is to use a mortality function such as l_{x+t} or ${}_tp_x$ as an integrating factor. This has been done by the authors in this paper. A fourth possibility is to use as an integrating factor a combined interest-mortality function such as $v^t l_{x+t}$, $v^t\ {}_tp_x$, or D_{x+t}. These lead to the usual equations for premiums and reserves.

If one allows basic terms such as the premium, amount of insurance, or rate of interest to vary with time, one can develop an indefinite number of special formulas. Some of these are seen in the paper by Fraser, Miller, and Sternhell, "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance" (TSA, XXI [1969], 343), and its discussion.

The authors have given some new illumination to the relationships among our familiar actuarial functions, and we are grateful to them.

JAMES C. HICKMAN AND JOSEPH J. GAYDA:

The mathematics of life contingencies is a subject of singular importance to life actuaries. Indeed, it may be viewed as the intellectual cornerstone of their profession. The subject is a blend of ideas from the mathematics of compound interest and from probability and statistics. Although the main ideas were developed many years ago, the creation of new approaches to classical results and the discovery of hidden relationships between life contingencies and allied topics has provided new insights and stimulation for generations of actuaries. The authors have contributed to this continuing effort to enrich the mathematics of life contingencies.

In keeping with the authors' spirit in providing us a novel path to traditional results, it seems appropriate to augment their paper with a discussion that relates some ideas from the theory of order statistics and multiple life actuarial functions. It is felt that, by understanding the relationship between order statistics and multiple life functions, actuarial students will gain valuable insights into both topics. In addition, a statistical approach to the development of actuarial functions serves to emphasize rather than suppress the random nature of losses in an insurance system.

I. SOME FUNDAMENTAL DISTRIBUTIONS

A set of random variables $X_1, X_2, \ldots, X_n$ is said to be a random sample if the X's are mutually independent in the statistical sense and identically distributed. Thus the joint probability density function of the random variables that make up a random sample is given by

$$h(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) \, ,$$

where $f(x)$ is the common probability density function of the n identically distributed random variables. If the n random variables are mutually independent in the statistical sense, but not necessarily identically distributed, the joint probability density function is given by

$$h(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) \, , \tag{1}$$

where $f_i(x_i)$ is the probability density function associated with X_i.

From this point on in this discussion we will assume that the random variables $X_1, X_2, \ldots, X_n$ are of the continuous type and have probability density functions $f_i(x_i) > 0$, $a_i < x_i < b_i$, zero elsewhere. Now we let $Y_n = \max(X_1, X_2, \ldots, X_n)$, $Y_1 = \min(X_1, X_2, \ldots, X_n)$, and in general Y_i is set equal to the ith random variable from among $X_1, X_2, \ldots, X_n$ when the n random variables are arranged in order of ascending magnitude. The joint probability density function of $Y_1, Y_2, \ldots, Y_n$, where $X_1, X_2, \ldots, X_n$ constitutes a random sample, is given by

$$k(y_1, y_2, \ldots, y_n) = n!\prod_{i=1}^{n} f(y_i) \, , \quad a < y_1 < y_2 < \ldots < y_n < b \, . \tag{2}$$

The set of Y's are called the order statistics in this case.

If the random variables $X_1, X_2, \ldots, X_n$ are mutually independent in the statistical sense but do not have identical distributions, the joint probability density function of $Y_1, Y_2, \ldots, Y_n$ is given by

$$k(y_1, y_2, \ldots, y_n) = \Sigma f_i(y_i) f_j(y_2) \ldots f_m(y_n) \, , \tag{3}$$

where the sum is taken over all $n!$ permutations of the subscripts on the individual probability density functions.

These fundamental distribution results may be obtained by a formal application of the change-of-variable technique. In fact, several widely used textbooks on mathematical statistics [1, chap. 4; 3, chap. 10] take this approach. However, with less rigor, the results may be appreciated by noting that the probability density of a sample point in the ordered sample space is obtained by adding the probability density at each of the $n!$ points in the original sample space that map into the same ordered sample point.

The probability density function of Y_k, the kth-order statistic, is given by

$$g_k(y_k) = \{n!/[(k-1)!(n-k)!]\}[F(y_k)]^{k-1}[1-F(y_k)]^{n-k}f(y_k)\,, \qquad a < y_k < b\,, \quad (4)$$

$$= 0 \qquad \text{elsewhere}\,,$$

where

$$F(y_k) = \mathrm{P_r}(Y_k \leq y_k) = \int_a^{y_k} f(x)dx$$

is the cumulative distribution function. This result may be obtained formally by integrating out the $n-1$ variables $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ from equation (2). Alternatively, and with much less rigor, equation (4) may be seen to represent the product of the probability that $k-1$ members of the random sample are less than y_k, the probability that $n-k$ members of the random sample are greater than y_k, and the probability density at the point y_k, times the number of points in the original sample space that will yield a particular value of y_k.

When the original X's are mutually independent, but not necessarily identically distributed, we have

$$g_k(y_k) = \Sigma \prod_{\text{terms}}^{k-1} F_l(y_k) \prod_{\text{terms}}^{n-k} [1 - F_m(y_k)]\, f_i(y_k)\,, \qquad a_i < y_k < b_i\,, \quad (5)$$

$$= 0 \qquad \text{elsewhere}\,,$$

where, for each i ($i = 1, 2, \ldots, n$), the sum extends over all $\binom{n-1}{k-1}$ partitions of the random variables into those less than y_k and those greater than y_k. In analogy to the situation in the simpler case when the random variables are identically distributed, this result may be obtained by integrating out $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ from equation (3). The result may be justified in an informal fashion, however, by observing that the probability density of y_k is obtained by multiplying the probability that $k-1$ random variables are less than y_k, the probability that $n-k$ are greater than y_k, and the probability density at y_k and then adding the resulting expression for all sample points in the original sample space that yield the same value of y_k.

II. ACTUARIAL APPLICATIONS

We recall that the probability density function for the random variable time until death, for a life that has already lived to age x, is given in actuarial notation by

$$f(t|x) = {}_tp_x\mu_{x+t}$$

and that the associated cumulative distribution function is given by

$$F(t|x) = 1 - {}_tp_x = {}_tq_x\,.$$

Consider a set of n lives, ages $x_1, x_2, \ldots, x_n$, and assume that their respective times until death are mutually independent random variables of the continuous type. The survival function associated with the status that ends on the kth death, $k = 1$, or $2, \ldots$, or n, is given in actuarial notation by

$$\Pr\,(\text{Time until the } k\text{th death} > t) = {}_tp_{\overset{n-k+1}{\overline{x_1x_2\,\ldots\,x_n}}}\,,$$

and the probability density function of time until the kth death is given by

$$-\frac{d}{dt}\left({}_tp_{\overset{n-k+1}{\overline{x_1x_2\,\ldots\,x_n}}}\right) = \Sigma\left(\prod^{\substack{k-1\\ \text{terms}}} {}_tq_{x_i}\right)\left(\prod^{\substack{n-k\\ \text{terms}}} {}_tp_{x_j}\right) {}_tp_{x_l}\mu_{x_l+t}\,, \tag{6}$$

where the sum extends over all possible partitions of the $n - 1$ remaining lives between the first and second continued products, while each of the n lives takes its turn in the role of x_l. Equation (6) is the actuarial version of equation (5).

If $k = 1$, equation (6) reduces to the probability density function of the time until the first death. We have in this case

$$-\frac{d}{dt}\left({}_tp_{x_1x_2\,\cdots\,x_n}\right) = \prod_{i=1}^{n} {}_tp_{x_i}\left(\sum_{l=1}^{n}\mu_{x_l+t}\right), \tag{7}$$

which is the usual form of the probability density function of the time until the "death" of the joint life status.

If $k = n$, equation (6) reduces to the probability density function of the time until the death of the last survivor. We have

$$-\frac{d}{dt}\left({}_tp_{\overline{x_1x_2\,\ldots\,x_n}}\right) = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} {}_tq_{x_j}\;{}_tp_{x_i}\mu_{x_i+t}\right). \tag{8}$$

Those who seek to relate equation (6) to expressions in Jordan [2, chap. 10] may set $n = 3$ and $k = 2$. This yields, following Jordan,

$$\begin{aligned}
-\frac{d}{dt}\left({}_tp_{\overset{2}{\overline{x_1x_2x_3}}}\right) &= -\frac{d}{dt}\left({}_tp_{x_1x_2} + {}_tp_{x_1x_3} + {}_tp_{x_2x_3} - 2\,{}_tp_{x_1x_2x_3}\right)\\
&= {}_tp_{x_1x_2}\mu_{x_1+t:x_2+t} + {}_tp_{x_1x_3}\mu_{x_1+t:x_3+t} + {}_tp_{x_2x_3}\mu_{x_2+t:x_3+t}\\
&\qquad - 2\,{}_tp_{x_1x_2x_3}\mu_{x_1+t:x_2+t:x_3+t}\,.
\end{aligned}$$

This result is identical with that obtained by making appropriate substitutions in equation (6),

$$-\frac{d}{dt}\left({}_tp_{\frac{2}{x_1x_2x_3}}\right) = (1 - {}_tp_{x_2})\,{}_tp_{x_3}\,{}_tp_{x_1}u_{x_1+t} + (1 - {}_tp_{x_3})\,{}_tp_{x_2}\,{}_tp_{x_1}u_{x_1+t}$$
$$+ (1 - {}_tp_{x_1})\,{}_tp_{x_3}\,{}_tp_{x_2}u_{x_2+t} + (1 - {}_tp_{x_3})\,{}_tp_{x_1}\,{}_tp_{x_2}u_{x_2+t}$$
$$+ (1 - {}_tp_{x_1})\,{}_tp_{x_2}\,{}_tp_{x_3}u_{x_3+t} + (1 - {}_tp_{x_2})\,{}_tp_{x_1}\,{}_tp_{x_3}u_{x_3+t}\,.$$

At this point, recalling the well-known analogy [2, chap. 14] between multiple decrement and joint life models, it is natural to inquire if, in fact, there are two statistical models. Perhaps one model is contained in the other. The answer to this question is that the two models differ in a fundamental way. The multiple decrement model deals with the joint distribution of the two random variables time until death, a continuous-type random variable, and the discrete random variable cause of death. The joint life model starts with the distribution of n independent random variables of the continuous type, and interest is centered on finding the distribution of the time until the first death. To gain further insight into the differences between the models, we note that the functions

$${}_tp_{x_i} = \exp\left(-\int_0^t \mu_{x_i+s}ds\right), \qquad i = 1, 2, \ldots, n$$

are each conditional survival functions, but

$${}_tp_x'^{(i)} = \exp\left(-\int_0^t \mu_{x+s}^{(i)}ds\right)$$

does not necessarily approach zero as t increases, and therefore it is not necessarily a survival function. To put it crudely, a life does not necessarily die, with probability 1, from some specified cause.

III. NET PREMIUMS

For notational convenience we let $f(t|n, k)$ denote the probability density function exhibited in equation (6). Net premiums, using the continuous model, will now be obtained using the *principle of equivalence*. That is, we will require that the present expected value of future losses, due to the random nature of time until death, be zero at the time that the insurance or annuity contract is issued.

Note that the general program for determining net premiums, which is illustrated in the examples that follow, is as follows: (1) Set down from general considerations an appropriate loss function. (2) Determine the probability density function of time until death. This may often be conveniently done by first determining the associated survival function and then taking the negative derivative of this survival function. (3) Set the expected value of the loss function equal to zero and solve for the net premium.

From this point on in this discussion all premium and reserve functions will pertain to a policy paying a unit benefit at the time of the kth death from among n lives in the insurance examples, and paying at an annual rate of 1 until the kth death in the annuity example. Solely to simplify the notation the subscripts $\overset{n-k+1}{_{x_1 x_2 \ldots x_n}}$ will be omitted.

A. *Immediate Life Annuity*

The loss to the insurer on a continuously paid life annuity, which will be paid until the instant of the kth death from among the n lives, is given by ${}_aL(T) = \bar{a}_{\overline{T|}} - \bar{a}$, where $\bar{a}$ is the net single premium and T is the continuous random variable time until the kth death. Invoking the principle of equivalence, we have

$$\bar{a} = E[\bar{a}_{\overline{T|}}] = \int_0^\infty \bar{a}_{\overline{t|}} f(t|n, k)dt \, .$$

B. *Single Premium Insurance*

The loss to the insurer due to the random nature of the time until the kth death from among the n lives, whose times until death are assumed to be mutually independent, is given by

$${}_AL(T) = v^T - \bar{A} \, .$$

In this expression $\bar{A}$ is the net single insurance premium and T is once more the time until the kth death. Invoking the principle of equivalence, we have

$$\bar{A} = E[v^T] = \int_0^\infty v^t f(t|n, k)dt \, .$$

It is now immediate, for fixed n and k, that

$$E[v^T + \delta\bar{a}_{\overline{T|}}] = \bar{A} + \delta\bar{a} = 1 \, .$$

C. *Annual Premiums*

The loss to the insurer, due to the random nature of the time until the kth death, involved in issuing an annual premium continuously paid insurance which will pay a unit benefit at the instant of the kth death is given by

$${}_PL(T) = v^T - \bar{P}\bar{a}_{\overline{T|}} \, .$$

Once again T is the random variable time until the kth death, and we assume that the times until death of the n lives are mutually independent in the statistical sense. $\bar{P}$ denotes the annual premium rate, and it is determined by the principle of equivalence to be

$$\bar{P} = E[v^T]/E[\bar{a}_{\overline{T}|}] = \bar{A}/\bar{a} .$$

IV. VARIANCES OF LOSS FUNCTIONS

The variances of the loss functions which were defined in Section III may be adopted as crude measures of the risk, due to the random nature of time until death, which is assumed by the insurer in issuing these contracts. In each case

$$\begin{aligned} \text{Var}\ (L) &= E[L^2] - (E[L])^2 \\ &= E[L^2] . \end{aligned}$$

A. *Immediate Life Annuity*

$$\text{Var}\ [{}_aL(T)] = (\bar{A}' - \bar{A}^2)/\delta^2 ,$$

where

$$\bar{A}' = E[v^{2T}] .$$

B. *Single Premium Insurance*

$$\text{Var}\ [{}_AL(T)] = \bar{A}' - \bar{A}^2 .$$

C. *Annual Premium Insurance*

$$\text{Var}\ [{}_pL(T)] = (\bar{A}' - \bar{A}^2)/(\delta\bar{a})^2 .$$

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(AUTHORS' REVIEW OF DISCUSSION)

JOHN A. FIBIGER AND STEPHEN G. KELLISON:

The authors are indeed grateful for the excellent discussions submitted by Professors Morrill and Nesbitt and by Professor Hickman and Mr. Gayda. Perhaps it is only fitting that a paper with dual authorship should have discussants appearing in pairs also.

Professors Morrill and Nesbitt have essentially recast the results of the paper from the viewpoint of a stationary population. This perspective does provide an interesting alternative insight into the concept of expected values. Mr. Charles T. P. Galloway has communicated similar ideas in a letter to the authors.

Although not directly discussing the main thesis of the paper, Professor Hickman and Mr. Gayda have made a valuable addition to the literature. Statistical approaches to the development of actuarial functions have not traditionally been utilized to a great extent in the study of life contingencies. However, these approaches do tend to underscore the fundamental random nature of traditional actuarial functions.

The authors are indebted to Mr. David G. Halmstad for suggesting the following references for those readers wishing to explore in greater depth the statistical variation of traditional actuarial functions about their mean values.

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TRANSACTIONS OF SOCIETY OF ACTUARIES
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A NEW LOOK AT GAIN AND LOSS ANALYSIS

ARTHUR W. ANDERSON

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BARNET N. BERIN:

The situation is improving for the serious student of pension mathematics. Most pension actuaries acknowledge that the existing actuarial literature is weak. Mr. Anderson's paper is a helpful contribution. The approach is a bit difficult, however, and may discourage the careful reading that it deserves.

The following synopsis of pension mathematics differs from the paper in several important respects: it assumes that the pension gain and loss analysis can be developed naturally and in a relatively straightforward manner; it assumes that a general approach is entirely possible (something Mr. Anderson appears to doubt); and it introduces the concept of a valuation check.

On one point, I would take serious exception to Mr. Anderson's paper. There is a defined actuarial gain under the aggregate cost funding method, and this is shown below.

Introduction

Pension mathematics is concerned largely with the valuation of pension plans and with the determination of accrued liabilities, one-year costs, and the actuarial gain. A necessary and important by-product is a range of company costs, the payment of which leads to the accumulation of assets.

Each of the important funding methods in use today can be defined in terms of accrued liabilities which can be tied together between successive durations, by exact formulas. If we trace a group of individuals between successive durations, we are led to the concept of liability gains and losses, which, in turn, are reduced by asset gains and losses to produce an over-all net actuarial gain or loss. (Both liability gains and losses and asset gains and losses are based upon a comparison of actual experience with expected experience.) The immediate-gain funding methods recognize this amount in full, while the spread-gain funding methods recognize a portion of this amount.

Experts in the field have long recognized the importance of the actuarial gain in interpreting valuation results and in providing valuable insight into pension problems but have questioned whether these results could be developed economically and whether there was sufficient interest in these results on the part of employers. The results can now be obtained economically, as a result of a significant improvement in computers, particularly in recent years. Interest in pension plan costs has been expressed not only by pension clients but by accounting firms, by government agencies, by unions, and by the public. There is no longer any question of the need to respond.

Taking account of recent developments, this synopsis will indicate how to analyze an immediate-gain funding method and a spread-gain funding method and will indicate systems whereby the valuation may be checked.

Preliminaries to the Valuation

Most pension valuations cover a one-year period, from the prior valuation date to the current valuation date. To do a current valuation, all participants at the prior valuation date must be identified as to their status at the current valuation date, and new entrants must be introduced. Status refers to a class designation such as active, terminated without vesting, terminated with vesting, dead, disabled, or retired.

To make this discussion manageable, subsequent equations will use age $x - 1$, at duration 0, for the prior valuation date and age x, at duration 1, for the current valuation date. The actuarial assumptions will involve interest and mortality discounts only. The more complex cases, including the effect of employee contributions, will be discussed later. It is not possible to do more than sketch the details in this presentation; those who have been exposed to pensions, in the more technical sense, will be able to follow all the results and to expand upon the material included. The notation and formulas will be kept as simple as possible.

An Immediate-Gain Valuation (Accrued Benefit Cost Method or Unit Credit Cost Method)

If we use (AB) for accrued benefit and (FSB) for the exact future service benefit over the year, an active participant's accrued liability can be tied together as follows:

$$(AB)_{x-1}\frac{N_{65}^{(12)}}{D_{x-1}}(1+i) + (FSB)\frac{N_{65}^{(12)}}{D_{x-1}}(1+i) + q_{x-1}(AB)_x\frac{N_{65}^{(12)}}{D_x} = (AB)_x\frac{N_{65}^{(12)}}{D_x}\,.$$

A retired participant's accrued liability can be tied together as follows:

$$(AB)_{65}\frac{N_{x-1}^{(12)}}{D_{x-1}}(1+i) - (AB)_{65}(1+\tfrac{13}{24}i) + q_{x-1}(AB)_{65}\left(\frac{N^{(12)}}{D_x}+\tfrac{11}{24}\right)$$

$$= (AB)_{65}\frac{N^{(12)}}{D_x}\,.$$

If we sum these equations for all participants included in the valuation at duration 0, and introduce the cost for new entrants on both sides of the equation, we have, at duration 1,

$$L_0(1+i) + FSC_0(1+i) + EM - {}^{i}EPP + L_1^N$$

$$= L_1^A + L_1^R + L_1^T + L_1^D + L_1^N\,.$$

Stated in words, the accrued liability at duration 0 with one year's interest, plus the future service cost at duration 0 with one year's interest, plus the expected mortality, less the expected pension payments with appropriate interest—indicated by the superscript i—plus the future service cost for new entrants at duration 1, is equal to the accrued liability at duration 1 for actives, for retireds, for terminated employees, for deceased employees, and for new entrants.

The liability gain or loss is

$$L_0(1+i) + FSC_0(1+i) - [L_1^A + L_1^R + L_1^N]$$

$$= L_1^T + (L_1^D - EM) - L_1^N + {}^{i}EPP\,.$$

There are several things to note:

1. The term in brackets on the left-hand side is the accrued liability which will be shown in the valuation report.
2. The first three terms on the right-hand side represent the gain from terminations of employment, the gain or loss from mortality, and the loss from new entrants.
3. The last term on the right-hand side, the expected pension payments with interest, will be compared with a similar asset item, as we shall see immediately below.

The Development of Assets

In a generalized situation, capable of simple modifications to fully reflect an insured or trusteed approach, assets may be tied together from duration 0 to duration 1 as follows:

$$A_0(1+i) + {}^{i}C - {}^{i}B - {}^{i}E + IG = A_1\,,$$

or assets at duration 0 with a full year's interest at the valuation interest rate plus contributions less benefits less expenses, all with appropriate interest at the valuation interest rate—indicated by the superscript i—plus the interest gain or loss equals assets at duration 1.

The asset gain or loss is

$$A_0(1+i) + {}^iC - A_1 = -IG + {}^iB + {}^iE .$$

Since assets reduce liabilities, the equation above will be taken negatively, so that the terms on the right-hand side will appear in their customary sense.

The Immediate Actuarial Gain

If we deduct the asset gain or loss from the liability gain or loss, we have each of the individual items of gain or loss, with positive signs indicating gains and negative signs indicating losses. (In more complex situations the signs are very helpful.)

The separate development of both liability gains and losses and asset gains and losses to develop the over-all net actuarial gain or loss, instead of reliance upon the traditional development of an expected unfunded liability to be compared with an actual unfunded liability (the difference being the actuarial gain), is most important, since the liability gain or loss can now be checked efficiently and since the asset gain or loss is very nearly self-checking. This does much to eliminate the unexplained residual item of gain or loss which made actuarial gain and loss analysis so difficult in the past: the individual items of liability gain or loss and asset gain or loss are developed directly and then combined to yield an over-all net gain or loss.

The same over-all net actuarial gain is produced as under the traditional approach, since the method illustrated above is equivalent to taking the unfunded liability at duration 0 with a full year's interest at the valuation interest rate, adding the future service cost at duration 0 with a full year's interest, deducting the contributions with interest to produce an expected unfunded liability at duration 1, and then deducting the actual unfunded liability at duration 1 to find the over-all net actuarial gain.

Valuation Check

The above theory can be applied to ensure accuracy in the valuation. Before starting the valuation, be sure that the data at duration 1 are consistent with the data at duration 0, except for the insertion of status at duration 1; the introduction of new entrants; and, in some cases, data changes to obtain the correct benefit. The following items should be calculated at duration 1, the current valuation date, for each status code separately.

A. For nonretired employees at ages less than or equal to age 65:
 1. The accrued liability at duration 0.
 2. The exact future service cost at duration 0, where the one-year benefit is the difference between the accrued benefit at duration 1 and the accrued benefit at duration 0.
 3. The expected mortality.
 4. The accrued liability at duration 1.
 5. A check step equal to $[(1) + (2)](1 + i) + (3) - (4)$. This should be a very small positive or negative amount for each employee. With sufficient decimal places in the rates, it will be zero.
 6. The estimated future service cost at duration 1.

B. For nonretired employees aged 66 or over and for retired participants:
 1. The accrued liability at duration 0.
 2. The expected pension payments with appropriate interest.
 3. The expected mortality.
 4. The accrued liability at duration 1.
 5. A check step equal to $(1)\ (1 + i) - (2) + (3) - (4)$. This should be a very small positive or negative amount for each employee. With sufficient decimal places in the rates, it will be zero.

Normally, other items of interest will also be included.

It is important to redetermine item A(1) and item B(1) to be certain that the starting accrued liability agrees with last year's ending accrued liability and to account for any differences from last year's accrued liability due to data changes which are assumed, here, to require corrections at the start of the plan year. (There is an alternative approach involving a correction at duration 1 which is similar in effect.) Any such difference with interest will become an item of actuarial gain or loss.

Item A(2) is quite important. At duration 0 we usually know only the estimated future service benefit for the coming plan year. The calculation of item A(2) gives us an important item of experience:

$$-(FSC_0' - FSC_0)(1 + i)\ ,$$

or the loss due to the excess of the actual future service cost over the estimated future service cost with interest. This result follows at once from the development of the liability gain and loss.

If Item A(5) and item B(5) are quite small in grand total, as they should be, the valuation is correct and can be completed with confidence. If this is not the case, the item 5 totals for each status should be checked separately for significant nonzero results, and finally item 5 should be scanned for each participant within any suspect status.

A systematic review of this type can be done very quickly. If something is wrong, the cause can usually be pinpointed and the valuation results corrected or redone. (A computer program can be adjusted very quickly to remove a source of error and the proper results furnished.)

Another Immediate-Gain Valuation (Individual Level Cost Method with Supplemental Liability or Entry Age Normal Cost Method)

If we replace the accrued benefit cost method's definition of accrued liability and future service cost with the individual level cost method's definition of accrued liability and normal cost, all the preceding formulas and procedures apply in a straightforward manner. (In what follows, by "individual level cost method," we mean "with supplemental liability.")

This funding method is important because it links the immediate-gain valuation and the spread-gain valuation. If the individual level cost method's normal cost factor, expressed as a dollar cost for each active participant or as a percentage of each active participant's salary, is assumed constant for all active participants, we are led at once to the aggregate level cost methods.

A Spread-Gain Valuation (Aggregate Level Cost Method without Supplemental Liability Method or Aggregate Cost Method)

Since the aggregate level cost method is an adaptation of the individual level cost method with a constant normal cost factor, we know that

$$NCF_0 = \frac{TL_0 - A_0}{PV_0}\,, \quad [TL_0 - NCF_0 PV_0] - A_0 = 0\,,$$

or that the total liability for active participants and for retired participants less the present value of future normal costs less the assets, that is, the unfunded liability, equals zero. A feature of this method is the determination of a normal cost factor, at duration 1, applicable to the next year which makes the unfunded liability at the valuation date zero. This is the heart of the method: If we use NCF_0 at duration 1, the unfunded liability is positive if there is an over-all net actuarial loss and negative (i.e., a surplus) if there is an over-all net actuarial gain.

If we return to the traditional method of developing the actuarial gain and apply it to this method—a valid approach, since we are dealing with a variation of the individual level cost method—the unfunded liability at the start of the year is zero, and, if we assume that the normal cost with one year's interest is exactly equal to the company's contribution with interest, as ideally it should be, we find that the expected unfunded liability is equal to zero, so that the negative of the actual unfunded liability using NCF_0 is equal to the actuarial gain:

$$-[TL_1 - NCF_0PV_1] + A_1 = \text{Actuarial gain} .$$

At this point, NCF_1 is determined, so that the unfunded liability at duration 1 is zero.

If we add and subtract NCF_1PV_1 immediately above, we are led to an important result of considerable practical value.

$$NCF_1 = NCF_0 - \frac{\text{Actuarial gain}}{PV_1} .$$

There are several things to note:

1. The accrued liabilities for certain employees, typically younger employees and often new entrants, may be negative: the present value of future normal costs, generated by a constant normal cost factor, applied to all active participants, may exceed the present value of projected benefits. These liabilities should be retained as negative amounts—no problem for the computer—in developing the items of liability gain and loss.
2. Accrued liabilities should be calculated, for each participant, using NCF_0 and developed forward from duration 0 to duration 1 exactly as if this were the individual level cost method described above.
3. The gain or loss for new entrants is based upon NCF_0.
4. The difference between last year's normal cost with one year's interest and the company's contribution with interest is an item of experience and part of the actuarial gain or loss.

Valuation Check

The preliminaries should be handled as above, for an immediate-gain valuation, and NCF_0 redetermined at duration 1, as of duration 0, to establish the correctness of the starting point. The expected unfunded liability should be equal to zero. The dollar amount of net actuarial gain should be calculated directly, by obtaining all the positive and negative components of actuarial gain or loss, and should be shown to equal

$$(NCF_0 - NCF_1)PV_1 ,$$

from the result obtained immediately above.

Another Spread-Gain Valuation (Aggregate Level Cost Method with Supplemental Liability Method or Entry Age Normal Frozen Initial Liability Funding Method)

We can follow the same procedure as in the aggregate level cost method without supplemental liability, since assets above are replaced by assets plus any remaining initial unfunded liability which has not been amortized. The initial unfunded liability will decrease to zero by a series of payments, which at the same time increases assets, so that the initial unfunded liability may be considered to be fixed or frozen. The important point is that this method differs from the prior method only in the definition of what constitutes "assets," so that the technique described above is applicable.

More Complex Cases (Including Ancillary Benefits)

In this brief review we mention only the treatment of employee contributions, turnover assumptions, salary-scale assumption, and ancillary benefits such as a disability benefit or a widow's benefit.

1. *Employee contributions.*—While the one-year cost is reduced, by the effect of employee contributions, and while the assets of the plan are increased by employee contributions, a liability must be established to recognize that these contributions, with interest at a specified rate, are payable upon termination of employment and upon death prior to retirement. (Death after retirement can be treated directly by use of an appropriate annuity form.) The preretirement death benefit cost and accrued liability can be handled by the calculation of life insurance type, commutation functions, and the reserve between duration 0 and duration 1 tied together by formulas similar to those connecting successive terminal life insurance reserves.

By assuming that the valuation interest rate is equal to the guaranteed interest rate applicable to employee contributions, it is possible to significantly reduce the detail work. The resulting overstatement of cost and liabilities is usually minor, provided that the excess of the valuation interest rate over the rate applied to employee contributions is not large.

2. *Turnover assumptions.*—Although the behavior of these rates is much different from that of mortality rates, decreasing to zero at some age prior to normal retirement age, once they are introduced into the determination of the basic valuation rates they can be handled in a manner similar to mortality rates in the formula connecting successive reserves.

3. *Salary-scale assumptions.*—This is a necessary complication in a final salaried plan valuation. The developments in such a valuation are similar to those shown above. The principal difference is that we project benefits and contributions to normal retirement age by introducing a salary-scale rate, say K, and by using both actual salaries $(AS)_{x-1}$, $(AS)_x$, and an expected salary $(AS)_{x-1}(1 + K)$. For the individual level

cost method with supplemental liability (or entry age normal cost method), the accrued liability is developed as follows:

$$\left[PB_{65}\frac{N_{65}^{(12)}}{D_{x-1}} - NCF_{EA}(AS)_{x-1}\frac{{}^{s}N_{x-1} - {}^{s}N_{65}}{{}^{s}D_{x-1}}\right](1 + i)$$

$$+ NCF_{EA}(AS)_{x-1}(1 + i)$$

$$+ q_{x-1}\left[PB_{65}\frac{N_{65}^{(12)}}{D_x} - NCF_{EA}(AS)_{x-1}(1 + K)\frac{{}^{s}N_x - {}^{s}N_{65}}{{}^{s}D_x}\right]$$

$$= PB_{65}\frac{N_{65}^{(12)}}{D_x} - NCF_{EA}(AS)_{x-1}(1 + K)\frac{{}^{s}N_x - {}^{s}N_{65}}{{}^{s}D_x}.$$

There are two approaches: add the present value of the increase in projected benefits, as a level premium determined at attained age x, to the normal cost; or redetermine the accrued liability and the normal cost. The first approach is more common.

In a spread-gain valuation we obtain the accrued liability at duration 0 based upon the appropriate salary at duration 0, the accrued liability at duration 1 based upon the estimated salary at duration 1, and the accrued liability at duration 1 based upon the actual salary at duration 1. The difference between the last two accrued liabilities represents the loss from salaries in excess of those assumed under the valuation assumption.

4. *Ancillary benefits.*—One of the first decisions is whether or not a specific decrement should be assumed in the basic rate structure employed in the valuation. For example, should a disability rate be assumed and rates appropriately discounted, or should commutation functions of the life insurance type be developed for the widow's benefit? Some of the considerations involved in this decision are the probable incidence of the event and the expected one-year claims; the relationship between the accrued liability before and after the event occurs; the size of the fund; and the size of the actuarial gains, components of the actuarial gains, and trend of the actuarial gains.

If certain considerations are favorable—that is, there are few expected claims; the accrued liability after the event is not significantly greater than before the event; there is a large fund; and there is a history of actuarial gains with consistent gains from certain sources, such as interest gains—it may be sufficient to terminal-fund this benefit or to develop one-year term costs which are accumulated at the valuation interest rate and are used at the time of terminal funding to provide any additional liability that may be required.

DONALD A. LOCKWOOD:

I read Mr. Anderson's fine paper with interest, since our office has always believed in (and, fortunately, many times achieved) a mathematical calculation of gains and losses by individual sources with a tolerance of 0.

We stress gain analysis (*a*) for purposes of checking for errors or faulty data in the annual valuation; (*b*) for keeping records of the validity of the individual assumptions; and (*c*) for reporting and tabulating the individual gains in our report to clients.

I believe that Mr. Anderson has done a commendable job, both in his thoroughness and in his ingenious notation. I think that he has gone astray, however, in his gain analysis under aggregate funding methods. Although his formulas are mathematically correct, their application can often lead to peculiar results.

This misdirection began with his formulas (65) and (66), where he expresses the change in unit normal cost (or accrual rate) in terms of the unit normal cost at the end of year, U_1, and the present value of future compensation based on salaries at the beginning of the year, D_0. In my opinion, these factors should properly have been U_0 and D_1, respectively, and formula (66) should read

$$\Delta U = \frac{1}{D_1}(\Delta N - U_0\Delta D) .$$

This formula is derived from the following identity:

$$\frac{N_0}{D_0} = \frac{N_1 - \Delta N}{D_1 - \Delta D} = \frac{N_1 - \Delta N + \left(\frac{N_1 - \Delta N}{D_1 - \Delta D}\right)\Delta D}{D_1}$$

or

$$U_0 = U_1 - \frac{1}{D_1}(\Delta N - U_0\Delta D)$$

or

$$\Delta U = \frac{1}{D_1}(\Delta N - U_0\Delta D) .$$

This formula has the following advantages as compared to Mr. Anderson's formula:

1. The spread factor becomes

$$SF = \frac{1}{\sum_{A_1} {}_1S_j\left(\frac{{}^sN_{(x+1)_j} - {}^sN_{y_j}}{{}^sD_{(x+1)_j}}\right)} ,$$

and gains or losses are spread over future salaries of those active members as of the current valuation date rather than over expected future salaries of beginning-of-the-year members based upon the prior valuation results. This ties the spread factor to figures appearing in the annual valuation report to the client, namely, the ratio of the normal cost to the item appearing in the actuarial balance sheet, the "Present Value of Future Normal Costs," or even to the present value of future payroll figure appearing in the unit normal cost calculation, assuming that this calculation is shown in the report.

2. The individual gain or loss elements related to the unit normal cost will be based upon the unit normal cost at the beginning of the year rather than at the end of the year. Then, if during the year there are heavy termination gains, or marked asset fluctuation as we have recently experienced causing sizable gains or losses, or severe salary increases, these items will not affect the calculation of the other gain or loss elements for the year. Otherwise, the other gain or loss elements would be severely distorted.

3. The gain or loss for new entrants will be related to the unit normal cost at the beginning of year. This again minimizes distortion when there are heavy gains or losses for the year in specific areas.

I would also like to comment on Mr. Anderson's formula for the gain from interest. This is the first element in his formulas (21), (40), (56), and (70). If Mr. Anderson's formula (21), for example, were broken into two elements, f_1 and f_2, where f_1 is the gain from fund investments and is equal to

$$I - iF_0 - I_K + \tfrac{13}{24}iB$$

and f_2 is the interest gain on pension payments saved and is equal to

$$\tfrac{13}{24}i\Big(\sum_{B_0}{}_0AB + \sum_{R_0}B - B\Big),$$

then it would be possible to calculate the actual earned interest rate j of the fund for the year, by the following simple formula:

$$j = i\,\frac{1}{1 - f_1/I},$$

where I is the interest earnings defined by the asset valuation method.

GERALD RICHMOND:

Arthur Anderson has written an excellent paper (limited in presentation, however, to a fixed retirement age) that should substantially advance the science of gain and loss analysis. I should first like to supplement his excellent theoretical development with some practical considerations. My pension department has been using the set theory developed to calculate components of the actuarial gain. It is important to have a worksheet on which the activity in each set of data is recorded for analysis. Our computer program accepts employee data cards on which current salary and the benefits from the prior year's valuation are punched. It produces the normal cost and past service liability for benefits of both this and the prior year (both of these are posted to the worksheet from which the salary loss can be calculated) for the individual funding methods and single premiums correspondingly for aggregate methods. Moreover, all terminations and changes in status (for example, from active to retired) are recorded on the worksheet along with the basic actuarial data; the changes in status are shown in both last year's and this year's status. We avoid the necessity of calculating the expected release seriatim in advance for active lives under normal retirement age as suggested in Anderson's paper, for individual funding methods, by use of the following relationship (a similar relationship involving single premiums is used for aggregate methods):

$$\sum_{A_0} PSL_{(x+1)_j} - \sum_{A_0} (PSL_{x_j} + NC_{x_j})(1+i) = \text{Expected release}\ ,$$

where PSL is the past service liability for the prior year's benefit and NC is the normal cost for the prior year's benefit.

Similar relationships can be used for the subsets B and R to obtain expected releases (a term including interest and a term involving benefits expected to be paid are introduced). The worksheet is designed to make it easy to add the liabilities for the prior year's benefits for terminations and changes in status to that of the survivors in the subset to arrive at $\Sigma PSL_{(x+1)_j}$. In general, the storing of the prior year's benefit on the punch card makes it possible to avoid any advance calculations for next year's gain and loss analysis. With this worksheet and with formulas for each component of the gain and loss (basically as developed in the article), we have found it possible for nonactuaries to calculate the actuarial gain with zero tolerance.

Second, with regard to unit credit contributory funding in Section II, I would prefer to define NC_0 retrospectively for calculating UAL as

$$NC_0' = \sum_{A_0} \left[\widetilde{UB}_j \frac{N_{y_j}^{(12)}}{D_{x_j}} (1.02) + C \frac{l_{x_j} - l_y}{l_{x_j}} \right] .$$

The employer is usually told to pay the estimated employer normal cost plus *actual* employee contributions. The use of actual employee contributions is desirable in calculating UAL, because it will give the correct past service payment by the employer, $K - NC'_0$, and not distort calculation of any special past service base dependent on prior years' past service contributions (the special past service base is basically present unfunded liability plus the sum of prior years' past service contributions in excess of interest on the unfunded liability). Insofar as the gain and loss analysis is concerned, this merely increases the aggregate gain G and reduces the salary loss logically, I believe, by $(C - \widetilde{C})$. Under the author's definition of G, $(C - \widetilde{C})$ is reflected as a past service payment rather than a gain and, in years when the employer pays the maximum past service payment (10 per cent of the past service base), makes it appear that the maximum past service payment has been exceeded by $(C - \widetilde{C})$.

Finally, in his section on aggregate funding methods, the author uses a spread factor

$$\frac{(1+i)\sum_{A_0} {}_0S_j\left(\frac{{}^sN_{x_j} - {}^sN_{y_j}}{{}^sD_x} - 1\right)}{\sum_{A_0} {}_0S_j},$$

which has ΣS_0 in the denominator. The numerator of the factor shown above for spreading component gains is the present value of future salaries of the expected survivors at time 1 from time 0 of subset A if their salaries increased according to salary scale or mathematically (omitting subscripts j):

$$(1+i)\sum_{A_0} {}_0S\left(\frac{{}^sN_x - {}^sN_y}{{}^sD_x} - 1\right) = \sum_{A_0} p_x \frac{s_{x+1}}{s_x}\, {}_0S\, \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}.$$

The "spread factor" used by the author is a mathematical quantity (factor) used in explaining or deriving retrospectively the change in U from last year to this year. However, the spread factor for spreading gains over future years may be different: the author's factor *was* used to spread the gains in the past year but would be used *next* year only if salaries increased according to scale, terminations occurred as expected, and new entrants had the same spread factor.

It is possible to define the aggregate gain, albeit limited to the numerator (rather than component gains), as "the expected present value of future service costs minus the actual present value of future service costs." For simplicity, let us limit the discussion to subset A. Then we have the gain equal to

$$\left(\sum_{A_0} {}_0B \frac{N_y^{(12)}}{D_x} - UL_0 - F_0 - NC_0\right)(1+i) - \left(\sum_{A_1} {}_1B \frac{N_y^{(12)}}{D_{x+1}} - UL_1 - F_1\right). \tag{1}$$

But

$$UL_1 = (UL_0 + NC_0)(1+i) - K - I_K . \tag{2}$$

Substituting expression (2) in formula (1), we have

$$\left(\sum_{A_0} {}_0B \frac{N^{(12)}}{D_x} - F_0\right)(1+i) - K - I_K - \left(\sum_{A_1} {}_1B \frac{N_y^{(12)}}{D_{x+1}} - F_1\right). \tag{3}$$

For aggregate cost formula (3) may be stated directly, but we have shown that it applies even with a supplemental liability. Thus, for all aggregate methods, the gain may also be stated to be the expected unfunded present value of all future benefits minus the actual unfunded present value of all future benefits. This dollar gain which is present in N_1 is to be spread by the reciprocal of the average temporary annuity

$$\left(\sum_{A_1} {}_1S \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \Big/ \sum_{A_1} {}_1S ,$$

which is an average temporary annuity weighted by salaries. Trowbridge ("Fundamentals of Pension Funding," *TSA*, Vol. IV) used an average temporary annuity weighted by lives. It is also appropriate, if desired, to use an average temporary annuity weighted by normal costs (based on an entry age normal valuation),

$$\left(\sum_{A_1} NC_1 \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right) \Big/ \sum_{A_1} {}_1NC .$$

To simplify the remainder of this discussion, let the salary index be unity at all ages (let there be no salary scale). An interesting aggregate relationship (similar to the individual relationships the author develops) highlighting the concept of "spreading the gain" can be developed from a manipulation of the formula for the dollar (not unit) normal cost, NC, for the frozen initial liability cost technique:

$$NC_1 = \Big(\sum_{A_1} B\frac{N_y^{(12)}}{D_{x+1}} - UL_1 - F_1\Big) \Big/ \Big[\Big(\sum_{A_1} {}_1NC\,\frac{N_{x+1}-N_y}{D_{x+1}}\Big) \Big/ \sum_{A_1} {}_1NC\Big]$$

$$= \frac{\sum_{A_1} {}_1NC\,\frac{N_{x+1}-N_y}{D_{x+1}}}{\sum_{A_1} {}_1NC\,\frac{N_{x+1}-N_y}{D_{x+1}}}$$

$$+ \frac{\Big\{\Big[\sum_{A_1}\Big(B_1\frac{N_y^{(12)}}{D_{x+1}} - {}_1NC\,\frac{N_{x+1}-N_y}{D_{x+1}}\Big) - F_1\Big] - UL_1\Big\}}{\Big(\sum_{A_1} {}_1NC\,\frac{N_{x+1}-N_y}{D_{x+1}}\Big) \Big/ \sum_{A_1} {}_1NC}$$

or

$$\sum_{A_1} {}_1NC + \{\ldots\}\,.$$

In words, this is the entry age normal tabular normal cost plus (entry age normal unfunded liability minus "frozen" unfunded liability, that is, UL) times the reciprocal of the average temporary annuity (the spread factor). This shows that the lifetime loss or gain remaining to be spread (loss if the entry age normal unfunded liability exceeds UL, gain if it does not) is the difference between an entry age normal unfunded liability reflecting all gains or losses and a frozen unfunded liability UL, "frozen" in the sense that all gains or losses are excluded from it (UL by definition is the entry age normal unfunded liability if all gains and losses are zero).

This approach may be what the author refers to as the analogue method using individual premium funding methods. I have found it also effective, however, in explaining the spread of the one-year gain in the numerator (the change in the lifetime gain), coupled with a comment on any changes in the average temporary annuity spread factor due to changes in age-service-salary distribution of the population. Thus one may develop the one-year gain in terms of present values of unfunded benefits and state that it is spread each year by the reciprocal of the average temporary annuity in the denominator, or one may develop the individual level premium analogue. The former has the merit of incorporating the total salary loss in the one-year loss, while the latter shows part of this salary loss as an increase in the tabular normal cost and only part as an increase in the lifetime loss under the plan (as is true of entry age normal).

It is quite true that these methods analyze the numerator and denominator separately, but the spread factor developed by the author may change from year to year (and thus the rate at which the lifetime gain is being spread) and also require analysis. Will the exact analysis of the change in the accrual rate at time 1 work at time 2 when both the first- and second-year gain are being spread? I submit that it will, but only in an artificial sense. In using the accrual rate adjusted for the spread of the gain in deriving the component gains for one year, the author creates artificial one-year gains due to the difference between the accrual rate adjusted and unadjusted for the spread of the gain. Thus, for example, if there is an aggregate gain in year 1, but a zero aggregate in year 2, the accrual rate at time 2 will be greater than at time 1 because a smaller lifetime gain remains to be spread. The use of the adjusted accrual rate in the component gains will show a second-year loss because the adjusted accrual rate is less than the unadjusted, when really there has been no one-year loss in terms of expected versus actual single premiums in the numerator. Thus the author incorporates into his one-year gains the effect of spreading prior year gains in that expected experience varies with the adjusted accrual rate and does not truly limit the analysis to the one-year gain. This is desirable to determine whether actual experience has been following expected experience. In effect, I am saying that when there is a lifetime gain (or loss) being spread, the accrual rate expected at time $t+1$ need not be the accrual rate at time t, and thus the change in the accrual rate need not be due entirely to the prior year's experience only (see Appendix).

The numerator-only approach may not identify the effect of each deviation of actual from expected experience on the accrual rate directly, but then neither does the author's method, for it requires further detailed analysis of the impact of the spreading of the lifetime gain on the accrual rate. The author's method appears more powerful than limiting the definition of gain to the numerator but would apparently require a great deal of time and manual calculation or a special computer program, since the gain items are not routinely produced as part of the valuation; gain items in the numerator-only approach are limited to one year's experience and would be routinely produced by the valuation and the author's set theory easily applied to analyze them. Recursion relationships involving single premiums would be used. Moreover, the gains or losses in the numerator reflect the true impact on costs of the pension plan, while changes in the denominator merely affect the rate at which they are funded (the incidence of costs over time). Both approaches do, however, require separate analysis of the impact of the spreading of the lifetime gain on the accrual rate. A rough measure of this impact under the numerator-only approach is the change in the amount of the lifetime gain (excluding the latest year's experience) being spread, multiplied by the spread factor. The author can make a similar analysis under his approach; he will, I believe, find himself analyzing the numerator to determine the lifetime gain remaining to be spread and thus what the accrual rate would be with (and without) the spread of the lifetime gain.

Finally, the analysis of the change in the accrual rate, rather than the gains in the numerator, calls for an analysis of how the accrual rate for aggregate cost funding and attained age normal funding approaches the accrual rate for frozen initial liability funding (see Trowbridge, "Fundamentals of Pension Funding," referred to above). My manipulation of the formula for the unit normal cost applies to frozen initial liability strictly but may also be applied to aggregate cost. For attained age normal, however, the individual premium analogue requires a unit credit valuation for the original past service credits and entry age normal valuation for future service credits, which is compared to UL, where UL is the unit credit unfunded if all gains and losses are zero. The author's gain component for new entrants offers remarkable possibilities for analysis of this kind of change in the accrual rate because the tabular accrual rate for new entrants is an entry age normal accrual, while the accrual rate for aggregate cost (assuming past service credits) or attained age normal for the initial covered group is greater than this, reflecting an implicit past service payment. Thus the first generation of new entrants continually creates "funding gains" to move the accrual rate toward the entry age normal accrual rate. If the author can further refine his analysis along the lines I have suggested, he will then truly have a more powerful technique than that of analyzing gains in the numerator (measuring their impact on dollar costs or the accrual rate by multiplying by a dollar or unit spread factor) and commenting separately on the impact of the change in the entry age-service-salary distribution and the impact of the spreading of the lifetime gain on the accrual rate.

APPENDIX

Let

$$\widetilde{U}_t = \text{Expected accrual rate at time } t;$$

$$EU_t = \text{Entry age normal unfunded liability at time } t;$$

$$\widetilde{(EU_t - UL_t)} = \text{Expected } (EU_t - UL_t)\ ;$$

$$SF_t = 1 \Big/ \sum S_t \frac{N_t - N_y}{D_t} \quad \text{or} \quad 1 \Big/ \sum NC_t \frac{N_t - N_y}{D_t}\ ;$$

$$\widetilde{SF}_t = \text{Expected } SF_t\,.$$

THEOREM. $\widetilde{U}_2 \neq U_1$ *if* $(EU_1 - UL_1) \neq 0$.

Now,

$$U_1 = U_0 + (EU_1 - UL_1)SF_1\,, \tag{A1}$$

noting that $EU_1 - UL_1 \neq 0$ as assumed;

$$\widetilde{U}_2 = U_0 + (\widetilde{EU_2 - UL_2})\widetilde{SF}_2\,; \tag{A2}$$

$$\widetilde{SF}_2 = SF_1\,; \tag{A3}$$

$$\begin{aligned}(\widetilde{EU_2 - UL_2}) = (EU_1 - UL_1)(1 + i) \\ - (EU_1 - UL_1)SF_1(1 + i)\,.\end{aligned} \tag{A4}$$

Substituting equations (A1), (A3), and (A4) in equation (A2), we have

$$\widetilde{U}_2 = U_1 + \{(EU_1 - UL_1)[i - SF_1(1 + i)]\}SF_1\,. \tag{A5}$$

Therefore, $\widetilde{U}_2 \neq U_1$, if $(EU_1 - UL_1) \neq 0$, since $SF_1 \neq 0$ and $[i - SF_1(1 + i)] < 1.0$, as the reader can readily verify in practice. (*Note.*—$\overline{EU_1 - UL_1}$ is the aggregate lifetime loss if $EU_1 - UL_1$ is positive and the aggregate lifetime gain if $EU_1 - UL_1$ is negative.) In words, $\widetilde{U}_2$ differs from U_1 by the amount of the lifetime gain amortized in the past year times the spread factor.

CHRISTOPHER C. STREET:

Mr. Anderson has shown us how a pension fund gain and loss may be developed by algebraic methods, and he has included examples of the common funding methods. The gain for the aggregate method turns out to have the same form as for individual entry age normal, except that the contribution rate or unit normal cost is constant for all employees and there is a term for new entrants. Of particular interest is the form of the "spread factor" which relates the gain to the change in the unit normal cost. The spread factor used in some analyses is the reciprocal of the present value of future salary as calculated at the end of the valuation year for actual survivors and current salaries. Mr. Anderson's spread factor, however, may be described as the reciprocal of the present value of expected future salary. This value is calculated as of the end of the valuation year for the expected survivors of those active at the beginning of the year and at their expected salaries. This result has particular significance in a year in which there has been a change in valuation assumptions or a general salary increase. In the author's notation (see his eq. [69]):

$$SF = \frac{1}{\sum_{A_0} p_x\, {}_0S\, \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}} = \frac{1}{(1+i)\sum_{A_0} {}_0S\left(\frac{{}^sN_x - {}^sN_y}{{}^sD_x} - 1\right)} .$$

Mr. Anderson indicates that a change in any one of the assumptions underlying his example could change the form of the spread factor. Part 1 of this discussion demonstrates that this result is perfectly general for the aggregate funding method, with or without supplemental liability. It reconstructs Mr. Anderson's development without bringing in any plan or employee status particulars, detailed gain and loss formulas, or recursion formulas. Provision made for expected future new entrants would change the expression for the gain but not the spread factor. The formulas are still valid if no salary scale is used, since this is equivalent to assuming a flat salary scale where $s_x = s_{x+1} = \ldots = 1$. I will use the author's notation where possible and will define only new symbols. Part 2 of this discussion relates Mr. Anderson's results to previous attempts at analysis of the gains under aggregate funding methods, in particular to the pertinent parts of Mr. Dreher's classic paper on gain and loss analysis and to discussions of this paper.

Everything follows from the definitions of the expected values given below. The superscript E will indicate an expected value, and the superscript A (or the lack of any superscript) an actual or known value. I will use $PVTB$ to indicate the present value of total benefits and $PVFS$ for the present value of future salary. The subscripts 0 and 1 will indicate summation over either all lives or all active lives at the beginning and end of the valuation year, respectively. Expected values at time 1, however, are to be considered as summations over those lives in the valuation at time 0. The basic definitions follow:

$$PVTB_1^E = (1+i)PVTB_0 ; \tag{1}$$

$$UL_1^E = UL_1^A = (1+i)(UL_0 + NC_0) - K - I_K ; \tag{2}$$

$$F_1^E = (1+i)F_0 + K + I_K ; \tag{3}$$

$$PVFS_1^E = (1+i)(PVFS_0 - S_0) , \tag{4}$$

or, in the author's notation,

$$(1+i)\sum_{A_0} {}_0S\left(\frac{{}^sN_x - {}^sN_y}{{}^sD_x} - 1\right) .$$

Expected benefit payments, omitted from the above definitions of F_1^E and $PVTB_1^E$, could be included without affecting the results.

The relation $U_1^E = U_0$, which will be used later in Part 2, can easily be derived from equations (1)–(4):

$$U_1^E = \frac{PVTB_1^E - (UL_1^E + F_1^E)}{PVFS_1^E}$$

$$= \frac{(1+i)[PVTB_0 - (UL_0 + NC_0 + F_0)]}{(1+i)(PVFS_0 - S_0)}$$

$$= \frac{PVTB_0 - (UL_0 + F_0) - NC_0}{PVFS_0 - S_0}$$

$$= \frac{NC_0}{S_0}, \quad \text{since} \quad \frac{NC_0}{S_0} = \frac{PVTB_0 - (UL_0 + F_0)}{PVFS_0};$$

$$U_1^E = U_0 \ldots . \tag{5}$$

Differences between actual and expected values are elements of the gain and loss. For the moment, however, we will consider the terms G^N and G^D to be arbitrary quantities implicitly defined by equations (7) and (9) below:

$$PVTB_1^A - (UL_1^A + F_1^A) = PVTB_1^E - (UL_1^E + F_1^E) - G^N \tag{6}$$

$$= (1+i)[PVTB_0 - (UL_0 + NC_0 + F_0)] - G^N , \tag{7}$$

$$PVFS_1^A = PVFS_1^E + \frac{G^D}{U_1} \tag{8}$$

$$= (1+i)(PVFS_0 - S_0) + \frac{G^D}{U_1} . \tag{9}$$

Part 1

Now, following Mr. Anderson,

$$\left.\begin{aligned} U_1 &= \frac{N_1}{D_1} = \frac{N_0 + \Delta N}{D_0 + \Delta D} = \frac{N_0 + \Delta N - U_1 \Delta D}{D_0} \\ &= U_0 + \frac{1}{D_0}(\Delta N - U_1 \Delta D) ; \end{aligned}\right. \tag{10}$$

$$\begin{aligned} \Delta N &= \Delta PVTB - \Delta(UL + F) \\ &= i[PVTB_0 - (UL_0 + F_0)] - (1+i)NC_0 - G^N \qquad \text{(from eq. [7])} \\ &= iN_0 - (1+i)NC_0 - G^N \\ &= iD_0U_0 - (1+i)S_0U_0 - G^N ; \end{aligned} \tag{11}$$

$$\Delta D = \Delta PVFS$$

$$= iPVFS_0 - (1 + i)S_0 + \frac{G^D}{U_1} \qquad \text{(from eq. [9])} \tag{12}$$

$$= iD_0 - (1 + i)S_0 + \frac{G^D}{U_1} ;$$

$$\Delta U = \frac{1}{D_0} (\Delta N - U_1 \Delta D) \qquad \text{(from eq. [10])}$$

$$= \frac{1}{D_0} [iD_0U_0 - (1 + i)S_0U_0 - G^N - iD_0U_1$$

$$+ (1 + i)S_0U_1 - G^D] \qquad \text{(from eqs. [11] and [12])}$$

$$= -i\Delta U + (1 + i) \frac{S_0}{D_0} \Delta U - \frac{G^N + G^D}{D_0} ;$$

$$\Delta U(1 + i) \left(1 - \frac{S_0}{D_0}\right) = - \frac{G^N + G^D}{D_0} = - \frac{G}{D_0} ,$$

where $G = G^N + G^D$;

$$\Delta U = - \frac{G}{(1 + i)(D_0 - S_0)}$$

$$= - \frac{G}{(1 + i)(PVFS_0 - S_0)} \tag{13}$$

$$= - \frac{G}{PVFS_1^E}$$

(compare the author's eq. [70]). The spread gain is normally defined as the decrease in the unit normal cost. If we now define the spread factor as the reciprocal of the present value of expected future salary, we see immediately that G must be equal to the gain, since

$$G \cdot SF = -\Delta U \qquad \text{or} \qquad G^N + G^D = -\Delta U \cdot PVFS_1^E ;$$

G^N and G^D may be thought of as gains in the numerator and denominator, respectively.

In the pure aggregate cost method (without supplemental liability) the development would go like this:

$$PVTB_1^A - F_1^A = PVTB_1^E - F_1^E - G^N$$

$$= (1 + i)(PVTB_0 - F_0) - K - I_K - G^N$$

$$= (1 + i)(PVTB_0 - F_0) - (1 + i)NC_0 \tag{7a}$$

$$+ [(1 + i)NC_0 - K - I_K] - G^N ,$$

$$
\begin{aligned}
\Delta N &= \Delta PVTB - \Delta F \\
&= i(PVTB_0 - F_0) - (1+i)NC_0 \\
&\qquad\qquad - [K + I_K - (1+i)NC_0] - G^N \qquad (11a) \\
&= iN_0 - (1+i)S_0U_0 - [K + I_K - (1+i)NC_0] - G^N .
\end{aligned}
$$

Comparing equations (11) and (11a), we can see that the development from this point on is the same, provided that we increase the gain by the excess of contributions over normal cost, adjusted for interest at the valuation rate.

Part 2

Going back to equations (6) and (8) and noting that $G = G^N + G^D$, we can now write the gain explicitly as

$$
\begin{aligned}
G^N &= [PVTB_1^E - (UL_1^E + F_1^E)] - [PVTB_1^A - (UL_1^A + F_1^A)] , \\
G^D &= U_1(PVFS_1^A - PVFS_1^E) .
\end{aligned}
$$

From equations (7) and (2) we obtain

$$
\begin{aligned}
G^N &= (1+i)[PVTB_0 - (UL_0 + NC_0 + F_0)] \\
&\qquad - [PVTB_1^A - (1+i)(UL_0 + NC_0) - K - I_K - F_1^A] \qquad (14) \\
&= (1+i)(PVTB_0 - F_0) - K - I_K - (PVTB_1 - F_1) ,
\end{aligned}
$$

which corresponds to Mr. Dreher's definition of the gain (*TSA*, XI, 604, eq. [1]), where, for the aggregate method, $AL = PVTB$. Mr. Anderson comments correctly that Dreher analyzes only the numerator of the expression of unit normal cost. For example, Dreher's aggregate gain from salary changes reflects only the effect of such changes on the present value of total benefits ($PVTB$). In the numerical example on page 624 of Dreher's paper the reconciliation of the change in the unit normal cost is possible because, as it happens, the ratio of the actual and expected salaries is very close to unity. Consequently, $G^N/PVFS_1^A \cong G/PVFS_1^E$. Dreher analyzes $G^N/PVFS_1^A$ rather than $G/PVFS_1^E$. It is not difficult to prove that this produces an overstatement of the gain as reflected by the drop in the contribution rate equal to $U_0(PVFS_1^E/PVFS_1^A - 1)$. Numerically this works out to be 0.03 per cent. The effect of including a level expected expense in the contribution rate provides an offset of 0.02 per cent to this overstatement, and the net difference of 0.01 per cent is masked by rounding.

Mr. Holcombe, in his discussion of Dreher's paper (*TSA*, XI, 642), suggests a more refined approach which involves a breakdown of each source of gain between the numerator and the denominator and the calculation of the marginal effects on the unit normal cost of each source of gain taken in some arbitrary sequence. His formula for the "accrual rate" $(AR)^A_{t+1}$ follows directly from equations (6) and (8) above:

$$U_1 = \frac{PVTB_1^A - (UL_1^A + F_1^A)}{PVFS_1^A}$$

$$= \frac{[PVTB_1^E - (UL_1^E + F_1^E)] - G^N}{PVFS_1^E + G^D/U_1} . \tag{15}$$

Another approach to equation (15) makes use of the equality $U_1^E = U_0$ (proved above):

$$U_1 = U_0 - \frac{G}{PVFS_1^E} \quad \text{(from eq. [13])}$$

$$= U_1^E - \frac{G}{PVFS_1^E} \quad \text{(from eq. [5])}$$

$$= \frac{PVTB_1^E - (UL_1^E + F_1^E) - G}{PVFS_1^E}$$

$$= \frac{PVTB_1^E - (UL_1^E + F_1^E) - G^N}{PVFS_1^E + G^D/U_1}$$

as in equation (15). This formulation has the defect that the effect of each source of gain on the decrease in the unit normal cost depends on the order of calculation, whereas Mr. Anderson's analysis is independent of this order.

PAULETTE TINO:

In this discussion I shall single out the case of the aggregate funding method. The derivation and analysis of the gain under this method—and they exist—differ in no ways from the derivation and analysis of the gain when other methods are used. It is only necessary to see that the accrued liability on which the decrements are operative is, under this method, defined as the present value of future benefits minus the present value of future normal costs. For example, the accrued liability at time 1, assuming the salary increased according to the salary scale, equals

$$_0B \frac{N_y^{(12)}}{D_{x+1}} - U_0 \, {}_0S \frac{s_{x+1}}{s_x} \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}} .$$

Note that U_0 is used, as general reasoning would suggest, and not U_1 (as in formula [70] of the paper). Mathematical developments based on the principle that a gain on an item is always the difference between the actual and expected values of that item will confirm that intuition. This is fortunate, for otherwise it would be necessary to perform a valuation in two passes through the computer, the first to obtain the items necessary for the computation of U_1 and the second to obtain the components of the gain. Another gratifying consequence of using U_0 is that the spreading factor becomes the present value of future salaries, which also would be suggested by general reasoning.

Mr. Anderson was led to the use of U_1 through equation (64), namely,

$$U_1 = \frac{N_1}{D_1} = \frac{N_0 + \Delta N}{D_0 + \Delta D}.$$

The gain on U_1 should be measured by consideration of the deviation of numerator and denominator from their expected values and not from their original values. (In no case is the gain in the step-rate method the difference between the unfunded liabilities at time 0 and at time 1.) The proper starting formula is

$$U_1 = \frac{N_1}{D_1} = \frac{N_0(1+i) - NC(1+i) - \Delta' N}{D_0(1+i) - (\Sigma_0 S)(1+i) - \Delta' D} = U_0 - \frac{\Delta' N - U_0 \Delta' D}{D_1},$$

where $\Delta' N$ and $\Delta' D$ are strictly gain items. This will be illustrated by a simple example in the following paragraphs.

Let us take a simple case. There are two decrements, interest and mortality. No salary scale is involved. All employees are aged 65 and under. There is no termination of employment, no new entrants.

We define, at time k,

TL_k = Present value of projected benefits;

UL_k = Unfunded liability;

F_k = Assets;

TA_k = Temporary annuity

$$= \frac{N_{x \text{ at } k} - N_y}{D_{x \text{ at } k}};$$

EM_{TL} = Expected mortality related to TL

$$= \sum_{A_0} q_x \, {}_0B \, \frac{N_y^{(12)}}{D_{x+1}};$$

AM_{TL} = Actual mortality release due to deceased employees related to TL

$$= \sum_{M} {}_0B \, \frac{N_y^{(12)}}{D_{x+1}};$$

EM_{TA} = Expected mortality release related to TA

$$= \sum_{A_0} q_x \frac{N_{x+1} - N_y}{D_{x+1}} ;$$

AM_{TA} = Actual mortality release due to deceased employees, related to TA

$$= \sum_{M} \frac{N_{x+1} - N_y}{D_{x+1}} ;$$

C = Company contribution with interest;

IG = Interest gain realized by fund from time 0 to 1;

U_k = Normal cost rate, in this case a dollar amount per employee

$$= \frac{TL_k - UL_k - F_k}{TA_k} ;$$

F_k = Value of fund;

l_k = Number of active lives.

We have

$$TL_1 = TL_0(1 + i) + EM_{TL} - AM_{TL} ,$$

$$UL_1 = UL_0(1 + i) + l_0U_0(1 + i) - C ,$$

$$F_1 = F_0(1 + i) + C + IG ,$$

$$\begin{aligned} N_1 &= TL_1 - UL_1 - F_1 \\ &= (TL_0 - UL_0 - F_0)(1 + i) - l_0U_0(1 + i) \\ &\qquad - IG + EM_{TL} - AM_{TL} \\ &= N_0(1 + i) - l_0U_0(1 + i) - IG + EM_{TL} - AM_{TL} , \end{aligned}$$

$$TA_1 = TA_0(1 + i) - l_0(1 + i) + EM_{TA} - AM_{TA} .$$

Note that the expected value of U at time 1 is that value of U_1 obtained when all gains are set to zero, since we assume then that the actuarial assumptions are realized. We have

$$\text{Expected value of } U_1 = \frac{N_0(1 + i) - l_0U_0(1 + i)}{TA_0(1 + i) - l_0(1 + i)} . \qquad (1)$$

Since $N_0/TA_0 = l_0U_0/l_0 = U_0$, the expected value of U_1 is U_0, as would be expected in a level premium approach. For simplicity of thinking, let us rewrite equation (1) as

$$\text{Expected value of } U_1 = \frac{N'}{TA'} . \qquad (2)$$

Thus N_1 will be equal to N' minus related gains and TA_1 will be equal to TA' minus related gains.

The over-all gain measured as the difference between the actual and the expected values of U_1 is written as

$$\frac{N_1}{TA_1} - \frac{N'}{TA'} = \frac{N' - IG + EM_{TL} - AM_{TL}}{TA' + EM_{TA} - AM_{TA}} - \frac{N'}{TA'}$$

$$= \frac{-(IG + AM_{TL} - EM_{TL})TA' + N'(AM_{TA} - EM_{TA})}{TA_1 TA'}$$

$$= \frac{-(IG + AM_{TL} - EM_{TL}) + \dfrac{N'}{TA'}(AM_{TA} - EM_{TA})}{TA_1}$$

$$= \frac{-(IG + AM_{TL} - EM_{TL}) + U_0(AM_{TA} - EM_{TA})}{TA_1}$$

$$= \frac{-IG - [(AM_{TL} - U_0 AM_{TA}) - (EM_{TL} - U_0 EM_{TA})]}{TA_1}.$$

The numerator gives the components of the dollar gain in dollar amounts. TA_1 is the spreading factor.

We see that the decrease in normal cost rate is due to the interest gain and the mortality gain. The expression in brackets at the numerator *is* the dollar mortality gain. It is not different in structure from the mortality gain in unit credit cases as soon as $(AM_{TL} - U_0 AM_{TA})$ is recognized as an accrued liability (as defined in the first paragraph) and $(EM_{TL} - U_0 EM_{TA})$ as an expected mortality release obtained as usual by multiplying by q_x the accrued liability of all employees included in the previous valuation; that is,

$$EM_{TL} - U_0 EM_{TA} = q_x \sum_{A_0} \left({}_0B \frac{N^{(12)}}{D_{x+1}} - U_0 \frac{N_{x+1} - N_y}{D_{x+1}} \right).$$

Gain on New Entrants

The gain on new entrants is equal to

$$\sum_{NE} \left({}_0B \frac{N_y^{(12)}}{D_{x+1}} - U_0 \frac{N_{x+1} - N_y}{D_{x+1}} \right),$$

since they are expected to enter at the rate U_0.

Introduction of a Salary Scale

The development is kept as above, except that the expected salary, that is, ${}_0S_x(s_{x+1}/s_x)$ is substituted for the actual salary at time 1. The gain on salary scale, computed on the old active employees, is equal to

$$\sum ({}_1B - {}_0B) \frac{N_y^{(12)}}{D_{x+1}} - U_0 \sum \left({}_1S - {}_0S \frac{s_{x+1}}{s_x} \right) \frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}.$$

Contributory Cases

The analysis is made on the employer accrued liability.

DAVID S. WILLIAMS:

Mr. Anderson's timely paper illuminates two interrelated aspects of pension fund gain and loss analysis which have been perennial sources of annoyance to pension actuaries: the formidable notation involved and the difficulty of analyzing gains (losses) with precision.

To anyone who has confronted these problems en route to computerization of gain and loss analysis, it will be quickly apparent that the use of set theory and notation is invaluable in the construction of precise formulas. I fully agree also with Anderson's major thesis, that it is not only feasible but highly desirable to develop a consistent and comprehensive network of formulas which will accurately express the total gain in terms of its components.

Generalized Formulas

My concern is that some actuaries may be dissuaded from further investigation of this thesis because the introductory remarks in the paper seem to imply that generalized formulas are rather impractical. The following exposition will, I hope, serve to illustrate my contention that generalized formulas are not only practical but highly desirable. Using set notation and the Fackler accumulation process, one can readily develop gain and loss expressions which are independent of plan benefit structure and funding method.

Scheduling of Retirements

A secondary purpose of this exposition is the recognition of an important source of gain not dealt with by Dreher or Anderson, namely, that due to retirements which occur at other than the "normal" retirement date. It is admittedly simpler to construct the valuation model on the basis that "expected" retirement dates coincide with actual retirement dates, but this contradicts the essential proposition inherent in gain and loss analysis, that is, that the model be capable of construction at time 0. Because of the magnitude of the cash equivalents involved, this gain is too important to ignore. It could be handled in a manner analogous to that for other decrements, but in all but the largest plans it is more suitably handled by adopting an average retirement age and constructing a model on this basis.

Definitions and Assumptions

1. An initial valuation is performed at time 0 and a terminal valuation at time 1.
2. The superscript E denotes an "expected" quantity derived during the initial valuation; during the terminal valuation, this quantity will be revalued to reflect the "actual" experience of the plan year. Thus ${}_1AL^E$ is the expected value (determined at time 0) of the accrued liability ${}_1AL$ that will exist at time 1 with respect to a given member, if there is no unexpected change in member status. Note that "expected" is *not* used here in its probabilistic sense.
3. All changes in member status and all death benefit payments occur at the end of the plan year. All annuity payments, and all contributions and expenses, fall due at the beginning of the plan year.
4. Death is the only cause of termination. Withdrawal and disability would be treated in an analogous manner.
5. Pensions are in the form of immediate single life annuities with payments ceasing upon death. Pension payments are made (annually in advance) directly from the fund as they fall due.
6. The initial and terminal valuations are performed using the same benefit formulas and valuation parameters.

Actual versus Expected Retirements

We will use Anderson's set notation but will regard B_0 (the set of members retiring on the date of valuation) as being a subset of the set A_0 of all active members. Our set equations are:

$$A_1 = A_0 - T \cap A_0 - R + N ,$$

$$R_1 = R_0 - T \cap R_0 + R .$$

But actual retirements R may not coincide with expected retirements R^E, and the gain and loss model constructed at time 0 must reflect this. Accordingly, the subset of active members for whom the expected status at time 1 coincides with the actual status is

$$A_1 - N - R^E \cap A_1 = A_0 - T \cap A_0 - \overbrace{R^E \cap R}^{\text{Normal retirements}} - \overbrace{R \cap (A_0 - R^E)}^{\text{Early retirements}} - \overbrace{R^E \cap A_1}^{\text{Postponed retirements}} ,$$

and it follows that

$$R_1 = R_0 - T \cap R_0 + R^E \cap R + R \cap (A_0 - R^E) .$$

A member who was expected to retire just prior to time 0 and who elected to postpone his retirement will, for the purpose of the initial valuation, be expected to retire at time 1. If he does so, he will be included in the subset $R^E \cap R$.

Expected Experience Formulas

The traditional Fackler accumulation formulas may be written

$$_{t+1}V = {}_tV(1+i) + P(1+i) - q_t(1 - {}_{t+1}V)\,,$$

$$\ddot{a}_{t+1} = \ddot{a}_t(1+i) - (1+i) + q_t\ddot{a}_{t+1}\,.$$

Pension fund liabilities accumulate in an analogous fashion: for active and retired members, respectively, accrued liabilities are expected as of time 0 to develop as follows:

$$\sum_{A_0-R^E} {}_1AL^E = \sum_{A_0} {}_0AL(1+i) + \sum_{A_0} P^E(1+i) - \sum_{\text{plan}} L^E(1+i) - \sum_{R^E} {}_1AL^E + \sum_{A_0} q_t({}_1AL^E - DB^E)\,, \tag{1}$$

$$\sum_{R_0+R^E} {}_1AL^E = \sum_{R_0} {}_0AL(1+i) - \sum_{R_0} B(1+i) + \sum_{R^E} {}_1AL^E + \sum_{R_0} q_t\,{}_1AL^E\,, \tag{2}$$

where

P = Total plan-year contribution on behalf of a given active member, payable at the beginning of the plan year;

L = Total plan-year operating expenses, payable at the beginning of the plan year;

DB = Benefit payable in the event that a given active member dies during (i.e., at the end of) the plan year;

B = Annual pension payable at the beginning of the plan year to a given retired member;

i = Valuation interest rate (the actual rate earned on the fund being denoted by i').

Formulas (1) and (2) define the total accrued liability expected to exist at time 1, that is,

$$\sum_{A_0-R_0} {}_1AL^E\,,$$

if actual experience during the year conforms precisely to the valuation assumptions, *with the exception that no deaths are assumed to occur.*

Actual Experience Formulas

The actual development of pension fund liabilities during the plan year can be traced as of time 1 in terms of

$$G_A + \sum_{A_1} {}_1AL = \sum_{A_0} {}_0AL(1+i') + \sum_{A_0} P(1+i') - \sum_{\text{plan}} L(1+i') - \sum_{R} {}_1AL - \sum_{T\cap A_0} DB\,, \tag{3}$$

$$G_R + \sum_{R_1} {}_1AL = \sum_{R_0} {}_0AL(1+i') + \sum_{R_0} B(1+i') + \sum_{R} {}_1AL\,, \tag{4}$$

and the total gain G can be expressed as

$$G = G_A + G_R - (i'-i)\Big(\sum_{A_0+R_0} {}_0AL - {}_0F\Big)\,. \tag{5}$$

Resulting Gain and Loss Formula

Subtracting equation (1) from equation (3) and shuffling terms produces

$$G_A = \sum_{A_0} {}_1AL^E - \sum_{A_1+R} {}_1AL + (i'-i)\Big[\sum_{A_0} ({}_0AL + P) - \sum_{\text{plan}} L\Big] + \sum_{A_0} (P - P^E)(1+i) + \sum_{\text{plan}} (L^E - L)(1+i) - \sum_{T\cap R_0} DB - \sum_{A_0} q_t({}_1AL^E - DB^E)\,. \tag{6}$$

A similar operation involving equations (2) and (4) produces

$$G_R = \sum_{R_0} {}_1AL^E - \sum_{R_1-R} {}_1AL + (i'-i)\sum_{R_0} ({}_0AL - B) - \sum_{R_0} q_t\, {}_1AL^E\,. \tag{7}$$

Equations (6) and (7) provide expressions for G_A and G_R which can be substituted in equation (5), producing

Term	Source of Gain
$G = -(i'-i)\Big({}_0F - \sum_{R_0} P - \sum_{\text{plan}} L - \sum_{R_0} B\Big)$	I. Investment return

Term	Source of Gain
$+ \sum_{A_0}(P - P^E)(1 + i)$	II. Contributions
$+ \sum_{\text{plan}}(L^E - L)(1 + i)$	III. Expenses
$- \sum_{N} {}_1AL$	IV. New entrants
$+ \sum_{T \cap A_0} ({}_1AL^E - DB) - \sum_{A_0} q_t({}_1AL^E - DB^E)$	V. Preretirement mortality
$+ \sum_{T \cap R_0} {}_1AL^E - \sum_{R_0} q_t \, {}_1AL^E$	VI. Postretirement mortality
$+ \sum_{\substack{A_0 - T \cap A_0 \\ + R_0 - T \cap R_0}} ({}_1AL^E - {}_1AL)$	Balancing term

Let us further examine the balancing term. For each member of the subset $(R_0 - T \cap R_0)$, it is true by definition that ${}_1AL^E = {}_1AL$. The subset $(A_0 - T \cap A_0)$ may be rewritten as

$$A_1 - N + R = A_1 - N + R \cap R^E + R \cap (A_0 - R^E)$$

$$= (A_1 - N + R \cap R^E - R^E \cap A_1) + R^E \cap A_1 + R \cap (A_0 - R^E) ,$$

which indicates how the balancing term may be analyzed into its components:

Term	Source of Gain
$\sum_{\substack{A_1 - N + R \cap R^E \\ - R^E \cap A_1}} ({}_1AL^E - {}_1AL)$	VII. Salary change
$\sum_{R^E \cap A_1} ({}_1AL^E - {}_1AL)$	VIII. Postponed retirement
$\sum_{R \cap (A_0 - R^E)} ({}_1AL^E - {}_1AL)$	IX. Early retirement

Conclusions

This generalized gain and loss formula clearly indicates the form of the various gain components. Components VIII and IX are similar in form to Component VII, which leads me to wonder how often gains arising from nonscheduled retirements have found their way into the gain attributable to salary changes!

It should be noted that a member who postpones his retirement will be included in Component VIII but not in Component VII, although he ought to be represented in both. This and a number of other refinements can be introduced if desired; for simplicity's sake they have not been shown here.

I must apologize to Mr. Anderson for taking some advantage of him: I have produced a generalized formula only by using generalized symbols for plan benefits. However, most pension actuaries will surely find that a gain and loss approach which moves from the general to the particular offers considerable theoretical and practical advantages.

(AUTHOR'S REVIEW OF DISCUSSION)

ARTHUR W. ANDERSON:

With regard to analysis of aggregate methods, I tend to agree with Mrs. Tino and Messrs. Berin and Lockwood that it is intuitively more appealing to have the "gain" spread by a factor of $1/D_1$ and to have the "accrued liabilities" defined in terms of U_0. I therefore recommend to the reader Mr. Lockwood's rephrasing of my equations (65)–(70), although Mr. Street's discussion shows that my formulation has its own rationale. There is no question of *correctness* in choosing between the two enunciations, however, but simply a question of taste. The following example is interesting in this connection and may prove instructive for those with little practical acquaintance with gain analysis.

Consider a plan which pays a pension of $1,000 per year at age 65, and assume that there are just two participants at time 0: Employee A, aged 30, and Employee B, aged 45. Our actuarial assumptions are mortality Ga-1970 (Gompertized) and 5 per cent interest. The fund at time 0 was $1,000, and the unfunded liability $1,500. For the rest of this example, we will refer to the accompanying tabulation of factors.

x	q_x	$N_{65}^{(12)}/D_x$	$(N_x-N_{65})/D_x$
30	0.000729	1.529	16.734
31	0.000781	1.606	16.532
.			
.			
.			
45	0.002880	3.248	12.549
46	0.003165	3.421	12.162
.			
.			
.			
50	0.004667	4.220	10.432

At time 0 we compute U_0 as

$$U_0 = \frac{1{,}000(1.529) + 1{,}000(3.248) - 1{,}000 - 1{,}500}{16.734 + 12.549}$$

$$= \frac{2{,}277}{29.283} = 77.76\,,$$

$$NC = 2(77.76) = 156\,.$$

Between time 0 and time 1, Employee B quits and Employee C (aged 50) enters the plan. The fund manager reports the following: balance, time 0, \$1,000; contribution (at year end), \$200; interest, \$80; and balance, time 1, \$1,280. UL_1 is computed as follows:

$$UL_1 = 1{,}500(1.05) + 156(1.05) - 200 = 1{,}539\,.$$

Then U_1 is computed as

$$U_1 = \frac{1{,}000(1.606) + 1{,}000(4.220) - 1{,}539 - 1{,}280}{16.532 + 10.432}$$

$$= \frac{3{,}007}{26.964} = 111.52\,.$$

Let us now analyze the change in unit normal cost (from 77.76 to 111.52) by the method of equation (70) and then by the Tino-Berin-Lockwood formula, to see the difference between the two.

I. From equation (70):

$$\text{Spread factor} = \frac{1}{1.05(16.734 + 12.549 - 2)}$$

$$= \frac{1}{28.647} = 0.034907\,.$$

A. Gain from termination (all causes)

$$\begin{aligned}\text{Expected release} &= 0.000729[(1{,}000)(1.606) - 111.52(16.532)]\\ &\quad + 0.002880[1{,}000(3.421) - 111.52(12.162)]\\ &= 6\,;\end{aligned}$$

$$\begin{aligned}\text{Actual release} &= 1{,}000(3.421) - 111.52(12.162)\\ &= 2{,}065.\end{aligned}$$

Change in U due to termination from all causes

$$= -0.034907(2{,}065 - 6) = -71.86.$$

B. Loss from new entrants $= 1{,}000(4.220) - 111.52(10.432)$
$= 3{,}057.$

Change in U due to new entrants $= 0.034907(3{,}057) = +106.71.$

C. Gain from interest $= 80 - 0.05(1{,}000) = 30.$

Change in U due to interest $= -0.034907(30) = -1.05.$

II. From Mrs. Tino's formula:

$$\text{Spread factor} = \frac{1}{26.964} = 0.037086\,.$$

A. Gain from termination:

$$\begin{aligned}\text{Expected release} &= 0.000729[1{,}000(1.606) - 77.76(16.532)] \\ &\quad + 0.002880[1{,}000(3.421) - 77.76(12.162)] \\ &= 7;\end{aligned}$$

$$\begin{aligned}\text{Actual release} &= 1{,}000(3.421) - 77.76(12.162) \\ &= 2{,}475\,.\end{aligned}$$

Change in U due to terminations

$$= -0.037086(2{,}475 - 7) = -91.53.$$

B. Loss from new entrants $= 1{,}000(4.220) - 77.76(10.432)$
$= 3{,}409.$

Change in $U = 0.037086(3{,}409) = +126.42.$

C. Gain from interest $= 80 - 0.05(1{,}000) = 30.$

Change in $U = -0.037086(30) = -1.11.$

The results are summarized in the accompanying tabulation.

	Anderson	Tino-Berin-Lockwood
U_0	77.76	77.76
Change due to:		
Terminations	− 71.86	− 91.53
Interest	− 1.05	− 1.11
New entrant	+106.71	+126.42
Rounding error	− 0.04	− 0.02
U_1	111.52	111.52

Thus we see that the two methods of analysis are not algebraically equivalent, but each may be called "correct."

Mr. Richmond raises an interesting question, namely, while we all agree that if actual experience conforms to expected there will be no change in the unit normal cost, what do we mean by "expected" experience? In my equation (70), for example, the term

$$\sum_{A_0} q_x\left({}_0B\,\frac{N_y^{(12)}}{D_{x+1}} - U_1\,{}_0S\,\frac{s_{x+1}}{s_x}\,\frac{{}^sN_{x+1} - {}^sN_y}{{}^sD_{x+1}}\right)$$

would probably be referred to as the "expected release" of accrued liability on account of mortality (or terminations) among members of set

A_0. We would say that q_x is the probability that j ($\in A_0$) will die and the terms in parentheses represent the accrued liability associated with j. Mr. Richmond says that this is misleading, since our "accrued liability" is not based on the same assumptions on which j's accrued liability was determined in the initial valuation—the presence of U_1 (or, for that matter, U_0) in the expression means that our accrued liability has been adjusted to reflect actual experience since the inception of the plan. Therefore, our analysis does not provide an accurate measurement of the appropriateness of, say, the mortality assumption over a period of more than one year, because the expected releases of liability are changed from year to year to reflect actual experience. What he is saying, in essence, is that he does not like my arrangement of terms (or, by implication, those of Mrs. Tino and the others) in the final analysis formula. Unfortunately, he was unable to arrange the terms in such a way as to provide a reconciliation of U_1 with U_0 as well as to satisfy his own very strict philosophical requirements.

I wish respectfully to point out that many of the discussants' objections do not pertain directly to the substance of the paper, which is that, if one knows how two successive valuations are done, one has all he needs for a mathematically exact gain analysis; the objections are, rather, aimed at my arrangement and/or labeling of terms in the examples. I repeat emphatically that the arrangement and labeling of terms in the analysis formula are entirely arbitrary and subject to the whim and fancy of the actuary responsible. We have seen that there are at least two distinct but correct ways to construct an analysis for aggregate methods. Mrs. Tino and Mr. Berin, on the one hand, and Mr. Richmond, on the other, protest my statement that there is no "gain" defined for these methods as there is for individual methods—but they offer contradictory definitions of the "gain." Perhaps I should have been more precise and simply said that, while there is a generally accepted definition for the word "gain" under individual funding methods, there is none for aggregate methods, and any definition of the word for aggregate methods is at present arbitrary (and not really necessary in any case). Gain and loss analysis is nothing more or less than algebraic manipulation: we run the risk of taking our number games too seriously.

A remark is in order concerning my set notation: the sets T and R are intended to represent distinct categories at time 1. Someone asked me about an employee who retired during the year and then died; does he belong in T or R? The answer is: either one, as you like, but not both. For the purists, let me offer the following definitions of T and R in an attempt to resolve the apparent ambiguity (see eqs. [6]–[8]):

$$R = R_1 \cap (A_0 + B_0) ,$$

$$T = (A_1 + B_1 + R_1)^C \cap (A_0 + B_0) ,$$

where the superscript C denotes the complement of a set.

Mr. Williams appears to have misinterpreted my definition of set B_0. I defined set B_0 as consisting of those active employees who are at or over their assumed retirement ages. When we use an assumed retirement age rather than a table of probabilities of retirement, we imply that we expect that only those who have reached the assumed age will retire; retirements at earlier ages or members of B_0 who remain active for another year must be considered deviations from the expected. Thus my set B_0 appears to be the same as Williams' R^E.

Both Williams and Berin contend that a general approach to gain and loss analysis is possible, whereas they inferred that I was skeptical about any general approach. The difference of opinion is, I think, due to different use of the word "general." Both men have introduced notation which corresponds to a *typical* valuation; neither seems to have devised a completely *general* theory. To see that this is so, just try to interpret their "general" formulas in the particular case of a contributory plan where vesting, widow's, and disability benefits are being prefunded using a three-decrement service table, a separate disabled life mortality table, and a remarriage table, not to mention a salary scale and a select turnover table, etc., etc. The existence of a truly general gain and loss analysis theory must necessarily rest on a truly general theory of actuarial valuation, since we have shown that one is only an algebraic restatement of the other. When someone invents a general theory of valuations—general in the sense of being universally applicable—he will be just a stone's throw away from a general gain and loss theory. Until then, I should not feel confident in my analysis in any particular situation unless it was very like another already proved or unless I had proved it directly.

Finally, let me thank all those who contributed to the written discussion of the paper; their efforts have added substantially to the scope of the paper and have thrown light into some gray corners. Special thanks are owed Mr. Street, who relates my work with Mr. Dreher's and thereby helps to maintain continuity in the literature. I am happy for the profession that we have succeeded in getting so much of current thinking on the subject of gain and loss analysis into print.