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SIMULATION OF THE RUIN POTENTIAL OF NONLIFE INSURANCE COMPANIES

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ABSTRACT

A simplified stochastic model of a casualty insurance company consists of two independent and unchanging probability distributions. The first of these is the distribution of intervals between successive claims, and the second is the distribution of individual claim amounts. Financially the company may be pictured as accumulating a steady flow of risk-loaded premiums in its risk reserve and paying claims therefrom at intervals determined by the first probability distribution and in amounts determined by the second. Such a simplified model adapts very easily to simulation by a computer, and this is what the author has done with several selected pairs of probability distributions in order to calculate the probability of a casualty company's being ruined shortly after its establishment.

In the second part of the paper the aggregate annual claim outgo of a casualty company is assumed to have a gamma distribution and various rate-making strategies are examined by computer simulation of ten randomly chosen companies over a forty-year period. All the companies commenced business with a fairly substantial risk reserve, but, surprisingly, several failed during the period, even though they charged theoretically correct pure premiums. Standard experience-rating methods were found to be a poor protection against adverse chance fluctuations whose cumulative effects were often substantial.

INTRODUCTION

GENERATIONS of actuarial students here and in Great Britain were introduced to probability theory through Whitworth's *Choice and Chance*, the first edition of which appeared in 1867 and the fifth and last in 1901. Proposition LI of this work reads: "If an event happen at random on an average once in time t, the chance of its not happening in a given period τ is $e^{-\tau/t}$." Expressed in modern terminology this could be rewritten thus: "If events are occurring randomly and independently at a mean rate $1/t$ per unit time, the density function of the period τ between events is $t^{-1} e^{-\tau/t}$, the negative exponential distribution." Filip Lundberg, in his doctoral thesis of 1903, suggested that this concept might well apply to the distribution in time of successive claims made on an insurance company. Coupling this with his further assumptions (*a*) that the size of an individual claim would be independent of the time since the previous claim and (*b*) that the (probability) distribution function $P(\cdot)$ of these claims would be time invariant, we have the first complete formulation of a stochastic process homogeneous in time and with independent increments.

In order to apply this model to the financial development of a growing insurance company, it was necessary to overcome the difficulty that the rate of claim occurrence is likely to be a function of the size of the business. Lundberg achieved this by deforming the time scale in such a way that a unit of time always corresponds to the (gradually decreasing) expected interval between two successive claims. Probability statements about events within a certain interval of "time" must then be interpreted in terms of the number of expected claims that will have occurred in that period.

Two probability functions that are of immediate interest in the application of the Lundberg model are $F(y,t)$, the distribution function of the aggregate claim outgo during a given time interval t, and $u(x,t)$, the probability that an insurance company (risk business) commencing its operations with a capital and surplus (risk reserve) of x will survive the time interval t without having a negative risk reserve in the meantime. The mathematical and numerical development of $F(y,t)$ has been reviewed by Paul M. Kahn in Volume XIV of *TSA* and by John C. Wooddy in his official *Study Note on Risk Theory*. On the other hand, the derivation of formulas for $u(x,t)$ and the subsequent numerical calculations have proved elusive; readers may refer to John A. Beekman's papers in Volumes XX and XXI of *TSA* to see some of the mathematical difficulties. This probability was first considered by Lundberg in a path-breaking paper presented to the 1909 International Actuarial Congress, and it turns out that it is almost exactly the same as the probability of a man's having to wait a time of less than x to be served if he joins a waiting line t time intervals after the server was free. The latter probability was first considered by Erlang in 1909, and, although his assumption for $P(\cdot)$ (the distribution function of service times) was specialized to the uniform distribution, his asymptotic results for large t anticipated later actuarial formulas.

THE LUNDBERG MODEL

The present paper is essentially a contribution to the numerical evaluation of $u(x,t)$ without resort to the complicated mathematics that appear in the literature.[1] The method used is simulation of the Lundberg model on an electronic computer. The nonlife insurance company under consideration is supposed to have a risk reserve of $R(\tau)$ at time τ, which has been accumulated from a given initial value $R(0)$ by crediting risk-loaded premiums supposed to be received in a continuous stream and by debiting claims as they occur. Expense loadings in the premiums are supposed to meet expenses exactly. The chance of a claim's occurring in an infinitesimal interval of time $(\tau,\tau + d\tau)$ and $P(y)$ the probability distribution function of the independent random variable Y representing the amount of any individual claim are supposed to be known. In the simulation experiments described below the distributions of the interval of time between two claims and of the individual claim sizes have been assumed to be either negative exponential (designated M) or Pareto (designated XI), for reasons explained in Appendix I. These two distributions are frequently utilized in the literature of risk theory, and there are a number of articles demonstrating their aptness in practice. Readers are referred to the author's recently published text for details. If interclaim intervals are distributed negative exponentially and claim sizes follow the Pareto distribution, we designate the model as M/XI/1, the final unit being inserted to conform with a notation widely used in queueing theory.

The simulations of the Lundberg model all refer to an operational period of time t. Since the mean interval of time between claims has been chosen to be unity, this means that we are calculating the probability of staying in business until at least t claims are expected to have occurred. We have chosen the mean individual claim as the unit of money, so that, if $R(0) = x$, it means that the business commences its operations with a risk reserve of x times the mean individual claim. As a corollary of these conventions the aggregate net premium for collection during the interval t is t, and, on increasing it by a risk loading of $100\,\eta$ per cent, the aggregate risk-loaded premium, assumed payable uniformly throughout the interval, is $(1 + \eta)t$.

SIMULATION OF THE M/G/1 MODEL

The calculation of $u(x,t)$ under the Lundberg model proceeds as follows. Having selected a suitable $P(\cdot)$ and chosen values for x, η, and t:

1. Compute a (pseudorandom) negative exponential variate, t_1, and a variate from $P(\cdot)$, y_1.

[1] References to the works of authors mentioned herein are given at the ends of chapters 2–4 of *Stochastic Theory of a Risk Business* by H. L. Seal (New York: John Wiley, 1969).

2. Calculate the risk reserve at time t_1, namely,

$$R(t_1) = x + (1 + \eta)t_1 - y_1 .$$

If the result is negative, ruin has occurred and we start a new company. (We have now slightly changed the mathematical model described in Appendix I by allowing a company with zero risk reserve to continue in business.)
3. If $R(t_1) \geq 0$, we compute another pair of variates, t_2 and y_2, calculate $R(t_1 + t_2) = x + (1 + \eta)(t_1 + t_2) - (y_1 + y_2)$, and proceed as with $R(t_1)$.
4. Continue to compute pairs of variates (t_j,y_j) for $j = 3, 4, \ldots$, and calculate the corresponding $R(\cdot)$. If r is the smallest integer such that

$$\sum_{j=1}^{r-1} t_j \leq t < \sum_{j=1}^{r} t_j \quad \text{and} \quad R\left(\sum_{j=1}^{r-1} t_j\right) \geq 0 ,$$

we say that the risk business has not been ruined in the period $(0,t)$.

If this procedure is repeated for n companies, of which l are ruined in the period $(0,t)$, our estimate of $v(x,t) = 1 - u(x,t)$, the probability of ruin, is $\hat{v}(x,t) = l/n$, with an estimated standard error $[(l/n)(1 - l/n)/n]^{1/2}$.

THE CASE M/M/1

As a first experiment we took $t = 20$, assumed $P(y) = 1 - e^{-y}$, and used nine combinations of x and η each with $n = 60{,}000$.[2] The resulting values of $v(\cdot,20)$ with their estimated standard errors are shown in Table 1. On an IBM 7094/7090 the aggregate computer execution time needed for the nine values of $\hat{v}(\cdot,20)$ was one hour and 45 minutes.

We see from Table 1 the relatively small influence of an increase in the risk loading from zero to 10 per cent and the substantial decrease in ruin probability caused by increasing the risk reserve. These results conform with expectation: in the short run ruin is caused by quick depletion of the

[2] The author's own amateurish program in FORTRAN appears as Appendix II. I am very grateful to David G. Halmstad for pointing out an error in the original version of it. The four pseudorandom subroutines used in that program were based on the formula $R_{n+1} = CR_n$ (modulo 2^{35}) for $n = 1, 2, 3, \ldots$ with, respectively, the following:

	C (Decimal)	R_1 (Octal Scale)
1......	186,277	127432147741
2......	186,285	315457047615
3......	186,293	120343014221
4......	186,301	203514745101

risk reserve; it is only in the long run that a constant risk loading makes its presence felt.

THE CASE M/XI/1

It is well known that the negative exponential fails as a representation of the average individual's claim-size distribution, because it underestimates the probability of claims that are several times the mean claim. Suppose, however, it is assumed that $P(\cdot)$ is a mixed negative exponential, the parameter of mixing being the mean, namely,

$$P(y) = \int_0^y dx \int_0^\infty a e^{-ax} dF(a) ,$$

TABLE 1

ESTIMATES OF RUIN PROBABILITIES: $v(x,20)$

η	x		
	0	5	10
0.00	0.875±0.001	0.372±0.002	0.120±0.001
.05	.853± .001	.334± .002	.098± .001
0.10	0.831±0.002	0.298±0.002	0.083±0.001

which reduces to $1 - e^{-\beta y}$, the negative exponential with mean β^{-1}, when

$$F(a) = \begin{cases} 0 , & 0 < a < \beta , \\ 1 , & \beta < a < \infty . \end{cases}$$

Let us suppose that the mean claims of individuals whose claim size is distributed negative exponentially are themselves distributed in a gamma distribution with index m and scale c^{-1}. Then

$$F'(a) = \frac{c^m}{\Gamma(m)} a^{m-1} e^{-ca} \qquad (m > 0)$$

and

$$P(y) = 1 - \left(1 + \frac{y}{c}\right)^{-m} \qquad (0 < y < \infty)$$

with an infinite mean unless $m > 1$. When the over-all mean is finite, it equals $c/(m - 1)$, and, if this is chosen as unit, we have $c = m - 1$.

This Type XI distribution for $P(\cdot)$ has a very long tail, as may be judged from the fact that, when $m < 2$ the variance of Y is infinite. In the actuarial literature it has been designated as a "dangerous" distribution and has been fitted successfully to claim distributions on several occasions. We will utilize it with $m = 1.5$ so that $c = 0.5$. The computer program already used for the M/M/1 model applies with only slight modification, and we illustrate it on $v(10,20)$. Table 2 shows how substantially the ruin probability is increased in comparison with the M/M/1 model.

SIMULATION OF THE XI/XI/1 MODEL

As a final example of the simulation of the classic ruin theory model we use formula (6) of Appendix I for the distribution function (d.f.) of interclaim periods and the foregoing $P(\cdot)$ with $c = m - 1$ for that of individual claims. We choose $k = 20$ in formula (6), noting that when $k \rightarrow \infty$ the d.f. becomes the negative exponential, and retain 1.5 for the value of m above. We made only lengthy calculations (43 minutes on the IBM 7094) for the no-loading case $\eta = 0$, and the result is shown in the last line of Table 3. It is observed that the longer tail distributions of interclaim periods and claims amounts, respectively, each add about 0.06 to the probability of ruin before twenty expected claims occur.

TABLE 2

ESTIMATES OF RUIN PROBABILITIES: $v(10,20)$

η	M/M/1	M/XI/1
0.00	0.120±0.001	0.178±0.002
.05	.098± .001	.166± .002
0.10	0.083±0.001	0.163±0.002

TABLE 3

ESTIMATES OF RUIN PROBABILITIES: $v(10,20)$

Model	Estimate
M/M/1	0.120±0.001
M/XI/1	.178± .002
XI/XI/1	0.245±0.002

A natural distribution to assume for $U(\cdot)$, the d.f. of individual claim propensities underlying formula (6), would be the negative exponential, and this implies that $k = 1$. The negative binomial (formula [7] of Appendix I) has been fitted on a number of occasions to the distribution of the claims of individual automobile drivers (or policyholders) and k-values in the neighborhood of unity have resulted. Now for values of $k \leq 1$, the expected time to the first claim is infinite when formula (6) is employed, while for $1 < k \leq 2$ the mean time to the first claim is finite but the variance of the distribution is infinite. Thus the direct Monte Carlo calculation of $v(10,20)$ is impossible for $k \leq 1$ and can prove very costly if $1 < k \leq 2$. We have therefore contented ourselves with an approximate two-decimal accuracy ($N = 600$ instead of 60,000) for a set of smaller k-values. The results are shown in Table 4 and demonstrate, we think, how "dangerous" automobile insurance can be.

RUIN IN AN EXPERIENCE-RATED BUSINESS

There have been a number of criticisms of the theory that has been adumbrated in Appendix I and simulated in the foregoing pages. Some of them are based on the "fact" that the failure of a risk business is almost certainly caused by poor administration and not by a run of heavy claim years. The answer here is that even a perfectly run risk business can find itself forced into ruin through no fault of its own; this will be illustrated below.

A more serious criticism in our view is that the theory assumes that the

TABLE 4

ESTIMATES OF RUIN PROBABILITIES: $\psi(10,20)$

(Model XI/XI/1)

k	Estimate	Computer Execution Time (Minutes)
20	0.26±0.02	3
10	.34± .02	
5	.51± .02	
2	0.85±0.01	44

actual mean of the distribution of aggregate claims in a real-time period, namely, of $F(\cdot,t)$ in relation (10) of Appendix I, is known and invariant. This is certainly not true in practice, particularly since the importance of risk theory is for the brand new risk business with modest capital and without any of its own claim experience on which to base its premiums.

Another criticism of ruin theory is that the probability of ruin within a given period of "time" is only one of a number of factors the risk manager must bear in mind. Ruin seldom occurs as the result of a single bad year. It is a run of poor years that depletes the risk reserve to a point where ruin threatens. The commonest reaction under such circumstances would be to look for unprofitable contracts within the portfolio and eliminate or rerate them. But it should not be overlooked that runs of "bad luck" can occur in the purest of atmospheres.

In what follows we will abandon the theoretical superstructure of ruin theory and simulate the behavior of ten companies' risk reserves over forty-year periods under a number of hypotheses about management's policies for the sequential estimation of premiums.

A convenient basic model is to assume that the aggregate annual claim distribution is given by chi-square with 6 degrees of freedom (i.e., with a mean of 6). This distribution is summarized in Table 5, and it is observed that a company with a risk reserve of 16.81 and a net premium income of between 5 and 6 has about one chance in a thousand of being ruined within a year. We accordingly use 16.81 as the initial risk reserve for each of the ten hypothetical companies and note that in the literature the desirability and amount of stop-loss insurance are frequently determined by requiring a one-year probability of ruin not in excess of 1 in 1,000.

TABLE 5

x	Probability of Claims in Excess of x
5.35	0.5
10.64	.1
12.59	.05
16.81	.01
22.46	.001
27.86	0.0001

Twelve hundred pseudorandom numbers were generated by means of the relation ($n = 1,2,3, \ldots$) $R_{n+1} = 186{,}309\ R_n$ (modulo 2^{35}), with $R_1 = 4131062271$ (octal scale), and, after transformation to negative exponential variates, were added in triplets and multiplied by 2 to produce 400 variates from a chi-square distribution with 6 degrees of freedom. Successive sets of forty variates were then supposed to represent the forty years of experience of ten insurance companies.

We now suppose that management of such a new company has to decide on what terms it will seek business. It believes its capital of 16.81 to be more than adequate, and it considers that its collective experience and know-how is equivalent to ten years of presumptive claim experience with the new company. It estimates that an aggregate risk-loaded premium of 6 will be adequate and competitive. (We ignore any expense loading by assuming that it exactly covers the expenses of the year.) Management decides to allow for any upward or downward trend in aggregate claims by experience rating the aggregate premium in such a way that after t years' experience it will be ($t = 1, 2, \ldots$).

$$\pi(t+1) = \frac{10}{10+t}\,\pi(1) + \frac{1}{10+t}\sum_{j=1}^{t} X_j\,, \tag{1}$$

where $\pi(1) = 6$ and X_j is the aggregate claims of the jth business year. This type of Bayesian experience rating has been suggested by Franckx, and we note that, if there is no trend in the claims of successive years,

$$E\pi(t+1) = \frac{1}{10+t}[10\pi(1) + t\kappa_1] \qquad (\text{where } \kappa_1 = EX_j)$$

$$\rightarrow \kappa_1 \quad \text{as} \quad t \rightarrow \infty .$$

The results of this experience-rating policy are shown in Table 6, which required one second of computer execution time (and about half a minute for compilation). Although the managements of these ten companies had correctly estimated the net, or pure, risk premium, six of them were ruined in the forty-year period, one of them in the first year. At the foot of Table 6 we have shown the range of premiums produced by relation (1) for the ten companies during their years in business. While it is difficult to say how long a contractholder in a mutual company would be prepared to continue his policy during a period of rising premiums, the ranges experienced by these ten companies do not appear unreasonable. Oddly enough, a fairly small loading in the premium would have changed the picture substantially. If every premium had been arbitrarily increased by 0.6, namely, 10 per cent of $\pi(1)$, five of the six failing companies would have been saved from ruin, the sole company to be ruined under these circumstances being Company 6, which would have succumbed at the end of the fifteenth (instead of the first) year.

Table 6 is also interesting in that it shows that ruin may occur as the result of a longish run of aggregate claims not very seriously in excess of the expected (e.g., Companies 9 and 10) or because of one or two years of unusually large claims (e.g., Companies 4, 6, and 7).

Let us now suppose that these ten companies had been run by managements which had only had experience with a better class of risks. These managements estimated just as confidently as the first set that the aggregate annual premium $\pi(1) = 5$. The results of this assumption are shown in Table 7, where seven of the companies are ruined before the end of the forty-year period.

As would be anticipated, ruin in Table 7 tends to occur earlier than it did in Table 6, and the fortieth-year risk reserves are lower. And, not unexpectedly, the ruin of Company 8 occurred at a time when, in Table 6, it narrowly escaped this fate. In fact, the earlier appearance of ruin in Table 7 is caused in several instances by single, large claim totals which only succeeded in sharply reducing the corresponding risk reserves of Table 6.

An interesting feature of the premium ranges shown at the foot of

TABLE 6

RISK RESERVES OF TEN BUSINESSES THAT CONFIDENTLY ESTIMATED THE MEAN ANNUAL CLAIM OUTGO (CORRECTLY) AT 6.0 AND USED FORMULA (1) TO RECALCULATE THEIR PREMIUMS

END OF YEAR	COMPANY									
	1	2	3	4	5	6	7	8	9	10
1	18.43	18.21	19.48	14.09	17.06	Ruin	18.65	11.23	16.97	22.64
2	20.86	19.59	20.79	19.00	15.66		18.66	14.34	21.13	27.81
3	24.03	23.19	15.87	10.31	18.15		20.36	19.92	25.20	26.26
4	13.09	25.68	19.75	13.05	19.88		24.47	24.94	27.60	25.68
5	17.01	21.81	22.44	11.58	17.35		24.81	18.81	26.20	18.31
6	18.05	24.08	22.73	16.47	19.43		26.16	9.36	30.31	22.21
7	19.32	25.46	26.57	18.31	19.73		25.84	13.09	21.35	26.73
8	16.60	27.69	30.34	20.25	21.27		20.34	14.67	11.27	29.68
9	18.39	21.32	19.28	23.78	19.74		16.99	18.86	8.23	31.47
10	24.21	19.25	13.06	26.00	22.74		18.06	20.34	11.59	27.82
11	19.08	11.30	11.43	24.55	25.46		12.72	20.97	11.31	20.97
12	13.86	11.63	12.93	18.44	22.04		15.21	23.48	13.75	21.89
13	14.88	14.09	18.42	23.42	22.61		5.72	27.84	17.56	4.19
14	16.85	15.03	16.10	24.67	21.95		6.52	30.14	16.75	7.97
15	15.63	10.20	18.41	19.83	23.33		11.33	34.16	8.24	9.82
16	20.13	5.48	16.67	13.87	24.56		11.70	12.04	10.64	13.22
17	17.62	10.95	15.12	11.92	20.76		12.02	13.98	12.92	16.98
18	23.13	3.91	16.14	14.07	25.04		13.95	6.12	12.17	18.88
19	25.43	Ruin	18.36	5.20	27.40		0.21	3.01	16.23	20.57
20	25.90		20.99	Ruin	26.46		4.70	4.99	7.93	18.14
21	29.73		24.90		22.43		6.17	5.87	10.39	20.69
22	29.23		17.04		24.64		9.17	10.28	5.18	15.45
23	26.44		18.58		29.98		5.31	12.98	5.82	13.94
24	30.02		24.27		24.42		10.30	13.19	4.83	6.92
25	30.43		28.86		27.01		13.25	9.57	8.08	2.40
26	32.40		32.94		27.05		14.48	10.44	4.52	4.98
27	36.74		29.30		29.54		12.40	15.76	Ruin	Ruin
28	34.75		30.20		27.09		5.57	16.73		
29	33.28		33.51		21.44		Ruin	11.55		
30	34.51		33.79		22.61			16.55		
31	30.40		37.34		25.07			14.87		
32	33.33		39.34		24.56			18.71		
33	36.85		43.77		27.13			12.13		
34	32.98		39.94		30.53			17.45		
35	35.04		35.24		32.00			20.25		
36	35.68		25.57		33.83			22.89		
37	40.33		27.59		34.44			23.04		
38	43.86		26.47		38.90			22.62		
39	41.65		27.81		42.13			25.74		
40	44.54		27.79		44.73			29.01		
Premium range	5.14–6.19	5.21–6.63	5.12–6.09	5.59–6.51	5.18–6.09		5.30–6.46	5.27–6.46	4.99–6.54	4.92–6.19

TABLE 7

RISK RESERVE OF THE SAME TEN BUSINESSES WHEN THE MEAN ANNUAL CLAIM OUTGO WAS CONFIDENTLY ESTIMATED TO BE 5.0 (INSTEAD OF THE TRUE 6.0) AND RELATION (1) USED TO RECALCULATE THE PREMIUMS

END OF YEAR	COMPANY									
	1	2	3	4	5	6	7	8	9	10
1	17.43	17.21	18.48	13.09	16.06	Ruin	17.65	10.23	15.97	21.64
2	18.96	17.68	18.88	17.09	13.75		16.75	12.43	19.22	25.90
3	21.29	20.45	13.13	7.57	15.41		17.62	17.18	22.45	23.52
4	9.58	22.17	16.24	9.54	16.37		20.96	21.43	24.09	22.17
5	12.78	17.59	18.21	7.35	13.12		20.58	14.59	21.97	14.08
6	13.16	19.19	17.83	11.58	14.54		21.27	4.47	25.42	17.31
7	13.80	19.94	21.05	12.79	14.22		20.32	7.57	15.84	21.21
8	10.50	21.59	24.23	14.15	15.16		14.23	8.56	5.17	23.58
9	11.73	14.65	12.62	17.12	13.07		10.33	12.20	1.57	24.81
10	17.03	12.06	5.88	18.82	15.55		10.87	13.15	4.40	20.63
11	11.39	3.61	3.74	16.86	17.78		5.03	13.28	3.63	13.28
12	5.69	3.47	4.76	10.28	13.87		7.04	15.31	5.59	13.73
13	6.26	5.47	9.80	14.80	14.00		Ruin	19.22	8.94	Ruin
14	7.80	5.98	7.04	15.61	12.89			21.09	7.70	
15	6.16	0.73	8.99	10.36	13.86			24.69	Ruin	
16	10.26	Ruin	6.80	4.00	14.69			2.17		
17	7.37		4.86	1.66	10.51			3.73		
18	12.51		5.52	3.44	14.41			Ruin		
19	14.45		7.38	Ruin	16.42					
20	14.57		9.67		15.13					
21	18.06		13.24		10.77					
22	17.25		5.05		12.66					
23	14.15		6.28		17.69					
24	17.42		11.67		11.82					
25	17.54		15.97		14.12					
26	19.23		19.76		13.87					
27	23.29		15.84		16.08					
28	21.02		16.47		13.37					
29	19.29		19.52		7.46					
30	20.26		19.55		8.36					
31	15.91		22.85		10.57					
32	18.59		24.60		9.82					
33	21.87		28.79		12.15					
34	17.77		24.73		15.32					
35	19.60		19.81		16.56					
36	20.02		9.91		18.17					
37	24.46		11.71		18.57					
38	27.77		10.38		22.81					
39	25.35		11.52		25.83					
40	28.04		11.29		28.22					
Premium range	4.64–5.58	4.59–5.66	4.56–5.61	5.00–5.79	4.98–5.38		4.68–6.60	4.75–5.68	4.37–6.33	4.21–5.98

Table 7 is that none of the three companies that remained in business at the end of forty years had ever charged a premium as large as the "true" premium of 6.00. Only two of the seven failing companies made premium charges in excess of the true mean claim outgo, but this did not save them from ruin.

Table 7 has shown the effect of an initial premium estimated with great confidence. In Table 8 we have supposed that management had very little idea of the true net premium and estimated it as $\pi(1) = 5$ without giving it any weight once experience started to accumulate. We thus have to substitute zero for 10 throughout relation (1), with the result that early experience will have a strong effect on the size of the premium. Ruin now occurs earlier than it does in Table 7 on three occasions, at the same time on two occasions, one year later on one occasion, and not at all in Company 8, which manages to accumulate substantial reserves. These are surprising results when we consider that in Table 7 management persevered in its erroneous estimate of 5.0 by giving it a "weight" of ten years' observations. They emphasize the relatively small influence of minor premium changes in comparison with the violent fluctuations of chance.

The rating formula (1) treats all experience, actual and hypothetical, as of equal weight. If it is suspected that diverse factors may influence the claims so that real upward trends may be followed by a leveling-out or even by downward movements, it may be desired to give much more weight to the most recent experience. Consider, therefore, a special case of a formula suggested by Simberg, namely, $(t = 1, 2, 3, \ldots)$.

$$\begin{aligned}\pi(t+1) &= 0.5X_t + 0.5\pi(t) \\ &= 0.5\sum_{j=0}^{t-1} 0.5^j X_{t-j} + 0.5^t \pi(1) ,\end{aligned} \tag{2}$$

where the latest year's experience is regarded as being as important as all previous information.

The basic chi-square annual claim distribution used hitherto is now assumed to shift to the right by 0.05 each year, the first year having a mean 6.00 and the fortieth year having a mean 7.95. The 1,200 pseudorandom variates described above were used once again and relation (2) was applied with $\pi(1) = 5.0$. The risk reserves of Table 9 resulted. Although seven of the ten companies were ruined, the general picture is quite similar to Table 8, where the mean aggregate claims were constant throughout the forty-year period. The use of relation (2) in this case where a company's experience is always below the upward trend is intuitively better

TABLE 8

RISK RESERVES OF THE SAME TEN BUSINESSES WHEN THE MEAN ANNUAL CLAIM OUTGO WAS INITIALLY GUESSED TO BE 5.0 (INSTEAD OF THE TRUE 6.0) AND A RELATION SIMILAR TO (1) USED TO RECALCULATE THE PREMIUMS

END OF YEAR	COMPANY									
	1	2	3	4	5	6	7	8	9	10
1........	17.43	17.21	18.48	13.09	16.06	Ruin	17.65	10.23	15.97	21.64
2........	18.39	17.31	17.37	20.47	14.43		15.99	18.41	19.99	21.51
3........	19.81	19.71	10.69	10.98	17.40		16.85	25.24	22.25	15.16
4........	6.88	20.47	14.66	15.40	18.80		19.96	29.65	22.41	11.77
5........	11.28	14.86	16.72	14.71	15.72		18.82	22.17	18.89	2.40
6........	12.17	16.25	16.15	20.42	17.69		18.94	12.46	21.50	5.68
7........	13.21	16.66	19.25	22.43	17.69		17.45	16.95	10.86	9.28
8........	10.19	17.94	22.05	24.37	18.94		10.98	18.88	0.09	11.07
9........	11.90	10.59	9.89	27.76	17.05		7.17	23.26	Ruin	11.63
10........	17.55	8.02	3.34	29.65	19.83		8.01	24.66		6.78
11........	11.96	Ruin	1.72	27.78	22.20		2.42	25.15		Ruin
12........	6.56		3.29	21.37	18.33		4.91	27.50		
13........	7.61		8.80	26.30	18.63		Ruin	31.63		
14........	9.58		6.31	27.32	17.69			33.57		
15........	8.29		8.53	22.25	18.84			37.17		
16........	12.76		6.65	16.20	19.82			14.57		
17........	10.12		5.01	14.31	15.75			16.58		
18........	15.55		5.98	16.55	19.86			8.74		
19........	17.67		8.12	7.74	21.98			5.82		
20........	17.93		10.65	Ruin	20.76			8.03		
21........	21.55		14.42		16.48			9.10		
22........	20.80		6.35		18.53			13.66		
23........	17.77		7.81		23.67			16.46		
24........	21.16		13.41		17.85			16.72		
25........	21.35		17.84		20.27			13.15		
26........	23.10		21.71		20.10			14.11		
27........	27.20		17.83		22.40			19.50		
28........	24.94		18.53		19.74			20.48		
29........	23.22		21.65		13.91			15.31		
30........	24.23		21.71		14.95			20.37		
31........	19.91		25.04		17.28			18.70		
32........	22.65		26.80		16.63			22.56		
33........	25.97		30.98		19.06			15.97		
34........	21.88		26.89		22.30			21.34		
35........	23.75		21.96		23.61			24.15		
36........	24.21		12.08		25.26			26.77		
37........	28.66		13.97		25.70			26.89		
38........	31.97		12.71		29.97			26.44		
39........	29.54		13.92		33.00			29.53		
40........	32.22		13.77		35.39			32.76		
Premium range...	3.43–6.66	3.56–6.79	3.89–6.17	5.03–8.20	4.97–6.57		3.93–6.00	4.11–7.49	3.04–6.67	2.35–6.24

TABLE 9

RISK RESERVES OF THE SAME TEN BUSINESSES WHEN SUBJECTED TO AN UPWARD DRIFT OF 0.05 IN THE MEAN AGGREGATE CLAIM AND WHEN PREMIUMS ARE RECALCULATED BY RELATION (2) WITH $\pi(1) = 5.0$

End of Year	Company 1	2	3	4	5	6	7	8	9	10
1	17.43	17.21	18.48	13.09	16.06	Ruin	17.65	10.23	15.97	21.64
2	18.65	17.46	18.15	18.56	14.00		16.36	15.07	19.52	23.87
3	20.15	19.88	11.81	8.06	16.71		17.35	20.18	21.49	18.66
4	6.98	20.21	17.02	13.51	17.43		20.34	22.54	21.08	16.84
5	14.43	14.14	18.65	12.18	13.61		18.31	12.89	17.19	9.06
6	15.49	16.94	17.20	17.73	16.10		18.27	4.28	20.61	15.89
7	16.26.	17.53	20.00	17.71	15.64		16.62	12.51	9.47	20.12
8	12.68	18.71	21.50	17.86	16.61		10.55	14.65	2.19	20.89
9	15.20	10.77	7.58	19.58	14.06		9.31	18.37	4.98	20.22
10	20.53	10.73	4.83	19.26	17.20		12.87	17.68	12.57	14.49
11	12.48	4.67	7.68	15.49	18.58		8.26	16.52	12.84	8.20
12	8.08	9.49	12.10	8.84	13.21		13.47	17.80	15.63	12.45
13	11.84	13.99	18.33	16.27	14.32		4.17	20.35	18.46	Ruin
14	14.66	14.78	13.82	16.41	13.60		9.34	19.71	15.37	
15	12.91	9.39	16.05	10.40	15.21		15.92	21.15	6.04	
16	17.65	6.57	13.16	6.04	15.88		14.91	Ruin	11.90	
17	13.14	15.12	11.78	7.58	11.17		14.32		14.75	
18	18.76	7.03	13.56	12.33	16.71		15.59		13.17	
19	18.50	5.51	15.63	3.72	17.66		0.59		17.12	
20	16.57	14.11	17.11	Ruin	14.87		10.78		6.83	
21	18.94	5.52	19.16		10.30		12.96		12.11	
22	15.87	11.41	8.49		14.08		15.58		7.11	
23	11.97	Ruin	12.26		19.12		10.07		10.25	
24	16.26		18.30		10.84		16.00		10.16	
25	15.29		20.33		14.65		17.02		14.27	
26	16.33		20.92		14.02		15.85		9.57	
27	19.22		13.56		16.11		11.93		1.75	
28	14.40		14.27		12.23		5.12		5.52	
29	12.41		17.01		6.98		Ruin		12.79	
30	14.03		15.38		10.97				9.42	
31	9.48		17.79		14.25				7.25	
32	14.09		17.48		12.92				9.78	
33	17.01		19.76		15.27				11.05	
34	11.11		12.70		17.29				11.87	
35	13.95		8.15		16.40				12.38	
36	13.96		0.75		16.29				15.56	
37	17.93		8.49		15.02				12.44	
38	18.83		9.21		18.18				Ruin	
39	13.58		11.95		18.58					
40	15.96		11.94		18.16					
Premium range	3.14–9.92	3.30–11.45	2.66–13.03	3.62–11.55	4.11–9.92		3.24–16.96	2.14–11.26	2.66–12.53	1.47–9.46

than the rating procedure used in Table 8 (namely, relation [1] with 10 replaced by zero). Table 10 verifies this, but shows that certain chance configurations may produce a better result for relation (1) rating (e.g., Company 8). Note how much narrower the premium ranges are in Table 10 than in Table 9. It may be doubted whether the policyholders of Company 7 would have remained to see their premiums range from 3.24 to 16.96 in twenty-eight years.

An unrealistic feature of the foregoing illustrations is that we have ignored the "1 in 1,000" rule for stop-loss reinsurance after the first year. If management had been asked to "guess" the amount of the annual claim outgo that would be exceeded once in a thousand years—a much harder task than estimating the mean of the distribution—the resulting stop-loss reinsurance would possibly have saved some of the companies that were shown to fail. Lack of space prevents the simulation of this type of management policy with its difficult decisions about reinsurance premium estimation and loading. An interesting question would be whether a run of bad luck might result in the reinsurance of the whole portfolio. Nevertheless the foregoing set of tables clearly demonstrates the importance of chance in the operation of a risk business.

APPENDIX I

MATHEMATICAL REVIEW

The Erlang model for an orderly waiting line (queue) with a single server assumes that customers arrive randomly with independent interarrival times distributed negative exponentially and that customers' service times are independent positive random variables Y with distribution function $P(\cdot)$ independent of the arrival times. It is convenient to measure time in units equal to the expected interarrival time in real time units. As each customer joins the queue, the waiting time for service ahead of a potential new customer jumps upward by an amount Y, and between customer arrivals the waiting time decreases uniformly toward a lower bound of zero. We suppose that the server is idle at time $t = 0$ and write $W(t)$ for the (virtual) waiting time ahead of a customer joining the queue at time $t > 0$. Construct a function $\bar{W}(t)$ similar to $W(t)$ except that negative values are permitted during a server's idle periods, with the result that the next upward jump occurs from a negative, instead of a zero, value. Then $W(t)$ is equal to $\bar{W}(t)$ increased by the largest amount by which $\bar{W}(\tau)$ is below the t-axis for $0 < \tau \leq t$. Hence

$$W(t) = \bar{W}(t) - \inf_{0<\tau\leq t} \bar{W}(\tau) = \sup_{0<\tau\leq t} [\bar{W}(t) - \bar{W}(\tau)]$$

$$= \sup_{0<\tau\leq t} [Y(t) - t - \overline{Y(\tau) - \tau}] = \sup_{0<\tau\leq t} [Y(t) - Y(\tau) - \overline{t - \tau}] ,$$

TABLE 10

RISK RESERVES OF THE SAME TEN BUSINESSES WHEN SUBJECTED TO AN UPWARD DRIFT OF 0.05 IN THE MEAN AGGREGATE CLAIM AND WHEN PREMIUMS ARE RECALCULATED BY RELATION (1) WITH $\pi(1) = 5.0$ AND 10 REPLACED BY ZERO

END OF YEAR	COMPANY									
	1	2	3	4	5	6	7	8	9	10
1........	17.43	17.21	18.48	13.09	16.06	Ruin	17.65	10.23	15.97	21.64
2........	18.34	17.26	17.32	20.42	14.38		15.94	18.36	19.94	21.46
3........	19.68	19.58	10.56	10.85	17.27		16.72	25.11	22.13	15.03
4........	6.66	20.24	14.43	15.18	18.57		19.74	29.42	22.19	11.54
5........	10.93	14.51	16.37	14.36	15.37		18.47	21.82	18.54	2.05
6........	11.67	15.75	15.65	19.92	17.19		18.44	11.96	21.00	5.18
7........	12.54	15.98	18.57	21.76	17.02		16.78	16.28	10.19	8.60
8........	9.32	17.07	21.18	23.49	18.07		10.11	18.00	Ruin	10.19
9........	10.80	9.49	8.79	26.62	15.95		6.07	22.16		10.53
10........	16.20	6.67	1.99	28.30	18.48		6.66	23.31		5.43
11........	10.34	Ruin	0.09	26.16	20.57		0.80	23.53		Ruin
12........	4.63		1.36	19.44	16.41		2.98	25.58		
13........	5.36		6.55	24.05	16.38		Ruin	29.38		
14........	6.98		3.71	24.72	15.09			30.97		
15........	5.32		5.56	19.27	15.86			34.20		
16........	9.38		3.28	12.82	16.45			11.19		
17........	6.32		1.21	10.51	11.95			12.78		
18........	11.30		1.73	12.30	15.61			4.49		
19........	12.95		3.40	3.01	17.25			1.09		
20........	12.70		5.43	Ruin	15.54			2.80		
21........	15.80		8.67		10.73			3.35		
22........	14.50		0.05		12.23			7.36		
23........	10.90		0.94		16.80			9.58		
24........	13.68		5.93		10.37			9.24		
25........	13.25		9.74		12.17			5.05		
26........	14.35		12.96		11.35			5.36		
27........	17.77		8.40		12.97			10.08		
28........	14.81		8.41		9.61			10.36		
29........	12.37		10.80		3.06			4.46		
30........	12.63		10.11		3.35			8.77		
31........	7.53		12.66		4.91			6.32		
32........	9.47		13.62		3.45			9.38		
33........	11.97		16.98		5.06			1.98		
34........	7.03		12.04		7.45			6.49		
35........	8.03		6.23		7.88			8.42		
36........	7.58		Ruin		8.64			10.15		
37........	11.11				8.15			9.34		
38........	13.47				11.47			7.94		
39........	10.06				13.53			10.06		
40........	11.75				14.92			12.29		
Premium range...	3.48–6.73	3.64–7.74	3.92–6.42	5.06–8.24	5.30–6.59		4.00–7.96	4.19–7.52	3.12–7.62	0.26–7.19

where $Y(\tau)$ is the sum of the random variables Y caused by arrivals at the queue through time τ. Because of the assumption of independent increments in $Y(\tau)$ homogeneous in time, the distribution function of $W(t)$ is given by

$$\begin{aligned} P[W(t) \leq x] &= P\{\sup_{0<\tau\leq t} [Y(t-\tau) - \overline{t-\tau}] \leq x\} \\ &= P\{\sup_{0<\tau\leq t} [Y(\tau) - \tau] \leq x\} = P[\sup_{0<\tau\leq t} \bar{W}(\tau) \leq x] . \end{aligned} \tag{3}$$

The Lundberg model for a risk business assumes that contractholders make claims independently on the company at random instants determined by a negative exponential with unit mean and that a contractholder's claim amount is an independent positive random variable Y with a distribution function $P(\cdot)$ independent of the time of the claim. The risk reserve (capital and surplus) of the risk business is incremented by premiums assumed to be paid uniformly at a rate π_1 per unit of time (which unit is equal to the real-time interval during which one claim is expected) and is reduced instantaneously by the amount of any claim that occurs. If the risk reserve is x at time $t = 0$ and is $R(t)$ at time $t > 0$, the net decrease in the risk reserve by time t is

$$x - R(t) = Y(t) - \pi_1 t , \tag{4}$$

where $Y(t)$ is now the sum of the claim amounts that have been paid through time t. When $\pi_1 = 1$, the right-hand side of this relation is equivalent to the $\bar{W}(t)$ of the queueing model. If the risk reserve of a company becomes zero[3] or negative as the result of a claim, the risk business is said to be ruined. The probability that ruin will not occur during the period $(0,t)$ is thus

$$\begin{aligned} P[R(\tau) > 0 \text{ all } \tau \text{ in } 0 < \tau \leq t] &= P\{\sup_{0<\tau\leq t} [x - R(\tau)] \leq x\} \\ &= P\{\sup_{0<\tau\leq t} [Y(\tau) - \pi_1\tau] < x\} \end{aligned} \tag{5}$$

and is a generalization of relation (3) to the extent that π_1 is not necessarily equal to unity. In what follows we write $u(x,t)$ for the probability of relation (5), and

$$\pi_1 = p_1 + \eta_1 = p_1(1 + \eta) = 1 + \eta ,$$

where p_1 is the mean of the distribution of Y and η_1 is the so-called risk loading. If we agree to work in monetary units each equal to the mean

[3] The inclusion of zero does not change the probability of ruin and allows us to retain formal identity with the queueing model.

claim, $p_1 = 1$ and η is called the rate of risk loading. The risk reserve x must, of course, be expressed in the same monetary unit.

In the queueing literature a convenient notation has been introduced to indicate what assumptions have been made about the interarrival and service times and the number of servers. We have utilized this in our discussion of, and illustrations from, ruin theory. For example, the foregoing description of the Erlang-Lundberg model would be designated as M/G/1, indicating that claims are independent and occur randomly (M for Markov), that the distribution of claim amounts is general (G), and that the theory is that of the queueing model with a single server (1). A generalization of this model is to replace the negative exponential distribution of interclaim periods by some arbitrary distribution, and this is designated as GI/G/1. For example, the chi-square distribution with n degrees of freedom could be used for interclaim periods, and the model would then be designated by K_n/G/1. A further generalization is to assume that the times at which claims occur form a stationary point process in which the joint distribution of the numbers of events in any k fixed intervals ($k = 1, 2, 3, \ldots$) is invariant under translation of the time scale. This allows for certain types of dependence between the lengths of successive interclaim periods and is the most general type of model for which the formulas of ruin theory are valid.

A particular stationary point process is known as the birth (claim) process with stationary increments. It is characterized by a density function of τ, the interclaim period following the epoch of the nth claim, given by ($n = 1, 2, \ldots$).

$$\lambda_n(\tau) \exp\left[-\int_0^{\tau} \lambda_n(s)ds\right] ,$$

where, using standard notation for derivatives,

$$\lambda_n(\tau) = -\frac{p_0^{(n+1)}(\tau)}{p_0^{(n)}(\tau)} , \qquad p_0(\tau) = \int_0^{\infty} e^{-\lambda\tau} dU(\lambda) ,$$

and $U(\cdot)$ is a distribution function. In fact $p_0(\tau)$ is the probability that there will be no claim in any period of length τ, and $U(\cdot)$ may be interpreted as the d.f. of individual propensities to make claims. A special case of this process has been widely used in the actuarial literature, where it is known as the Pólya process. $U(\cdot)$ is then a gamma distribution with index k and scale k^{-1} (so that k/k, or unity, is the mean), and it is easy to show that this implies

$$\lambda_n(\tau) = \frac{k+n}{k+\tau} = \frac{1+n/k}{1+\tau/k}$$

and that the density of the interclaim period following the nth claim is

$$(1 + n/k)(1 + \tau/k)^{-k-n-1} \qquad (0 < \tau < \infty) ,$$

corresponding to a d.f.

$$1 - (1 + \tau/k)^{-k-n} . \tag{6}$$

(It can be shown that the d.f. of the interclaim period following an arbitrary origin of time is that of relation [6] with $n = 0$.) This distribution function is Type XI in Karl Pearson's family but is more commonly known by the name Pareto. The Pólya process for claims in conjunction with a general distribution of claim amounts is conveniently designated XI/G/1.

It is to be noted that the general birth process formula for the probability of n claims in a period t, namely,

$$p_n(t) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dU(\lambda) ,$$

leads to the negative binomial

$$p_n(t) = \binom{k+n-1}{n} \left(\frac{t}{t+k}\right)^n \left(\frac{k}{t+k}\right)^k \tag{7}$$

in the case of the Pólya process. The mean of this discrete distribution is t, showing that the time scale of relation (6) is equivalent to the number of expected claims. The calculation of $u(x,t)$ on the alternate models M/G/1 and XI/G/1 may thus be made strictly comparable by using a negative exponential with unit mean and relation (6), respectively. Although claims are not occurring uniformly throughout the interval t, premiums are supposed to be received uniformly at the rate $1 + \eta$ per unit of time.

We conclude this brief review of the hypotheses of ruin theory by stating the formulas that have been derived to calculate the probability of a risk business maintaining a nonnegative risk reserve throughout a period $(0,t)$. These formulas were based on the original Erlang-Lundberg model, namely, M/G/1, but they can be generalized without difficulty. Arfwedson's integro-differential equation for $u(x,t)$ is

$$\pi_1[u(x,t) - u(0,t)] - \int_0^x u(x-y,t)[1 - P(y)]dy = \frac{\partial}{\partial t} \int_0^x u(z,t)dz . \tag{8}$$

An alternative form due, independently, to Beneš and Prabhu is

$$u(x,t) = F(x + \pi_1 t,t) - \int_{\tau=0+}^{t} u(0,t-\tau)dF(x + \pi_1\tau,\tau) , \tag{9}$$

where

$$F(y,t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P^{n*}(y) , \tag{10}$$

where $P^{n*}(\cdot)$ is the d.f. of the aggregate of n claims, each with d.f. $P(\cdot)$, and λ is usually chosen as unity with an appropriate change in time scale. The "boundary" probability $u(0,t)$ appearing in both these equations is given explicitly by

$$u(0,t) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \int_0^{\pi_1 t} \left(1 - \frac{y}{\pi_1 t}\right) dP^{n*}(y) . \tag{11}$$

Neither (8) nor (9) is easy to evaluate numerically, and efforts have been made to calculate, instead, the bivariable Laplace transform of $u(x,t)$, namely,

$$\mu(r,s) = \int_0^{\infty} e^{-rx} dx \int_0^{\infty} e^{-st} u(x,t) dt . \tag{12}$$

Arfwedson's result is that

$$\mu(r,s) = \frac{1/r - 1/\rho(s)}{s + 1 - \pi_1 r - \pi(r)} , \tag{13}$$

where

$$\pi(r) = \int_0^{\infty} e^{-ry} dP(y)$$

and $\rho(s)$ is that single value of r that satisfies the equation

$$s = \pi_1 r + \pi(r) - 1 .$$

However, even if $\mu(r,s)$ could be calculated for a set of pairs of values of (r,s), there still remains the problem of its inversion. So far as we know no realistic numerical results have been calculated from any of these formulas, since they were published between eight and thirteen years ago.

On the other hand, Segerdahl was able to prove nearly thirty years ago that, with x fixed,

$$\frac{1 - u(x,t)}{1 - u(x,\infty)} \sim \Phi\left(\frac{t - \mu}{\sigma}\right) , \tag{14}$$

where $\Phi(\cdot)$ is the standard normal d.f., and μ and σ are functions of x and η and the first three moments of the two distributions $P'(y)$ and $e^{-\kappa y} P'(y)$, where κ is given by

$$1 - (1 + \eta)\kappa - \int_0^{\infty} e^{-\kappa y} dP(y) = 0 .$$

Since an asymptotic relation is available for $\psi(x) \equiv 1 - u(x,\infty)$, this means that $u(x,t)$ can be calculated for "large" x and t—though it is not known just how big these variables must be for given numerical accuracy.

There is thus a significant gap between the practically important cases where x and/or t are small and those where asymptotic theory becomes applicable. It is fortunate that the queueing and risk models can be simulated very easily on an electronic computer to produce values of $u(x,t)$ nearly correct in the third decimal place.[4] For values of η and x that lead to small values for $1 - u(x,t)$, the probability of ruin before time t, it appears that between one and two minutes of large computer time is required for each claim expected. It would thus be relatively expensive to calculate ruin probabilities extending well into the future. On the other hand, it is the short period which is of more interest to the *entrepreneur*, and inexpensive computer runs will satisfy him. The longer run will be of interest to the theoretician who wants to know when asymptotic theory will be adequate.

There are substantial advantages in simulating a risk business on a computer. For example, it is not necessary to confine interclaim periods to be intervals of a stationary point process. It is also possible to generalize $P(\cdot)$ to be an arbitrary function of the time at which a given claim occurs. Neither of these generalizations has been considered in the text.

APPENDIX II

```
C     FREQUENCY ØF RUIN ØF N RISK BUSINESSES IN A
C     PERIØD DURING WHICH T CLAIMS ARE EXPECTED.
C     READ-IN HAZARD (AMBDLA), MEAN CLAIM RECIPRØ-
C     CAL (ALPHA), PRØPØRTIØNATE RISK LØADING (ETA),
C     INITIAL RISK RESERVE (RRØ), TIME PERIØD CØN-
C     SIDERED (T), NUMBER ØF TRIALS (N).
      READ(5,1) JØBS
    1 FØRMAT (I5)
      WRITE (6,2) JØBS
    2 FØRMAT (1H1, 50X, 'NUMBER ØF JØBS'/1H0, 'JØBS = ',I5)
   10 READ (5,50) AMBDLA, ALPHA, ETA, RRØ, T, N
   50 FØRMAT (5F10.5,I5)
      WRITE (6,51) AMBDLA, ALPHA, ETA, RRØ, T,N
   51 FØRMAT (1H1,50X, 'INPUT PARAMETERS'/1H0,
```

[4] This degree of accuracy corresponds to a simulation of 60,000 risk businesses. Longer, and thus more expensive, computations could improve the accuracy of the result if this were desired.

```
     1 'AMBDLA = ', F10.5, 6X, 'ALPHA = ', F10.5,6X, 'ETA = ',
     2 F10.5,6X, 'RRØ = ', F10.5,6X, 'T = ', F10.5,6X,'N = ',I5)
       JØBNØ = 0
       IRUIN = 0
       IREPS = 0
       ICØUNT = 0
       ST = 0.0
       RR = RRØ
C      START A BUSINESS AND CALCULATE A RANDØM EX-
C      PØNENTIAL INTERCLAIM PERIØD TESTING TØ SEE IF
C      IT EXCEEDS T
   100 CALL GAS1(R)
       T1 = -ALØG(R)/AMBDLA
       ST = ST + T1
       IF ((ST-T).GE.0.0) GØ TØ 250
C      CALCULATE A RANDØM EXPØNENTIAL CLAIM
C      AMØUNT
       CALL GAS2(R)
       Y1 = -ALØG(R)/ALPHA
       ICØUNT = ICØUNT + 1
C      CALCULATE RESULTING RISK RESERVE AND REGIS-
C      TER RUIN
       RR = RR + (T1*AMBDLA*(1.0 + ETA)/ALPHA) - Y1
       IF (RR.LT.0.0) IRUIN = IRUIN + 1
       IF (RR.LT.0.0) GØ TØ 250
       GØ TØ 300
C      CALCULATE ANØTHER RANDØM EXPØNENTIAL IN-
C      TERCLAIM PERIØD AND CHECK WHETHER TØTAL
C      TIME ELAPSED EXCEEDS T
   150 CALL GAS3(R)
       T2 = -ALØG(R)/AMBDLA
       ST = ST + T2
       IF((ST - T).GE.0.0) GØ TØ 250
C      CALCULATE ANØTHER RANDØM EXPØNENTIAL
C      CLAIM AMØUNT
```

```
      CALL GAS4(R)
      Y2 = -ALØG(R)/ALPHA
      ICØUNT = ICØUNT + 1
C     CALCULATE RESULTING RISK RESERVE AND REGIS-
C     TER RUIN
      RR = RR + (T2*AMBDLA*(1.0 + ETA)/ALPHA) - Y2
      IF (RR.LT.0.0) IRUIN = IRUIN +1
      IF(RR.LT.0.0) GØ TØ 250
      GØ TØ 300
C     IF SUM ØF INTERCLAIM PERIØDS EXCEEDS T ØR IF
C     RISK RESERVE NEGATIVE RECØRD FREQUENCY ØF
C     RUIN AND START ANØTHER BUSINESS
  250 IREPS = IREPS + 1
      ICØUNT = 0
      ST = 0.0
      RR = RRØ
C     IF NUMBER ØF ITERATIØNS EQUALS N STØP
      IF (IREPS.EQ.N) GØ TØ 350
      GØ TØ 100
C     ØTHERWISE THE BUSINESS CØNTINUES AND
C     ANØTHER CLAIM IS AWAITED
  300 IF ((ICØUNT/2)*2.NE.ICØUNT) GØ TØ 150
      GØ TØ 100
C     RECØRD RESULTS AND PRINT-ØUT
  350 RELFRQ = FLØAT(IRUIN)/FLØAT(N)
      WRITE (6,400) RELFRQ
  400 FØRMAT(1H1,20X,'RELATIVE FREQUENCY ØF RUIN IN
    1 PERIØD T'/(8E15.6))
      JØBNØ = JØBNØ + 1
      IF (JØBNØ.EQ.JØBS) GØ TØ 500
      GØ TØ 10
  500 STØP
      END
```

DISCUSSION OF PRECEDING PAPER

JOHN A. BEEKMAN:

Professor Seal is to be congratulated on a fine contribution to the literature of collective risk theory and simulation theory. His techniques should be extremely useful in approximating ruin functions for short-time periods.

I feel it appropriate to make a few comments on some of the mathematical papers devoted to risk theory. In addition to creating and analyzing models for a risk business, they at times derived useful formulas which are very inexpensive to apply. Thus, in some cases, they produced formulas the application of which would take several hours of desk calculations as opposed to several hours of expensive computer time.

Let me illustrate from my own experience. Although some of the derivations in my *TSA* papers were a little involved, some of the results can be easily applied. Thus the approximation of the long-term ruin function

$$\psi(u) = \lim_{T\to\infty} \psi(u, T) ,$$

where $\psi(u, T)$ is equal to $v(u, T)$ in Dr. Seal's notation by a function of the incomplete gamma function (see my paper [*TSA*, Vol. XXI] and the forthcoming discussion by Professor Newton Bowers), can be performed by anybody with a knowledge of *undergraduate* statistics. Furthermore, in the examples considered, it has proved very accurate and superior to the classical 1926 Lundberg approximation. As evidence of this, let me recapitulate the four examples in my paper and my review of the discussions.

EXAMPLE 1: $P(x) = 1 - e^{-x}(x \geq 0)$. As pointed out in Mr. Bowers' discussion of my paper, the approximation gives the exact answer.

EXAMPLES 2 AND 3. In these two examples the approximate values of $\psi(u)$ computed by the method developed in that paper and discussion differ by only a small amount from the Lundberg approximation, as more correctly stated in my review of the discussions. Incidentally, the use of my form of the approximation rather than Bowers' form in the figures is not significant because of the large u values. In fact, the differences would have been smaller if Bowers' formula had been used.

EXAMPLE 4. Fire insurance distribution from Cramèr (referenced in my review of the discussions of the paper). My review points out a significant improvement in the Beekman-Bowers approximation over the

Lundberg approximation and, moreover, errors less than 3 per cent for a wide range of initial capital values, as compared with the exact values computed from an integral equation.

This is *not* to suggest that my and Professor Bowers' formula for $\psi(u)$ should be used to approximate $\psi(u, T)$. The differences can be significant for small T's. Thus the values of $\psi(u)$ comparable to Table 1 of Professor Seal's paper are shown in the following tabulation:

η	RUIN PROBABILITIES: $\psi(x)$		
	x		
	0	5	10
0.00......	1.000	1.000	1.000
.05......	0.952	0.749	0.591
0.10......	0.909	0.576	0.366

I believe that line 1 of the tabulation can be dismissed, because it has no practical significance. Furthermore, I believe that Dr. Seal's number of expected claims of only 20 is much too low for insurance companies and even low for self-insured employee fringe benefits. For this and other reasons given below, it would be very interesting to see (for $\eta = 0.05$ or 0.10) values of $\psi(x, T)$ for $x = 0$, 5, 10, and $T = 40$, 60, 80, 100. As Professor Seal points out, this distribution of claims $[P(x) = 1 - e^{-x}]$, where $x \geq 0$, "underestimates the probability of claims that are several times the mean claim." Therefore the approach of $\psi(u, T)$ to $\psi(u)$ as T becomes larger probably is slower than it would be in a more realistic case. Nevertheless, the calculations proposed would yield very interesting results, I feel. Considerable theoretical work on this problem has been done, as can be judged from pages 291–95 of C. O. Segerdahl's paper, "A Survey of Results in the Collective Theory of Risk."[1] A practical study of the type proposed would appear to be a worthy complement to the theoretical work. A graph showing the approach of $\psi(u, T)$ to $\psi(u)$ as T gets larger would be very interesting.

My main concern with simulation is its cost. A few thoughts about its potential errors, however, are in order for the unwary. You must be careful, as Dr. Seal was, to perform enough trials that your estimated standard error is small. You must also realize, as Dr. Seal does, that there is a positive probability that the true but unknown ruin probability will fall out-

[1] Ulf Grenander (ed.), *Probability and Statistics*, Harold Cramèr volume.

side any confidence band. Thus there is a 1 per cent chance that the true value will fall outside the band [Estimate −2.58 (Standard Error); Estimate +2.58 (Standard Error)]. This is different from numerically approximating an analytic expression for $\psi(u, T)$, such as is given for Dr. Seal's Example 1 in formula (2) (page 295 of the Segerdahl paper). The error analysis of such an expression could be phrased as follows: There is a 100 per cent chance that the error is less than so and so. The reader might wish to consult page 72 of my 1966 *Skandinavisk Aktuarietidskrift* paper for such an approximation. I must admit, however, that error analysis of analytic approximations may be difficult. Thus I am unable to give an error bound on my approximation of $\psi(u)$. I do feel, though, that it is quite accurate and have previously referred to four examples in that regard.

I am currently doing research on approximating $\psi(u, T)$, the finite time ruin function, in a fashion similar to that for $\psi(u)$. If successful, it would be considerably cheaper than simulation, as it would involve only a few desk calculations.

Aside from the cost aspect, I am a firm believer in computer solutions. I have done considerable work in approximating differential equations by difference schemes on a computer.

I hope that Professor Seal's excellent paper helps to spur the greater application of collective risk techniques.

(AUTHOR'S REVIEW OF DISCUSSION)

HILARY L. SEAL:

In answer to Professor Beekman's criticism, I should explain why I calculated the probability of ruin for this little casualty company over 20 time intervals.

In the first place, I take an interval of time as being the average time required to have a claim. It is odd that actuarial mathematics of the theory of ruin is identical with the mathematics developed by telephone engineers to find the length of time one must wait for the line to be connected. In telephone practice, time is measured in units equal to the average time between calls coming into the telephone exchange. This is analogous to our own measure of time as the average time required for a claim.

Why, then, did I use only 20 time intervals as the period over which to calculate the probability of ruin of a casualty insurance company? There were two reasons: (1) If you are going to go broke, you are going to go broke quickly! In other words, suppose that the probability of eventual ruin is, say, 1 in 100 but that the probability of being ruined before you have 20 claims is 80 per cent. The forever probability is useless in view of the latter. (2) To say that the number of 20 time intervals is small is a relative matter. Suppose the 20 claims happen to be 20 Boeings that have just been insured. These 20 claims would be crucial to company management.

Finally, I would like to say that I think Professor Beekman's idea of calculating the probability of ruin over increasingly larger intervals of time is excellent. These probabilities would show very clearly how quickly asymptotic theory takes over. (Telephone engineers call this theory "steady state" theory.) I think it is a very good idea to see at what point the two types of probabilities do link up.

I only drew one set of random numbers for all the companies, so that the same claim experience would operate in the different rate-making circumstances. I assumed that every company started in business with sufficient capital and surplus that it should be ruined in its first year with a probability of 1 in 1,000.