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EXPECTED PROFIT FORMULAS

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ABSTRACT

A five-factor expected profit formula is presented, with first-year lapse rate and amount of insurance as variables. The formula is appropriate for a single age-plan-sex cell.

INTRODUCTION

EXPECTED profit, however defined, has interested actuaries for many years. Usually expected profit for a new individual life insurance policy has been calculated as a function of issue age, plan, sex, mortality rates, interest rates, lapse rates, expense factors, amount of insurance, cash values, and the like. If, however, the expected profit is calculated for a single age-plan-sex cell, all of the variables but two may be thought of as constants. That is, once issue age, plan, and sex have been identified for a cell, then one set of mortality and interest rates, one set of expense factors, one set of cash values, and so on, will be used for the cell. Expected profit may then be expressed as a function of lapse rates and amount of insurance. This function may be reduced to a five-factor formula. (See "Theory" below.) The five-factor expression of expected profit may be helpful when one considers the profitability to the company of individual policies and the business produced by specific agents or agencies.

THEORY

There are many expected profit formulas available to the actuary today. A fairly simple one is shown below.

$$P/M = \frac{1}{A}\sum_{t=1}^{\omega-x}\{[(A\ {}_tB + {}_tF)(1 - {}_te) - A\ {}_te' - {}_te'']_{t-1}E'_x$$

$$- (A\ {}_tDB + C^d)(1 + \tfrac{1}{2}i_t)q_{[x]+t-1}\, v_t\ {}_{t-1}E'_x \qquad (1)$$

$$- (A\ {}_tCV + C^s)w_t\, v_t\ {}_{t-1}E'_x\}\ ,$$

where ${}_0E'_x = 1$ and ${}_tE'_x = v_t(1 - q_{[x]+t-1} - w_t)_{t-1}E'_x$.

Definitions:

P/M = Expected profit per $1,000 at issue;
A = Amount of insurance in thousands;
t = Policy year;
x = Issue age;
${}_tB$ = Basic annual premium per thousand for policy year t;
${}_tF$ = Policy fee per policy for policy year t;
${}_te$ = Expense per dollar of annual premium for policy year t;
${}_te'$ = Expense per thousand for policy year t;
${}_te''$ = Expense per policy for policy year t;
C^d = Cost per death claim paid;
C^s = Cost per surrender value paid;
${}_tDB$ = Death benefit per unit during policy year t;
${}_tCV$ = Cash value per unit at end of policy year t;
i_t = Interest rate for policy year t;
$v_t = 1/(1 + i_t)$;
$q_{[x]+t-1}$ = Mortality rate for policy year t for issue age x;
w_t = Lapse rate at end of policy year t.

For a single age-plan-sex cell this formula may be written as a function of lapse rates and amount of insurance, or

$$P/M = f(w_t, A) . \tag{2}$$

This is possible because all the other factors are fixed, once the age, plan, and sex of the cell are established.

If each renewal lapse rate (w_t) is assumed to be a function of the first-year lapse rate, so that

$$w_t = g_t(w_1) + h_t , \tag{3}$$

where g_t and/or h_t may be zero, then for a single age-plan-sex cell the expected profit per $1,000 at issue may be written as a function of two variables, first-year lapse rate and amount of insurance in thousands, or

$$P/M = f(w_1, A) . \tag{4}$$

For an $(\omega - x)$-year time period the terms of formula (1) may be rearranged to produce the following:

$$\begin{aligned} P/M &= a_0 + a_1w_1 + a_2(w_1)^2 + a_3(w_1)^3 + \ldots + a_{\omega-x}(w_1)^{\omega-x} \\ &\quad + \frac{1}{A}\left[b_0 + b_1w_1 + b_2(w_1)^2 + b_3(w_1)^3 + \ldots + b_{\omega-x}(w_1)^{\omega-x}\right] . \end{aligned} \tag{5}$$

If only the terms through the second order in the two variables (w_1, $1/A$) are used, the following formula results:

$$P/M = a + bw_1 + c(w_1)^2 + d\left(\frac{1}{A}\right) + e\left(\frac{w_1}{A}\right), \tag{6}$$

where a, b, c, d, and e are constants for each cell. In most cases very little accuracy is lost by dropping the third- and higher-order terms, so that the five-factor formula (6) reproduces the many-factor formula (1) or formula (5) to a surprising degree of accuracy.

If the band method is used to grade premiums by size, or if commissions, mortality, or other factors vary with amount bands, a separate five-factor formula is needed for each band within a cell.

APPLICATION

The five-factor formula may be simplified for certain uses. Formula (6), if applied to riders that carry neither policy fee nor per-policy expenses, reduces to a three-factor formula:

$$P/M = a' + b'w_1 + c'(w_1)^2 . \tag{7}$$

Also, if no lapse-rate variation is expected within a cell (perhaps for single premium policies), formula (6) becomes a two-factor formula:

$$P/M = a'' + d''\left(\frac{1}{A}\right). \tag{8}$$

Under this condition, formula (8) will reproduce formula (1) exactly.

The key to the five-factor formula is the derivation of the coefficients a, b, c, d, and e for each age-plan-sex cell. Several methods are possible. For example, if there is concern regarding the accuracy of formula (6), P/M could be calculated for many points within each cell and regression analysis used. Correlation coefficients in the high 0.90's might be expected for almost all cells. Table 1 illustrates results of this method and shows that the correlation coefficient for the example is 0.99993.

With another method, detailed present values are calculated for only five points within each cell and simultaneous equations are used to produce the coefficients. Table 2 illustrates results obtained using this method. The correlation coefficient for this example is 0.99986.

The five-factor formula sheds some light on policy-fee theory. For a given first-year lapse rate, the five-factor formula (6) becomes a two-factor formula (8). Expected profit per \$1,000 may decrease or increase with increasing A, depending on the sign of d''. In either case P/M will approach a'' asymptotically as A increases. If, for the lapse rate assumed,

d'' equals zero, then the expected profit per $1,000 is constant for all amounts, perhaps an indication that the policy fee is "perfect." Such a lapse rate may be obtained from formula (6) by setting the last two terms equal to each other with opposite signs and solving for w_1. That is,

$$d\left(\frac{1}{A}\right) = -e\left(\frac{w_1}{A}\right) \tag{9}$$

or

$$w_1 = -\frac{d}{e}. \tag{10}$$

TABLE 1

EXAMPLE OF REGRESSION ANALYSIS TECHNIQUE*

A	w_1	P/M		
		Detailed Formula	Regression Formula†	Difference
2	0.07	29.17	29.01	0.16
2	0.12	24.06	24.12	−0.06
2	0.15	21.12	21.26	−0.14
2	0.20	16.50	16.62	−0.12
2	0.30	7.91	7.80	0.11
5	0.07	20.92	20.89	0.03
5	0.12	17.56	17.53	0.03
5	0.15	15.61	15.58	0.03
5	0.20	12.49	12.47	0.02
5	0.30	6.74	6.72	0.02
10	0.07	18.11	18.18	−0.07
10	0.12	15.33	15.33	0.00
10	0.15	13.70	13.69	0.01
10	0.20	11.13	11.09	0.04
10	0.30	6.30	6.35	−0.05
50	0.07	16.00	16.02	−0.02
50	0.12	13.60	13.57	0.03
50	0.15	12.23	12.18	0.05
50	0.20	10.07	9.99	0.08
50	0.30	6.04	6.06	−0.02
300	0.07	15.48	15.57	−0.09
300	0.12	13.18	13.21	−0.03
300	0.15	11.89	11.86	0.03
300	0.20	9.77	9.75	0.02
300	0.30	5.94	6.00	−0.06
Total	....	350.85	350.85	0.00

* The figures are illustrative and do not necessarily represent actual experience.

† Regression formula: $P/M = 19.01751 - 52.79422w_1 + 31.25709(w_1)^2 + 34.20801/A - 101.97365w_1/A$. Correlation coefficient = 0.99993.

The five-factor formula has another interesting characteristic. For any given amount (A), there may be a realistic first-year lapse rate which produces a maximum or minimum P/M. Such a w_1 may be identified from formula (6) by setting the partial derivative of P/M with respect to w_1 equal to zero and solving for w_1. That is,

$$\frac{\partial(P/M)}{\partial w_1} = b + 2cw_1 + \frac{e}{A} = 0 \qquad (11)$$

TABLE 2

EXAMPLE SHOWING RESULTS OF SOLVING SIMULTANEOUS EQUATIONS FOR FIVE SELECTED POINTS

A	w_1	P/M		
		Detailed Formula	Simul. Eq. Formula†	Difference
2	0.07	29.17*	29.17	0.00
2	0.12	24.06	24.30	−0.24
2	0.15	21.12	21.44	−0.32
2	0.20	16.50	16.79	−0.29
2	0.30	7.91*	7.91	0.00
5	0.07	20.92	20.90	0.02
5	0.12	17.56	17.57	−0.01
5	0.15	15.61	15.64	−0.03
5	0.20	12.49	12.53	−0.04
5	0.30	6.74	6.72	0.02
10	0.07	18.11	18.14	−0.03
10	0.12	15.33	15.33	0.00
10	0.15	13.70*	13.70	0.00
10	0.20	11.13	11.10	0.03
10	0.30	6.30	6.32	−0.02
50	0.07	16.00	15.94	0.06
50	0.12	13.60	13.53	0.07
50	0.15	12.23	12.15	0.08
50	0.20	10.07	9.96	0.11
50	0.30	6.04	6.01	0.03
300	0.07	15.48*	15.48	0.00
300	0.12	13.18	13.16	0.02
300	0.15	11.89	11.83	0.06
300	0.20	9.77	9.73	0.04
300	0.30	5.94*	5.94	0.00
Total		350.85	351.29	−0.44

* Points used in simultaneous equations.

† Simultaneous equations formula: $P/M = 18.85022 - 51.39958w_1 + 27.73821(w_1)^2 + 34.74562/A - 102.59732w_1/A$. Correlation coefficient = 0.99986.

Note that in this example many other sets of five points might have been selected.

or

$$w_1 = \frac{-b - e/A}{2c}. \tag{12}$$

Application of the above principles to the example in Table 1 results in a first-year lapse rate, for constant P/M, of 0.34 (i.e., for "perfect" policy fee) and also results in the values shown in the accompanying tabulation.

Amount in Thousands (A)	First-Year Lapse Rate for Minimum P/M
2	1.66*
5	1.17*
10	1.01*
50	0.88
300	0.85

* Impossible.

Formula (6) may be used to develop expected profit for a policy and, from that, for the total business produced by an agent or an agency, provided that an expected first-year lapse rate is assigned to each policy. Expected lapse rates are discussed in an article by N. F. Buck entitled "First-Year Lapse and Default Rates" (*TSA*, Vol. XII). Expected first-year lapse rates may be based on such characteristics as issue age, sex, premium size, and premium mode, whether the policy is owned by an old or a new policyholder, and so on.

In conclusion, a simple five-factor formula for expected profits may be more convenient in certain analyses than the usual present value formula.

DISCUSSION OF PRECEDING PAPER

WILLIAM C. KELLIE:

1. *Derivation of Factors*

Five factors must be derived for each age-sex-plan cell; if the band system is used, these factors must be developed for each age-sex-plan combination within each band. These factors can be developed by applying linear regression analysis to several exact values or by determining exact values for five different points within a cell and solving the resulting set of simultaneous equations. This analysis would be required for each age-sex-plan cell within each band of policies. The author demonstrates the increased accuracy resulting from linear regression analysis, so that this approach would be preferred. Whether linear regression or the solution of simultaneous equations is used, a substantial investment is necessary to develop the needed factors. Also, these factors will change with changes in interest, mortality, and withdrawal rates.

2. *Application by Size of Company*

The large number of exact values required for derivation of the factors requires an efficient method of calculation, as by a computer. If an in-house computer is available (as is the case in most large companies), the investment required for this system might be better spent developing a computerized tool to generate exact present values. We at Connecticut General have included an internal rate of return calculation along with the book profit calculation.

3. *First-Year Withdrawal Rate*

Derivation of a first-year withdrawal rate to minimize present value of profit seems to have little use. Also, it would appear that a withdrawal rate of 100 per cent in the first year would always be the value that minimized present value of profit.

Dependence on first-year withdrawal rate may be questioned. The approach taken by Mr. Lewis assumes that the first-year lapse rate is significantly greater than lapse rates for renewal years or that the lapse rates for the first several years are high; if these relationships are absent, the approximation deteriorates. To be more specific, Mr. Lewis' formula (6) is derived from formula (5) by dropping the third- and higher-order terms. This implies that profits for these durations are negligible or that consideration for these profits is included in the factors *a–e*. If the decrements due to death and withdrawal are small during the first policy durations for a policy with a normal first-year surplus drain and later emergence of profits, then the value of these profits would not be negligible and may not be adequately reflected in the factors *a–e*. Also, in the description of formula (2), it is stated that the only remaining variables are w_t and A. If early withdrawals are small while later withdrawals are large, then variations in the later withdrawals may affect expected profits. In the extreme case where $w_1 = 0$, the expected profits for any given value of A are constant.

4. *Conclusion*

The approach is interesting and may be useful where computerized facilities for calculating exact present values of profit at issue are not readily available.

REA B. HAYES:

The five-factor formula for expected profit developed by James L. Lewis, Jr., does seem to convey more information than would a single expected profit calculation or asset share developed by conventional means. I have used the formula to study one interesting situation which may have some general appeal in studying margins for individual agencies. Given any set of asset share or expected profit formulas, what is the effect of varying only first-year persistency? The approach treats all renewal rates as constants. The more general case in which they are functions of the first-year lapse rate would be difficult to handle, although the case where only the first few are functions of the first-year lapse rate and the remainder are constants would probably be a manageable extension of the ideas expressed here.

In general, we split all items entering into the profit formula into first-year items and renewal items. All renewal items, whether income or outgo, can be valued or revalued by the following formula:

$$\frac{1 - q_{[x]} - w_1'}{1 - q_{[x]} - w_1} V ,$$

where $q_{[x]}$ and w_1 have the meaning defined by Lewis, w_1' is the variable first-year termination rate that we are considering, and V is any renewal item valued from the second year on, according to any scale of lapse and withdrawal rates being studied. In general, I took such rates as were defined in the paper from asset share and profit studies we had already made and then studied the variations caused by changes in w_1' from 0.1 to 0.6, useful outside limits to what one might expect from different agencies or agents.

Each first-year item was given separate consideration to decide how it would vary with w_1'. Some of our conclusions might be of interest. Death claims were considered independent, which is consistent with viewing the mortality rate as an absolute one in the formula and can be rationalized by saying that those who are about to die will not lapse. Lapses and surrenders would be considered proportional to the first-year lapse rate. Those companies who participate in the Life Office Management Association expense study will have reliable values for the expense of a surrender. Unfortunately, the LOMA study does not separate out the expense of a lapse. We guessed that it might be between one-fourth and one-half the expense of a surrender. Although the original Lewis formula does not include dividends, we treated them as items of outgo. This item requires special handling, depending on whether the dividend is contingent on payment of full second-year premium and on the company's practice with respect to mortuary dividends during the first year. On expense items, we recognized that there would be some partial savings with increasing lapse rates but that many expenses are incurred in full in connection with issue.

To illustrate the kind of results obtained, we show in the accompanying tabulation the formula for one age-plan cell (age 35, ordinary life plan) and the results produced for various values of w_1'.

FORMULA: $5.42 - 6.49w_1' - 12.79/A - 18.24w_1'/A$

	Value of First-Year Lapse Rate					
	0.1	0.2	0.3	0.4	0.5	0.6
P/M for $A=10$.......	3.31	2.48	1.65	0.82	−0.02	−0.85

(AUTHOR'S REVIEW OF DISCUSSION)

JAMES L. LEWIS, JR.:

Mr. Kellie's comments regarding the derivation of coefficients and the application by size of company are helpful. However, if one is making an agency profit analysis, it may be desired not to make changes in basic assumptions for five or ten years, in order to isolate agency trends from changes in assumptions.

Mr. Kellie has also suggested that the first-year withdrawal rate to minimize the present value of profit per unit is only a curiosity. Perhaps he is correct. However, this rate might be used as a testing tool. Also, I am not sure that a 100 per cent first-year withdrawal rate will always minimize the present value of profit per unit.

Mr. Kellie said that if early withdrawals are small while later withdrawals are large, then variations in the later withdrawals may affect expected profits. While this conclusion may be true, I am not sure that the logic leading to the conclusion is sound. It seems to me that the third- and higher-order terms dropped in formula (6) are not particularly related to profits for any specific duration.

Mr. Hayes has demonstrated an innovative use of the five-factor expected profit formula by converting it to a four-factor formula for the study of first-year persistency by agency. He pointed out that dividends may be included as items of outgo.

I hope that others will find the five-factor expected profit formula useful.