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UTILITY THEORY

New York Regional Meeting

I. How can utility theory be used as a tool in general management?
II. How can utility theory be applied in the management of the life insurance business?
III. What role does utility theory play in reinsurance?
IV. What light can utility theory shed on consumer behavior with respect to insurance?

CHAIRMAN EDWARD A. LEW: The basic notions of utility were so clearly set forth some two hundred and thirty years ago that the subject is perhaps best introduced with a little history. Specifically, there appeared in 1738 in the proceedings of the Imperial Academy of Sciences of what was then St. Petersburg a paper entitled "Exposition of a New Theory of the Measurement of Risk," by Daniel Bernoulli, a member of the famous Swiss family of distinguished mathematicians. This paper contained the following statements:

> No valid measurement of the value of a risk can be obtained without consideration being given to its utility, that is to say, the utility of whatever gain accrues to the individual or, conversely, how much profit is required to yield a given utility. However, it hardly seems plausible to make any precise generalizations, since the utility of an item may change with circumstances. Thus, though a poor man generally obtains more utility than does a rich man from an equal gain, it is nevertheless conceivable, for example, that a rich prisoner who possesses two thousand ducats but needs two thousand ducats more to repurchase his freedom will place a higher value on a gain of two thousand ducats than does another man who has less money than he. Though innumerable examples of this kind may be constructed, they represent exceedingly rare exceptions. We shall, therefore, do better to consider what usually happens, . . .

and Bernoulli went on to say that the utility resulting from any small increase in wealth would be inversely proportional to the quantity of goods previously possessed.

Bernoulli was trying to emphasize that the utility of money depended on a person's wealth and made much of the point that a small amount of money is more precious to a pauper than it is to a rich man. He proceeded to develop this concept of diminishing marginal utility, which has been for many years a key principle in economics. He also brought out, in discussing the St. Petersburg problem (betting with a coin to receive 2^{n-1} units if heads appeared first on the nth toss of the coin), that, even when the mathematical expectation of a prospective gain is infinite, its utility to a player of limited means is quite small.

What does this add up to? In my judgment, utility theory endeavors to explain or guide preferences in risk-taking in situations where the mathematical expectation does not provide an adequate criterion for making decisions. Another way of looking at this issue is that rational decisions in the face of uncertain risks involve objectives other than those maximizing the mathematical expectation of a gain.

Bernoulli was, as a matter of fact, also concerned with the question of why a man would insure a ship when the premium he was required to pay was substantially in excess of the mathematical expectation of the loss. He reasoned that rational people act so as to maximize not expected gain but expected utility. A person to whom diminishing marginal utility applies will, as a rule, be willing to pay for insurance more than the mathematical expectation of the loss in order to be freed from what was then referred to as the evil of uncertainty.

Modern theory of utility was developed by John Von Neumann and Oskar Morgenstern in their opus *The Theory of Games and Economic Behavior.*[1] On the basis of a few relatively simple theorems, they demonstrated the existence of a utility function whose expected value controls choices and showed that such a function was measurable. This was a key issue that was debated at some length.

In a more practical context, utility theory has been used recently to explore businessmen's attitudes to risk taking in situations which allow a number of alternative courses. These empirical approaches suggest that many decision makers—and that includes some of us—shun even carefully weighed risks and avoid decisions offering a good chance for large gains when there is a possibility of significant loss.

Utility theory has obvious applications in reinsurance, which hinges on the primary insurer's attitude to large risks. Other interesting applications of utility theory occur in connection with disaster insurance.

Currently an important new area for applications of utility theory lies in the behavior of insurance buyers. In the case of life insurance, the buyer's utility is affected by emotional factors, such as fear of death and the tendency to undervalue unwelcome probabilities. In the case of health insurance, the buyer's utility can be influenced by the offer of an appropriate deductible.

Professor John Hammond will give us a more detailed overview of utility theory, Professor Karl Borch will deal with applications of utility

[1] John Von Neumann and Oskar Morgenstern, *The Theory of Games and Economic Behavior* (Princeton, N.J.: Princeton University Press, 1947).

theory to the management of the insurance business, Professor Robert Miller will take up the role of utility theory in reinsurance more intensively, and Paul Kahn will expand on utility theory as a guide to insurance purchasing.

PROFESSOR JOHN S. HAMMOND III:* As the leadoff speaker in this series of four presentations, my assignment is to give you a broad overview of utility theory, including (1) what utility theory is, (2) how it is used in business, (3) important implications to practitioners and managers, and (4) some potential application areas in insurance. While little specifically will be said about insurance until the end of the paper, all the material is applicable to the use of utility in insurance.

I. WHAT UTILITY THEORY IS

What is our subject? Before saying what it is, let us state the context in which it applies. It is applicable to decision making under uncertainty or under risk—in other words, in those situations where the outcome of any course of action is unknown at the time a decision must be reached. Under such situations one must be concerned about a criterion for choice. How does one choose a course of action that is appropriately conservative?

To do so, one must have a procedure in mind for analyzing the decision problem. The significant steps include the following:

1. Defining the alternative courses of action
2. Defining the uncertainties associated with each course of action and assigning probabilities to them
3. Describing the ultimate consequences of each potential sequence of choices and outcomes (usually in economic terms)
4. Defining a risk-taking attitude
5. Choosing a course of action that is optimal, given the risks and risk-taking attitude

Our subject can now be defined: It is the precise definition of the desired risk-taking attitude and its appropriate incorporation into the decision. As can be inferred from the over-all procedure outlined above, utility theory is a part of the more general field called statistical decision theory [7, 9].

II. USING UTILITY THEORY

A. OVER-ALL PROCEDURE

Assuming that a problem has already been formulated in terms of decision options, probabilities, economics, and the like, let us focus on

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the activities required in using utility theory in its solution. Broadly speaking, they are the following:

1. Definition of a risk-taking attitude in the form of a utility curve
2. Using that curve to convert economic consequences into utilities
3. Selecting that course of action that has the highest expected utility

B. USING A UTILITY CURVE FOR DECISION MAKING

We will examine these steps out of order, first showing what to do with a utility curve after it is obtained and then concerning ourselves with the process of obtaining it. The discussion is aimed at enhancing your intuitive understanding of how utility theory captures a risk-taking attitude and incorporates it into a decision analysis and at giving you a feel for the major steps of the process. The exact procedural and mathematical details are explained elsewhere [3, 5, 8, 10].

The last two steps are the implementation of a result frequently attributed to Von Neumann [12]. Briefly, if $\tilde{x}$ represents the net economic consequence of a decision, which is uncertain at the time the decision is made, $u(x)$ represents the decision maker's utility function for consequences x, and $F(x)$ and $G(x)$ represent, respectively, the cumulative probability distributions of the uncertain outcome of two ventures between which a choice must be made, then the decision maker should prefer the first venture to the second if

$$\int u(x)dF(x) \geq \int u(x)dG(x) \; .$$

This result says that the decision maker should make that choice which maximizes his expected utility.

C. DEFINITION OF RISK-TAKING ATTITUDE

Examination of the first step should make clear the way in which the utility curve captures the decision maker's risk-taking attitude. We will start the explanation with an extremely simple example.

Suppose that you as a private individual owned the rights to the following attractive but mythical business venture: Depending on a certain business event that has a 50 per cent chance of occurring, you will make $50,000; if it fails to occur, you will lose $5,000.

What would the venture be worth to you? Its mathematical expectation is $22,500. Would you sell it for $22,000? Most probably your answer is yes. How about $20,000? Most of us, being somewhat conservative, would take an amount less than the expected value in order to be *sure* of gaining some significant sum.

It should be clear that the value you assign to the venture tells a great deal about your risk-taking attitude. For example, if you are willing to sell for $15,000, you are more averse to taking risk than someone who would hold out for $20,000. In other words, a sure amount of $15,000 is preferable to you to the risk associated with the venture, whereas the other fellow is saying that, if he is offered only $15,000 (or $17,500, or $19,900, for that matter), he would rather take his chances with the venture.

We can exploit this basic idea that the "value" of a venture is a measure of a man's risk-taking attitude to obtain a precise indication of his overall risk-taking attitude. The first requirement is that we give a fancy name to this value and a more precise definition. A certainty equivalent for a venture will be defined as the rock-bottom price that the decision maker will accept for the venture. The certainty equivalent is really an indifference point; if he were offered just a little less than it, he would prefer to hold on to the venture and run the risk associated with it; if he were offered just a little more than his certainty equivalent, he would consider selling out.

From the example, it is clear that the decision maker's certainty equivalent for a venture is not necessarily equal to the mathematical expectation of the venture. In fact, for conservative people it is less than the mathematical expectation. Decision theorists assign the name "risk premium" to the difference between the mathematical expectation and the certainty equivalent [6], where the word "premium" is used in a different sense from its use in insurance. One can say that a conservative man will have a positive "risk premium" for a given venture and the larger the "risk premium," the greater his conservativeness or aversion to risk.

An individual's utility curve can be derived by asking him to give his certainty equivalents for a few well-chosen, simple, two-outcome ventures and then using these to plot a smooth utility curve, which summarizes his risk-taking attitude toward all ventures whose consequences fall within the range covered by the curve. For example, if the decision maker is choosing among ventures whose possible consequences range between a loss of $5,000,000 and a gain of $10,000,000, his utility curve must encompass that range. The process of obtaining the curve (which we do not discuss in detail here) requires that careful checks be made before the curve is used, to insure that it actually represents the decision maker's intended risk-taking attitude.

Assessing a utility curve is a soul-searching process that is especially difficult the first time an individual attempts it. At first he may find that it is hard to decide on certainty equivalents for ventures, that there are some logical inconsistencies in his responses, and that his responses to the same question may vary noticeably from one time to the next. However, these difficulties are the very reason that formal consideration of a risk-taking attitude is important. One needs to know what his risk-taking attitude is, one does not want to have logical inconsistencies mar his decisions, and one does not wish his attitude to fluctuate wildly from day to day.[1] Fortunately, with some effort, the difficulties can be resolved, and one's ability to articulate his risk-taking attitude increases greatly with experience.

Before going on, let us mention one of several practical problems in the use of utility. A utility curve of the sort described here is a curve for economic consequences measured at a given point in time, which implies that one wants to judge the merits of a decision as of a single point in time. For example, one might wish to derive a utility curve for the net cash flows of a company between the present and December 31, 1969. When one is choosing among alternatives which have uncertain payoffs widely separated in time (in other words, when one wants to look at the merits of a decision at several points in time), the theory becomes much more complicated than what we have described here [1, 4]. Nonetheless, the simple theory gives considerable insight into the more complicated problem.

III. IMPORTANT IMPLICATIONS TO PRACTITIONERS AND MANAGERS

Having briefly described what utility is and how it fits into decision making under uncertainty, let us discuss some of the management implications of this important concept, each of which is of significance in insurance.

A. UTILITY IS A SUBJECTIVE MEASURE

It should be obvious from the example used that attitude toward risk is subjective, clearly a case of "one man's meat is another man's poison." It is impossible to specify a utility function for an individual that is in any sense "objective" or "correct." In fact, one notes a considerable difference among the utility curves of individuals [2, 11], which is in part a reflection of the fact that different people have different objectives and situations.

B. WHAT CAN WE LEARN ABOUT AN INDIVIDUAL FROM HIS UTILITY CURVE?

One can tell a great deal about the risk-taking attitude expressed by a decision maker's utility curve by a rather casual examination of the

[1] While one's attitude should not fluctuate greatly from day to day, change in attitude over longer time periods is appropriate as one's situation and opportunities change.

curve. There are three generically different types of curve, corresponding to three generically different postures toward risk. These are illustrated in Figure 1 and described in the following way:

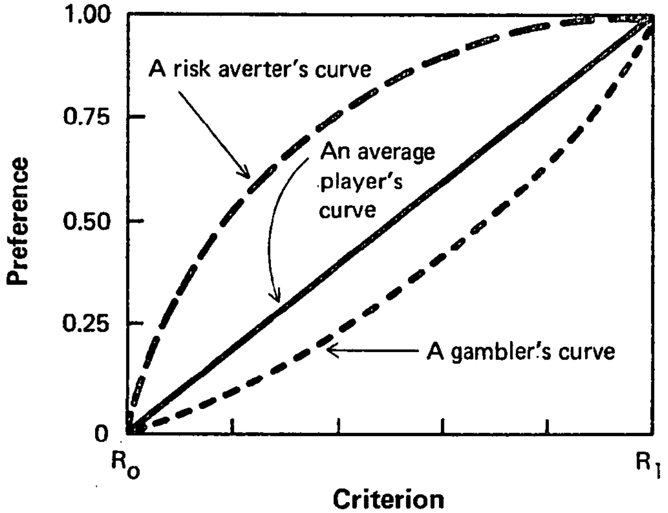

FIG. 1

1. *The conservative man.*—The conservative man's curve is characterized by the fact that it is concave as viewed from below. This is the type of curve observed most commonly in practice; most of us are conservative to a degree.

2. *The average player.*—This person's curve is a straight line, which means that he wishes to play the long-run averages. A linear preference curve is frequently found when a man makes decisions whose consequences are small compared to the total asset position of the company, as is frequently the case in large corporations. Maximizing expected utility when a utility curve is linear is equivalent to maximizing the mathematical expectation of the economics, and thus straight expected-value analysis suffices for such a person.

3. *The gambler.*—This person is the rarest of the three types; his utility curve is convex when viewed from below, and he is, in effect, willing to pay a premium above the mathematical expectation for the "thrill" of gambling or for other reasons.

Sometimes utility curves are observed that are a composite of two or more of the types just described. For example, a curve might show risk aversion in its upper region and gambling inclinations in its lower region, resulting in a curve shaped like an S.

In addition to being able to classify curves into generic types, one can also speak about the degree to which a curve falls into a given class. For example, among conservative person's curves there are degrees of conservatism. Roughly speaking, this is measured by the degree of concavity of the individual's utility curve. More strictly speaking, the degree of conservatism reflected by any point on a utility curve is measured by the negative of its second derivative divided by its first derivative [6].

Clearly, what a man's utility curve tells about his risk-taking attitude is useful information in hiring and placement with him within an organization and in motivating and dealing with him.

C. COMMUNICATING ATTITUDE TOWARD RISK IN A COMPANY

Just as individuals have risk-taking attitudes, so do corporations. A common difficulty in companies, however, is that top management has not clearly articulated a risk-taking posture for the company, and as a result there are probably as many different "company positions" as there are decision makers. At other times the risk-taking posture is stated implicitly in bits and pieces through institutional rules, such as "Don't insure an individual's life for an amount exceeding X." The difficulty with such rules is that they get promulgated individually over time and soon become accepted as "gospel." In this vaunted position no one checks to see whether they may be inconsistent, in the sense that some rules may be too conservative relative to others, or reviews them to see whether they still fulfill their intended purpose in light of the company's changing situation.

Utility theory, on the other hand, potentially offers a way out of some of these difficulties. Since it clearly articulates a firm's risk-taking attitude, it implicitly determines which risks to take and which to avoid or, on the other hand, at what price a risk is worth taking. It also provides a vehicle for examining some of the institutional rules that have stood for so many years, to ascertain whether they are consistent with one another, and in some instances obviates the need for them.

Poor communication of risk-taking attitude is often due in part to the fact that top management itself does not know precisely what its attitude is. In such a situation, assessing a curve should remove much ambiguity in the minds of top management and subsequently elsewhere in the company.

Thus it is quite proper and potentially beneficial for corporations to have utility curves, just as individuals do. Such a curve would represent the attitude of top management as representatives of the firm's stockholders. A firm's utility curve will depend upon a number of factors, one of the more important of which would be its size. For example, one would expect that in most cases as a firm gets larger it would become less risk-averse, because it can afford to take bigger risks.

D. DIVIDING THE RESPONSIBILITY FOR THE ELEMENTS OF A DECISION

If a corporate utility curve is promulgated, an interesting division of responsibility in decision making results. Part of the ingredients of a decision is supplied locally, and part is supplied from above. Since the individual manager is closest to the decision problem, and has access to the data and the judgment necessary to specify the alternatives, probabilities, and economics, it is probably most advantageous for these to be supplied by him. On the other hand, the risk-taking attitude is a matter of policy, and thus it is properly supplied from above. Fortunately, decisions whose impact is of sufficient magnitude to warrant formal consideration of the company's risk-taking attitude usually are made near the top of an organization, so that the ingredients of the decision usually come from sources reasonably close in an organizational sense.

E. MANAGERS CAUGHT IN A SQUEEZE

There is reason for top management to be concerned that the risk-taking attitude that the individual manager uses in making decisions on behalf of the company is not the one that top management would like him to use. In the majority of cases the individual utility curve reflects considerably more conservatism than the company would consider desirable. In an article in the *Harvard Business Review*, Ralph O. Swalm [11] displayed some curves which illustrate this point. He concluded, "Our managers are surely not the takers of risk so often alluded to in the classical defense of the capitalistic system."

I suspect that the trouble lies in the control systems that we have set up to reward and punish our managers. These systems are generally hard on managers if they have short-run failures; at least, managers think the systems work that way. Moreover, the added reward to the manager for a huge financial success is perceived to be a relatively small increment to the reward for simply doing a good job.

A very important reason for this bias is that the control systems highlight a financial loss but fail to show a potential lost profit on ventures that the manager has avoided. As a result, he plays it safe, avoiding opportunities which have even minutely small probabilities of significant failures. Chances for big successes are usually passed up, too, since they are usually accompanied by chances for significant failures.

Let me give an example from insurance. I understand that some, if not many, casualty insurance companies judge and reward their branch office managers on the basis of their expense ratios (expenses/written premiums) and their loss ratios (losses/earned premiums). While their companies may be multimillion dollar or billion dollar companies, these managers may be responsible for a portion of business ranging from a few million to many millions, depending on the regions they service. Because the ratios on which they are measured are short term and because amounts of money which are small for the company can nonetheless make or break a man's performance record, their behavior tends to be much more risk-averse than the companies would desire. In one instance with which I am familiar, an insurance company considerably increased the minimum size of risk at which reinsurance could be considered in its casualty and liability lines. They soon found that branch managers were finding all sorts of ways to circumvent the rule and "bootleg" reinsurance.

The important point is that merely specifying a corporate risk-taking attitude more clearly is not sufficient to ensure that the risk-taking behavior will conform to that intended. One must determine ways to reward managers in a manner that is consistent with the desired risk-taking attitude, or they will find ways to subvert the system.

F. RISK AND ATTITUDE TOWARD RISK: THE DANGER OF DOUBLE COUNTING

In decision making under uncertainty, we can think of the choice among alternatives as being determined by two distinct factors: risk (probabilities and economics) and attitude toward risk (utility). The importance of keeping these two factors apart is worth emphasizing, because decision makers are so often tempted to mix one with another. For example, sometimes people talk about "conservative" probability distributions, which in fact are a mixture of risk and attitude toward risk. At other times people tend consciously or unconsciously to assign unduly high probabilities to events where consequences are particularly attractive or unattractive.

It is absolutely essential that the decision maker think only about the chances of occurrence of an event when he is assessing a probability, paying no attention whatever to the desirability or undesirability of the consequences which might result if the event occurred. Similarly, in assessing a utility curve, it is absolutely essential that the decision maker think only about what his attitudes would be should he ever face an uncertainty of the sort for which he is being asked to assess a certainty equivalent. He should pay absolutely no attention to the chances that he will ever face such an uncertainty. In the absence of such precautions there will be double counting.

Another source of double counting which must be guarded against is conservative modeling, that is, making conservative assumptions and using conservative estimates to help ensure that resultant decisions are conservative. This is a common ad hoc practice which, if mixed with an analysis using utility, could improperly bias the results.

G. PRESCRIPTIVE VS. DESCRIPTIVE UTILITY

While many people, myself included, find that the main value of utility theory is to make better decisions and thus *improve* behavior, there are others, including Swalm, who feel its main value is as a device to *describe* or predict behavior. I would agree that utility theory offers a useful set of concepts and a vocabulary that sharpens the description of human behavior. However, I find utility theory much less useful as a predictive tool, because humans are prone to make logical errors and inconsistencies when making complex decisions without the aid of formal analysis. If individuals always made consistent decisions without formal analysis, there would be no need for the utility theory and the formal analysis! Thus I feel that, while utility theory has some use as a descriptive tool, its main value is as a prescriptive one.

IV. APPLICATIONS IN INSURANCE

Since the insurance business is pervaded by decisions under uncertainty, utility theory obviously has many potential applications. These applications can be usefully subdivided into five classes:

1. Deciding which risks to take, what exclusions to adopt, what limits of coverage to offer, and what premiums to charge
2. Investing the assets of the firm
3. Reinsurance
4. Helping the insurance agent formulate a strategy for calling on prospects, given uncertainty about whether the prospects will actually buy
5. Influencing consumer behavior regarding the purchase of insurance

The first three categories involve decisions requiring optimization of the company's behavior, using its own utility curve; the fourth involves optimizing the agent's behavior, given his curve; and the fifth involves the consumer's behavior in the face of his attitude toward risk. The first three classes of application areas clearly involve decision-making areas where the company's risk-taking attitude needs to be taken into account explicitly. In the fourth class it is the agent's attitude that matters.

With regard to the fifth there are many possibilities. The essential objective is to understand the customer better. If the insurance companies could better understand the customer's attitude toward risk, they would be in a better position to assess probabilities for the customer's response to various approaches for marketing insurance. This suggests research into consumer risk-taking attitudes. Obviously, most of the purchasers of insurance are risk-averse, for otherwise there would be no reason to pay loaded premiums (unless one's probabilities for collecting were greater than those of the insurance company). Precise evidence on consumer utility, for example, might influence the establishment of premiums, for clearly those risks for which the consumer has high-risk premiums are those where more loaded insurance premiums are likely to be acceptable. In addition, one might be able to relate risk-taking attitude displayed by people in the class. There are a number of possibilities which present themselves, given this better understanding. Comments on this subject by Professor Miller and Dr. Kahn will deal with applications to the reinsurance and to the consumer problem. The reader is referred to these for a more specific understanding of how utility theory fits into insurance decision making.

V. CONCLUSIONS

Utility theory offers a means of clearly articulating a risk-taking attitude and of incorporating it into decision analyses in a manner which ensures that the resultant decisions are optimum, given the defined attitude.

As a part of statistical decision theory, it provides a means of conceptualizing problems which should improve decisions, whether it is used formally or informally. I feel that it is of potential direct usefulness in the formal analysis of some insurance problems. In addition, and perhaps more importantly, it provides a vocabulary and a set of concepts which sharpen the discussion of some important management and policy problems in insurance. For example, the fact that it distinguishes between the risk and the attitude toward risk provides useful insights into important problems which were formerly confused by the fact that these separable issues were considered together in an ad hoc manner. Thus its usefulness does not depend exclusively upon its formal application. I feel that it holds great promise for the insurance industry in the years ahead.

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PROFESSOR KARL BORCH:* In this presentation I shall argue that the real difficulty in many decision problems is to spell out what we want to achieve. This may sound surprising and possibly disappointing. The typical decision maker in insurance is usually thought of as a company president or a high-level executive, who may seek the advice of experts. This decision maker will probably be taken aback if his experts tell him that he must make up his mind before they can set to work and give their advice. Experts do not, of course, behave in this way. They will usually admit that their task is to "sort out" the problem, leaving the decision to the judgment of the decision maker.

This division of labor may seem natural, but it may not always be very efficient. As an illustration, let us assume that a reinsurance company has to choose one of the following two contracts:

Contract 1 will give a loss of $10 million with probability p or a profit of $600,000 with probability $1 - p$.

Contract 2 will give a loss of $4 million with probability p or a profit of $200,000 with probability $1 - p$.

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In this situation the task of an expert is to estimate the probability p and possibly add some measure of the "credibility" attached to this estimate. He may, however, waste his time if the company's president has decided that he will not risk a loss of $10 million on a single contract, unless, of course, p is virtually zero. If the expert had known this from the outset, he could have used a very coarse estimator for the probability.

The example we have presented shows that the outcome of a decision in insurance cannot be described or "predicted" as a certain profit or loss. The best that we can do is to specify a probability distribution over the possible outcomes. If we can appeal to the Law of Large Numbers, it may be possible for practical purposes to ignore the shape of the probability distribution and consider only its first moment, that is, the expected profit or loss. The classical actuarial mathematics risk rests on the Law of Large Numbers. When this law does not apply, we will need a different mathematical apparatus, an apparatus which actuaries for more than a century have referred to as the "theory of risk."

THE UTILITY CONCEPT

From the observations in the preceding paragraph it follows that the typical decision problem in insurance consists in selecting the best probability distribution from an available set. In order to make decisions of this kind, we must have a rule which tells us when one probability distribution—over a range of profits—is better than another. A rule of this kind must necessarily be of a subjective nature. The rule will represent an "attitude to risk," and there is clearly no objective rule which can tell us which attitude is the right one to take.

A rule for choosing among probability distributions can be described in different ways. If, however, the rule is consistent in the precise sense of Von Neumann and Morgenstern [8], it is possible to describe the rule by specifying a *utility function*, $u(x)$, such that

$$\int u(x)dF(x) > \int u(x)dG(x) ,$$

if, and only if, the distribution $F(x)$ is considered better than $G(x)$.

The considerations above indicate that a general theory of insurance must, or at least could, be based on the utility concept. This has in fact been recognized for a long time. Almost one hundred-forty years ago Barrois [1] constructed a very complete theory of fire insurance, based on the particular utility function, $u(x) = \log x$, orginally used by Bernoulli [2]. It must, however, be admitted that the present popularity of the utility concept in insurance literature is due to the result by Von Neumann and Morgenstern mentioned above rather than to the half-forgotten studies of nineteenth-century actuaries.

The utility concept may be considered indispensable in theoretical work on insurance, but it does not seem to have found many applications to insurance practice. One explanation of this apparent paradox may be that presidents and executives of insurance companies find it difficult to specify the utility function which represents their preference-ordering over the set of attainable profit distributions. This is a real problem which we shall discuss in some detail in the following section. Another explanation may be that we have oversimplified the problem. Any decision problem in an insurance company certainly involves a choice among probability distributions, but it is not certain that these decisions or choices can be studied in isolation. In simple terms, any decision may depend on the whole situation of the company, and this situation may again depend on the choices which are expected to be available in the future. If some dependence of this kind is important in real life, we must dismiss the simple static decision problem as irrelevant and develop a dynamic theory. Some ways of doing this have been discussed in other papers [3, 4].

A SIMPLE EXAMPLE

To construct a simple example, let us assume that the utility function is a polynomial of second degree. The expected utility will then be a linear function of the two first moments of the probability distribution. This means that, when evaluating a profit distribution, the decision maker will consider only expected profits and the variance of profits. As a first approximation this decision rule may seem reasonably acceptable, and it is the basis of much of the earlier work on the theory of risk. A brief historical sketch and a number of references are given in another paper [3]. The rule has become very popular during the last decade, through the work of Markowitz [6], Tobin [7], and others. It is well known that the rule can lead to contradictions [5], but we shall not elaborate this point, since we are using the rule only to illustrate some more basic problems.

Let us now consider the following insurance contract:

Premium: \$110
Possible claim payments:
\$10,000 with probability 0.01 or
0 with probability 0.99

This contract will give an expected profit $E = \$10$, and the standard deviation of the profit is approximately $S = \$1{,}000$. If an insurance company sells n contracts of this kind, it will obtain a portfolio with

$$\text{Expected profit} = nE,$$
$$\text{Standard deviation} = \sqrt{n}\, S.$$

In the following we shall ignore administrative costs. We can safely do this by assuming that they are covered by a suitable loading of the premium. We shall, however, assume that it will cost the company $500,000 to bring this contract to the market, that is, to make it available to the public. The problem of the company is then to decide whether this contract should be launched on the market or not. It is obvious that this decision must depend on the number of contracts which the company expects to sell. To facilitate the decision, the company can prepare a table like the one following:

n (Number of Contracts Sold)	nE (Expected Profit)	$\sqrt{n}S$ (Standard Deviation of Profit)
0.......	−$ 500,000	$ 0
10,000.......	− 400,000	100,000
40,000.......	− 100,000	200,000
50,000.......	0	225,000
90,000.......	400,000	300,000
100,000.......	500,000	315,000
120,000.......	700,000	345,000
160,000.......	1,100,000	400,000

From the table we see that the company can be expected to "break even" if it can sell at least 50,000 contracts. It is, however, likely that the company will want to do better than just break even, if it decides to take the risk involved in launching the new contract. The risk is represented by the standard deviation, and it is easy to see that substantial losses can occur.

After gazing at this table for some time, the decision maker may decide that it is worthwhile to launch the new contract if sales will exceed 120,000. He may justify this decision by noting that the profit distribution must be approximately normal. Expected profit is more than twice the standard deviation, so that the operation is virtually certain to be profitable.

The decision that we have suggested does, in a sense, imply that the buck is passed on to the marketing department of the company. It is, however, obvious that market research can never predict sales with certainty. At best the outcome of such research can be a probability distribution over a set of possible sales. Let this distribution be $g(n)$, the probability that n contracts will be sold. We then have

$$\sum_{n=0}^{\infty} g(n) = 1 .$$

If the market research indicates that $g(n) = 0$ for $n < 120{,}000$, the marketing department can guarantee that sales will exceed 120,000 and it can then recommend that the new contract be launched. Normally a marketing department does not make statements in this form. A more likely statement would be

$$\sum_{n=0}^{120{,}000} g(n) < 0.05 .$$

This means that there is a probability of 0.95 that sales will exceed the level of 120,000, which the decision maker considered as a minimum after having studied the above table. The decision maker may then decide to launch the new contract.

There are reasons to believe that some decisions in insurance companies actually are made in a manner similar to the one we have indicated. This decision procedure has, however, some unsatisfactory aspects. There is, for instance, no obvious reason why uncertainty due to "sampling fluctuations," as expressed in the table, shall be treated in a manner different from that of the uncertainty about market reaction—represented by the distribution $g(n)$.

To study this question, it is convenient to express our argument in a slightly more general form. Let $F(x)$ be the cumulative probability distribution of profits from one single insurance contract with expectation E and standard deviation S. Profits from a portfolio of n such contracts will then have the probability distribution $F^{(n)}(x)$, which can be computed as the nth convolution of $F(x)$ with itself. If $g(n)$ is the probability that n contracts are sold, the distribution of profits from the resulting portfolio is

$$H(x) = \sum_{n=0}^{\infty} F^{(n)}(x) g(n) .$$

If $\lambda(t)$ is the characteristic function of $F(x)$, the characteristic function of $H(x)$ is

$$\gamma(t) = \sum_{n=0}^{\infty} [\lambda(t)]^n g(n) .$$

Differentiating twice and setting $t = 0$, we obtain

$$\gamma'(0) = \lambda'(0) \sum_{n=0}^{\infty} n g(n) ,$$

$$\gamma''(0) = [\lambda'(0)]^2 \sum_{n=0}^{\infty} n(n-1) g(n) + \lambda''(0) \sum_{n=0}^{\infty} n g(n) .$$

Since $\lambda'(0) = E$ and $\lambda''(0) = S^2 + E^2$, we have for the portfolio:

Expected profit: NE,
Standard deviation: $(NS^2 + E^2T^2)^{1/2}$,

where N and T are, respectively, mean and standard deviation of $g(n)$.

Through these manipulations we have reduced our problem, so that it now consists in simply deciding if the pair $[NE, (NS^2 + E^2T^2)^{1/2}]$ is acceptable or not; that is, if it is better than the pair (0, 0).

It appears that a decision of this kind is often considered very difficult in practice. It seems at least that many executives are reluctant to make the decision, without asking for "more information." Such information will usually consist of further studies, which conceivably may reduce the standard deviation and which are certain to cost money, and hence reduce expected profits. Before asking for additional information of this kind, one should at least be sure that it really will make the decision easier.

In real life an insurance company may have to choose among many different actions, each leading to specific profit distributions, represented by pairs (E_1, S_1), (E_2, S_2). . . . The decision maker may find it difficult to pick the best pair from this set, and his easy way out is to ask that the whole set be recalculated. This means, however, only that the difficulties are postponed. Sooner or later the decision maker must formulate some rule as to when one ES-pair is better than another.

GENERALIZATIONS

In our example we have assumed that only the two first moments of the profit distribution were considered by the decision maker. This implied that a preference-ordering over a set of probability distributions could be represented by an ordering over a set of ES-pairs.

The assumption is obviously unrealistic, but it served to illustrate our main point. This point stands out even more clearly when we try to generalize the model. If the decision maker feels that other properties of the profit distribution should be taken into account, he will probably find it difficult to explain exactly how these properties (skewness, "tails," etc.) affect the decision. The real problem is, of course, to specify the utility function which represents the decision maker's preference-ordering over a set of probability distributions.

In real life the problem is even more difficult, as we cannot usually ignore the *time element*. At a given point of time, the future profits of an insurance company can only be described by a stochastic process: $x_1, x_2, \ldots, x_t, \ldots$. The decisions made by the management will influence this process, and the problem is to steer the process so that it develops in the most desirable way. In order to solve this problem, the company's management must have a preference-ordering over a set of stochastic processes. An ordering of this kind cannot be represented by a simple utility function.

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PROFESSOR ROBERT B. MILLER:* We are all well aware of the fact that the actuary is called upon to play many roles in his practice. I want to say a few words about two such roles and how the concept of utility may relate to them. The roles are those of statistician and decision maker (or perhaps adviser to a decision maker). As a statistician the actuary is called upon to estimate the distributions of risks faced by his company, for without such estimates intelligent decisions about the business of taking risks could hardly be made. As a decision maker (or adviser to a decision maker) the actuary is called upon to bring the results of his statistical analyses and all other relevant information to bear on the vital decisions which can make or break his company.

To see how utility might enter the picture sketched above, let us consider just one problem that our actuary might be asked to tackle. Suppose that he has been asked to consider the advisability of purchasing reinsurance for a certain line of business. Let us denote the amount of total claims from this line of business for the coming year by X. Of course, X is a random variable, and we assume that its distribution de-

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pends on an unknown parameter θ (θ may denote a collection of parameters). By the principle of equivalence, the total net premium that the company should charge is just the expected value of X, that is,

$$P(\theta) = \underset{X|\theta}{E}(X) .$$

Note that I have used the symbol $X|\theta$ to denote the fact that the distribution of X depends on θ. (I plan to speak only in terms of net premiums, although it is easy enough to replace these with gross premiums.)

A reinsurance agreement can be thought of as a function $T(X)$ of X, which tells what amount of claims the reinsurance company will cover if the total amount of claims is X. Of course, the net reinsurance premium will be the expected value of $T(X)$, that is,

$$\pi_T(\theta) = \underset{X|\theta}{E}[T(X)] .$$

Thus if the original company experiences total claims X and has reinsurance coverage $T(X)$, its asset position will be

$$P(\theta) - [X - T(X)] - \pi_T(\theta) .$$

Let us suppose that the company can express its preference for money via a utility function u. Then its utility of the above asset position is

$$u\{P(\theta) - [X - T(X)] - \pi_T(\theta)\} .$$

Of course, the actuary does not know in advance what value X will have, so he bases his decisions on the expected value of the utility, that is, on

$$U(T, \theta) = \underset{X|\theta}{E}\langle u\{P(\theta) - [X - T(X)] - \pi_T(\theta)\}\rangle .$$

This is the fundamental expression for decision making, because it expresses the company's attitude toward the facing-the-risk position defined by θ and T. Supposing, just for a moment, that θ is known and that the company can choose from a finite collection of reinsurance agreements $T_1, \ldots, T_k$, the best advice the actuary can give is to compare the k numbers

$$U(T_1, \theta) , \quad \ldots , \quad U(T_k, \theta)$$

and purchase the agreement which corresponds to the largest of these numbers. In doing so, the company will "maximize its expected utility."

It goes without saying that the value of θ is typically not known, and this fact leaves uncertain not only what reinsurance to purchase but even what premium to charge. In this situation the actuary, acting as a statistician, will seek to estimate θ as accurately as possible and then use the estimated value of θ as if it were the true value for decision-making purposes. It can be shown [1, pp. 180–81] that, if θ is estimated very efficiently in a certain technical sense, the procedure outlined above will enable the actuary to choose the best possible reinsurance agreement. By the way, the utility function u enters into the technical definition of efficiency, so that, theoretically at least, utility does have an impact on the estimation problem. But even if technical efficiency is not achievable in practice, and it often may not be, it can be striven for. While the estimation problem is notoriously difficult, it seems to me that we who work in insurance are fortunate to have vast amounts of data and past experience to aid us in discovering the true underlying distributions of risk. Thus we may be able to get closer to maximum efficiency than is possible in some other fields.

Suppose that the actuary has found an efficient estimate of θ, call it $\hat{\theta}$. Then the total net premium that the company should charge is $P(\hat{\theta})$, and the net premium for reinsurance agreement T will be $\pi_T(\hat{\theta})$. Finally, the actuary's best advice now is to choose the agreement which corresponds to the largest of the numbers

$$U(T_1, \hat{\theta}) , \quad \ldots , \quad U(T_k, \hat{\theta}) .$$

In the first part of this talk I have tried to show how the concept of utility might enter into a practical insurance problem. Needless to say, the picture has been simplified—in particular, we have not considered how the solution to our problem might be affected by the way in which we handle other lines of business. In a sense I have presented the solution in a vacuum. Nevertheless, I hope that it will stimulate your thinking.

Now I would like to spend a few minutes discussing some recent theoretical developments and some of their possible implications for decision making. As is often the case in theoretical discussions, one must make some rather severe assumptions. For our problem we make the following assumption: that the company sets the amount of net reinsurance premium that it is willing to pay and chooses only from agreements with this net premium, which we denote by π. With this assumption and the assumption that θ is estimated by $\hat{\theta}$, the company wishes to find a reinsurance agreement which maximizes

$$U(T, \hat{\theta}) = \underset{X|\theta}{E} \langle u\{P(\hat{\theta}) - [X - T(X)] - \pi\} \rangle . \qquad (1)$$

The problem of choosing the optimal reinsurance agreement in this context was first considered by Borch [2] and later by Kahn [3], both of whom chose T to minimize the variance of the claims retained by the original company, that is, Var $[X - T(X)]$. Minimizing variance is equivalent to maximizing $U(T, \hat{\theta})$, assuming that u is a second-degree polynomial. The T that does the trick is a stop-loss agreement, namely,

$$T(X) = \begin{cases} 0 , & X < a , \\ X - a , & X \geq a , \end{cases}$$

where a is chosen so that $\underset{X|\hat{\theta}}{E}[T(X)] = \pi$. Thus, if total claims are less than a, the reinsurer pays nothing, but the amount of total claims above a is covered completely by the reinsurer. This is the form of reinsurance agreement that the original company considers optimal, given our assumptions. Professor Arrow of Stanford [4] showed that this same T maximizes the expected utility in display (1), when u is any risk-averse utility function, that is, u is strictly concave from below.

What are some possible implications of this result? First of all, stepping out of the reinsurance context for a moment, let us think of the original company as an individual seeking casualty or health insurance coverage. This result seems to say that, if the individual was a risk averter, he would be happiest buying a plan with a deductible, the amount of the deductible being determined by the amount of premium he was willing to pay. In fact, Arrow was considering federally funded health plans when he proved the theorem stated above. Speaking of the government leads us to a second possible implication. If, as has been proposed, the federal government acts as a reinsurer of risks arising from natural catastrophes, such as the floods we are experiencing in the Midwest now, then the plan which would be in the best interests of the companies involved may be a stop-loss type of plan.

I want to note that the question of the type of agreement that the reinsurer prefers to sell has also been considered in the literature. Suppose the reinsurer has utility function v and that he sells reinsurance agreement T for premium π. His expected utility, assuming θ is estimated by $\hat{\theta}$, is

$$V(T, \hat{\theta}) = \underset{X|\hat{\theta}}{E}\{v[\pi - T(X)]\} ,$$

and, of course, he wants to sell the T that maximizes his expected utility. Vajda [5] and Hickman and Zahn [6] have provided proofs of the fact that, if v is a second-degree polynomial and if T satisfies a very natural regularity condition, the optimal T is a proportional agreement, namely,

$$T(X) = (\pi/P)X .$$

This is the type of agreement the reinsurer would like to sell if his objective is to minimize the variance of his risk. I [7] have been able to show that this proportional agreement is optimal when v is any risk-averse utility function. This and Arrow's results point up an interesting conflict of interest between the original company and the reinsurer.

In closing, I would like to say that while, until now, discussions about utility have belonged mainly to theoreticians, perhaps we are entering a time when practitioners will also take up this stimulating topic.

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DR. PAUL M. KAHN: If the concept of utility is helpful in analyzing the decisions that insurance companies make, it should also be helpful in decision making by the consumer of insurance, whether it be of life insurance, of health insurance, or of automobile insurance. Some attention has been paid to applying utility ideas to the problems of the insurance companies themselves, such as the types of reinsurance, premiums for these reinsurance contracts, and dividend distribution policies, but relatively little attention has been given to the decision-making process of the consumer. It is not as yet feasible to use utility theory as a tool to predict his insurance behavior exactly or to advise him precisely what it should be, but it can possibly provide clues on what are and what are not rational courses of action, which are, after all, part and parcel of the decision-making process.

RISK AVERSION

Before describing some utility applications to consumer problems, let me first explain the term "risk aversion," for it will appear frequently. If a person has a utility function $u(x)$ which is concave downward, that is, such that $u''(x) < 0$, he will prefer having an amount with certainty to a fair gamble with that amount as expected payoff. On the other hand, a person with a convex utility function, that is, one such that $u''(x) > 0$, is said to have a risk preference or to be a gambler. For a fuller discussion of risk aversion and preference, see Dr. Hammond's presentation. One measure of risk aversion is the function $r(x)$,

$$r(x) = -\frac{u''(x)}{u'(x)},$$

which has the property that it is a decreasing function of x if and only if for every risk an individual's cash equivalent (the amount for which he would exchange the risk) is larger and the amount that he would be willing to pay for insurance is smaller, the larger his wealth [10].

APPLICATIONS OF UTILITY THEORY

Some economists have already given attention to insurance consumers' applications of utility. Dr. Borch, to cite only one example, has investigated the problem of how much reinsurance premiums should be [2].

A recent paper by Dr. Borch's colleague, Mr. Jan Mossin, is a lucid discussion of some of the consumer's problems in buying insurance [9]. Mr. Mossin considers the problem of how much an individual should pay for insurance coverage. He points out that the maximum premium that one should be willing to pay depends on his wealth and that the larger his wealth, the lower this maximum, provided only that he has a decreasing aversion to risk. In simpler terms, the wealthier a person is, the less he should be willing to pay for insuring his property.

For a simple example, let us assume that an individual has a piece of property of value L, that his other assets are worth A, that the probability that his property will be completely damaged is p, and that it will suffer no damage be $1 - p$. He may insure this property for a premium π. The buyer's problem is to determine the maximum premium that he should be willing to pay.

If he buys no insurance, his final wealth Y_1 can be described as

$$Y_1 = \begin{cases} A & \text{with probability } p\,; \\ A + L & \text{with probability } 1 - p\,. \end{cases}$$

The expected utility of this situation is

$$E[U(Y_1)] = u(A)p + u(A + L)(1 - p)\,.$$

If he buys insurance, however, and pays the premium π, his final wealth will be

$$Y_2 = A + L - \pi \text{ with certainty}\,,$$

and the expected utility of this situation is

$$E[U(Y_2)] = u(A + L - \pi) .$$

He should then be willing to pay up to that amount π which makes the two utilities equal,

$$u(A)p + u(A + L)(1 - p) = u(A + L - \pi) .$$

Any smaller premium π provides him a higher utility than he would have if he did not buy insurance, and any larger premium would result in a utility below that of his not insuring at all.

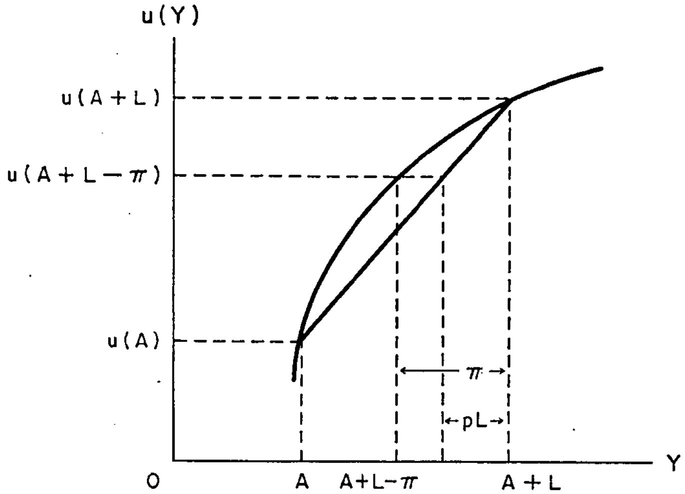

FIG. 1

This maximum premium is a function of the probability of loss p, the value of the property itself L, and the worth of his other assets A. If the buyer is averse to risk (i.e., $u''(x) < 0$), this maximum premium is greater than the actuarial net premium or expected loss, pL. This can be seen from Figure 1, due to Mossin.

This maximum premium π increases as both p and L increase, but Mossin has shown that it decreases as A, the value of the other assets, increases and that this fact follows as a consequence of the decreasing risk aversion.

Another related problem considered by Mossin is the optimal coverage for a given premium. In this case, let us assume, as before, that an individual has a risk property with value L and other assets with value A. He wishes to insure his risk property up to an amount C, which cannot exceed its value L. If the premium rate is denoted by π, he will pay a premium πC, and if there is a loss X he will receive an amount equal to $(C/L)X$. His assets will then be expressible as

$$Y = A + L - \left(1 - \frac{C}{L}\right)X - C\pi .$$

To choose the optimum value of C, the amount of insurance, is to maximize the expected utility of Y, the outcome, with respect to the probability distribution of X, provided only that the insurance amount, though positive, does not exceed the value of the property.

Let us differentiate the expected utility $E[U(Y)]$ and choose the value of C for which the first derivative is zero:

$$\frac{dE[U(Y)]}{dC} = E\left[U'(Y)\left(\frac{X}{L} - \pi\right)\right],$$

$$\frac{d^2E[U(Y)]}{dC^2} = E\left[U''(Y)\left(\frac{X}{L} - \pi\right)^2\right].$$

The second derivative will be negative if we assume that the individual is risk-averse. Hence there will be a unique value of C, which maximizes the expected utility.

In this particular case, let us consider full coverage, that is $L = C$. The first derivative of $E[U(Y)]$ becomes

$$U'(A + L - L\pi)\left[\frac{E(x)}{L} - \pi\right],$$

which is positive, thereby giving an optimum only if $E(x) > L\pi$. Mossin shows that full coverage would be optimum only if the premium were less than the expected claim, a result unlikely in practice. Hence, in this type of situation, we can say that full coverage is never optimal. But people do buy full coverage.

Any insight into how large a percentage of coverage would in fact be optimal depends on the claim distribution and the individual's actual utility function. Mossin does derive, however, the general result that the larger the individual's wealth, the lower the optimal coverage. This conclusion depends only on the individual's being averse to risk, but it is otherwise independent of his actual utility function.

Let us turn our attention to the types of coverage generally sold by life insurance companies.

In a 1963 paper, "Uncertainty and the Welfare Economics of Medical Care" [1], Kenneth Arrow compares certain characteristics of the medical care industry with those of welfare economics and points up the connection between the uncertainty underlying the incidence and treatment of disease and the special problems of medical care economics. He seeks to answer the question, What factors besides utility can help explain the medical care industry? The paper is particularly useful for its breadth in analyzing these factors. In this paper, he incidentally furnishes a result quite similar to those of Mossin cited above. This is that the optimal health insurance policy from the buyer's standpoint is full coverage for all claims in excess of a fixed-dollar deductible amount. He, like Mossin, assumes only that the individual is risk-averse. Another result which he cites is due originally to Dr. Borch. That is, if both the insurance company and the buyer are risk-averters and if we exclude consideration of costs other than the claim cost, any "optimal" policy must be on a coinsurance basis.

Since it has been established that a policy providing for full coverage with a deductible is optimal under the assumption of the individual's risk aversion, let us consider whether we can learn something about what may be an optimal deductible amount. Here again, Mossin has given us some insight.

Let us assume that an individual with A assets purchases a medical policy with a deductible amount S and full coverage for amounts in excess of S. If X denotes the amount of the losses in a year, say, and if $Z(X, S)$ denotes the amount paid by the company, then Z may be expressed as

$$Z(X, S) = \begin{cases} 0 & \text{if } X \leq S\,, \\ X - S & \text{if } X > S\,. \end{cases}$$

The premium $\pi(S)$ depends upon the deductible amount S and can be represented as

$$\pi(S) = (1 + \lambda)E[Z(X, S)]\,,$$

where λ represents the loading factor. Then the final outcome Y to the individual after the experience in the year is

$$Y = A - \pi(S) - X + Z(X, S)\,.$$

If the random variable X representing the claim amount has density $f(x)$, the expected utility of this insurance arrangement is

$$E[U(Y)] = \int_0^S U(A - \pi - x)f(x)dx + U(A - \pi - S)\int_S^\infty f(x)dx\,.$$

The problem now is to choose S so as to maximize this expected utility. Mossin argues that S must be strictly positive, for, if it were zero, that is, if there were no deductible, the first derivative would be positive and there could not be a maximum. On the other hand, if S

is positive, then either $E[U(Y)]$ is an increasing function of S, so that the optimal deductible S is infinite, and there is no insurance, or the optimal S is finite. That there is a nonzero, finite deductible amount which will maximize the expected utility is a reasonable conjecture which seems to fit well with observations, but as yet we are not assured that it is more than a conjecture, at least from a strictly mathematical point of view.

If we assume that the optimal deductible amount is in fact finite, then we are able to say something useful about the deductible S as a function of the individual's wealth. With the general condition that the consumer has decreasing risk aversion, then the larger his wealth, the larger the optimal deductible amount. This result should be no surprise to actuaries, since companies often issue policies with larger deductibles only to applicants with larger incomes. That this application of utility ideas agrees with the actuary's common sense may serve both to confirm our trust in our judgment and to make us slightly more comfortable in the use of a new tool.

AREAS FOR FURTHER RESEARCH

These examples show that some investigation into applying utility theory to insurance has been started. But the problems remaining are basic and nontrivial, and we list some of them.

So far we have not considered very deeply the shape of an individual's utility function. A utility function is concerned with one's attitude toward money and risk, but it may also be thought of as a reflection of consumption preferences. One's feelings about several other different factors enter into the shape of his utility curve, factors such as one's preference for liquidity, the effect of the outcomes on one's tax position, and, particularly for a firm, the aspect of public relations.

A related problem was brought out in a series of recent exchanges in *Management Science* between Morris Hamburg, of the University of Pennsylvania, and William Matlock, of the University of Pittsburgh, on the one hand, and Robert Hayes, of the Harvard Graduate School of Business Administration, on the other [4, 5]. One of the questions in apparent dispute was how to assess a utility function—whether to do so in terms of the loss to which the insured is exposed and from which insurance is to protect him, at least in part, or in terms of the buyer's total asset position, including in this phrase savings, fixed assets, current income, and possibly even future expectations. In this case, the disagreement was more apparent than real.

A more fundamental difficulty on which these gentlemen really disagreed was in finding a proper criterion for evaluating alternative insurance contracts. Hayes used the expected utility of the changing asset position, which is the approach of Mossin and others described earlier. Hamburg and Matlock, on the other hand, based their discussion on the marginal expected utility corresponding to a rise in the liability limit. Hayes closes the exchange by reflecting that, since two people can look at the same problem from roughly the same point of view and come to widely different conclusions, we need more research in the field. He asks, "How *do* people make insurance decisions? What factors are important to them? How do they evaluate the risk involved, and do they have the information to do this properly?"

A corollary to these questions is the current controversy over the definition of the cost of life insurance. What criteria may a consumer use in choosing among competing insurance policies or among various combinations of insurance with savings?

The major aspect of this area for utility theory is how we can realistically, but meaningfully, work with a series of consumptions, each related to a different time period; that is, how can we choose between present and future consumption? If C_0 represents consumption for the current period, C_1 represents consumption for the next period, and so on, then the multiperiod consumer should seek to maximize a utility function $U(C_0, C_1, \ldots, C_n)$ for several successive periods. In a 1966 paper [7] Mr. Liviatan suggests that the stream of future consumption $C_1, \ldots, C_n$ could be replaced by a single index X, and the problem could then be to maximize a two-dimensional utility function $V(C_0, X)$. He discusses two possible approaches to this index. One, due to Leontief [6], would represent the "future" by the perpetual (equal) stream of future consumption; the other, suggested by Dewey [3], would use the wealth planned for the next period. There is also some work in progress by Richard Meyer at Harvard on this problem of present vs. future consumption and of analyzing a multiperiod sequence of decisions [8]. The parallel with the situation of a life insurance prospect or owner facing the decisions of whether to buy a policy, to continue premium payments, or to surrender for cash, is striking. These remarks serve merely to point up that several of these problems vital to the insurance industry are receiving some attention and to suggest that some among our profession might find an acquaintance with this area in order.

Earlier, we referred to utility theory as a tool, but a tool ideally suited to an actuary's kit. The actuary makes decisions affecting the safety and profitability of a risk enterprise. Faced with the mortality and morbidity statistics of the past, he must make decisions under uncertainty as to the mortality and morbidity experience of the future. It is with the economics of uncertainty that an actuary is concerned, in the last analysis. Dr. Borch has made available to us an excellent introduction into the insights and analysis necessary to come to grips with the *Economics of Uncertainty* [2]. He illustrates very clearly that utility is the natural language of risk economics, as probability is the natural language of physics. He is concerned with decision making under uncertainty, which is the basic job of an actuary.

Because of its apparently central place in the future development of actuarial theory, the Research Committee has selected decision theory as the topic for the Fourth Annual Research Conference, cosponsored by the corresponding committee of the Casualty Actuarial Society with the assistance of faculty members from the Harvard Graduate School of Business Administration. It is in decision theory that many of the topics which have occupied us in the last few years can be subsumed—utility theory, the way to quantify choice; Bayesian statistics and credibility theory, the way to quantify insight into probabilities when we have no clear knowledge of them; and game theory, a way of structuring decision making, especially when one finds that he is not alone in the world.

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MR. F. ALLEN SPOONER: I would like to emphasize one point with respect to Paul Kahn's presentation. When one combines utilities of several outcomes to get the expected utility, it is important to use the right probabilities. For an analysis from the consumer's viewpoint, these are the probabilities the consumer associates with the events, which are probably not the true probabilities of the outcomes. For insurance, the consumer is likely not to be very close to the statistics indicating the probabilities, and your results can be thrown off badly if this is not taken into account.

MR. GEROLD W. FREY: Mr. Spooner's observation should also be emphasized with regard to Professor Miller's subject. The reinsurer may make a different estimate—probably much safer—from the estimate that the insurer makes.

I have a question for Professor Hammond. You said that you are an adherent of the prescriptive theory of utility, in contrast to others who are descriptive adherents. You also referred to human error, which is heavily involved in decision making in complex situations. Do you not just relegate the problems of human errors to the estimates that you make in the utility curve?

PROFESSOR HAMMOND: I would like to start out by answering your question in terms of decision theory and then in terms of utility theory. I see decision theory as some logical glue that holds together the various elements of a decision and thus helps to avoid human error. The idea is to "divide and conquer," to focus your undivided attention on the elements one by one and then to combine them. If you look at the probabilities alone, at the economics alone, at the attitude toward risk alone, and at the structure of the problem alone, one by one, in a way that makes you very conscious of what you are doing, you avoid logical slips and flaws. You avoid making a decision that is inconsistent with your beliefs and attitudes.

And now let me focus on the utility theory. You would be surprised at how inconsistent people are when they first assess a utility curve. For example, they make certain statements which would allow you to turn them into money pumps. They say, in effect, that they prefer A to B and B to C and C to A. You make a lot of money from a guy like that, if he really believes what he says.

The important point is that you need a way of policing yourself and checking for flaws in reasoning and eliminating them. Utility theory in particular and decision theory in general offer a way of doing so and thus reducing what you term human error.

PROFESSOR MILLER: In my discussion of reinsurance, I would like to point out that the evaluation of the parameter θ does not preclude the fact that the reinsurer might have a different idea about what its value is. This can be recognized by using the actual gross premiums for the various agreements being considered. Essentially there is no problem about how the reinsurer tests the value of θ.

MR. ROBERT F. LINK: I have a question related to the design of deductibles, limits of coverage, or other splitting of medical bills between the insured and the insurer in major medical plans. Does utility theory address itself to the fact that, when you spread the risk on a medical plan, the consumer is less inclined to see a doctor than he would be if his insurance company were bearing the whole risk?

PROFESSOR HAMMOND: The heart of the problem that you express lies in the probabilities used in the analysis in addition to the utilities. When, as the insurer, you are analyzing the problem of whether to share the risks or whether to carry all the risks yourself or what percentage the consumer should bear, you have to come up with good probability distributions on claim behavior as well as make use of utility theory.

MR. PAUL H. JACKSON: Is there not an assumption here that there is a utility curve? Is it not possible that, if you were helping to assess a utility curve and you posed, let us say, twelve different questions to a board chairman which were all exactly the same but in different terms, he would give you a scattering of answers?

PROFESSOR HAMMOND: The answer is yes. If you ask the question in twelve different ways and he gives you different answers, depending on how you asked the question, then he is in trouble, because he is being inconsistent with himself. Under the circumstances, it becomes a problem of confronting him with the fact that he is inconsistent and of helping him get at his true attitudes.

Most people have not had the experience of delving into their attitudes toward risk in quite the same way as we have been describing it here. This is a new way of thinking about it for the average person. Perhaps an analogy is helpful. Assessing a curve for the first time is rather like trying to write a paper in a newly learned foreign language. The first time around you probably will not say what you really mean, and, when this is pointed out to you and you make changes, you may still be somewhat off the mark. After a few iterations, you finally get what you want. After becoming more comfortable with the language, you can get what you want on the first or second draft.

It is worth pointing out that it is proper to get different answers to different questions if they are really asking different things. For example, a certainty equivalent for a venture, where the stakes are small, will be close to its mathematical expectation, whereas for a very risky gamble we would expect greater risk aversion. Differences of this sort are to be expected.