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PROJECTIONS OF VARIABLE LIFE INSURANCE OPERATIONS

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THE introduction of variable life insurance and annuities has made it necessary for actuaries to give more intensive thought to the possible outcomes of operations which involve not only the usual factors of mortality, expenses, and interest but also changes in the value of the equity investments that support the variable contracts. This calls for projections of financial operations that include price fluctuations in the stock market as a basic factor.

Such projections may be made either by simulating the behavior of the stock market or by developing analytical expressions to represent the fluctuations of market prices. More specifically, we can construct a model of the stock market that simulates the fluctuations of a common stock portfolio, and we can then incorporate the sets of values so generated into a model of the financial operations of a company marketing variable life insurance. Alternatively, we can visualize these fluctuations in the form of a mathematical function and then use this function in a risk-theory setting to gain information about the effects of such fluctuations on the financial operations of a company.

The traditional model-office calculations do not explicitly include measures of the fluctuations in the aggregate earnings which usually arise from variations in mortality, withdrawals, expenses, or agency input. The introduction of equity price changes into the financial operations of an insurance company renders the task of developing financial projections very complex, so that we are inevitably led to approach the problem first through simulation, then through simplified mathematical analysis, and finally through a combination of the two.

This paper discusses an example showing how simulation has been used to develop financial projections for variable life insurance with a guaranteed minimum death benefit and how recourse to mathematical analysis based on risk-theoretic considerations can enable us to see the impact of stock price changes more clearly. These projections reveal the effects of stock price fluctuations interacting with the more common elements, such as mortality and persistency, on the financial results of a variable life insurance operation. As an important example, this analysis has measured the extreme sensitivity of the earnings of a variable life insurance company to a combination of unfavorable persistency with investment performance. It has also revealed that the cost of a minimum guaranteed death benefit under a variable life insurance policy may vary widely with investment performance. Two other major uses of these projections were analyzing the effect of variable life insurance sales on the cash flow of the parent and its ability to invest in fixed-dollar securities (described by Walker [9]) and determining the gross premium level to provide a predetermined rate of return to the parent. This paper describes these findings.

Each of the two methods, simulation and analysis, has certain advantages. With simulation we can examine the operations of an enterprise too complex to be formulated in a set of equations. Even if the operations could be so formulated, the equations might well be too difficult to solve or to use in financial projections. With the aid of a computer model, we can simulate financial operations in great detail under a broad range of assumptions as to mortality, expenses, and investment experience, and study interactions between these factors. The principal advantages of the analytic approach are that it provides more precise measures of the fluctuations of financial results, that it focuses attention on the nature of these interactions, and that it is usually less expensive than simulation.

SIMULATION WITH COMPUTER MODEL

The computer model of a life insurance company selling only variable life insurance and ancillary benefits was developed to project financial operations in order to determine the feasibility of marketing variable life insurance through a subsidiary of an established life insurance company and to examine various policy designs and pricing structures. For this purpose it was necessary to have projections of the subsidiary's financial operations under several assumptions as to mortality, withdrawal, expenses, agency input, and investment performance.

Stock Market Model

A computer model, somewhat similar to that of Turner [8], was designed to generate sets of share values to reflect a range of the investment experience of a common stock portfolio. Two assumptions were needed for this model: (1) an estimate of the long-term trend of stock prices and (2) an estimate of the range of fluctuations around the trend line. The assumption about the basic long-term average rate of capital appreciation was made in consultation with our economists and was set at 6.5 per cent per annum. The assumption about fluctuations around this trend line was based on a graduated distribution of the month-to-month changes in the value of Standard and Poor's Price Index of 425 industrial stocks traded on the New York Stock Exchange at the end of each month from January, 1947, to January, 1970. Our economists believe that the fluctuations in stock market prices over the next few decades or so are not likely to be radically different from those during the years following World War II.

A random number generating program was used to determine individual sample values of the fluctuations based on the distribution just mentioned. If m represents the mean of the distribution of monthly percentage changes and y a sample value from the same distribution, then the value for a particular month was calculated from the value for the previous month by multiplication by the factor $(1.065)^{1/12} + y - m$. The subtraction of the mean m effectively removes the particular upward bias inherent in the postwar data selected and provides a set of fluctuations around the 6.5 per cent trend line. The sets of monthly share values furnished by the stock market model constitute the input into the computer model of the operations of the variable life insurance subsidiary.

Assumptions concerning the Policy

The company offers a fixed premium, variable benefit, whole life insurance policy with assets at least equal to the reserves invested in a separate equity account. The policy participates fully in the investment earnings of the separate equity account but does not share in the mortality or expense experience of the company. The policy is designed so that, if the death benefit is increased because of favorable past investment performance, it will not decrease below its then current level as long as the net investment return is at least equal to the assumed interest rate (AIR), such as 3 per cent or $3\frac{1}{2}$ per cent. In particular, the death benefit remains level if the net investment return is exactly equal to the AIR. Net annual investment earnings above the AIR increase the death benefit, and the increased death benefit continues as long as the AIR is earned. Conversely, when the net investment earnings are below the AIR, the deficit produces a reduction in the death benefit, which reduced benefit remains constant as long as the AIR is earned. The policy guarantees a minimum death benefit equal to the initial insurance amount, but there are no minimum guarantees with respect to cash surrender values. Details of this contract are given in Walker's paper to the National Conference on Variable Life Insurance [9].

Assumptions concerning Company Operations

The assets of the subsidiary are invested wholly in common stocks, except for cash, which is held to the larger of $100,000 and 1 per cent of the total assets. All stock transactions in the separate account are effected and dividends received as of the end of each month. Realized capital gains (or losses) are calculated as the product of (i) the average gain (or loss) per share for the entire portfolio, as measured by the difference between market and book values at time of sale, and (ii) the number of shares sold.

Start-up expenses account for most of the parent company's investment in the subsidiary, which is initially $10 million in paid-in capital and surplus. Year-end surplus in the subsidiary is maintained at a minimum of $3 million by transfers from the parent, if necessary, in accordance with New York State requirements. Surplus in excess of the larger of $10 million and 5 per cent of the subsidiary's assets is paid annually to the parent.

Reserves are held on the Commissioners Reserve Valuation Method. When the death benefit in the absence of the guarantee would be less than the guaranteed benefit, a special reserve is set up at the end of the year by a transfer of funds from surplus. When the market recovers, these funds are transferred back. Because certain federal income tax questions were unresolved, the model assumes present tax rates and a tax reserve recognizing net unrealized capital gains. To the extent that taxes payable on capital gains or income are greater than assumed, the cost of the benefits provided is increased.

Variation in Experience Parameters and Fund Charges

For projections of company operations under many conditions, the model permits the variation of several experience parameters by using different input data for different simulations. These include mortality, persistency, the annual inflation rate in per-policy expenses, and the volume of sales. One set of values for these parameters was selected as reasonable in our judgment and was termed "standard," and we also tested several variations to identify those parameters which most influenced financial results.

The standard mortality assumption was that mortality would follow the parent company's current select and ultimate experience on standard lives. The variations of the mortality assumption were that mortality would be 95 per cent and 102 per cent of current experience. The standard withdrawal rates were similar to the parent company's current withdrawal experience. The variation was a scale of rates 50 per cent higher.

Expenses enter the model in three ways. The initial start-up costs are translated into monthly overhead expenses, which decrease annually to zero by the seventh year of operation. The per-policy expense rates used follow the parent's current expense rates modified to take account of the expense savings of not paying dividends. The standard assumption was made that expense rates before adjusting for changing average size increased by 5 per cent per annum, and the variation assumed was 3 per cent. Increasing expense levels also require revising premium rates and policy fees in the eleventh and twenty-first years.

The standard assumption for sales volume reflects a quick buildup, but with the rate of increase in annual sales tapering off after the sixth year of operations. A more optimistic variation and a less optimistic one were tested as alternatives, with faster and slower buildups, respectively.

A monthly charge is made, equal to a percentage of the average value of the fund to cover investment expenses and the cost of expense and mortality guarantees. The model was tested with two fund charges, denoted *a* and *b*, with *b* $\frac{1}{4}$ per cent higher than *a*.

Investment Performance Assumptions

Fifty simulations of sets of monthly values over thirty years were generated by the stock market model, five sets of which were chosen to represent different patterns of investment performance, from very poor to highly successful. The "asset yields" for each of these five sets of 30-year share values were calculated with the standard set of experience parameters. "Asset yield" is the average annual rate of return (capital gains and dividends) on assets invested in the separate account. This is the annual rate of interest which, when applied to the net amount of cash flowing into the subsidiary's separate account, will accumulate to the subsidiary's total assets. These five sets are labeled A, B, C, D, and E in decreasing order of asset yield (13.5, 10.9, 9.1, 7.8, and 4.4 per cent, respectively).

SOME IMPORTANT FINDINGS

Yield to the Parent

The computer model produced the set of average annual yields to the parent under several different outcomes, which are of great importance when contemplating a variable life insurance subsidiary. The yield is the interest rate equalizing (1) the present value of capital paid in by the parent and (2) the present value of the dividends to the parent and of the subsidiary's surplus at the end of the thirtieth year. This thirty-year yield is conservative. It fails to reflect adequately the full yield on the business issued in this period, because current accounting practices require the immediate chargeoff of high first-year expenses and because it does not recognize earnings on these policies beyond thirty years. These yields are displayed in Table 1 for each of the five sets of share values under the standard set of experience assumptions and under several variations.

Table 1 provides data on how sensitive the yield is to different levels of investment performance, mortality, and other factors. Under most assumptions the yield varies only slightly with stock market performance, increasing in the standard case from 4.4 per cent for the very unfavorable experience of set E to 5.8 per cent for the very favorable experience of set A. These yields become very significant as a measure of financial results

TABLE 1

YIELD TO PARENT COMPANY OVER THIRTY YEARS
(Fund Charge *b*)

Assumptions	Set of Share Values				
	A	B	C	D	E
Standard	5.8%	5.2%	4.9%	4.1%	4.4%
Low mortality, standard otherwise	6.7	5.9	5.7	4.9	5.3
High mortality, standard otherwise	5.6	4.9	4.6	3.8	4.1
Low expense rates, standard otherwise	10.4	9.3	9.5	9.1	9.3
High withdrawal rates, standard otherwise	− 8.8	−14.2	−14.7	−23.5	−33.9
Slow sales buildup, standard otherwise	3.1	3.0	2.4	1.5	1.8
Fast sales buildup, standard otherwise	7.2	6.2	6.0	5.4	5.6

if persistency is poor and as a measure of the interaction between persistency and investment performance. If both these elements are unfavorable, there is a severe, negative effect on the parent company's return on its investment in the subsidiary; although it is intuitively clear that this combination would be unfavorable, it is imperative to have some measure of how bad it would really be. Management must proceed now to appraise the likelihood of this contingency. Further tests as to the sensitivity of the yield to poor persistency and investment performance might be warranted, such as studying persistency as a function of stock market fluctuations.

A drop from 5 to 3 per cent in the annual increase in per-policy expense rates doubles the yield and illustrates the sensitivity of the yield to the expense assumption. The yield is only moderately sensitive to mortality experience, and a 5 per cent decrease in mortality rates experienced increases the yield from 10 to 20 per cent.

To gauge the understatement of these yields noted earlier, the model was run for thirty years but with only ten years of issue. The yield with share value set C and the standard assumptions is 9.2 per cent on this block of business, as compared with 4.9 per cent from Table 1, where the issues of the first ten years are combined with subsequent issues in the thirty years of operations. What is most important is that, although at a higher level, these yields on ten years of issue exhibit the same patterns generally as do those in Table 1.

Cost of Guaranteed Minimum Death Benefit

The death benefit under these variable life insurance policies reflects the investment experience of the equity account, subject to a minimum guarantee equal to the initial face amount. In the absence of this guarantee, the death benefit would fall below the initial death benefit if the investment return is sufficiently poor. For any group of policies the cost of this guarantee is the present value of the excess, if any, of the minimum guarantee over the natural benefit, that is, what the benefit would be without a guarantee. Simulation was applied to this problem in two ways, one using the company model, the other in a more direct fashion.

The operations of the company for thirty years were simulated using the model with each of the five sets of share values, with the standard assumptions and the variations tested earlier in calculating yield, and with each of the two fund charges considered. The net single premiums for this benefit by issue age but for all issue years combined were calculated using a 5 per cent discount rate. (These risk premiums are maintained in a fixed-dollar account.) Comparison of the ratios of these costs to the first-year premium shows that the cost of this guarantee when measured in this way is not sensitive to expense levels or sales volume and is only slightly sensitive to market performance.

Table 2 shows the costs expressed as a "net single premium" for the minimum death benefit per $1,000 of first-year premium to illustrate their order of magnitude. These figures are based on the standard assumptions, the "average" investment performance (set C), and both fund charges. These figures indicate the level of risk premiums under the assumption that this benefit be self-supporting over a period of several years and reflect this averaging process. What is needed is a distribution of these costs, but the relative expense of running the company model forces us to proceed with a more direct calculation.

To approximate the distribution of the claim costs under the minimum guarantee, we simulated the stock market model to produce 100 sets of monthly fluctuations over 55 years. The simulated investment performance was then used to produce 100 values of the claim costs using a mortality table representative of the parent company's current experience. These 100 values furnish a distribution of the claim costs. Several interesting statistics of this distribution, such as the mean, median, standard deviation, skewness, maximum, and 90th percentile are exhibited in Table 3 for the distribution of these costs per $1,000 of first-year premium.

TABLE 2

NET SINGLE RISK PREMIUMS FOR MINIMUM DEATH BENEFIT GUARANTEE PER $1,000 OF FIRST-YEAR PREMIUM
(All Years of Issue Combined)

Fund Charges	Issue Age			
	25	35	45	55
a..........	$0.10	$0.14	$0.35	$0.71
b..........	0.11	0.16	0.40	0.80

TABLE 3

DISTRIBUTION OF NET SINGLE RISK PREMIUMS FOR MINIMUM DEATH BENEFIT GUARANTEE PER $1,000 OF FIRST-YEAR PREMIUM
(Based on Separate Years of Issue)

Issue Age	Mean	Median	Standard Deviation	Skewness	Maximum	90th Percentile
25.......	$ 2.15	$0.06	$ 6.35	4.5	$ 42.17	$ 7.40
35.......	4.47	0.07	14.32	4.7	94.80	13.95
45.......	8.97	0.16	27.20	4.4	168.00	32.20
55.......	15.07	0.40	42.50	4.1	251.99	44.79

The most pronounced characteristic of this cost distribution is its skewness. (Recall that a symmetric distribution like the normal has a zero skewness coefficient.) Half of the 100 simulated values of the net single risk premium are zero or very small, but the mean value is several times the median, and the largest observed value is greater than the mean by several standard deviations. Table 3 shows that, although in most cases no significant loss will occur, there remains a small but positive chance that a very large loss will occur. This situation is similar to catastrophe-type coverages in casualty insurance. The average value of the claim cost, that is, the net single premium, is not a very good indication of a charge to make for this benefit, while the 90th percentile value has the useful property that it should be sufficient about 90 per cent of the time. As Bühlmann has recently pointed out [2], the equivalence principle is not a common tool in casualty-type coverages.

RISK-THEORY MODELS

General Risk-Theory Models

If the fluctuations in the financial operations of a variable life insurance company can be effectively studied by the methods of risk theory or, more naturally, by a synthesis of risk theory and simulation rather than by either alone, certain significant advantages accrue. The reliability and variation of the statistics produced can be better measured. The important parameters can be better isolated. The interactions between variables can be studied more precisely. We shall then have better estimates of the likelihood of severe losses, whether through bad investment performance, heavy mortality, or other unfavorable elements.

The usual risk-theory approach to studying the operations of an insurance company requires knowledge of the random process describing the frequency of claim occurrence and knowledge of the distribution of individual claim amounts. This individual claim amount distribution is not traditionally a function of time. For variable life insurance, however, these claims are a function of the time-dependent random process which describes stock price movements. The modification of the risk-theory approaches to incorporate this new process is as yet an intractable problem.

The difficulties in adapting the customary risk-theory model to variable life insurance are twofold. In the first place, the common assumption that the amount of successive claims be independent does not seem reasonable for variable life insurance under which the death benefit varies daily, say, and the underlying assets are valued daily also. Successive claims depend on the then current values of the common stock portfolio, and these are not independent. If the death benefit varies less often than the portfolio is valued, this difficulty may be reduced. The solution to this problem probably lies in the direction of incorporating a diffusion-type Markov process into the collective-risk-theory model, but that is beyond the scope of this paper.

Another complication is that the reserves do not increase smoothly as they are assumed to do in usual collective-risk-theory models, except for the discontinuities related to claim occurrence, but they must reflect the random nature of the stock market, which is by no means smooth.

To give a simple example of the application of the analytic approach to a problem arising in designing a variable life insurance policy which is at the same time both practical and solvable, we analyze the cost of a minimum guaranteed death benefit under a variable reduced paid-up insurance nonforfeiture option. This is equivalent to considering the same type of guarantee under a single premium whole life policy. The cost of this benefit is derived using both simulation and an analytic curve to represent the underlying stock market behavior, and these two techniques are compared. As a first step in such a program, we must choose a mathematical curve to represent stock price changes.

Stock Market Models

In deriving an analytic function to represent stock market price changes we should review the sizable American literature on the construction and criticism of models of stock price behavior, a problem first studied by Bachelier at the turn of the century [1]. James Hickman has summarized these models under three types [5]:

a) The classical random walk model, where the price changes are independent and identically distributed random variables. A special case is a random walk model with an upward drift.
b) A model wherein the expected price for any period is the price in the previous period, that is, the expected price change is zero. This model makes no assumption regarding independence or identical distributions and is called a martingale.
c) A model similar to model *b*, but with an upward drift. This is called a submartingale.

Much of the literature tests the advantages of various trading rules and analyzes price changes over long periods. There appears to be substantial agreement that there are theoretically advantageous trading rules but that the predictive value is sufficiently small that the effect of stock transfer costs, even in the absence of commissions on purchases and sales, annuls any practical, commercial use of these rules. These models do, however, support the reasonableness of representing price fluctuations by a mathematical function.

Approximations to Price Fluctuations

Bachelier's early work, later confirmed by Osborne [7], suggested the normal distribution to represent price changes. On the other hand, more recent studies have indicated that large price changes are in fact observed to occur more frequently than is suggested by the normal distribution. This research points out that price fluctuations follow a bell-shaped curve, but a curve with thicker tails than those of the normal distribution.

To provide a better fit, Mandelbrot [6] suggested the stable Paretian distributions, a class of distributions generally with thick tails but with the disturbing property of infinite variance except in the special, limiting case of the normal. The other well-known member of this class is the Cauchy distribution. The stability of these distributions means that if stock market transactions are spread reasonably uniformly over time and if daily price changes are independent and identically distributed according to a stable Paretian law, then changes over weeks and months also follow a stable Paretian law of precisely the same form but with a different origin and scale. These distributions are the only limiting distributions for sums of independent, identically distributed random variables. Fama [4] has presented much convincing evidence from price changes of several stocks on the New York Stock Exchange to support Mandelbrot's hypothesis, certainly insofar as the accuracy required for our purposes here is concerned.

Use of Stable Paretian Distributions and Log-Normal Distributions

The importance of the Paretian distributions to represent stock price fluctuations is that there is a small but significant probability that substantial losses may occur under variable life insurance with a guaranteed minimum death benefit or a guaranteed cash value at one or more durations. The risk of offering this guarantee is similar to those of stop-loss reinsurance or of the insurance of jumbo jets in that there is a small but not infinitesimal chance of a very large loss.

The family of stable Paretian distributions has the inconvenient property that there are no known closed, analytic functions for the densities except for a few special cases, including the normal and the Cauchy distributions. Except in these cases, numerical approximations must be used (see Fama [4]). Although the weight of evidence favors the stable Paretian distribution for representing stock market price changes rather than the normal or even the better-fitting log-normal (i.e., the natural logarithm of price changes is normally distributed), the log-normal may still be helpful for several reasons. It provides some check on results derived from simulations. It is also far easier to use than the general stable Paretian distribution. To illustrate these points, we give an example of the application of both simulation and mathematical analysis to a comparatively simple problem noted earlier.

In the variable life insurance policy described earlier, the question arises as to the cost of guaranteeing a minimum death benefit under a variable reduced paid-up nonforfeiture benefit. This problem is easier than that of determining the cost on the minimum guaranteed death benefit under the basic policy because the formula for the face amount for each policy year is a simple function of the face amount for the preceding policy year and the investment rate of return during that year and because no account need be taken of premium payments. It may be recalled that this problem is the same as that of determining the cost of a guaranteed minimum death benefit under a single premium variable life insurance policy.

Let i_t represent the net annual investment return. Let F_t represent the face amount of the reduced paid-up benefit during the tth policy year. Then, if i is the AIR,

$$F_{t+1} = \frac{1 + i_t}{1 + i} F_t .$$

We proceed to find the cost of guaranteeing a minimum death benefit equal to the initial amount of the reduced paid-up benefit by using simulation and by using the log-normal distribution to represent annual

TABLE 4

DISTRIBUTION OF NET SINGLE RISK PREMIUMS FOR MINIMUM REDUCED PAID-UP DEATH BENEFIT GUARANTEE PER $1,000 OF INITIAL PAID-UP DEATH BENEFIT

(Fund Charge b)

Option Age	Mean	Standard Deviation	Skewness	Maximum	90th Percentile
45.......	$0.89	$ 2.82	5.8	$ 22.77	$ 2.61
55.......	2.21	6.63	5.5	51.31	6.88
65.......	4.58	12.75	5.0	91.32	14.74
75.......	7.99	19.98	4.3	131.70	29.25

fluctuations in the stock market. The cost is defined as the present value, at 4 per cent interest and with mortality following the parent company's current ultimate experience, of the excess, if any, of the minimum benefit over the benefit that would be payable in the absence of a guarantee.

Simulation

The stock market model discussed above was used to simulate 100 times the underlying investment performance over twenty years; the mean, standard deviation, skewness, maximum, and the 90th percentile of the cost of the guarantee under each of the 100 simulations are shown in Table 4 per $1,000 of guaranteed paid-up benefit for ages 45, 55, 65, and 75 at which this nonforfeiture benefit is elected.

The distribution of the net single risk premiums for this benefit exhibits clearly the extreme skewness which appears characteristic of the cost of a wide variety of guarantees under equity-linked life insurance and annuity products. The mean value is relatively small, the median is in fact zero (and hence is omitted from the table), the standard deviation is fairly large, skewness is extremely pronounced, and the largest observed value is several times the mean. Although the cost for these guarantees may be expected to be moderate in most cases, the rare heavy loss presents a hazard requiring consideration.

Table 4 demonstrates that in most cases claim costs should be moderate, but there remains a small but not negligible probability that the claim cost will be very high. The distribution of these 100 simulated claim costs has a long right tail. As a practical matter the claim cost may be sufficiently high to suggest careful consideration before granting this benefit, at least without some explicit charge.

Analysis

The cost of this benefit can also be calculated under the assumption that the natural logarithm of the percentage annual change in stock prices follows a normal distribution. This distribution is not quite so realistic as the stable Paretian distribution but is chosen here to illustrate the computation because it is straightforward to work with and is often close enough for most practical purposes.

Let I_t be the random variable representing the net investment return in the tth year. Let $X_t = 1 + I_t$ and $Y_n = X_1 X_2 X_3 \ldots X_n$. Let F_n be the random variable representing the natural face amount in the nth policy year. Then, if $F_1 = 1$,

$$F_{n+1} = v^n Y_n .$$

Our basic assumption is that $\log X_t$ is normally distributed with mean μ and variance σ^2. Then $\log Y_n$ is normally distributed with mean $n\mu$ and variance $n\sigma^2$, and $\log F_{n+1}$ is normally distributed with mean $n(\mu - \delta)$ and variance $n\sigma^2$, since

$$\begin{aligned} \log F_{n+1} &= \log (Y_n v^n) \\ &= \log Y_n + n \log v \\ &= \log Y_n - n\delta . \end{aligned}$$

Let Z_t be the random variable representing the claim under the guaranteed minimum death benefit in the tth policy year. Then (if $F_1 = 1$)

$$Z_t = \begin{cases} 1 - F_t & \text{if} \quad F_t < 1 \\ 0 & \text{if} \quad F_t \geq 1 . \end{cases}$$

We use the fact that $\log X_t$ is normally distributed (as is therefore also $\log F_t$) to derive the expected value of Z_t; then the net single premium π_x for the guarantee when the policy is placed under the variable reduced paid-up option at attained age x is expressible as

$$\pi_x = \sum_{t=1}^{\omega-x} v^t \ {}_{t-1}p_x \ q_{x+t-1} E(Z_t) \ .$$

In the Appendix it is shown that

$$E(Z_n) = \Phi(a_n) - \exp\left[n\left(\frac{\sigma^2}{2} + \mu - \delta\right)\right]\Phi(a_n - \sqrt{n}\,\sigma) \ ,$$

where $a_n = -n(\mu - \delta)/\sqrt{n}\,\sigma$ and Φ represents the cumulative distribution function of the standard normal variate.

TABLE 5

EXPECTED ANNUAL CLAIM COSTS FOR A MINIMUM GUARANTEE OF $1,000 PER $1,000 OF INITIAL REDUCED PAID-UP INSURANCE FOR THE FIRST TWENTY YEARS AFTER OPTION ELECTED

Year t	$E(Z_t)$	Year t	$E(Z_t)$
1	$ 0.00	11	$7.72
2	19.97	12	6.76
3	20.00	13	5.92
4	18.46	14	5.17
5	16.59	15	4.53
6	14.74	16	3.99
7	13.01	17	3.48
8	11.45	18	3.06
9	10.04	19	2.67
10	8.80	20	2.34

From analyzing the stock market model, we find the mean and variance of X_t to be approximately 1.0902 and 0.0131, respectively, after taking account of the fund charge. From this it follows that the mean and variance of $\log X_t$ are 0.0809 and 0.0110, respectively. With these values in the expression for $E(X_t)$ and with either tables of the normal distribution or a method of numerical quadrature for an electronic calculator, we can calculate the values of $E(Z_t)$ and π_x directly as shown in Tables 5 and 6.

The expected annual claim cost under the variable reduced paid-up benefit with a minimum guaranteed death benefit decreases fairly steadily, from $20.00 per $1,000 of initial death benefit in the third year after the reduced paid-up option is elected, to $2.34 in the twentieth year and to $0.20 in the fortieth year. These costs will vary with the AIR, the distribution of annual stock market price changes assumed and the fund charge. The net single premiums given in Table 6 are the present values under interest and mortality of the annual claim costs in Table 5. These premiums are very close to those in Table 4, derived by simulation. As expected, they are slightly lower, since the assumption of normality is not conservative and large claims would not arise as often as under the distribution from the stock market model with fluctuations around a predetermined mean based upon actual post-World War II experience.

Table 6 highlights the possibility that the costs for this guarantee may experience large deviations. Both the standard deviations and the skewness coefficients are larger under the log-normal assumption than under

TABLE 6

NET SINGLE RISK PREMIUM FOR A GUARANTEE OF $1,000 PER $1,000 OF INITIAL REDUCED PAID-UP INSURANCE

Option Age	Mean	Standard Deviation	Skewness
45........	$0.89	$10.60	15.8
55........	2.18	16.54	9.9
65........	4.49	23.47	6.8
75........	7.75	30.33	5.0

simulation. This phenomenon reinforces the caution needed in pricing guarantees on equity-linked contracts, quite apart from any consideration of reserve questions (see Coates [3]).

This example is useful for several reasons. In the first place, it estimates the cost for a minimum guaranteed death benefit under a variable reduced paid-up nonforfeiture; the cost is sufficiently high that some companies may pause before granting this type of benefit, at least without explicit charge. This example illustrates the use of an analytic function representing stock price fluctuations to project experience under a variable life benefit, albeit a simple one. It also provides a comparison between the simulation and analytic approaches.

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APPENDIX

We derive an expression for the expected value of the excess, if any, of the minimum guaranteed benefit under the variable reduced paid-up option over the value of this benefit in the absence of a guarantee under the assumption that the logarithm of the annual changes in the value of the underlying investments is normally distributed. Using the notation introduced in the text,

$$Z_n = \begin{cases} 0 & \text{if} \quad F_n > 1 \\ 1 - F_n & \text{if} \quad F_n < 1 , \end{cases}$$

where $\log F_n$ is normally distributed with mean $n(\mu - \delta)$ and variance $n\sigma^2$.

We calculate first the probability that Z_n is zero:

$$\text{Prob}\,(Z_n = 0) = \text{Prob}\,(F_n \geq 1) = \text{Prob}\,(\log F_n \geq 0)$$

$$= \frac{1}{\sqrt{2\pi n}\,\sigma} \int_0^\infty \exp\left\{-\frac{1}{2n\sigma^2}[f_n - n(\mu - \delta)]^2\right\} df_n .$$

Setting

$$w = \frac{1}{\sigma\sqrt{n}}[f_n - n(\mu - \delta)] ,$$

we have

$$\text{Prob}\,(Z_n = 0) = \frac{1}{\sqrt{2\pi}} \int_{a_n}^{\infty} e^{-w^2/2} dw = 1 - \Phi(a_n)\,,$$

where $a_n = -\sqrt{n}(\mu - \delta)/\sigma$.

For $Z_n > 0$, we have

$$\begin{aligned} H(z) &= \text{Prob}\,(Z_n < z) = \text{Prob}\,[1 - \min\,(1, F_n) < z] \\ &= \text{Prob}\,[1 - z < \min\,(1, F_n)] \\ &= \text{Prob}\,(1 - z < F_n) \\ &= \text{Prob}\,[\log\,(1 - z) < \log F_n] \\ &= \frac{1}{\sqrt{2\pi n}\,\sigma} \int_{\log(1-z)}^{\infty} \exp\left\{-\frac{1}{2n\sigma^2}[f_n - n(\mu - \delta)]^2\right\} df_n\,. \end{aligned}$$

Then

$$\begin{aligned} h(z) &= H'(z) \\ &= \frac{1}{\sqrt{2\pi n}\,\sigma(1 - z)} \exp\left\{-\frac{1}{2n\sigma^2}[\log\,(1 - z) - n(\mu - \delta)]^2\right\} \end{aligned}$$

Therefore,

$$\begin{aligned} E(Z_n) &= 0\cdot\Phi(a_n) + \int_0^1 zh(z)dz \\ &= \frac{1}{\sqrt{2\pi n}\,\sigma} \int_0^1 \frac{z}{1 - z} \exp\left\{-\frac{1}{2n\sigma^2}[\log\,(1 - z) - n(\mu - \delta)]^2\right\} dz\,. \end{aligned}$$

This can most clearly be reduced to the desired form by a series of three transformations.

Set $s = \log\,(1 - z)$; then $ds = -dz/(1 - z)$, and

$$E(Z_n) = \frac{1}{\sqrt{2\pi n}\,\sigma} \int_{-\infty}^{0} (1 - e^s) \exp\left\{-\frac{1}{2n\sigma^2}[s - n(\mu - \delta)]^2\right\} ds\,.$$

Set $t = [s - n(\mu - \delta)]/\sqrt{n}\,\sigma$; then

$$\begin{aligned} E(Z_n) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a_n} \{1 - \exp\,[t\sqrt{n}\,\sigma + n(\mu - \delta)]\} \exp\,(-\,t^2/2)dt \\ &= \Phi(a_n) - \frac{1}{\sqrt{2\pi}} \exp\,[n(\sigma^2/2 + \mu - \delta)] \int_{-\infty}^{a_n} \exp\,[-\tfrac{1}{2}(t - \sqrt{n}\,\sigma)^2]dt\,. \end{aligned}$$

Set $v = t - \sqrt{n}\,\sigma$; then

$$E(Z_n) = \Phi(a_n) - \exp[n(\sigma^2/2 + \mu - \delta)]\Phi(a_n - \sqrt{n}\,\sigma) .$$

We give here the derivation, due to Bowers, for the expressions for the second and higher moments of Z_n used to calculate the standard deviation and the skewness coefficient in Table 6.

Let C be the random variable equal to the present value of the claim payment Z_t if death occurs in the tth year after reduced paid-up option is elected. In the notation introduced earlier,

$$C = \sum_{t=1}^{\omega-x} v^t\, {}_{t-1}p_x\, q_{x+t-1} Z_t .$$

Let us introduce the indicator random variable I_t which takes on the value 1 if a death occurs in year t and 0 if death does not occur in year t; that is,

$$I_t = \begin{cases} 1 & \text{with probability} \quad {}_{t-1}p_x\, q_{x+t-1} \\ 0 & \text{with probability} \quad 1 - {}_{t-1}p_x\, q_{x+t-1} . \end{cases}$$

This random variable has the following simple properties:

$$I_t^2 = I_t\,, \quad E(I_t) = E(I_t^2) = {}_{t-1}p_x\, q_{x+t-1}\,, \quad E(I_sI_t) = 0 \quad \text{for} \quad s \neq t .$$

The present value of claim costs, C, is now expressible as

$$C = \sum_{t=1}^{\omega-x} v^t I_t Z_t$$

and

$$C^2 = \left(\sum_{t=1}^{\omega-x} v^t I_t Z_t\right)^2 = \sum_{t=1}^{\omega-x} v^{2t} I_t Z_t^2 + 2\sum_{s\neq t} v^{s+t} I_s I_t Z_s Z_t .$$

It now follows that

$$E(C^2) = \sum_{t=1}^{\omega-x} v^{2t}\, {}_{t-1}p_x\, q_{x+t-1} E(Z_t^2) .$$

By an argument similar to that adduced above for $E(Z_n)$, it follows that $E(Z_n^2)$ can be expressed in terms of a standard normal distribution:

$$E(Z_n^2) = \frac{1}{\sqrt{2\pi n}\,\sigma} \int_0^1 \frac{z^2}{1-z} \times \exp\left\{-\frac{1}{2n\sigma^2}[\log(1-z) - n(\mu-\delta)]^2\right\} dz .$$

Set $s = \log(1-z)$; then

$$E(Z_n^2) = \frac{1}{\sqrt{2\pi n}\,\sigma} \int_{-\infty}^{0} (1 - 2e^s + e^{2s}) \exp\left\{-\frac{1}{2n\sigma^2}[s - n(\mu-\delta)]^2\right\} ds .$$

If we successively let $t = [s - n(\mu - \delta)]/\sqrt{n}\,\sigma$ and $v = t - 2\sqrt{n}\,\sigma$, we find

$$E(Z_n^2) = \Phi(a_n) - 2\exp\left\{\frac{n}{2}[\sigma^2 + 2(\mu - \delta)]\right\}\Phi(a_n - \sqrt{n}\,\sigma)$$

$$+ \exp\left\{\frac{n}{2}[(2\sigma)^2 + 4(\mu - \delta)]\right\}\Phi(a_n - 2\sqrt{n}\,\sigma) .$$

Similarly, for third and higher moments,

$$E(Z_n^r) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\exp\left\{\frac{n}{2}[(\sigma j)^2 + 2j(\mu - \delta)]\right\}\Phi(a_n - j\sqrt{n}\,\sigma) .$$

DISCUSSION OF PRECEDING PAPER

JOHN M. BOERMEESTER:

Paul Kahn has offered a most intriguing and informative paper concerning the possibilities of operating a variable life insurance company. The only trouble with the paper is that it seems to raise too many questions in the mind of the reader. Perhaps the author has overestimated our ability to interpret his astute observations. Understandably, he may have felt a restraint in some areas in exposing certain delicately guarded corporate assumptions. In any event, I would like to make a few comments and, incidentally, ask a few questions, in the hope that the author might be able to answer them when he replies to the discussants:

1. A graduation was obtained of the distribution of the monthly changes (as well as annual changes) for the Standard and Poor's price index of industrial stocks. If this graduation for the monthly changes is in a mathematical form, what are the parameters?
2. It would be interesting to know the distribution of the model-office issues by number of contracts, plan, age, sum insured, and premium.
3. A market strategy evidently was used in simulating the cash flow. What were the assumptions regarding the timing and volume of sales?
4. A "special" reserve was established by transfer when the death benefit fell below the guaranteed amount. How was this "special" reserve calculated?
5. Five sets of investment performances were used for illustration, 9.1 per cent being the average. What position did the four other rates occupy in the distribution of possible investment return?
6. Dividends were assumed to be paid to the parent. What are the features of the dividend program?
7. A few more details of the variable policy would be welcome to the uninitiated reader. For example, how often are the death benefits adjusted?

Finally, I would like to point out, in the area of pricing guarantees, that an estimate of the 90th percentile of a true distribution obtained from a sample of one hundred simulations could be quite far from the true value, particularly if the underlying distribution has a long tail. For example, in connection with the value of \$2.61 for age 45, shown as an estimate of the 90th percentile in Table 4, one can say only with 95 per cent confidence that the true value lies in the range bounded by the 83d and 96th values of the ordered statistics of the hundred simulations. Unfortunately, the author states that the cost of performing a larger number of simulations would be high. However, if a thousand were performed, then one could say with 95 per cent confidence that the true value of the 90th percentile lies in a relatively much more compressed range within the ordered simulated values. For those who are interested, a basis for making statements of this type concerning confidence may be found in *Introduction to Mathematical Statistics*, by Hogg and Craig.

JOHN C. FRASER:

In reading Dr. Kahn's paper, we were startled by the figures in Table 1 indicating the serious losses to the parent company that would result from a 50 per cent increase in the rates of withdrawal.

Our tests at New York Life indicate that an increase in the rates of withdrawal is not a serious problem on variable life insurance—no more so than on fixed-dollar life insurance. Perhaps this is because our proposed policy is participating, and under such conditions the initial expense deficit can be amortized more rapidly. On the other hand, perhaps the $10 million paid in capital to the Equitable subsidiary is what is causing the problem. We do not plan to use a subsidiary and as a result would not incur anything like a $10,000,000 start-up cost.

JOHN B. CUMMING:

Although Dr. Kahn and I work for the same company, I was not involved in the project on which he reports. Thus my remarks are those of an interested, although unknowledgeable, layman. Dr. Kahn has prepared a lucid and thought-provoking paper.

A particular problem of special interest, mentioned by Dr. Kahn in passing, is the correlation between investment performance and persistency. It seems certain that there will be some correlation between these two, but we have no comparable historical evidence to provide a basis for projecting that correlation. Although a study of mutual fund redemptions might provide some evidence, the life insurance element in the contract is likely to change the results. One would hope that the presence of life insurance would improve persistency relative to that of a mutual fund.

It would be of interest to have Dr. Kahn's thoughts about the effect on yield to the parent company, if policyholders cash out in a falling market and buy, or reinstate their contracts, in a rising market. To what extent is it practical to study this problem? With computers it should be feasible to vary lapse rates as a function of the market. However, in the absence of better knowledge of the nature of that variance, the results might be so speculative as to call into question the validity of the projection itself.

FRANK P. DI PAOLO:

An insurance company can develop a competitive and salable variable insurance product if it is able to obtain better than Standard and Poor's (or Dow Jones) investment results. In the case of a common stock portfolio, however, better-than-average performance is generally associated with higher-than-average risk, which in turn may result in a higher probability of ruin if the company guarantees a minimum death benefit and/or cash values.

In a study of mutual funds, Professor Irwin Friend[1] proved that portfolios producing better-than-average returns have higher variances and beta coefficients—both measures of volatility. In fact, the mean variance of the group of portfolios that Professor Friend classified as "high risk" was $2\frac{1}{2}$ times the mean variance of the "low-risk" group, while the mean return of the "high-risk" group was only 11 per cent better for the period January, 1960—March, 1964, and 36 per cent better for the period April, 1964—June, 1968.

Now, if the investment manager of Dr. Kahn's model variable insurance company will be able to obtain a mean return slightly better than the Standard and Poor's but with a substantially higher variance, the cost of the mortality and/or asset share guarantee could increase materially. For example, if we were to increase the mean and standard deviation of log X_t by 10 per cent and 50 per cent, respectively, the values of $E(Z_t)$ given in Table 5 would be about twice as large in the early years and about five times in the twentieth year. A 10 per cent increase in the mean of log X_t corresponds to an increase in the expected annual return of about 10 per cent.

RICHARD Q. WENDT:

Dr. Kahn is to be congratulated for presenting an excellent paper on the subject of variable life insurance. I am sure that this paper will help to illuminate some of the darkness in this new actuarial field.

I will confine my comments on the paper to the subject of the projection of investment performance of the company's portfolio. As Dr. Kahn states, the simulation model appears to be highly sensitive to variations in investment performance.[2]

It appears that the simulation model implicitly assumes that the

[1] Irwin Friend, Marshall Blume, and Jean Crockett, *Mutual Funds and Other Institutional Investors* (New York: McGraw-Hill Book Co., Inc., 1971).

[2] The views expressed herein are those of the author and do not necessarily reflect the views of the Securities and Exchange Commission or of the author's colleagues on the staff of the commission.

investment performance of the insurance company's portfolio will duplicate the performance of the stock market, and it is this assumption upon which I will focus my comments. The assumption is apparently based on the belief that any well-diversified common stock portfolio will closely follow the performance of the stock market. However, an analysis of the investment performance of mutual funds shows that, in fact, it is relatively rare for the performance of actual common stock portfolios to duplicate market performance. There are two important factors, in addition to stock market performance, which control the investment performance of any portfolio of common stocks: the volatility of the portfolio relative to the market and the ability of the investment manager. The effect of the portfolio's volatility on investment performance has become well known in recent years; the mutual funds which did so much better than the bull market of 1968 performed much worse than the bear market of 1969. It was not uncommon for a mutual fund with an aggressive investment policy to increase and decrease twice as much as the stock market during those years.

The recent *Institutional Investor Study Report* of the Securities and Exchange Commission analyzed the performance of 236 mutual funds over the period 1960–69 and of 80 insurance company separate accounts over the period 1965–69.[3] The analysis was conducted by determining the alpha and beta parameters for the regression model

$$R_F - R_{RF} = \alpha + \beta(R_M - R_{RF}) ,$$

where R_F is the return on the fund portfolio; R_{RF} is the return on a risk-free asset, for example, 30-day Treasury Bills; and R_M is the return on the market portfolio, for example, Standard and Poor's 500 Index.

The quantities R_F, R_{RF}, and R_M were computed on a monthly time period and took account of capital appreciation and dividend income (dividends were assumed to be reinvested). The dependent variable in the regression is the "risk premium" on the fund portfolio—the difference between the fund's return and the return on a risk-free asset; the independent variable is the "risk premium" of the market portfolio. The beta parameter is a measure of the volatility of the portfolio relative to market performance, while the alpha parameter, the risk-adjusted performance of the fund, can be thought of as a measure of the ability of the investment manager. It should be noted that the mutual fund

[3] The description of the methodology used is given in the Appendix to Sec. F of chap. 4 of the *Report;* the results of the analysis of mutual fund performance are given in Sec. I of chap. 4; and the results of the analysis of separate account performance are given in Sec. F(5) of chap. 6.

performance was determined net of expenses and without regard to the amount of cash held in the fund, whereas the simulation model is based on the performance, before expenses, of the amount actually invested.

The Institutional Investor Study concluded that there was a strong relationship between investment objectives and volatility. Table 1 of this discussion shows the distribution of funds according to volatility and investment objective for 125 of the 236 mutual funds, with complete data for the period January, 1960—December, 1969. Of the 93 funds

TABLE 1*

RELATIONSHIP BETWEEN STATED INVESTMENT OBJECTIVES AND MUTUAL FUND VOLATILITY

(125 Funds for 1960-69 Period)

Volatility Range	Capital Gain	Growth	Investment Objective: Growth Income	Investment Objective: Income	Total
0-0.4	0	0	0	3	3
0.4-0.8	0	5	18	12	35
0.8-1.0	2	7	33	2	44
1.0-1.2	5	21	4	0	30
Over 1.2	8	5	0	0	13
Total	15	38	55	17	125

* Source: Securities and Exchange Commission. *Institutional Investor Study Report*, Table IV-105.

with an investment objective of either growth or growth and income, 65 (70 per cent) had volatility between 0.8 and 1.2; the remaining 30 per cent differed from the market volatility by more than 20 per cent. These results are consistent with the results for the entire group of 236 mutual funds with nine or more observations during the January, 1965—December, 1969, period.

The study's analysis of separate account performance included separate accounts registered under the Investment Company Act of 1940 and nonregistered separate accounts, both on a commingled and on an individual employer basis. Of the 80 separate accounts studied, 54 (68 per cent) had a volatility between 0.8 and 1.2. The 10 separate accounts with a volatility greater than 1.2 had an average volatility of 1.36, or 36 per cent higher than the market volatility.

Analysis of the results presented by the Institutional Investor Study shows that the volatility of the portfolio should not be implicitly assumed to be 1.00; rather, there should be an explicit assumption of portfolio volatility, taking into account the investment objectives of the portfolio. On the other hand, it would not be unreasonable to explicitly assume a portfolio volatility of 1.00, if that were consistent with the desired investment objective.

However, the assumption of a zero or positive alpha value may be more optimistic than the situation warrants. The study reported that, of the 125 mutual funds with complete data available between 1960 and 1969, 47 per cent had a negative alpha value. This would appear to be a relatively high percentage, but the percentage of negative alpha values for the 1960–64 period is even greater: 58 per cent of the 125 funds with complete data during the period had a negative alpha value. Over the 1965–69 period, 30 per cent of the 125 mutual funds had a negative alpha value. The results reported for the 236 mutual funds with at least nine observations for ten years are consistent with these figures.

An indication of the level of alpha values before deduction for expenses can be obtained by utilizing the distribution of alpha values reported by the study. If it is assumed that deductions for expenses from each mutual fund amounted to 1 per cent of net assets, then approximately 22 per cent of the 125 mutual funds had negative alpha values, on a gross basis, for the 1960–69 period.

Whereas the volatility of the portfolio may in large measure be controllable by the investment manager, the analysis of the alpha parameters shows that in a significant portion of the mutual funds the investment managers have not been able to attain an alpha value greater than zero. One objection to this analysis of investment management might be that risk-adjusted performance is a relatively new concept and fund advisers were not concerned with maximizing alpha values during the 1960–69 period. However, this objection is only partially valid. Although the risk-adjusted performance concept has only recently been developed in a mathematical context, previously an investment adviser who attempted to maximize his "reward" for a given level of risk would also be attempting to maximize the alpha value for his portfolio, even if he did not think about what he was doing in these terms.

Because of the "extreme sensitivity of the earnings of a variable life insurance company to a combination of unfavorable persistency with investment performance" and the wide variations in the cost of the minimum guaranteed death benefit due to investment performance, it would seem appropriate to consciously include alpha and beta parameters in the simulation model to project the investment performance of the portfolio with respect to the market performance. As a further modification, the simulation model could be used to determine the most favorable portfolio volatility level and the effect of different alpha values on the earnings of the company.

ANNA M. RAPPAPORT:

I enjoyed this paper very much. The author should be commended for using risk theory to provide a new and creative approach to the pricing of what appear at first glance to be benefits which cost very little.

Very often actuaries ignore, or arbitrarily add, a nominal premium for benefits which have a low expected value. I encourage people to take a long, hard look at the techniques used by Dr. Kahn and to consider their possible application to pricing of other benefits in individual policies.

DALE F. ETHINGTON:

The stock market model used by the author would appear to yield a reasonable simulation of stock prices. As such, it is useful in determining possible financial experience of the subsidiary. However, great care must be taken in interpreting statistics, such as those found in Table 3, based on 100 sets of investment experience, each over 55 years, generated by this model. In studies I have done on minimum benefit guarantees, results have been a function of the average rate actually earned as well as fluctuations around that rate. That is, if the average rate over 50 years is 4 per cent, the cost is likely to be higher than if the average rate over the same period is 6 per cent.

An alternate method would be to group the simulations based on average yield over the period selected. Probabilities could be assigned to these yield groups independently, based on economic analysis and judgment. This method has the advantage that it separates the average rate of return and fluctuations around that rate into separate elements. Thus, if the yield is within, say, 1 per cent of the assumed mean rate, then a cost distribution based solely on market fluctuations can be examined. Further, the consequences of earning more or less than the assumed mean rate can be examined. The disadvantages of this approach are (1) that more simulations are needed to get reliable distributions at the high and low ends of average yield and (2) that probabilities must be determined for each yield group.

The proposed method may not give results different from or better than the author's. However, if the method used in the paper is to be followed, a presentation of the distribution of average yields and a cost distribution based on a range of yields around the assumed average rate would aid considerably in interpreting and understanding the results.

WILLIAM A. BAILEY:

Dr. Kahn presented a very stimulating paper. I would like to present a table which corresponds to Table 5 in Dr. Kahn's paper. My table (Table 1 below) is based on Mr. DiPaolo's Table 2 on page 552 of Volume XXI of the *Transactions* and assumes that the amount of the minimum guaranteed death benefit for year t is based on the value of the equity fund at duration $t - 1$. I am not sure what assumed interest rate (AIR) Dr. Kahn used in calculating his Table 5. My figures have been calculated using a method of implicit enumeration. Frequency distributions of the

TABLE 1

EXPECTED ANNUAL CLAIM COSTS FOR A MINIMUM GUARANTEE OF $1,000 PER $1,000 OF INITIAL REDUCED PAID-UP INSURANCE FOR YEARS AFTER OPTION ELECTED

Year t	Kahn's $E(Z_t)$	Bailey's $E(Z_t)$		
		AIR = 0%	AIR = 4%	AIR = 10.62%
1	$ 0.00	$ 0.00	$ 0.00	$ 0.00
2	19.97	26.53	37.58	61.06
3	20.00	24.92	42.61	86.23
5	16.59	18.69	43.43	121.94
6	14.74	15.86	42.31	136.04
11	7.72	6.73	33.89	191.76
20	2.34	...	...	...
21	...	1.22	19.61	268.07
40	0.20	...	...	...
41	...	0.04	6.39	373.07
51	...	0.01	3.67	412.01

accumulated value of a single investment of $1.00 after periods of 1, 12, 24, 48, 60, 120, 240, 480, and 600 months are available on request. Mr. DiPaolo's Table 2 assumes an annual upward drift of 10.62 per cent, whereas Dr. Kahn assumes about 9.02 per cent. Note that this would tend to make my $E(Z_t)$ less than Dr. Kahn's, whereas the reverse is actually the case.

Essentially, two ways of measuring the risk involved in issuing a single policy with a guaranteed minimum death benefit are as follows:

1. Calculate a frequency distribution of $v^t Z_t$, where this function represents the present value (discounted at interest only) of the guaranteed minimum death benefit *payable once* at the time this particular life insured dies; the probability associated with a particular value of $v^t Z_t$, say, $v^t Z$, would be ${}_{t-1|}q_x h_t(Z)$, where ${}_{t-1|}q_x$ is the probability of the life insured's dying in the tth year and $h_t(Z)$ is the probability that a value of

Z will be assumed by the variable Z_t for the tth year. The mean of this frequency distribution is

$$\sum_{t=1}^{\omega-x} \sum_{\text{All } Z} [v^t Z \; {}_{t-1|}q_x h_t(Z)] = \sum_{t=1}^{\omega-x} v^t \; {}_{t-1|}q_x \Big[\sum_{\text{All } Z} Z h_t(Z) \Big] .$$

2. Calculate a frequency distribution of

$$\sum_{t=1}^{\omega-x} v^t \; {}_{t-1|}q_x Z_t \, ,$$

where this function represents the present value (discounted for interest and mortality) of the *cost of coverage* equal to the guaranteed minimum death benefit *each year* during which the life insured is expected to be alive; the probability associated with a particular value Z of Z_t is $h_t(Z)$. The mean of this frequency distribution is

$$\sum_{t=1}^{\omega-x} v^t \; {}_{t-1|}q_x \Big[\sum_{\text{All } Z} Z h_t(Z) \Big] .$$

Thus each way produces the same mean; however, the variances (or other moments) are not equal. That is, approaches (1) and (2) produce different frequency distributions, even though the means are equal. I believe that Tables 4 and 6 in Dr. Kahn's paper are based on approaches (2) and (1), respectively, and therefore do not represent the same random variable.

The random variable C, in Dr. Kahn's Appendix, is defined both as

$$C = \sum_{t=1}^{\omega-x} v^t \; {}_{t-1}p_x q_{x+t-1} Z_t \qquad \text{(A)}$$

and

$$C = \sum_{t=1}^{\omega-x} v^t I_t Z_t \, , \qquad \text{(B)}$$

where

$$I_t = \begin{cases} 1 & \text{with probability} \quad {}_{t-1}p_x q_{x+t-1} \\ 0 & \text{with probability} \quad 1 - {}_{t-1}p_x q_{x+t-1} \, . \end{cases}$$

It would be clearer if I_t were denoted as a function $I_t(T)$ of the random variable T which represents the year of death; $I_t(T) = 1$ if $T = t$ and $I_t(T) = 0$ if $T \neq t$. Then definition (B) of the random variable C relates to way (1) of measuring the risk involved in issuing a single policy with a guaranteed minimum benefit, while definition (A) relates to way (2). Since definitions (A) and (B) of C are not identical, this causes some confusion both in the Appendix and in understanding the basis of Tables 4 and 6. I hope that the author will clarify this matter in his reply.

There is another point concerning the Appendix which needs clarification. If Z_n, the amount of guarantee for year n, depends on the investment experience of the first $(n - 1)$ years only, then formulas for $E(Z_n^r)$ in the Appendix are actually formulas for $E(Z_{n+1}^r)$, since they are based on $Y_n = X_1X_2 \ldots X_n$ and not on $Y_{n-1} = X_1X_2 \ldots X_{n-1}$. This may or may not require some correction of the values in Table 6.

Table 4 represents measures (mean, standard deviation, and so on), of the possible outcomes where a single policy is involved. The extension to more than one policy issued at different ages for different amounts in different calendar years has required simulation techniques. However, I predict that use of simulation (that is, a Monte Carlo approach involving the generation of random numbers) to project such things as variable life insurance operations will soon be rendered obsolete. Replacing simulation will be methods of implicit enumeration. Problems will be structured as one-, two-, or even three-dimensional random walks, provided that the process is Markovian. Some problems which appear to be non-Markovian can be structured as Markov processes by suitably defining the "states" involved. For example, instead of using simulation techniques to calculate Dr. Kahn's Table 4, it is possible to define a Markov process wherein the "states" would be defined by two variables; for example, (1) the total present value cost of the coverage equal to the guaranteed minimum death benefit through duration t and (2) the value of the equity account at duration t. Although we would presumably only be interested in the first variable, the second is required in order to determine the appropriate transition probabilities.

Another example would be a return of premium benefit under disability income policies. Here the "states" of the Markov process would be defined by the following variables: (1) active lives and disabled lives at duration t; (2) if disability exists, length of time since date of disablement; and (3) total benefits paid through duration t. The first and second are needed to enable us to calculate transition probabilities; the third is the variable whose frequency distribution we really want. Absorbing barriers can be imposed to take account of the fact that no return of premium benefit will be payable if the total of benefits paid has exceeded some portion of the premiums.

In the process we will be able to produce complete frequency distributions based on empirical data, without the need to fit analytical curves to existing data. Although measures such as standard deviations, skewness, and so on, will be produced as by-products, we will no longer have to be content with merely having these measures; the whole frequency distribution will be available to us, enabling us to answer many different questions about the financial risks involved.

(AUTHOR'S REVIEW OF DISCUSSION)

PAUL MARKHAM KAHN:

Mr. Boermeester has requested detailed specifications of the company model. (1) The graduation of the monthly index values was in the form of a histogram and was not approximated by an analytic function. (2) The model was based on one plan of insurance (variable ordinary life), with 40 per cent sold at age 25, 40 per cent at age 35, 15 per cent at age 45, and 5 per cent at age 55. (3) All funds except for cash were invested in equities. The portfolio turnover rate was $1\frac{1}{4}$ per cent monthly (15 per cent annually), and monthly dividends were assumed at a rate of $3\frac{1}{2}$ per cent applied to the product of (*a*) the total number of shares at the beginning of the month and (*b*) an average value per share over the preceding several months. (4) The special reserve was equal to the amount necessary to bring the reserve to the level of the reserve on a fixed-dollar policy for the guaranteed amount. (5) The other rates are given in the text. (6) Surplus in excess of 10 per cent of the assets was paid to the parent. (7) The death benefit varies annually.

Mr. John Fraser expressed surprise at the unfavorable yields to the parent company if withdrawal rates are high (from −8.8 per cent to −33.9 per cent). He suggests that this finding may result from the parent's $10 million start-up cost and that the results would not necessarily hold for a company writing variable life insurance directly rather than through a subsidiary. During the first two decades of operations the parent may expect to put into the subsidiary $150 million, mainly for new business strain. If variable life insurance is written directly, this cost would to some extent be camouflaged, since a "yield to the parent" would not be calculated explicitly. It is hard to see that this cost could be avoided, however. This raises the related question of reserve strengthening if a prospective reserve approach is taken. If investment performance turns out to be very bad over a prolonged period, some recent studies with the model indicate a potential strengthening of $500 million for a subsidiary with $6 billion in assets. These are serious matters.

Mr. John Cumming, in noting the sensitivity to bad lapse experience and bad investment performance of the yield-to-the-parent figures in Table 1, suggests that we test the model with lapse rates which are a function of the market. This would have been done except for the cost.

Mr. DiPaolo gives some figures which suggest that the cost of guaranteed minimum death benefits is extremely sensitive to the insurer's investment policy as defined by the standard deviation of the annual changes in the value of its portfolio. For example, if the mean and standard deviation of the logarithm of the annual rate of return are increased by 10 per cent and 50 per cent, respectively, the expected values of the annual claim costs as given by Table 5 would increase by a factor of 2 in the early years and by a factor of 5 in the twentieth policy year.

Mr. Wendt's discussion highlights the standard deviations of portfolio rates of return which are met in practice.

The author agrees with Mrs. Rappaport's suggestion that the techniques described in this paper could be used for pricing benefits which are commonly ignored or handled by a nominal premium.

Mr. Ethington and Mr. Boermeester question the sufficiency of one hundred simulations for pricing guarantees. I agree that more simulations would be desirable, but the cost element played the dominant role in selecting the number.

Mr. William Bailey notes that the standard deviations for the guaranteed minimum death benefits in Table 6 derived from mathematical analysis are larger than those in Table 6 for the simulation case. A difference between these two approaches, which should have been noted in the paper, is that the formula used for the standard deviations given in the Appendix for the analytic case takes account of variation due to both market fluctuations and time of death, while for the simulation case only the variation arising from investment performance is considered.

It should be noted that the random variable Z_n in the Appendix, with mean $n(\mu - \delta)$, should refer to the claims in the $(n + 1)$st policy year.

The author expresses his thanks to the discussants for their stimulating comments.