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THE NEW YORK LIFE VARIABLE LIFE INSURANCE DESIGN ON A DAILY BASIS

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ABSTRACT

This paper develops the theoretical formula for the face amount under the New York Life variable life insurance design adapted to a daily basis. In addition, a formula is given, for use in practice, which represents an approximation to the theoretical formula, and a discussion is included of the adequacy of that approximation. Also derived is a formula for the daily net premium, and a table of these premiums is appended.

INTRODUCTION

The basic theory underlying the New York Life's proposed variable whole life insurance policy was introduced in the paper "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance" by John C. Fraser, Walter N. Miller, and Charles M. Sternhell in Volume XXI of the *Transactions of the Society of Actuaries*. For simplicity almost all of the theory in that paper was presented on a traditional functions basis. In practice, however, the New York Life intends to have the face amount of its proposed policy vary daily. The development of the face amount formula appropriate for a daily basis is the focus of this paper.

I wish to thank John C. Fraser, Walter N. Miller, and Gordon D. Shellard, who reviewed the paper and offered a number of constructive suggestions.

DERIVATION OF THE THEORETICAL FORMULA FOR THE DAILY FACE AMOUNT

The recursive formula for the face amount for a whole life policy, developed by using traditional functions as in equation (6) of the paper cited, is

$$F_t = F_{t-1}\left(\frac{{}_{t-1}V_x + P_x/F_{t-1}}{{}_{t-1}V_x + P_x}\right)\left(\frac{1+i'_t}{1+i}\right), \tag{1}$$

where

F_{t-1} and F_t = Face amounts at the end of the $(t-1)$st and tth policy years, respectively;

$_{t-1}V_x$ = Terminal reserve at the end of policy year $t - 1$ for a whole life policy issued at age x;

P_x = Net level annual premium for a whole life policy issued at age x, providing a death benefit at the end of the year of death;

i'_t = Actual net annual investment rate earned by the separate account during the tth policy year;

i = Annual interest rate assumed in the calculation of net premiums and reserves.

Since the underlying theory presented in the paper is general, however, the above formula for the annual face amount will also be the theoretically correct recursive formula for the daily face amount, provided that we reinterpret the symbols accordingly.

Thus, after slightly modifying the symbols of equation (1) to make them more compatible with a daily basis, we have, as the theoretical formula for the daily face amount,

$$F_{t+r/365} = F_{t+(r-1)/365}\left[\frac{_{t+(r-1)/365}V^{(365)}(\bar{A}_x) + {}^d\Pi_x/F_{t+(r-1)/365}}{_{t+(r-1)/365}V^{(365)}(A_x) + {}^d\Pi_x}\right] \times \left[\frac{1 + i'_{t+r/365}}{1 + i^{(365)}/365}\right], \tag{2}$$

where

$F_{t+(r-1)/365}$ and $F_{t+r/365}$ = Face amounts at the end of the $(r - 1)$st and rth days, respectively, of the $(t + 1)$st policy year;

$_{t+(r-1)/365}V^{(365)}(\bar{A}_x)$ = Reserve at the end of the $(r - 1)$st day of the $(t + 1)$st policy year for a whole life policy with daily premiums and immediate payment of claims;

${}^d\Pi_x$ = Net level daily premium for a whole life policy issued at age x, providing an immediate death benefit;

$i'_{t+r/365}$ = Actual net *daily* investment rate earned by the separate account on the rth day of the $(t + 1)$st policy year;

$i^{(365)}/365$ = *Daily* interest rate assumed in the calculation of net premiums and reserves.

Thus $i^{(365)}/365$ is equal to $(1 + i)^{1/365} - 1$, where i is the equivalent assumed *annual* interest rate.

The terms in equation (2) that require further analysis are ${}^d\Pi_x$ and

${}_{t+(r-1)/365}V^{(365)}(\bar{A}_x)$, representing the daily net premium and daily reserve, respectively. That is, in order to utilize equation (2) for actual calculation of face amounts, we must have explicit expressions for both these terms. An expression for ${}^d\Pi_x$ is derived in the Appendix, and a table of ${}^d\Pi_x$ on a 1958 CSO 3 per cent net level basis for issue ages 0–75, male and female, is included. To determine an expression for ${}_{t+(r-1)/365}V^{(365)}(\bar{A}_x)$, consider the basic equation of equilibrium introduced as equation (1) in the paper cited:

$$({}_{t-1}V_x + P_x)(1 + i) = q_{x+t-1}(1 - {}_tV_x) + {}_tV_x \,, \tag{3}$$

where all symbols have their customary meanings. Equation (3) represents the relation between successive *annual* terminal reserves for a whole life insurance policy payable at the end of the year of death and is thus based on *traditional* functions.

Consistent with the logic followed previously in transforming the annual face amount formula into the daily face amount formula, equation (3) will also hold as the equation of equilibrium connecting successive *daily* reserves, provided that we reinterpret the symbols accordingly. We must note, however, that the "1" in the expression $(1 - {}_tV_x)$ in equation (3) represents the benefit paid at the end of the year of death. For the daily case, the analogue of the "1" must represent the benefit paid at the end of the day of death. Since the benefit we are considering is, however, an immediate death benefit, as opposed to the traditional death benefit reflected in equation (3), the "1" is no longer appropriate. Instead, what we require is the equivalent value at the end of the day of a benefit of 1 paid immediately at death during the day, or, equivalently, we need the value of K in the following equation:

$$\bar{A}^{\,1}_{y:\overline{1/365}|} = K A^{(365)\,1}_{y:\overline{1/365}|} \,, \tag{4}$$

where

$\bar{A}^{\,1}_{y:\overline{1/365}|}$ = A one-day insurance on a life aged y, payable at the moment of death during that day;

$A^{(365)\,1}_{y:\overline{1/365}|}$ = A one-day insurance on a life aged y, payable at the end of that day if death occurs during that day.

From basic principles, we have

$$\bar{A}^{\,1}_{y:\overline{1/365}|} = \int_0^{1/365} v^s \, {}_sp_y \, \mu_{y+s} ds \,, \tag{5}$$

where all symbols have their customary meaning.

Assuming a uniform distribution of deaths over the entire year, we have

$${}_sp_y \, \mu_{y+s} \doteq q_y \,, \tag{6}$$

and therefore

$$\bar{A}^{\;1}_{y:\overline{1/365}|} \doteq \int_0^{1/365} v^s \, q_y ds \, , \tag{7}$$

or

$$\bar{A}^{\;1}_{y:\overline{1/365}|} \doteq q_y \int_0^{1/365} v^s ds \, . \tag{8}$$

The integral in equation (8) represents a continuous annuity certain for one day, $\bar{a}_{\overline{1/365}|}$. Thus

$$\bar{a}_{\overline{1/365}|} = \frac{1 - v^{1/365}}{\delta} \, . \tag{9}$$

Substituting this value for the integral in equation (8), we have

$$\bar{A}^{\;1}_{y:\overline{1/365}|} \doteq \left(\frac{1 - v^{1/365}}{\delta}\right) q_y \, . \tag{10}$$

On the other hand, also from basic principles, we have

$$A^{(365)\;1}_{y:\overline{1/365}|} = v^{1/365} \; {}_{1/365}q_y \, , \tag{11}$$

where ${}_{1/365}q_y$ is the probability that a life aged exactly y will die in the next day.

By the assumption of a uniform distribution of deaths over the entire year, we have

$${}_{1/365}q_y \doteq \tfrac{1}{365}(q_y) \, . \tag{12}$$

Therefore,

$$A^{(365)\;1}_{y:\overline{1/365}|} \doteq v^{1/365}(\tfrac{1}{365})q_y \, . \tag{13}$$

Substituting the values for $\bar{A}^{\;1}_{y:\overline{1/365}|}$ and $A^{(365)\;1}_{y:\overline{1/365}|}$ from equations (10) and (13), respectively, into equation (4) and solving for K, we have

$$K \doteq \frac{[(1 - v^{1/365})/\delta]q_y}{v^{1/365}(\tfrac{1}{365})q_y} \, . \tag{14}$$

Using the relationship

$$v^{1/365} = \frac{1}{1 + i^{(365)}/365} \, , \tag{15}$$

equation (14) simplifies to

$$K \doteq \frac{i^{(365)}}{\delta} \, . \tag{16}$$

The expression for K in equation (16) corresponds to the familiar i/δ in the annual case, and the derivations of the two expressions are entirely analogous.

Returning now to the transformation of equation (3), we have finally, after reinterpreting symbols and substituting the value of K just derived, the following equation as the equation of equilibrium connecting successive daily reserves:

$$\begin{aligned}&[{}_{t+(r-1)/365}V^{(365)}(\bar{A}_x) + {}^{d}\Pi_x]\left[1 + \frac{i^{(365)}}{365}\right] \\ &= {}_{1/365}q_{x+t+(r-1)/365}\left[\frac{i^{(365)}}{\delta} - {}_{t+r/365}V^{(365)}(\bar{A}_x)\right] + {}_{t+r/365}V^{(365)}(\bar{A}_x) .\end{aligned} \tag{17}$$

Since an expression for ${}^{d}\Pi_x$ is derived in the Appendix, as previously noted, the final term that must be evaluated before equation (17) can be used to generate daily reserves is ${}_{1/365}q_{x+t+(r-1)/365}$. By definition,

$${}_{1/365}q_{x+t+(r-1)/365} = \frac{l_{x+t+(r-1)/365} - l_{x+t+r/365}}{l_{x+t+(r-1)/365}} , \tag{18}$$

where $l_{x+t+(r-1)/365}$ and $l_{x+t+r/365}$ have their customary meanings, and ${}_{1/365}q_{x+t+(r-1)/365}$ is as previously defined in equation (11).

Assuming a uniform distribution of deaths throughout a policy year, we have

$$l_{x+t+(r-1)/365} \doteq l_{x+t} - \left(\frac{r-1}{365}\right) d_{x+t} \tag{19}$$

and

$$l_{x+t+r/365} \doteq l_{x+t} - \left(\frac{r}{365}\right) d_{x+t} . \tag{20}$$

Substituting the above values in equation (18) and simplifying, we have

$${}_{1/365}q_{x+t+(r-1)/365} \doteq \left(\tfrac{1}{365}\right) \frac{d_{x+t}}{l_{x+t} - [(r-1)/365]d_{x+t}} , \tag{21}$$

or

$${}_{1/365}q_{x+t+(r-1)/365} \doteq \left(\tfrac{1}{365}\right) \frac{q_{x+t}}{1 - [(r-1)/365]q_{x+t}} . \tag{22}$$

Thus we have, finally, that equation (2) in conjunction with equations (17) and (22) and equation (A14) of the Appendix provide the means for calculating theoretical daily face amounts.

THE NEW YORK LIFE APPROXIMATION TO THE THEORETICAL DAILY FACE AMOUNT FORMULA

The theoretical daily face amount formula derived in the preceding section entails the use of daily reserve factors. For practical reasons it was felt preferable to find a sufficiently close approximation to the theoretical formula to obviate the need for using such daily reserve factors.

The method of approximation decided upon was to use, in place of each different daily reserve factor during a given policy year, a constant value that would closely approximate the average daily reserve factor during that policy year. Specifically, the values ${}_{t+(r-1)/365}V^{(365)}(\bar{A}_x)$ $(1 \leq r \leq 365)$ are replaced by the value

$$ {}_tV_x^* = \tfrac{1}{2}[{}_t\bar{V}(\bar{A}_x) + {}_{t+1}\bar{V}(\bar{A}_x)] $$

throughout the $(t+1)$st policy year.

Thus the formula for the daily face amount which the New York Life intends to use in practice for its proposed variable whole life insurance policy is

$$ F_{t+r/365} = F_{t+(r-1)/365}\left[\frac{F_{t+(r-1)/365}({}_tV_x^*) + {}^d\Pi_x}{F_{t+(r-1)/365}({}_tV_x^*) + F_{t+(r-1)/365}({}^d\Pi_x)}\right] \times \left[\frac{1 + i'_{t+r/365}}{1 + i^{(365)}/365}\right]. \tag{23} $$

In order to judge the adequacy of the approximation, daily face amounts produced by the above formula and by the theoretical formula were computed and compared for issue ages 15, 25, 35, 45, 55, and 65; for assumed constant separate account net annual investment rates 0, 3, 6, and 9 per cent; and for all policy anniversaries throughout the lives of the respective contracts for each issue age.

Results of these tests indicated that the approximation selected by the New York Life was a very good one. Thus, for the 0 and 6 per cent net investment rates, except for two or three durations at the outset and a few extreme durations corresponding to attained ages 95 and higher, the face amount differences between the two formulas are less than \$1 per \$1,000 of initial face amount. At the 9 per cent net investment rate, the absolute dollar differences per \$1,000 initial face amount are still quite low, being less than \$2 at all points except for roughly the same pattern of exceptional durations cited above. Furthermore, even for an extreme attained age where the absolute dollar difference is significantly greater than \$2, it should be borne in mind that the applicable face amount at that point is many times larger than the initial face amount. Accordingly, for the 9 per cent rate, the dollar difference expressed as a percentage of the applicable face amount is in general less than one-tenth of 1 per cent, just as is generally the case for the lower rates. Of course, when the net investment rate is 3 per cent, equal to the assumed reserve interest rate, the face amounts produced by both formulas are identical and constant at the initial face amount.

Overall, then, the differences in face amounts produced by the New York Life and by the theoretical formulas are very small and are well within the limits that equity would demand.

APPENDIX

DERIVATION OF THE DAILY NET PREMIUM

The daily net premium for a whole life insurance policy with immediate payment of claims is represented by ${}^d\Pi_x$. Thus

$$ {}^d\Pi_x = (\tfrac{1}{365})P^{(365)}(\bar{A}_x) \tag{A1} $$

and

$$ {}^d\Pi_x = (\tfrac{1}{365})\frac{\bar{A}_x}{\ddot{a}_x^{(365)}} . \tag{A2} $$

Since ${}^d\Pi_x$ remains constant throughout the life of the policy, any bias in the expression for ${}^d\Pi_x$ has the potential for producing discrepancies in the face amount that may be greatly compounded because of the large number of iterations of the daily face amount formula. Therefore, it was felt desirable and practicable to calculate it as accurately as possible.

Accordingly, a formula was sought for $\ddot{a}_x^{(365)}$ that would be more accurate than the usual approximation based on $\ddot{a}_x^{(m)} \doteq \ddot{a}_x - [(m-1)/2m]$. Also, refining the usual approximation by adding the next term of the series, $-[(m^2-1)/12m^2](\mu_x+\delta)$, is not productive, since the μ_x in that term must itself be approximated.

Thus, starting from first principles, we have

$$ \ddot{a}_{x:\overline{1|}}^{(365)} = (\tfrac{1}{365})\sum_{t=0}^{364} v^{t/365}\ {}_{t/365}p_x , \tag{A3} $$

where ${}_{t/365}p_x$ is the probability that a life aged x will survive t days. Assuming a uniform distribution of deaths throughout a year, we have

$$ {}_{t/365}p_x \doteq 1 - \left(\frac{t}{365}\right) q_x . \tag{A4} $$

Substituting the above value in equation (A3), we obtain

$$ \ddot{a}_{x:\overline{1|}}^{(365)} \doteq (\tfrac{1}{365})\sum_{t=0}^{364} v^{t/365}\left[1 - \left(\frac{t}{365}\right) q_x\right] , \tag{A5} $$

or

$$ \ddot{a}_{x:\overline{1|}}^{(365)} \doteq (\tfrac{1}{365})\sum_{t=0}^{364} v^{t/365} - \left[\frac{q_x}{(365)^2}\right]\sum_{t=0}^{364} t v^{t/365} . \tag{A6} $$

The first summation in equation (A6) represents an annuity due of 1 per day for 365 days. The second summation represents an immediate annuity for 364 days, with the first payment equal to 1 and each successive payment increasing by 1. In the case of both annuities the effective interest rate per day is $i^{(365)}/365$. From compound interest theory, we have

$$\left(\tfrac{1}{365}\right)\sum_{t=0}^{364} v^{t/365} = \frac{d}{d^{(365)}}\,. \tag{A7}$$

For the second annuity, on the basis of compound interest theory, we have

$$\begin{aligned}\sum_{t=0}^{364} t v^{t/365} &= \frac{1}{i^{(365)}/365} + \frac{1}{[i^{(365)}/365]^2} \\ &\quad - \left[\!\left[\frac{(365)^2 v}{d^{(365)}} + v\left\{\frac{1}{i^{(365)}/365} + \frac{1}{[i^{(365)}/365]^2}\right\}\right]\!\right].\end{aligned} \tag{A8}$$

After simplifying, we obtain

$$\sum_{t=0}^{364} t v^{t/365} = (365)^2 v\left[\frac{i - i^{(365)}}{i^{(365)} d^{(365)}}\right]. \tag{A9}$$

Substituting the values from equations (A7) and (A9) in equation (A6), we have

$$\ddot{a}^{(365)}_{x:\overline{1}|} \doteq \frac{d}{d^{(365)}} - q_x v\left[\frac{i - i^{(365)}}{i^{(365)} d^{(365)}}\right]. \tag{A10}$$

If the above annuity is extended to a life annuity, and if we assume a uniform distribution of deaths separately for each year of life, then, by summing the right-hand side of equation (A10), we arrive at the following:

$$\ddot{a}^{(365)}_x \doteq \sum_{s=0}^{\omega} v^s\, {}_sp_x\left\{\frac{d}{d^{(365)}} - q_{x+s} v\left[\frac{i - i^{(365)}}{i^{(365)} d^{(365)}}\right]\right\}. \tag{A11}$$

Introducing commutation functions, we have

$$\ddot{a}^{(365)}_x \doteq \frac{d}{d^{(365)}}\frac{N_x}{D_x} - \left[\frac{i - i^{(365)}}{i^{(365)} d^{(365)}}\right]\frac{M_x}{D_x}. \tag{A12}$$

Evaluating the above expression for $i = 3$ per cent, we have

$$\ddot{a}^{(365)}_x \doteq \frac{0.9854050N_x - 0.5035931M_x}{D_x}. \tag{A13}$$

Substituting the value of $\ddot{a}^{(365)}_x$ from equation (A13) in equation (A2) and simplifying, we have, finally,

$${}^d\Pi_x \doteq \left(\tfrac{1}{365}\right)\frac{\bar{M}_x}{0.9854050N_x - 0.5035931M_x}. \tag{A14}$$

Equation (A14) was used to calculate the daily net premiums given in Table A1.

TABLE A1

Table of Daily Net Premiums Per $1,000 Initial Face Amount
(1958 CSO 3 Per Cent Net Level Basis)

Age	Daily Net Premium	Age	Daily Net Premium	Age	Daily Net Premium
	Male		Male—*Continued*		Male—*Continued*
0	$0.015508	35	$0.046349	70	$0.249151
1	0.015366	36	0.048241	71	0.263464
2	0.015744	37	0.050236	72	0.278683
3	0.016161	38	0.052341	73	0.294941
4	0.016601	39	0.054558	74	0.312424
5	0.017065	40	0.056893	75	0.331312
6	0.017553	41	0.059351		
7	0.018066	42	0.061940		Female*
8	0.018606	43	0.064670		
9	0.019171	44	0.067551	0	$0.014183
				1	0.014082
10	0.019763	45	0.070593	2	0.014418
11	0.020381	46	0.073807	3	0.014793
12	0.021025	47	0.077203	4	0.015188
13	0.021693	48	0.080794		
14	0.022386	49	0.084591	5	0.015605
				6	0.016044
15	0.023103	50	0.088608	7	0.016506
16	0.023845	51	0.092858	8	0.016991
17	0.024613	52	0.097357	9	0.017499
18	0.025409	53	0.102125		
19	0.026235	54	0.107181	10	0.018030
				11	0.018583
20	0.027095	55	0.112546	12	0.019159
21	0.027991	56	0.118244	13	0.019758
22	0.028927	57	0.124299	14	0.020379
23	0.029907	58	0.130733		
24	0.030933	59	0.137574		
25	0.032010	60	0.144852		
26	0.033140	61	0.152595		
27	0.034328	62	0.160839		
28	0.035575	63	0.169621		
29	0.036887	64	0.178980		
30	0.038266	65	0.188953		
31	0.039717	66	0.199580		
32	0.041246	67	0.210889		
33	0.042856	68	0.222907		
34	0.044556	69	0.235650		

* The daily net premium for a female above age 14 is identical with the corresponding daily net premium for a male three years younger.

DISCUSSION OF PRECEDING PAPER

JOHN M. BOERMEESTER:

Mr. Scher's paper is a most welcome addition to the technical literature on variable life insurance, particularly to the area concerned with the daily valuation of the variable life face amounts payable under a specific contract of the New York Life design.

First, it might be worthwhile to underscore the fact that the evaluation of the theoretical formula given in the paper involves the assumption of a uniform distribution of deaths.

The author discusses and supports the adequacy of his approximation of the reserve factor. Perhaps he might give similar support of his decision to reject the Woolhouse approximation for $\ddot{a}_x^{(365)}$.

The author arrives at the equation of equilibrium given in formula (17) by a circuitous route. He states just before his equation (2) that "the . . . formula for the annual face amount will also be the theoretically correct recursive formula for the daily face amount, provided that we reinterpret the symbols accordingly." The problem is how to reinterpret the symbols. This problem, fortunately, can be avoided by developing the equation of equilibrium directly from first principles. Doing so produces a relatively very short proof. We start with the recursive relationship

$$_{t+(r-1)/365}V^{(365)}(\bar{A}_x) + {}^d\Pi_x = \bar{A}_{x+t+(r-1)/365} - {}^d\Pi_x a^{(365)}_{x+t+(r-1)/365} . \tag{1}$$

We first express the single premium value in the following manner:

$$\begin{aligned}\bar{A}_{x+t+(r-1)/365} = & \int_0^{1/365} v^s \, {}_s p_{x+t+(r-1)/365}\mu_{x+t+(r-1)/365+s}ds \\ & + \int_{1/365}^{\omega} v^s \, {}_s p_{x+t+(r-1)/365}\mu_{x+t+(r-1)/365+s}ds .\end{aligned} \tag{2}$$

Assuming a uniform distribution of deaths, equation (2) becomes

$$\begin{aligned}\bar{A}_{x+t+(r-1)/365} = & \, 365 \, {}_{1/365}q_{x+t+(r-1)/365}\int_0^{1/365} v^s ds \\ & + v^{1/365} \, {}_{1/365}p_{x+t+(r-1)/365}\bar{A}_{x+t+r/365}\end{aligned} \tag{3}$$

$$\begin{aligned} = v^{1/365}\Big[& {}_{1/365}q_{x+t+(r-1)/365} \frac{i^{(365)}}{\delta} \\ & + {}_{1/365}p_{x+t+(r-1)/365}\bar{A}_{x+t+r/365}\Big] .\end{aligned} \tag{4}$$

We next express the annuity value as

$$a^{(365)}_{x+t+(r-1)/365} = v^{1/365}\,{}_{1/365}p_{x+t+(r-1)/365}\ddot{a}^{(365)}_{x+t+r/365}\,. \tag{5}$$

Substituting the values of expressions (4) and (5) in equation (1), we now obtain

$$\begin{aligned}{}_{t+(r-1)/365}V^{(365)}(\bar{A}_x) + {}^{d}\Pi_x = v^{1/365}\Big[&\frac{i^{(365)}}{\delta}\,{}_{1/365}q_{x+t+(r-1)/365} \\ &+ {}_{1/365}p_{x+t+(r-1)/365}\bar{A}_{x+t+r/365} - {}^{d}\Pi_x\,{}_{1/365}p_{x+t+(r-1)/365}\ddot{a}^{(365)}_{x+t+r/365}\Big]\,.\end{aligned} \tag{6}$$

Rearrangement of the terms of equation (6) will immediately reproduce the author's equation of equilibrium (17).

IAN M. CHARLTON:

Mr. Scher's paper is an interesting extension of the original paper of Messrs. Fraser, Miller, and Sternhell entitled "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance" (*TSA*, XXI, 343). Particularly interesting is the method he suggests for handling the obvious problem in benefit calculations caused by the daily investment of the net premium and application of investment experience. Depending on the tolerance permitted in drafting policy language, the description of ${}_tV^*_x$ could become awkward.

In order to provide for PALIC's intention to offer variable life on a limited payment basis as well as on a whole life basis, and to anticipate a valuation period less frequent than daily, it was decided to develop a generalized approach which would permit adherence as closely as possible to traditional formulas, to allow accommodation to whatever frequency of recognition of investment experience and/or premium payment frequency may be required, and to be as specific as necessary in policy form descriptions.

The formulas assuming a "$p$thly" recognition of premiums and investment experience follow. These can be reconciled to Mr. Scher's formulas, provided that the adjustment for immediate payment of claims is omitted. Their derivation is appended to this discussion.

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}[i^2v^{(p+1)/p}] + (1 - D_{x+n}/D_x)[p(1 - v^{1/p}) - iv^{1/p}]}{[p(1 - v^{1/p})]^2}$$

$$A^{(p)}_x = \frac{iA_x}{p[(1+i)^{1/p} - 1]}\,,$$

$${}_nP^{(p)}_x = A^{(p)}_x \div \ddot{a}^{(p)}_{x:\overline{n}|}\,.$$

For integral year duration t,

$$ {}_t^nV_x^{(p)} = A_{x+t}^{(p)} - {}_nP_x^{(p)}\ddot{a}_{x+t:\overline{n-t}|}^{(p)} . $$

For interim year duration $t + s/p$,

$$ {}_{t+s/p}^{\ \ n}V_x^{(p)} = \frac{1}{l_{x+t+s/p}} \left\{ l_{x+t+(s-1)/p}\left[{}_{t+(s-1)/p}^{\ \ n}V_x^{(p)} + \frac{{}_nP_x^{(p)}}{p} \right](1+i)^{1/p} - d_{x+t+(s-1)/p} \right\} , $$

where $t + s/p < n$. $F_{t+s/p}^B$, the face amount at the beginning of the valuation period which ends $t + s/p$ years after issue, is given by

$$ F_{t+s/p}^B = F_0^B + [F_{t+(s-1)/p}^E - F_0^B]\left[\frac{{}_{t+(s-1)/p}^{\ \ n}V_x^{(p)}}{{}_{t+(s-1)/p}^{\ \ n}V_x^{(p)} + {}_nP_x^{(p)}/p} \right] , $$

where F_0^B is the initial amount and $F_{t+(s-1)/p}^E$ is the face amount at the end of the interim period $t + (s-1)/p$.

$$ F_{t+s/p}^E = F_{t+s/p}^B \left[\frac{1 + (I - m)}{1 + i} \right]^{1/p} , $$

where $I - m$ is the net investment rate after all deductions and i is the assumed interest rate for reserves, both on an annualized basis. The formula for F^B can be translated into the following policy language and used as the death benefit during any valuation period:

The Variable Sum Insured at the beginning of, and during, any subsequent valuation period is equal to the sum of (1) the Initial Sum Insured and (2) the result obtained by multiplying (*a*) the difference (positive or negative) between the Variable Sum Insured at the end of the preceding valuation period and the Initial Sum Insured by (*b*) the ratio, for the Initial Sum Insured, of the terminal reserve for the preceding valuation period to the initial reserve for the current valuation period.

The Variable Sum Insured at the end of any subsequent valuation period equals the Variable Sum Insured at the beginning of such valuation period multiplied by the net investment factor for the valuation period.

The above formulas can be adapted to the Commissioners Reserve Valuation Method (CRVM). The applicable formulas for a whole life plan are

$$ \beta_x^{(p)} = P_x^{(p)} + \left(\frac{M_{x+1}}{N_{x+1}} - \frac{M_x - M_{x+1}}{D_x} \right) \div \ddot{a}_x^{(p)} , $$

$$ \alpha_x^{(p)} = \beta_x^{(p)} - \frac{p[\beta_x^{(p)} - P_x^{(p)}]\ddot{a}_x^{(p)}}{(1-v)/(1-v^{1/p}) - q_x\{[v^{1/p} - pv + (p-1)v^{(p+1)/p}]/p(1-v^{1/p})^2\}} . $$

These formulas make the accepted assumption that the excess allowed is based on annual frequencies regardless of the frequency of p. As would be expected, resulting increases in the death benefit using the CRVM produce lower values than under the net level reserve method.

The accompanying tabulation shows, for a 7 per cent net investment rate, how the amount of insurance would change for a person aged 35, assuming a monthly investment period. The reserve basis is the 1958 CSO Table (age nearest, $3\frac{1}{2}$ per cent). As can be seen, no meaningful change in the amount of insurance can be seen under the CRVM until well into the second policy year. Under the net level method, results are immediate. As might be anticipated, the change in amount of insurance under the CRVM occurs one year later than under the net level.

Policy Year	Month	Net Level Reserve Method	CRVM
1	1	$1,000	$1,000
1	3	1,003	1,001
1	7	1,008	1,002
2	1	1,015	1,001
2	7	1,023	1,008
3	1	1,031	1,016
4	1	1,048	1,032
5	1	1,065	1,048
20	1	1,388	1,361

Regulation permitting, we intend to make a specific transfer of net premiums to the separate account on a monthly basis, our valuation period being a calendar month. We intend to withdraw from the separate account the cost of insurance on a monthly basis—also the amounts released because of lapses. We intend to reconcile to the sum of the applicable terminal reserves for the remaining in-force on a monthly basis.

APPENDIX

Using the assumption of uniform distribution of deaths throughout a year of age, we have

$$p\ddot{a}^{(p)}_{x:\overline{1}|} = \frac{1}{l_x}\left\{l_x + \frac{v^{1/p}[(p-1)l_x + l_{x+1}]}{p} + \frac{v^{2/p}[(p-2)l_x + 2l_{x+1}]}{p} + \ldots + \frac{v^{(p-1)/p}[l_x + (p-1)l_{x+1}]}{p}\right\}$$

$$= \frac{1}{l_x}\left\{l_x\left[1 + \left(\frac{p-1}{p}\right)v^{1/p} + \left(\frac{p-2}{p}\right)v^{2/p} + \ldots + \frac{v^{(p-1)/p}}{p}\right]\right.$$

$$\left. + l_{x+1}\left[\frac{v^{1/p}}{p} + \frac{2v^{2/p}}{p} + \ldots + \left(\frac{p-1}{p}\right)v^{(p-1)/p}\right]\right\}$$

$$= \frac{1}{l_x}\left\{\frac{l_x}{p}[p + pv^{1/p} + pv^{2/p} + \ldots + pv^{(p-1)/p}]\right.$$

$$\left. - \frac{(l_x - l_{x+1})}{p}[v^{1/p} + 2v^{2/p} + \ldots + (p-1)v^{(p-1)/p}]\right\}$$

$$= [1 + v^{1/p} + v^{2/p} + \ldots + v^{(p-1)/p}]$$

$$- \left(\frac{l_x - l_{x+1}}{l_x}\right)\left(\frac{1}{p}\right)[v^{1/p} + 2v^{2/p} + \ldots + (p-1)v^{(p-1)/p}] \, ,$$

$$\ddot{a}^{(p)}_{x:\overline{1}|} = \frac{1}{p}\left(\frac{1-v}{1-v^{1/p}}\right) - q_x\left\{\frac{v^{1/p} - pv + (p-1)v^{(p+1)/p}}{[p(1-v^{1/p})]^2}\right\} ,$$

and

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \ddot{a}^{(p)}_{x:\overline{1}|} + vp_x\ddot{a}^{(p)}_{x+1:\overline{1}|} + v^2\,{}_2p_x\ddot{a}^{(p)}_{x+2:\overline{1}|} + \ldots + v^{n-1}\,{}_{n-1}p_x\ddot{a}^{(p)}_{x+n-1:\overline{1}|} \, .$$

Let us set

$$\frac{1-v}{1-v^{1/p}} = G \, , \quad \frac{v^{1/p} - pv + (p-1)v^{(p+1)/p}}{p(1-v^{1/p})^2} = J \, .$$

Then

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \frac{G}{p}(1 + vp_x + \ldots + v^{n-1}\,{}_{n-1}p_x)$$

$$- J(q_x + vp_xq_{x+1} + \ldots + v^{n-1}\,{}_{n-1}p_xq_{x+n-1})$$

$$= \frac{G}{p}\ddot{a}_{x:\overline{n}|} - \frac{J(1+i)}{p}A^{\,1}_{x:\overline{n}|} \, .$$

Use

$$A^{\,1}_{x:\overline{n}|} = A_x - \frac{D_{x+n}}{D_x}A_{x+n}$$

$$= (1 - d\ddot{a}_x) - \frac{D_{x+n}}{D_x}(1 - d\ddot{a}_{x+n})$$

$$= 1 - \frac{D_{x+n}}{D_x} - d(\ddot{a}_{x:\overline{n}|}) \, .$$

Then

$$p\ddot{a}^{(p)}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|}[G + J(1+i)(d)] - J(1+i)\left(1 - \frac{D_{x+n}}{D_x}\right)$$

$$= \ddot{a}_{x:\overline{n}|}(G + iJ) - J(1+i)\left(1 - \frac{D_{x+n}}{D_x}\right)$$

and

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|}\left\{\frac{1-v}{p(1-v^{1/p})}+\frac{i[v^{1/p}-pv+(p-1)v^{(p+1)/p}]}{[p(1-v^{1/p})]^2}\right\}$$

$$-(1+i)\left(1-\frac{D_{x+n}}{D_x}\right)\left\{\frac{v^{1/p}-pv+(p-1)v^{(p+1)/p}}{[p(1-v^{1/p})]^2}\right\},$$

which reduces to

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}[i^2v^{(p+1)/p}]+[p(1-v^{1/p})-iv^{1/p}](1-D_{x+n}/D_x)}{[p(1-v^{1/p})]^2}.$$

Assuming uniform distribution of deaths throughout the year of age,

$$(l_x-l_{x+1/p}) = (l_{x+1/p}-l_{x+2/p}) = \ldots = [l_{x+(p-1)/p}-l_{x+1}] = \frac{1}{p}\,d_x\,,$$

$$A^{(p)}_x = v^{1/p}\frac{1}{p}\frac{d_x}{l_x}+v^{2/p}\frac{1}{p}\frac{d_x}{l_x}+\ldots+v^{1+1/p}\frac{1}{p}\frac{d_{x+1}}{l_x}+\ldots$$

$$=\frac{1}{p}\,(v^{1/p}+\ldots+v^{p/p})\left(\frac{d_x+vd_{x+1}+\ldots}{l_x}\right)$$

$$=\frac{v^{1/p}}{p}\left(\frac{1-v}{1-v^{1/p}}\right)(1+i)A_x$$

$$=\frac{i}{p[(1+i)^{1/p}-1]}\,A_x\,.$$

HAROLD CHERRY:

Mr. Scher has written a very interesting and timely paper on the practical application of a design for a variable life insurance policy on a daily basis. The purpose of this discussion is to suggest an alternative definition of the daily net premium which pérmits it to be expressed exactly in a simple form and which results in values very close to those based on Mr. Scher's definition.

Mr. Scher's daily net premium is payable in advance at the beginning of each day and provides for immediate payment of claims. If we define an alternative premium which, in addition, provides for refund of a portion of the premium at death, the expression for the daily net premium becomes particularly simple.

First, let us review the more familiar annual case. It has been shown in the actuarial literature a number of times in the past that the exact net annual premium payable in advance at the beginning of each policy year, providing for immediate payment of claims and for an appropriate premium refund benefit, is

$$\text{Net annual premium} = \frac{d}{\delta}\,\bar{\Pi}_x\,, \tag{1}$$

where $\bar{\Pi}_x = \bar{A}_x/\bar{a}_x$; $\bar{\Pi}_x$ is the net annual premium providing for immediate payment of claims and payable continuously. (The technically correct symbol for $\bar{A}_x/\bar{a}_x$ is $\bar{P}(\bar{A}_x)$, but for convenience we are using $\bar{\Pi}_x$.)

The analogue of equation (1) in the daily case, using the symbol ${}^d\Pi'_x$ for the daily net premium to distinguish it from Mr. Scher's ${}^d\Pi_x$, is

$$ {}^d\Pi'_x = \frac{1}{365}\frac{d^{(365)}}{\delta}\bar{\Pi}_x = \frac{1}{365}\frac{d^{(365)}}{\delta}\frac{\bar{M}_x}{\bar{N}_x}. \tag{2} $$

Formulas (1) and (2) are based on the assumption that an "appropriate" premium refund benefit is provided. In the annual case, in order for equation (1) to be exact, the premium refund benefit must be defined as the net annual premium multiplied by $\ddot{a}_{\overline{1-t}|}$, where t is the fraction of the year from the last premium due date to the moment of death. This refund benefit is slightly larger than one based on a linear factor of $(1 - t)$ applied to the premium. The analogue in the daily case is a refund benefit equal to the net daily premium multiplied by $365\ddot{a}^{(365)}_{\overline{1/365-t}|}$, which is slightly larger than a benefit based on a linear factor of $(1 - 365t)$.

If the alternative definition of the net premium suggested in this discussion is used, the formulas for terminal reserves also have a particularly simple form. It can be shown that the terminal reserve for a policy under this alternative definition is exactly equal to the reserve on the same date for a similar policy but with premiums payable continuously (sometimes referred to as the "fully continuous" reserve). Thus, in the annual case, the exact terminal reserve on the tth policy anniversary is given by

$$ {}_tV = \bar{A}_{x+t} - \bar{\Pi}_x\bar{a}_{x+t}. \tag{3} $$

In the daily case, the exact terminal reserve at the end of the rth day of the $(t+1)$st policy year is given by

$$ {}_{t+r/365}V = \bar{A}_{x+t+r/365} - \bar{\Pi}_x\bar{a}_{x+t+r/365}. \tag{4} $$

All the above formulas are exact for a policy providing the benefits described, regardless of the distribution of deaths over each year of life. In practice, continuous functions are usually approximated by assuming a uniform distribution of deaths over each year of life; this assumption is also used by Mr. Scher in his derivation of the net daily premium. Thus it would be interesting to compare his formula for the net daily premium with formula (2) of this discussion based on the usual approximations for continuous functions. The premium as given by Mr. Scher's equation (A2) is

$$ {}^d\Pi_x = \left(\frac{1}{365}\right)\frac{\bar{A}_x}{\ddot{a}^{(365)}_x}, $$

and the expression for $\ddot{a}_x^{(365)}$ based on a uniform distribution of deaths is given by his equation (A12):

$$\ddot{a}_x^{(365)} \doteq \frac{d}{d^{(365)}} \frac{N_x}{D_x} - \left[\frac{i - i^{(365)}}{i^{(365)} d^{(365)}}\right] \frac{M_x}{D_x}.$$

Thus his formula for the daily net premium is

$$^d\Pi_x \doteq \left(\frac{1}{365}\right) \frac{\bar{M}_x}{[d/d^{(365)}]N_x - \{[i - i^{(365)}]/i^{(365)} d^{(365)}\} M_x}. \qquad (5)$$

To obtain an expression for Mr. Scher's daily net premium which is easily compared with the suggested alternative, we substitute the following for M_x in equation (5):

$$M_x = D_x - dN_x = vN_x - N_{x+1}; \qquad (6)$$

simplifying, we obtain

$$^d\Pi_x \doteq \left[\frac{d^{(365)}}{365}\right] \frac{\bar{M}_x}{\{[i^{(365)} - d]/i^{(365)}\} N_x + \{[i - i^{(365)}]/i^{(365)}\} N_{x+1}}. \qquad (7)$$

The alternative expression for the daily net premium is given by equation (2) of this discussion. On the basis of the assumption of a uniform distribution of deaths, we can substitute the following for $\bar{N}_x$ in equation (2):

$$\bar{N}_x \doteq \left(\frac{\delta - d}{\delta^2}\right) N_x + \left(\frac{i - \delta}{\delta^2}\right) N_{x+1}. \qquad (8)$$

The alternative formula for the daily net premium then becomes

$$^d\Pi'_x \doteq \left[\frac{d^{(365)}}{365}\right] \frac{\bar{M}_x}{[(\delta - d)/\delta]N_x + [(i - \delta)/\delta]N_{x+1}}. \qquad (9)$$

Comparing formula (7) with formula (9), we can see that the right-hand sides of the equations are the same, except that δ takes the place of $i^{(365)}$ in formula (9). Of course, δ is extremely close in value to $i^{(365)}$ ($\lim_{m\to\infty} i^{(m)} = \delta$), so that numerical values for premiums based on formulas (7) and (9) are very close to each other.

We would expect the premium value given by formula (9) to be slightly higher than that given by formula (7), since formula (9) provides for a premium refund while formula (7) does not. This is illustrated by the comparison, shown in the accompanying tabulation, of daily net premiums

Age	Scher's Premium ${}^{d}\Pi_x$ (1)	Alternative Premium ${}^{d}\Pi'_x$ (2)	Excess of Col. 2 over Col. 1 (3)
0	\$0.015508	\$0.015508	....
10	0.019763	0.019764	\$0.000001
20	0.027095	0.027095	....
30	0.038266	0.038267	0.000001
40	0.056893	0.056894	0.000001
50	0.088608	0.088612	0.000004
60	0.144852	0.144862	0.000010
70	0.249151	0.249182	0.000031

per \$1,000 to six decimal places for a whole life policy on a male life, based on the 1958 CSO Table at 3 per cent.

PETER T. LE CLAIR:

Mr. Scher's paper provides an insight into the practical aspects of handling a fixed premium variable life insurance policy.

It should be noted that the formula for evaluating $\ddot{a}^{(365)}_{x:\overline{1}|}$, presented in the Appendix to the paper, can be generalized for any value of p as follows:

$$\ddot{a}^{(p)}_{x:\overline{1}|} = \frac{d}{d^{(p)}} - q_x v\left[\frac{i - i^{(p)}}{i^{(p)}d^{(p)}}\right],$$

where $d^{(p)} = p(1 - v^{1/p})$ and $i^{(p)} = p[(1 + i)^{1/p} - 1]$. This is based on the assumption of a uniform distribution of deaths throughout the year.

For any values of p and n,

$$\ddot{a}^{(p)}_{x:\overline{n}|} = \ddot{a}^{(p)}_{x:\overline{1}|} + (vp_x)[\ddot{a}^{(p)}_{x+1:\overline{1}|}] + \ldots + (v^{n-1}\,{}_{n-1}p_x)[\ddot{a}^{(p)}_{x+n-1:\overline{1}|}].$$

Assuming a uniform distribution of deaths throughout each year of age, we have

$$\begin{aligned}\ddot{a}^{(p)}_{x:\overline{n}|} &= \left[\frac{d}{d^{(p)}}\right](1 + vp_x + \ldots + v^{n-1}\,{}_{n-1}p_x)\\ &\quad - \left[\frac{i - i^{(p)}}{i^{(p)}d^{(p)}}\right](v)(q_x + vp_xq_{x+1} + \ldots + v^{n-1}\,{}_{n-1}p_xq_{x+n-1})\\ &= \left[\frac{d}{d^{(p)}}\right](\ddot{a}_{x:\overline{n}|}) - \left[\frac{i - i^{(p)}}{i^{(p)}d^{(p)}}\right](A^{1}_{x:\overline{n}|}).\end{aligned}$$

Thus, for an n-payment life policy on which death claims are paid immediately and premiums are invested, and the separate account is valued p

times a year, the annualized net premium is

$$ {}_nP^{(p)}(\bar{A}_x) = \frac{\bar{A}_x}{[d/d^{(p)}](\ddot{a}_{x:\overline{n}|}) - \{[i - i^{(p)}]/i^{(p)}d^{(p)}\}(A^{\;1}_{x:\overline{n}|})} $$

$$ = \frac{\bar{M}_x}{[d/d^{(p)}](N_x - N_{x+n}) - \{[i - i^{(p)}]/i^{(p)}d^{(p)}\}(M_x - M_{x+n})}. $$

In the special case where $p = 365$ and $n = \omega - x$, this reduces to

$$ P^{(365)}(\bar{A}_x) = \frac{\bar{M}_x}{[d/d^{(365)}](N_x) - \{[i - i^{(365)}]/i^{(365)}d^{(365)}\}(M_x)} $$

and

$$ {}^d\Pi_x = \frac{1}{365} P^{(365)}(\bar{A}_x) $$

$$ = \left(\frac{1}{365}\right) \frac{\bar{M}_x}{[d/d^{(365)}](N_x) - \{[i - i^{(365)}]/i^{(365)}d^{(365)}\}(M_x)}, $$

which yields formula (A14) of the paper.

If death claims are paid at the end of the valuation period in which the death occurs, rather than immediately, the annualized net premium for an n-payment life policy is

$$ {}_nP^{(p)}(A_x^{(p)}) = \frac{A_x^{(p)}}{[d/d^{(p)}](\ddot{a}_{x:\overline{n}|}) - \{[i - i^{(p)}]/i^{(p)}d^{(p)}\}(A^{\;1}_{x:\overline{n}|})}. $$

Under the assumption of uniform distribution of deaths throughout each year of age,

$$ A_x^{(p)} = \left(\frac{1}{p}\right)\left(\frac{d_x}{l_x}\right)(v^{1/p} + \ldots + v) + \left(\frac{1}{p}\right)\left(\frac{d_{x+1}}{l_{x+1}}\right)(vp_x)(v^{1/p} + \ldots + v) $$

$$ + \ldots + \left(\frac{1}{p}\right)\left(\frac{d_{\omega-1}}{l_{\omega-1}}\right)(v^{\omega-x-1}\,{}_{\omega-x-1}p_x)(v^{1/p} + \ldots + v) $$

$$ = \left(\frac{1}{p}\right)(v^{1/p} + \ldots + v)(1 + i)(A_x) $$

$$ = \left(\frac{1}{p}\right)\left[\frac{1 - v}{(1 + i)^{1/p} - 1}\right](1 + i)(A_x). $$

But since $i^{(p)} = p[(1 + i)^{1/p} - 1]$ and $1 - v = i/(1 + i)$, this reduces to

$$ A_x^{(p)} = \left(\frac{i}{i^{(p)}}\right) A_x. $$

The equation of equilibrium, analogous to formula (17) of the paper, for this situation is

$$[{}_{t+(r-1)/p}^{\;n}V^{(p)}A_x^{(p)} + {}_n\Pi^{(p)}]\left[1+\frac{i^{(p)}}{p}\right]$$

$$= {}_{1/p}q_{x+t+(r-1)/p}[1 - {}_{t+r/p}^{\;n}V^{(p)}A_x^{(p)}] + {}_{t+r/p}^{\;n}V^{(p)}A_x^{(p)}\,,$$

where

$${}_n\Pi^{(p)} = \frac{1}{p}\,{}_nP^{(p)}(A_x^{(p)})\,.$$

After the end of the premium-paying period, of course, the ${}_n\Pi^{(p)}$ term in the equation of equilibrium is treated as zero.

(AUTHOR'S REVIEW OF DISCUSSION)

EDWARD SCHER:

I would like to thank Messrs. Boermeester, Charlton, Cherry, and LeClair for their discussions of the paper.

Mr. Boermeester notes that I discussed and supported the adequacy of the reserve factor approximation used in equation (23), and he wonders whether similar support might be in order concerning use of the expression for $\ddot{a}_x^{(365)}$ developed in equation (A12), as opposed to the usual $\ddot{a}_x - (m-1)/2m$, based on the Woolhouse expansion.

In the former case an approximation is being substituted for a precise theoretical expression, and therefore some justification is warranted. In the latter case, however, the Woolhouse approximation, although very extensively employed, uses only the first few terms of the expansion, and it is thus, even by its own lights, an approximation. On the other hand, the expression in equation (A12), developed on the basis of the assumption of a uniform distribution of deaths, is precise. As stated in the paper, "Since ${}^d\Pi_x$ remains constant throughout the life of the policy, any bias in the expression for ${}^d\Pi_x$ has the potential for producing discrepancies in the face amount that may be greatly compounded because of the large number of iterations of the daily face amount formula. Therefore it was felt desirable and practicable to calculate it as accurately as possible."

The accompanying tabulation shows the differences between the values of $\ddot{a}_x^{(365)}$ produced by the expression in equation (A12) and those produced by the Woolhouse approximation. Note that the Woolhouse approximation consistently overstates the value of $\ddot{a}_x^{(365)}$, which could have been

x	Values for $\ddot{a}_x^{(365)}$		Difference [(1)−(2)]
	Based on Expression in Equation (A12) (1)	Based on Woolhouse Approximation (2)	(3)
15	26.323	26.326	−0.003
25	24.248	24.251	−0.003
35	21.517	21.520	−0.003
45	18.076	18.079	−0.003
55	14.157	14.161	−0.004
65	10.150	10.154	−0.004
75	6.645	6.649	−0.004

anticipated, since the first term of the Woolhouse expansion that is omitted is negative.

Mr. Boermeester provides a slightly different method of arriving at equation (17). Essentially what he has done is to define the daily reserve from first principles in terms of net single premiums for an insurance and an annuity at nonintegral ages. He then develops expressions connecting successive such net single premiums, based on the assumption of a uniform distribution of deaths. Finally, substitution of these derived expressions in his original equation leads to the equation of equilibrium connecting successive daily reserves, equation (17) of the paper.

The method used in the paper was to begin with equation (3), the equation of equilibrium connecting successive annual reserves. From there, by analogy, one can jump directly to the final equation connecting successive daily reserves, equation (17), if one feels secure enough to general reason the $i^{(365)}/\delta$ term, which appears in equation (17). I felt that most readers would welcome the details of the derivation of $i^{(365)}/\delta$.

The relative brevity claimed by Mr. Boermeester for his method will be found on analysis to depend entirely on his omission of those steps that derive the term $i^{(365)}/\delta$, which I have included.

In general, the relationships developed in the paper are on a daily basis, which is of prime interest to my company. Mr. LeClair and Mr. Charlton have provided some interesting and worthwhile extensions of these results by generalizing to the case where premiums are paid and investment experience is reckoned on a $p$thly basis. The functions evaluated are $\ddot{a}_{x:\overline{n}|}^{(p)}$, ${}_nP^{(p)}(\bar{A}_x)$, ${}_nP^{(p)}(A_x^{(p)})$, $A_x^{(p)}$, and the associated $p$thly reserves and face amounts.

The formulas derived by Mr. Charlton may be rendered somewhat less formidable in appearance if his definitions of the quantities G and J are recognized to be equivalent to $p[d/d^{(p)}]$ and $v\{[i - i^{(p)}]/i^{(p)}d^{(p)}\}$, respectively. With this in mind, it is more easily seen that Mr. Charlton's formulas agree with Mr. LeClair's and with those in the paper.

Also of interest is that portion of Mr. Charlton's discussion dealing with the Commissioners Reserve Valuation Method. He provides formulas for the CRVM premiums on a $p$thly basis, and he presents figures verifying that the use of a CRVM reserve basis produces face amounts that are less responsive than those produced by the use of a net level reserve basis. At the beginning of his discussion, it is not clear what awkwardness Mr. Charlton is referring to in his comment on policy language drafting.

Mr. Cherry suggests an alternative definition of the daily net premium that involves a premium refund at death feature. Since he emphasizes that his premium is exact, it should perhaps be pointed out that the premium defined in the paper is no less exact. It might also be noted that if a premium refund at death benefit is contemplated, then the premium is no longer uniquely defined, since any specified premium can be rendered "exact" by an appropriately defined premium refund.

As pointed out by Mr. Cherry, the terminal reserves generated by his premiums are equal to the "fully continuous" terminal reserves, a suggested advantage. At nonintegral durations, however, these reserves do not lend themselves to calculation any more easily than those generated by the daily net premium defined in the paper. Since, in order to avoid using daily reserve factors that change each day, we must perforce introduce some approximation, the imputed advantage would appear academic in any event.

As Mr. Cherry's analysis and figures show, there is little numerical difference between the two premiums, so that it does not seem worthwhile to conceptually complicate the definition of the daily net premium by introducing an involved premium refund benefit.