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THE INTEREST RATE ASSUMPTION AND THE MATURITY STRUCTURE OF THE ASSETS OF A LIFE INSURANCE COMPANY

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BACKGROUND AND SUMMARY

INTEREST rates are now at historically high levels, and, as a result, the range of interest rates that responsible actuaries are assuming for the future has widened dramatically. Ten years ago an assumption of 4.5 per cent for nonparticipating rate calculations was an average high assumption compared with the previous twenty-five years' earnings. The same 4.5 per cent is now deemed by many to be the lowest level to which rates might fall in the foreseeable future.

The problem and the importance of interest earnings rates and assumptions about them are increased by the current movement in the financial analyst and accounting professions to adjust the statutory earnings of life insurance companies. These adjustments will have the effect of reducing the penalty on earnings that statutory accounting exacts when high interest assumptions are used. This would eliminate the margins that might protect us if our assumptions prove too optimistic.

Since there is, in fact, no generally accepted theory for the long-term prediction of interest rates and neither actuaries, accountants, economists, financial analysts, nor politicians have any demonstrated accuracy in this field, a new approach must be adopted which will either allow the prediction of this unpredictable or make the accuracy of such a prediction less crucial. Such an approach does exist, and its development depends upon the co-ordination of the investment and actuarial functions to a degree not common in American actuarial practice.

In this paper I shall

1. Argue that the interest rate assumption should concern itself with the long-term rate at which funds can be invested in the future.
2. Demonstrate that, once such an assumption has been made about the investment environment, the crucial factor becomes the kind of investment the company has actually made—most particularly, the maturity pattern of the investments.
3. Present the theory of immunization of investment to changes in interest rates and provide an extensive series of examples and formulas used in its implementation. Using this theory, we can determine the maturity pattern

of our assets which will minimize or eliminate the effects of future changes in interest rates.
4. Provide examples of immunization in a model office where sales history and present asset distribution are similar to those of the life insurance industry of the United States. To the extent that the model actually represents the industry, some inferences can be drawn concerning the investment posture of the industry.
5. Present some implications for investment behavior, product development, the bases for interest rate assumptions, valuation, and adjusted earnings.
6. Present unsolved problems and answer some previously stated objections.

Any discussion of portfolio management for a life insurance company must go back to Redington, Haynes and Kirton, and Bailey and Perks. Redington, in particular, is responsible for the basic concept and approach. However, the following work differs in some important respects from Redington's. It should therefore be viewed as an adaptation of his work, as well as that of other British authors, to American conditions.

In England this policy of matching asset maturities with the developing payment requirements is extensively used, and the surplus of a company is reduced if reasonable matching does not exist. However, most companies are not in a fully immunized position most of the time. Rather, this idealized condition is one from which the companies depart for specific reasons at specific times. I would think that in the United States also it should be an integral part of all long-range investment portfolio planning. British conditions seem to differ from those in this country because of ready availability of government perpetuals and a lack of dated maturity bonds—the reverse of our condition. On the insurance side, British practice avoids guaranteed cash values and the many options against the company that we delight in.

INTRODUCTION

About fifteen years ago I had my first opportunity to set rates for life insurance policies. My training had prepared me well for the selection of appropriate mortality and lapse assumptions. I was given expense assumptions by my superior which seemed to bear a reasonable relationship to the recent operation of the company, and my course of study with the Society of Actuaries had me eager to start assigning clerks to calculate asset shares.

One factor was missing: What interest rate should be used? I asked my superior, and he suggested that I contact the investment department. It was a small company, so I knew where that was, and, in fact, I was acquainted with one of the investment officers. I wandered over and asked him what he thought. He explained to me that he and his department were very busy trying to make money for the company and showed me some very impressive charts with earnings and price-earnings ratios of the Standard and Poor's indexes. We had a pleasant discussion about the stock market and the future of IBM. The investment officer then excused himself with the comment that his department had no idea of and no interest in long-term interest rates but was occupied with next year's statement yield and capital gains. He finally advised me to ask the actuary about long-term interest rates, as he was an actuary and should know. I avoided explaining to him that the syllabus did not include a section on interest rates that would be effective during each of the next thirty or forty years, since I did not wish the profession to be downgraded by a confession of nonomniscience. Additionally, my immediate superior clearly knew less about this subject than I did.

Further investigation through reinsurers and so on indicated that elsewhere the same condition prevailed. After discussions with personnel in the investment and actuarial departments of other companies, I finally came to the conclusion that the relationship between these two departments in small companies is limited to an occasional game of golf. In large companies it is often limited to the knowledge that they are both listed in the same internal phone book. This situation may be acceptable if interest rates are stable and at a moderate level. When, as at present, they are high, moderately fluctuating, and subject to a precipitous drop, this lack of co-ordination could seriously impair the future of a company. It is to the values of such co-ordination that I address myself, and in the following discussion I will attempt to demonstrate the importance of knowing and considering the investment function.

We will first consider the use of asset shares where the assumed interest rate is for specific new investments and reinvestment is at the new-money rate which applies in the year of reinvestment. This is reasonable, for if money is invested it is invested in something particular which has a particular yield and a maturity date. The common assumption of a yield on assets is actually a combination of new-money yield rates and a pattern of maturity and reinvestment. After we have illustrated the effects of different maturity patterns upon asset shares, I shall go on to the more complex theory of immunization and its consequences.

ASSET SHARES WHERE ASSUMED INTEREST RATE IS FOR NEW INVESTMENTS ONLY

The calculations below are based on the following series of assumptions:

Plan: endowment at age 95; Rate \$19.44/\$1,000.
Male age: 40.

Mortality: 1955–60 Select and Ultimate Table adjusted to age last birthday.
Expenses: $10.29/$1,000 first year; $0.50/$1,000 renewal.
Lapses: Moorhead S.
Interest:
6 per cent, years 1–5.
5.5 per cent, years 6–10.
5 per cent, years 11–15.
4.5 per cent, years 16–20.
Cash values: grading up to CRVM reserve after twenty years.
Per cent expenses
Year 1, 98 per cent.
Year 2, 15.5 per cent.
Year 3, 13.0 per cent.
Years 4–10, 10.0 per cent.
Years 11–20, 5.0 per cent.

Normally, asset shares are developed by formulas which use the percentage of policies remaining in force as a divisor, so that each line is kept in terms of policies remaining in force. Since in this case we are interested in actual dollars available for investment, we will reverse the usual process and multiply each factor by l_t, the number of policies remaining, so that we get a projection of actual cash flows to and from the company in each future year. Let

$$l_t = l_{t-1}(1 - q_t^w - q_t^d + q_t^d q_t^w) \ , \quad l_0 = 1.0 \ ;$$

$$\text{Claims cost} = 1{,}000 \times l_t \times q_t^d \ ;$$

$$\text{Cash value} = CV_t \times q_t^w \times l_t \ ;$$

$$\text{Flat expenses} = \text{Flat expense rate} \times l_t \ ;$$

$$\text{Premium less per cent expenses} = l_t \times P(1 - E_{\%t}) \ .$$

In Table 1 column 1 is premium less expenses, claims, and cash value costs—the net result of the insurance operation lacking the investment element. This money is presumed available for investment at the beginning of the year. We are, in this illustration, ignoring the possibility of investing the portion for death claims or cash values for a part of the policy year. This "balance to investment" is an interesting item in its own right, since it is not only negative for the first year but, surprisingly, is negative for years after the sixteenth year. This perhaps emphasizes the importance of the investment element, for after the sixteen years the operation depends upon the investment to provide a positive cash flow.

The next three columns are cash from previous investment (our investment income), cash for new investment (sum of investment income and

TABLE 1

ASSET SHARES WITH SEVERAL REINVESTMENT ASSUMPTIONS

Duration	Interest Rate	Balance to Investment (1)	Alternate 1: Cash from Previous Investment (2)	Alternate 1: Cash for New Investment (3)	Alternate 1: Total Investment (4)	Alternate 2: Cash from Previous Investment (5)	Alternate 2: Cash for New Investment (6)	Alternate 2: Total Investment (7)	Investment in Pure Discount $[(1)\times(1+i)^{21-t}]$ (8)
1	6 %	(−)11.24	0	(−)11.24	(−) 11.24	0	(−)11.24	(−)11.24	(−)36.05
2	6	11.52	(−)0.67	10.85	(−) 0.39	(−)0.67	10.85	(−) 0.39	34.85
3	6	10.18	(−)0.02	10.16	9.77	(−)0.02	10.16	9.77	29.06
4	6	9.57	0.59	10.16	19.93	0.59	10.16	19.93	25.77
5	6	8.69	1.20	9.89	29.82	1.20	9.89	29.82	22.08
6	5.5	7.84	1.79	9.63	39.45	1.79	9.63	9.63	17.50
7	5.5	7.11	2.17	9.28	48.73	2.32	9.43	19.06	15.05
8	5.5	6.36	2.68	9.04	57.77	2.84	9.20	28.26	12.76
9	5.5	5.62	3.18	8.80	66.57	3.34	8.96	37.22	10.68
10	5.5	4.83	3.66	8.49	75.06	3.84	8.67	45.89	8.70
11	5	4.62	4.13	8.75	83.81	4.36	8.98	8.98	7.53
12	5	3.85	4.19	8.04	91.85	4.76	8.61	17.59	5.97
13	5	3.08	4.59	7.67	99.52	5.19	8.27	25.86	4.55
14	5	2.32	4.98	7.30	106.82	5.60	7.92	33.78	3.26
15	5	1.64	5.34	6.98	113.80	6.00	7.64	41.42	2.20
16	4.5	0.32	5.69	6.01	119.81	6.38	6.70	6.70	0.40
17	4.5	(−) 0.38	5.39	5.01	124.82	6.68	6.30	13.00	(−) 0.45
18	4.5	(−) 1.12	5.62	4.50	129.32	6.97	5.85	18.85	(−) 1.28
19	4.5	(−) 1.76	5.82	4.06	133.38	7.23	5.47	24.32	(−) 1.92
20	4.5	(−) 2.39	6.00	3.61	136.99	7.47	5.08	29.40	(−) 2.50
21	4.5	....	6.16	....	....	7.70	....	....	....
Total	....	....	....	....	....	....	....	....	158.16

operations), and total investment. Obviously, in the first year there is a negative balance to investment, a zero cash flow from previous investments, and there are negative new investment and total investment items. According to our assumptions, the investment is made at the beginning of the policy year. We now make an assumption that becomes crucial in later work—namely, that there is a positive cash flow into the company from some other source. Since the flow exists and the cash loss on this policy prevents an investment, we charge against this policy exactly the negative investment that the loss on this policy involves.

In our first set of investment assumptions we assume that all money is invested in a bank or other institution which may change its interest rate annually. This means that 6 per cent is earned only during the first five years, 5.5 per cent during years 6–10, and so on. The cash value of this policy at the end of twenty years is $358, which, for the portion of policies remaining, will correspond to a reserve of $161.01 in an annual statement. Using the assumptions described in our interpretation of the investment procedure, we will have assets of $143.15 toward that liability—a most unsatisfactory result.

Now examine the next three columns, and consider an alternative interpretation of the investment assumptions. Perhaps the meaning is that money received each year will be invested in annual coupon bonds maturing at the end of the twentieth policy year. The same coupon will be received in each subsequent year, so that the 6 per cent yield is received in each year for the net investment of the first five years and correspondingly for investments of each of the other five-year periods. In this case, instead of earning $72.49 in interest over the period, we would earn $83.57. Instead of earning 4.5 per cent in the last year, we earn 5.25 per cent, and instead of having $143.15 to offset a $161.01 liability, we have $154.23. We are now $6.78, or 4.4 per cent of assets, in the hole instead of $17.86, or 12.5 per cent.

Let us consider further alternative investments. The last column goes a little further and assumes that each year we invest the net result of operations in total discount bonds having no coupon at the assumed rate. The numbers shown in the column represent the accumulation of the net of operation at the assumed rate of interest to the end of the twentieth year. This is the assumption that all interest is reinvested at the same rate as the original investment upon which it was earned. We now have assets at the end of twenty years of $158.16, $2.85, or less than 2 per cent, under our reserve.

From our last assumption, let us modify our second example by investing in an obligation that will mature at the end of the thirtieth policy year—ten years later than before. In this case we have, at the end of our twenty-year period, not cash but a series of bonds maturing in ten years: $2.98 at 6 per cent, $45.89 at 5.5 per cent, $41.42 at 5 per cent, and $29.40 at 4.5 per cent. Now $29.40 in a 4.5 per cent bond in a 4.5 per cent environment is worth $29.40, but each of the other investments should now sell at a premium which, in total, amounts to $8.86. In this last case we now have assets of $163.06, compared to our liability of $161.01, or a profit of $2.05. This represents an increase in assets of about 14 per cent over our original plan of investing, which gave a net result of $143.15. More dramatically, it amounts to an increase of investment income from $72.49 to $92.40, or 27 per cent.

What conclusions can we draw from these simple examples?

1. Assumptions about future interest rates are not as simple as they seem, because the way the corporate treasurer actually invests may be more important than the investment climate. If we invested from the beginning at 4.5 per cent, we would end up with assets of $133.26, only $9.89 less than our first pattern of investments. This is about half the difference created by varying the pattern of investments.
2. Investing so that the interest is reinvested at a high rate is the most effective procedure.
3. Investing for long periods can create the most favorable results, but it seems to depend on the environment assumed and on the investment in a most peculiar way. When interest rates are low, the value of our older investments is increased, so that, depending upon how we have invested, reduced interest rates may be considered good.

This last point about the interrelation of the pattern of our investments and the environment is the point of the remainder of our argument. Before we conclude this section, however, let me point out that the examples treated are not trivial. Corporate treasurers do have the option of placing varying amounts of money in savings and loans, short-term industrial mortgages, current coupon bonds, and very long discount bonds, so that two companies may make the same assumption about the future of interest rates, and, depending upon the pattern of investments and their maturities, one may do well while the other is in deep distress.

The Interest Rate of a Going Concern

We must now spend some time on what was referred to as a "crucial assumption." That was that money is available from another source to pay the cash losses of the first year on any particular policy form. This allowed us temporarily to avoid the problem of defining the interest rate we are dealing with. We have said that the only interest rate assumptions we would make would be with respect to the rate at which money could be invested at some future date. The problem is, of course, that there is not one interest rate at any point in time but very many.

Rates on income-producing securities will differ, depending upon differences in risk, liquidity, size, special supply-and-demand factors, legality, sinking fund characteristics, taxability, appreciation possibilities on common stocks or convertible bonds and the corresponding income penalty we pay for this possibility, call features, and finally the maturity date.

One major function of the life insurance company investment officer is to properly assess the great variety of yields available on different securities. This means that he has some standards of differences in yield that are appropriate for these varied securities. If we deal with a going concern, then, since short-maturity securities are produced within our portfolio by the simple passage of years, he need concern himself only with long-term (twenty years or more) yields. Since particular yields must be related in his mind to some risk-free yield with all the distorting factors eliminated, let us intend to use that idealized rate from our following work, reducing all yields to this underlying rate.

BASIC THEORY OF IMMUNIZATION

Examination of Table 1 makes it clear that shortly (actually, in the twenty-sixth year) the total fund will start to decrease. If our securities mature in exactly the needed amounts to overcome the deficiency from operation, we have matched our assets and liabilities exactly. Immunization is the process devised by Redington for achieving the effects of exact matching without having to try to find the variety of needed maturities.

If we look in retrospect at our endowment at age 95 contract from date of inception to date of maturity, we have

$$\sum_{t=1}^{95-x} l_t\left(P_t - r_tP_t - E_t - \frac{F_tq_{x+t-1}}{1+J/2} - q_t^{\omega}\frac{CV_t}{1+J}\right)(1+i)^{95-x-t} - \$1{,}000\,l_{95} \geq 0\,,$$

where

P_t = Premium for year t;
r_t = Per cent expense rate in year t;
E = Per $1,000 expense in year t;
F = Face amount;
CV_t = Cash value;
q_t = Mortality rate in year t;
q^{ω} = Withdrawal rate in year t;
i = Interest rate;
J = Short-term interest rate.

Let us combine the terms of the equation to

$$\sum_{t=1}^{95-x} B_t(1+i)^{95-x-t} \geq 0 ,$$

where B_t is the balance of cash from the operation in year t. If P is the exact premium necessary to balance our first equation out to zero, then, after the first year,

$$\sum_{t=2}^{95-x} B_t(1+i)^{95-x-t} > 0$$

and

$$\sum_{t>2}^{95-x} B_t(1+i)^{95-x-t} < 0 .$$

At the end of the first year there is a negative balance which is overcome with some margin by the second and perhaps the third premium payment. For each year thereafter, there is a necessary balancing item to bring the equation to zero, and this represents whatever assets have been accumulated on account of this block of business. Let us now refer to the cash flow of investment or from investment as income or maturities in year t as A_t. Then, at any point in time,

$$\sum_{t}^{95-x} B_t(1+i)^{95-x-t} = \sum_{t}^{95-x} A_t(1+i)^{95-x-t} .$$

This peculiar form was developed to emphasize that only one interest rate is assumed, and that is the rate at which the company actually makes its investments in future years. No conflict is involved in the fact that the investment income from A_t is perhaps at a rate different from i; i is the rate at which any such income can be reinvested. It is not necessary that i be the same for all durations.

Moving to a more familiar form, we see that, dividing by $(1+i)^{95-x}$,

$$\sum_{t}^{95-x} B_t v^t = \sum_{t}^{95-x} A_t v^t ,$$

or

$$-\Sigma B_t v^t + \Sigma A_t v^t = 0 .$$

Our objective in this is to minimize the change in value of the item on the left because of a change in interest rate. But this implies that we want the first derivative of this function with respect to i to be equal to zero:

$$\frac{d}{di}(-\Sigma B_t v^t + \Sigma A_t v^t) = +\Sigma t B_t v^{t+1} - \Sigma t A_t v^{t+1} = 0 .$$

Dividing by v, we obtain

$$+\Sigma t B_t v^t = +\Sigma t A_t v^t .$$

The difference in sign simply indicates the direction of flow. This function, the first moment of cash flow, has been previously referred to by Macaulay as "duration." Our condition actually amounts to a minimum or a maximum value of our original equation. If we have a choice, we would prefer that the condition

$$-\Sigma B_t v^t + \Sigma A_t v^t = 0$$

give a minimum value for the left side of the equation, for we would then be in a position to develop a valuation profit if interest rates change in either direction.

Again, the condition can be formulated mathematically by using the calculus. Taking the second derivative of our function, we obtain

$$\frac{d^2}{di^2}(-\Sigma B_t v^t + \Sigma A_t v^t) = -\Sigma t(t+1)B_t v^{t+2} + \Sigma t(t+1)A_t v^{t+2}.$$

If this is greater than zero, then our function is a minimum rather than a maximum or a point of inflection. We should therefore try to make this second derivative positive, that is,

$$-\Sigma t(t+1)B_t v^t > -\Sigma t(t+1)A_t v^t\ ;$$

since

$$\Sigma t\ B_t v^t = \Sigma t\ A_t v^t\ ,$$

then

$$\Sigma t^2 A_t v^t > \Sigma t^2 B_t v^t\ .$$

Our criteria for immunization are

$$\Sigma t B_t v^t = \Sigma t A_t v^t$$

and

$$\Sigma t^2 B_t v^t < \Sigma t^2 A_t v^t\ .$$

Dividing both equations by

$$\Sigma B_t v^t = \Sigma A_t v^t\ ,$$

we obtain

$$\frac{\Sigma t B_t v^t}{\Sigma B_t v^t} = \frac{\Sigma t A_t v^t}{\Sigma A_t v^t}\ , \qquad D_{1B} = D_{1A}\ ,$$

or the durations of the assets and the balance from operations must be equal. For the second moment of t,

$$\frac{\Sigma t^2 B_t v^t}{\Sigma B_t v^t} < \frac{\Sigma t^2 A_t v^t}{\Sigma A_t v^t}$$

or

$$D_{2B} < D_{2A}\ .$$

CALCULATION OF D_1 AND D_2 FOR VARIOUS ASSETS

Calculation of D_1 and D_2 for the balance from operations is, in general, restricted to actually running down the cash flow projection. In the case of the more common assets, however, several formulas are available for their calculation. The following formula for D_1 is that of Macaulay, to whom I have referred previously; it applies to many types of insurance company investments:

$$D_1 = \frac{R}{R-1} - \frac{QR + n(1 + Q - QR)}{R^n - 1 - Q + QR},$$

where

F = "Face" value of the bond in dollars, that is, the "principal" sum in dollars;

I = Number of dollars paid periodically, that is, number of dollars called for by one coupon;

P = Number of dollars paid for the bond, that is, the "price" in dollars;

n = Number of periods the bond has to run, that is, number of periods to maturity;

R = Periodic rate of "yield" (e.g., if the bond is selling to yield 4 per cent per annum, $R = 1.02$ [under the semiannual convention bond tables]);

Q = Ratio of the face value of the bond to a coupon payment, that is, $Q = F/I$;

D_1 = "Duration" of the bond in periods.

The following special cases are of some interest.

1. If $F = 0$, as in the case of a mortgage with the periodic payment I,

$$D_1 = \frac{R}{R-1} - \frac{n}{R^n - 1}.$$

2. If $I = 0$ (a single payment),

$$D_1 = n.$$

3. As $R \to \infty$, $D \to 1$.
4. If $n = 1$, $D_1 = 1$.
5. If $n = \infty$, as for British consols or Canadian Pacific 4's,

$$D_1 = \frac{R}{R-1}.$$

The following is the corresponding formula for D_2, where PV is the present value of the obligation:

$$D_2 = \frac{1}{R^n(PV)}\left\{\frac{I}{(R-1)^3}[R^{n+2} + R^{n+1} - (n+1)^2R^2 + (2n^2 + 2n - 1)R - n^2] + n^2F\right\}.$$

The simplifications corresponding to those above are as follows:

1. If $F = 0$, as in the case of an amortizing mortgage, where $PV = (I/R^n)(R^n - 1)/(R - 1)$,

$$D_2 = \frac{R^{n+2} + R^{n+1} - (n + 1)R^2 + (2n^2 + 2n - 1)R - n^2}{(R^n - 1)(R - 1)^2}.$$

2. If $I = 0$, $D_2 = n^2$.
3. As $R \to \infty$, $D_2 \to 1$.
4. If $n = 1$, $D_2 = 1$.
5. If $n = \infty$, as for a perpetuity or a common stock,

$$\frac{R^2 + R}{(R - 1)^2} = \frac{R(R + 1)}{(R - 1)^2}.$$

In general, D_1's and D_2's combine linearly:

$$D_{1(A+B)} = \frac{(PV_A)D_{1A} + (PV_B)D_{1B}}{PV_A + PV_B},$$

where PV_A and PV_B are the present values of A and B. Similarly,

$$D_{2(A+B)} = \frac{(PV_A)(D_{2A}) + (PV_B)(D_{2B})}{PV_A + PV_B}.$$

Two special cases deserve comment. The first is that of securities where there is partial repayment of principal during the term, and the second is that of common stocks.

Securities where There Is Partial Repayment of Principal during the Term

The first case occurs routinely in amortizing home mortgages and in those bonds that use a sinking fund. In the calculations which follow, I assume that on home mortgages one-nth of them are paid in full at the end of each of the n payment periods, along with the periodic payment on each remaining mortgage. This is in reasonable accord with the data I have seen on this. These payoffs probably correspond to a family's moving and may be influenced by the relationship between the interest rates on the existing mortgage and those in the open market on new mortgages, as well as by any prepayment penalties. We should expect mortgagees to try to select against us. In the case of the sinking fund bond, I have assumed that one-nth of the face amount is paid off at the end of each period with interest on the remaining balance. I know of no studies of the forms of sinking fund obligations, and I believe that variations are so great that my calculations are merely illustrations of a hypothetical situation.

Formulas for present values and D_1 and D_2 can be developed for these obligations, but the formulas are very sensitive and require that calculations be made to more significant digits than are convenient. The illustrative calculations were done by actual evaluation of the amounts for each individual payment.

Growth Stocks

Valuation of common stocks is, of course, a subject on which there is much study but few generally accepted principles. We shall here continue the common approach of valuing income payments and shall leave for others the valuation of securities that pay little or no dividend but depend for their attractiveness upon increases in earnings or price of the stock.

The naïve approach is to assume that growth at some rate, g per cent per period, continues indefinitely. If $G = 1 + g$, then we may replace $1/R$ by G/R in our formulas for perpetuities, and we obtain

$$PV = \frac{IG}{R - G}, \qquad D_1 = \frac{R}{R - G}, \qquad D_2 = \frac{R(R + G)}{(R - G)^2}.$$

Even with g as low as 2 or 3 per cent, the effects of indefinite growth produce dramatically high values compared with fixed-income securities. Since growth at a rate higher than 2 or 3 per cent cannot be assumed to continue for a very long period, these formulas do not apply to rapid-growth situations. As is obvious, if G is greater than R, then all values become infinite, and if G is almost equal to R, then all values become too large to be meaningful.

A pattern for growth that is more reasonable, and yet can be handled, first appeared in the classic *Theory of Investment Value*, by John Burr Williams, originally printed in 1938. It is assumed that dividends in the most recent years were I_0 and that they will increase with a compounded growth rate of g per year for m years, so that, during this period,

$$I_t = I_0 \times (1 + g)^t,$$

and, after m years,

$$I_m = I_0 \times (1 + g)^m.$$

Thereafter the dividend will increase more slowly:

$$I_t = 2I_m - I_0(1 + g)^{2m-t}$$

for t greater than or equal to m.

A characteristic of this pattern is that an inflection point occurs at $t = m$, and $I_{2m} = 2I_m - I_0$, so that the same dollar increase occurs from $t = 0$ to $t = m$ as from $t = m$ to $t = 2m$. Thereafter the dividend asymptotically approaches $2I_m$, so that $I_\infty = 2I_m$. While this pattern is somewhat arbitrary, it at least allows meaningful answers to be developed for plausible assumptions as to growth. It is also possible to work out formulas for companies that are somewhere along in their growth cycle.

Values for D_1 and D_2 can be found most conveniently by a combination of formula and direct calculation, using the formula to calculate the values for a dividend following the pattern $I_t = 2I_m - I_0(1+g)^{2m-t}$ for all durations and developing the difference for the first m years:

$$PV = 2I_m v\left(\frac{1}{1-v}\right) - (1+g)^{2m} I_0\left(\frac{v}{1+g}\right)\left(\frac{1+g}{1+g-v}\right) + \sum_{t=1}^{m} v^t\{I_t - [2I_m - I_0(1+g)^{2m-t}]\}\ ,$$

$$D_1 = \frac{1}{PV}\left[\!\left[2I_m v\left(\frac{1}{1-v}\right)^2 - (1+g)^{2m} I_0\left(\frac{v}{1+g}\right)\left(\frac{1+g}{1+g-v}\right)^2 + \sum_{t=1}^{m} t v^t\{I_t - [2I_m - I_0(1+g)^{2m-t}]\}\right]\!\right],$$

$$D_2 = \frac{1}{PV}\left[\!\left[2I_m v\left[\frac{1+v}{(1-v)^3}\right] - (1+g)^{2m} I_0\left(\frac{v}{1+g}\right) \times \left[\frac{(1+g+v)(1+g)^2}{(1+g-v)^3}\right] + \sum_{t=1}^{m} t^2 v^t\{I_t - [2I_m - I_0(1+g)^{2m-t}]\}\right]\!\right].$$

Specimen Values of Various Assets

In Tables 2–4 we shall show a series of present values of assets using various valuation interest rates. Of particular interest may be the results when the coupon rate of the investment differs markedly from the valuation rate. In these calculations we have assumed that bonds will be called if the valuation rate is 1 per cent or more below the coupon rate. The intent of these tables is to illustrate a range of investments including the following:

1. New issues—utilities (five-year call) and industrials (ten-year call) with coupons around the current level (7 or 8 per cent).
2. Discount bonds—those with coupons well below current interest levels—and premium bonds where the coupon is above the valuation rate.
3. Sinking fund bonds. These are actually rather common, although the provisions differ quite widely. They have been little recognized as a separate item, and the effect on the value of an obligation is usually ignored by investors entirely.
4. Long-term noncallable bonds, of which there are a few around.
5. The Canadian Pacific 4's—the only perpetuity on the New York Bond Exchange, although perpetuities are an investment staple for English companies.

6. Common stocks, using a simple and a sophisticated model.
7. Home mortgages, valued both on a conventional basis and with consideration of the effects of early repayments.

If we examine the table of present values (Table 2), the comparative effects of a change in the level of interest rates and durations are striking.

TABLE 2

PRESENT VALUES OF VARIOUS ASSETS

Description of Security	4%	5%	6%	7%	8%
$1,000 20-year bond, 5-year call, at					
101.5, 3% coupon	$ 863.22	$ 748.97	$ 653.28	$ 572.90	$ 505.18
103.0, 5% coupon	1,069.22	1,000.00	884.43	786.45	703.11
105.0, 7% coupon	1,175.76	1,126.58	1,079.86	1,000.00	901.04
107.0, 9% coupon	1,281.99	1,229.73	1,180.04	1,132.79	1,087.84
$1,000 20-year bond, 10-year call, at					
102.0, 5% coupon	1,095.22	1,000.00	884.43	786.45	703.11
103.0, 7% coupon	1,265.46	1,174.20	1,091.00	1,000.00	901.04
$1,000 50-year bond, no call, at					
3% coupon	784.51	633.86	526.02	446.89	387.37
5% coupon	1,215.49	1,000.00	842.00	723.44	632.42
Canadian Pacific 4's, $40 coupon, perpetuity	1,000.00	800.00	666.67	571.43	500.00
$1,000 20-year sinking fund, no call, at					
5% coupon	1,079.03	1,000.00	929.64	866.82	810.56
7% coupon	1,237.08	1,148.97	1,070.35	1,000.00	936.85
Common stock, 3%, perpetual growth, $20 dividend	2,060.00	1,030.00	686.67	515.00	412.00
Common stock, Williams, 10% for 10 years, $15 dividend	1,338.53	996.86	778.23	628.54	520.83
20-year mortgage:					
5%	1,089.03	1,000.00	921.18	851.23	789.01
7%	1,279.41	1,174.77	1,082.17	1,000.00	926.90
20-year mortgage (repayments):					
5%	1,061.20	1,000.00	944.22	893.27	846.64
7%	1,189.70	1,120.53	1,057.52	1,000.00	947.39
Policy loans, 5%	1,009.62	1,000.00	990.57	981.31	972.22

In the case of a 5 per cent five-year callable bond, a decrease in rate from 5 to 4 per cent raised the value by only 6.5 per cent, while a change to 6 per cent decreased the value 11.6 per cent. The value of the common stock, according to the naïve growth assumption, doubles when the valuation rate goes from 5 to 4 per cent and decreases by 33 per cent when it goes from 5 to 6 per cent. Other assets obviously fall within the general range of these extremes, except for mortgages assuming repayment and policy loans, which are the most stable.

TABLE 3

D_1 VALUES IN YEARS OF VARIOUS ASSETS

Description of Security	4%	5%	6%	7%	8%
$1,000 20-year bond, 5-year call, at					
101.5, 3% coupon	14.74	14.28	13.80	13.30	12.79
103.0, 5% coupon	4.51	12.86	12.37	11.87	11.37
105.0, 7% coupon	4.37	4.36	4.34	11.05	10.57
107.0, 9% coupon	4.26	4.24	4.22	4.20	4.19
$1,000 20-year bond, 10-year call, at					
102.0, 5% coupon	8.10	12.86	12.37	11.87	11.37
103.0, 7% coupon	7.70	7.60	7.49	11.05	10.57
$1,000 50-year bond, no call, at					
3% coupon	23.21	20.43	17.94	15.77	13.93
5% coupon	21.18	18.76	16.62	14.76	13.18
Canadian Pacific 4's, $40 coupon, perpetuity	26.00	21.00	17.67	15.29	13.50
$1,000 20-year sinking fund, no call, at					
5% coupon	7.92	7.63	7.35	7.09	6.83
7% coupon	7.69	7.41	7.14	6.89	6.64
Common stock, 3%, perpetual growth, $20 dividend	104.00	52.50	35.33	26.75	21.60
Common stock, Williams, 10% for 10 years, $15 dividend	33.81	28.17	24.26	21.37	19.13
20-year mortgage:					
5%	8.72	8.40	8.09	7.78	7.49
7%	8.72	8.40	8.09	7.78	7.49
20-year mortgage (repayments):					
5%	6.06	5.86	5.67	5.48	5.30
7%	6.11	5.91	5.71	5.52	5.34
Policy loans, 5%	1.00	1.00	1.00	1.00	1.00

Examination of the tables of D_1 and D_2 (Tables 3 and 4) confirms the theoretical conclusion that D_1 and D_2 are closely related to the sensitivity of the assets to changes in interest rate.

Test of the Theory of Immunization with Assets

The accompanying tabulation shows a comparison of the values of several asset portfolios at several interest rates. The first is a single asset of $1,531 face amount, a twenty-year 3 per cent coupon bond, with $D_1 = 13.8$ and $D_2 = 241$ and with $PV = \$1{,}000$ at 6 per cent. The first combination portfolio is $917.55 of a fifty-year 5 per cent noncallable bond and $192.72 of a twenty-year 9 per cent bond callable in five years, with $D_1 = 13.8$ and $D_2 = 377$ at 6 per cent. The second combination is $1,151.77 in face amount of Canadian Pacific 4's and $234.36 of policy loan assets; $D_1 = 13.8$ and $D_2 = 466$ at 6 per cent. The third combination is \$542.99 of our common stock with perpetual 3 per cent growth and \$633.00 of policy loans, with $D_1 = 13.8$ and $D_2 = 918$ at 6 per cent.

Interest Rate	Single Asset	First Combination	Second Combination	Third Combination
4%..........	\$1,322	\$1,362	\$1,388	\$1,758
5%..........	1,147	1,155	1,156	1,192
6%..........	1,000	1,000	1,000	1,000
7%..........	877	882	888	901
8%..........	773	790	804	839

The theory is, in this example, confirmed, and we find that if the D_1's are equal, the portfolios vary in a similar pattern with changes in interest rates, with the portfolio having the higher D_2 having a greater value for interest rates other than the base rate. This has been interpreted to imply

TABLE 4

D_2 VALUES IN YEAR 2 OF VARIOUS ASSETS

Description of Security	4%	5%	6%	7%	8%
\$1,000 20-year bond, 5-year call, at					
101.5, 3% coupon.........	263.23	252.31	240.99	229.36	217.49
103.0, 5% coupon.........	21.67	215.64	204.25	192.82	181.45
105.0, 7% coupon.........	20.76	20.66	20.56	171.89	161.24
107.0, 9% coupon.........	20.01	19.89	19.78	19.66	19.54
\$1,000 20-year bond, 10-year call, at					
102.0, 5% coupon.........	74.79	215.64	204.25	192.82	181.45
103.0, 7% coupon.........	69.45	68.23	66.99	171.89	161.24
\$1,000 50-year bond, no call, at					
3% coupon................	834.63	688.66	562.09	456.03	369.50
5% coupon................	708.55	586.44	482.25	395.64	325.03
Canadian Pacific 4's, \$40 coupon, perpetuity..........	1,326.00	861.00	606.56	452.02	351.00
\$1,000 20-year sinking fund, no call, at					
5% coupon................	92.39	87.13	82.11	77.35	72.84
7% coupon................	87.99	82.95	78.16	73.62	69.33
Common stock, 3%, perpetual growth, \$20 dividend.....	21,528.00	5,460.00	2,461.56	1,404.38	911.52
Common stock, Williams, 10% for 10 years, \$15 dividend.	1,865.91	1,277.44	940.36	726.61	581.29
20-year mortgage:					
5%.....................	108.41	102.38	96.56	90.96	85.58
7%.....................	108.41	102.38	96.56	90.96	85.58
20-year mortgage (repayments):					
5%.....................	57.12	54.07	51.18	48.43	45.83
7%.....................	57.95	54.86	51.92	49.13	46.48
Policy loans, 5%............	1.00	1.00	1.00	1.00	1.00

that investing should always try to maximize the D_2. This would also, however, maximize the problem of portfolio rearrangements in going from one interest rate to another, and it also means that, if immunization has not been reached (which it probably will not be), losses with adverse changes in valuation rate are maximized. Overall, it is probably best always to move toward equalization of D_1 and D_2.

Actually, some insight into what is going on may be obtained by looking at the roots of the polynomial equation the coefficients of which represent the difference in cash flows between the two streams.

$$PV = 0 = a_1 v + a_2 v^2 + \ldots + a_n v^n .$$

If only a_1 and a_n are different in sign from the other coefficients, then the equation has only two positive roots, and if the derivative is zero when $PV = 0$, those roots are equal.

AN ILLUSTRATIVE MODEL OFFICE

In the following sections, I shall show the effects of changes in interest rates on both the assets and the future benefit and premiums of a model office. Some conclusions about appropriate investment policy can be drawn. Since what we are doing is also the calculation of gross premium reserves needed under various valuation interest assumptions, the difference between the value of the assets and the gross premium reserve is the gross premium surplus. We can therefore arrive at some indication concerning the use of special interest rates in the calculation of adjusted earnings, with the corresponding effect on the development of profits.

The choice of the components of a model office depends upon the objective to be accomplished. In this case we wish

1. To illustrate the mathematical effects of immunization.
2. To show that immunization should be considered by many companies.
3. To show the relation between immunization and the development of earnings and surplus.
4. To make it possible for many companies to draw reasonable conclusions about their own probable condition.
5. To use assumptions which are related to current conditions.
6. If possible, to structure the office so that plausible inferences may be drawn about a wider area than the office being illustrated.

The Assets

In the development of the asset distribution, I will follow the asset distribution of the insurance industry at the end of 1970 (see accompanying tabulation). Assumptions as to the yield and nature of the assets are,

Kind of Asset	Face Amount of Asset	Distribution within Category
Government security	$ 11,068	20-year, 3% bond
Corporate bonds	73,098	$50,000, 20-year, 5%; $23,098, 20-year, 7%
Common stock and real estate equity	21,740	This is market value, so assume 4.5 per cent return on market to get present values, half in 3% perpetual growth and half in Williams 10% for 10 years
Mortgages	74,375	$50,000 at 5%, not repayment; $24,375 at 7%, not repayment
Policy loans	16,064	All 5%
Total	$196,345	

of course, my own. The earnings rate on this portfolio is 5.3 per cent, the same as the average rate for the industry in 1970. Table 5 gives the values according to our tables for these amounts and this distribution of assets. The values were calculated both with and without common stock to emphasize the importance of these securities and to allow those who do not agree with the basis used to value common stock and compute its D_1 and D_2 to recompute them and make their own adjustments. The values for equities follow from the original assumptions, and varying those assumptions will, of course, change the result.

TABLE 5

MODEL-OFFICE ASSET VALUES

	Valuation Rate				
	4%	5%	6%	7%	8%
Statement value	$196,345.00	$196,345.00	$196,345.00	$196,345.00	$196,345.00
Government bonds	9,554.12	8,289.60	7,230.50	6,340.86	5,591.33
Corporate bonds	80,618.70	76,021.74	69,164.11	62,420.50	55,967.72
Common stock	94,031.91	57,698.83	42,172.31	33,092.30	27,060.76
Mortgages	85,639.62	78,635.02	72,436.89	66,936.50	62,043.69
Policy loans	16,218.54	16,064.00	15,912.52	15,763.76	15,617.74
Total present value	$286.062.89	$236,709.19	$206,916.33	$184,553.92	$166,281.24
D_1	27.89	16.01	12.40	11.48	10.31
D_2	4,123.63	847.24	403.56	280.10	212.40
Without common stock:					
Statement value	$174,605.00	$174,605.00	$174,605.00	$174,605.00	$174,605.00
Present value	$192,030.98	$179,010.36	$164,744.02	$151,461.62	$139,220.48
D_1	6.58	8.67	8.24	8.87	8.42
D_2	70.50	119.98	111.07	126.18	116.91

The Liabilities

For purposes of an illustrative model office, I have worked with only one plan of insurance, a nonparticipating endowment at age 95 issued at age 40 (premium $19.95), and one annuity, immediate at age 65. The same plan of insurance is presumed to have been issued throughout the history of the office since 1918. The sales assumptions follow and are roughly proportional to the sales of the life insurance industry since 1918. Lapses are Moorhead S, except that in the first model we assume a 50 per cent lapse ratio at attained age 65; this is Model A. This high lapse at age 65 is intended to balance somewhat the lack of any endowments and also to recognize that many individuals purchase life insurance with the expectation of utilizing settlement options at age 65. A second model

TABLE 6

MORTALITY AND CASH VALUES

Year	Mortality	Cash Value	Year	Mortality	Cash Value
1	1.33	0	11	6.06	180
2	1.76	10	12	6.76	200
3	2.15	29	13	7.59	219
4	2.55	49	14	8.49	239
5	2.86	69	15	9.33	259
6	3.37	87	16	11.53	279
7	3.76	105	17	12.66	298
8	4.20	124	18	13.93	318
9	4.69	142	19	15.33	338
10	5.33	161	20	16.88	358

with pure Moorhead S lapses is Model B. Mortality is assumed to follow the 1955–60 Select and Ultimate Table adjusted for age last birthday. Cash values are a current scale equaling the 3 per cent CRVM reserve after twenty years. Per cent expenses are assumed to be 98 per cent the first year, 15.5 per cent the second, 13 per cent the third, 10 per cent for years 4–10, and 5 per cent thereafter. Per $1,000 expenses are assumed to be $10.29 per $1,000 for the first year and $0.50 per $1,000 thereafter. Table 6 shows the detailed claim and cash value assumptions for twenty years of the contract.

Life insurance of only one plan and age is used, rather than several plans with four or five ages, for the following reasons:

1. I think a clearer and more easily understood picture is presented. With a variety of plans and ages, the separate effects of each would not be evident and the cause of a particular effect could not be discerned.

2. The calculations are greatly simplified and reduced in quantity.
3. It does relatively little violence to facts to consider only current sales. Some sort of whole life always provides the largest sales, but now term and riders have increased while limited pay and endowments have melted away, so the average has to be whole life.
4. The plans that have been ignored would not greatly affect the model, since a twenty-year endowment has a D_1 of almost twenty years, retirement income is much higher, but term has little reserve at all to be matched.
5. No model can really represent the industry, and one that follows one company probably is not applicable to any other.
6. Failure to include dividends has no effect, since that would simply offset premium.
7. The simple model actually represents the industry to a surprising degree. Using the sales and lapse assumptions, we get the comparison shown in the accompanying tabulation.

	Premium	In Force	Reserves and Liabilities
Industry—1970 (1971 *Fact Book*)	$21,679	$731,097	$115,400
Model A—1971 50% lapse rate at 65	15,087	909,804	108,694
Model B—1971	15,417	934,882	130,527

The lack of endowments, retirement income, and the higher premiums of participating insurance are all evident, but to a lesser extent than we would have imagined. The in-force figures imply that industry lapses have been considerably higher than Moorhead S over the years. The reserve per $1,000 confirms the lack of higher premium plans and the use of lower valuation interest rates. In our model office, cash value is used as a measure of liability, since it actually corresponds to the amount we might be called upon to pay out and also because it is a better measure of amounts that might actually be available for investment than the reserve. Overall, I believe that the combination of one plan of insurance and this one annuity will understate the D_1 for the combined payouts of the industry, but I know of no way to prove this. In the case of the annuity, mortality is a-1949 projected for thirty years. Only the net benefit payout is considered (expenses are ignored).

Following the sales assumptions (Table 7) are the projected cash flows for the two life insurance models (Tables 8 and 9). These presume issues from 1918 through 1970, but the cash flows start in the following year, that is, 1971. All calculations are thus toward the position to be reached at the end of 1971. In the calculations that follow, we are working on a scale of one-millionth of corresponding values for the insurance industry.

Discounting the cash flows, we get the values shown in Table 10 for our two life insurance model offices. For annuities the average reserve per $1 of income in the industry might be around $10.40, and, if we apply that to the $48,600,000,000 reserves in the industry, we get for our model $4,670 of annual income. Table 11 is a summary of values of D_1 so far,

TABLE 7

SALES ASSUMPTIONS

Year	Sales Assumption	Year	Sales Assumption
1970	$132,980	1943	$ 8,022
1969	123,500	1942	7,041
1968	112,266	1941	7,935
1967	103,545	1940	7,022
1966	95,987	1939	6,886
1965	89,643	1938	6,745
1964	79,430	1937	7,593
1963	68,862	1936	7,314
1962	61,259	1935	7,550
1961	58,888	1934	7,363
1960	56,183	1933	6,786
1959	55,138	1932	7,896
1958	50,839	1931	10,161
1957	48,937	1930	11,905
1956	38,941	1929	12,305
1955	32,207	1928	11,654
1954	26,824	1927	10,777
1953	24,908	1926	10,508
1952	21,579	1925	10,060
1951	19,000	1924	8,764
1950	18,260	1923	8,273
1949	15,848	1922	6,720
1948	15,787	1921	6,248
1947	16,131	1920	7,634
1946	16,244	1919	6,369
1945	10,577	1918	3,520
1944	9,184		

along with additional values developed by combining our two life insurance models with the annuity in proportion to their gross premium reserves. Examination of Table 11 indicates that, even though the assumptions were chosen to shorten the D_1 of the liabilities (by assuming immediate annuities only, for example) and to lengthen the D_1 of the asset, the disparity is very substantial for interest rates over 6 per cent. If the effects of equities are discounted, the D_1 of the assets is so short that no effective protection is possible against changes in interest rates downward. The asset model is poised for a substantial rise in interest rates from their present historically high levels. I do not believe that most investment

TABLE 8

MODEL A

Year	Premium (Less % Expenses)	Other Expenses	Claims	Cash Value Cost	Cash Value Liability
1	$15,087	$1,784	$ 5,751	$ 4,042	$108,694
2	16,426	454	6,032	4,285	118,968
3	15,812	433	6,325	4,410	129,251
4	15,310	417	6,633	4,562	138,627
5	14,797	402	6,950	4,884	146,870
6	14,309	387	7,274	5,050	154,144
7	13,839	373	7,599	5,352	160,522
8	13,379	359	7,928	5,698	165,960
9	12,923	345	8,258	5,908	170,498
10	12,476	331	8,598	6,382	173,961
11	12,025	317	8,919	6,939	176,171
12	11,480	302	9,207	7,736	176,887
13	10,912	287	9,450	7,841	176,692
14	10,350	273	9,682	8,125	175,490
15	9,783	258	9,891	8,124	173,479
16	9,225	243	10,099	8,245	170,538
17	8,667	228	10,202	8,325	166,736
18	8,113	214	10,281	8,822	161,742
19	7,546	199	10,301	9,543	155,275
20	6,955	183	10,241	10,226	147,378
21	6,345	167	10,094	10,589	138,379
22	5,727	151	9,873	11,030	128,229
23	5,099	134	9,560	11,546	116,887
24	4,460	117	9,148	12,244	104,215
25	3,803	100	8,626	12,775	90,415
26	3,137	82	8,009	1,679	86,899
27	2,931	77	8,014	1,612	83,116
28	2,729	72	7,989	1,539	79,093
29	2,531	66	7,932	1,463	74,857
30	2,337	61	7,838	1,383	70,462
31	2,149	56	7,705	1,300	65,942
32	1,966	51	7,534	1,216	61,347
33	1,790	47	7,336	1,130	56,714
34	1,621	42	7,113	1,044	52,074
35	1,459	38	6,862	958	47,474
36	1,304	34	6,576	872	42,932
37	1,158	30	6,256	788	38,502
38	1,020	26	5,905	706	34,216
39	891	23	5,526	627	30,117
40	771	20	5,122	551	26,239
41	661	17	4,699	479	22,602
42	560	14	4,265	412	19,233
43	469	12	3,832	350	16,146
44	388	10	3,406	294	13,356
45	316	8	2,985	243	10,866
46	253	6	2,570	197	8,689
47	199	5	2,162	157	6,819
48	153	4	1,775	123	5,248
49	115	3	1,419	94	3,950
50	84	2	1,099	70	2,894
51	59	1	819	50	2,051
52	40	1	582	35	1,395
53	25	0	387	22	892
54	13	0	144	12	508
55	5	0	104	5	217

TABLE 9

MODEL B

Year	Premium (Less % Expenses)	Other Expenses	Claims	Cash Value Cost	Cash Value Liability
1	$15,417	$1,793	$ 6,900	$2,909	$119,644
2	16,781	464	7,228	3,174	130,527
3	16,192	443	7,566	3,337	141,381
4	15,712	427	7,914	3,495	151,319
5	15,220	413	8,271	3,631	160,299
6	14,759	399	8,646	3,751	168,355
7	14,318	385	9,028	3,857	175,704
8	13,893	372	9,426	3,950	182,357
9	13,482	359	9,844	4,027	188,234
10	13,083	347	10,284	4,087	193,430
11	12,695	334	10,735	4,128	197,866
12	12,232	322	11,190	4,152	201,546
13	11,775	310	11,657	4,157	204,396
14	11,322	298	12,128	4,147	206,497
15	10,874	286	12,602	4,122	207,772
16	10,430	275	13,087	4,084	208,221
17	9,988	263	13,485	4,034	207,868
18	9,550	251	13,874	3,975	206,788
19	9,116	240	14,244	3,908	204,916
20	8,686	229	14,588	3,844	202,246
21	8,257	217	14,899	3,787	198,782
22	7,831	206	15,170	3,716	194,550
23	7,407	195	15,390	3,630	189,579
24	6,987	184	15,555	3,532	183,908
25	6,571	173	15,667	3,420	177,574
26	6,161	162	15,730	3,298	170,670
27	5,757	151	15,740	3,166	163,240
28	5,360	141	15,692	3,024	155,337
29	4,971	131	15,578	2,874	147,018
30	4,591	121	15,394	2,717	138,387
31	4,221	111	15,133	2,554	129,510
32	3,862	101	14,798	2,388	120,485
33	3,516	92	14,409	2,220	111,386
34	3,184	84	13,971	2,050	102,273
35	2,866	75	13,478	1,881	93,238
36	2,562	67	12,916	1,713	84,318
37	2,274	60	12,287	1,548	75,618
38	2,003	52	11,597	1,387	67,201
39	1,750	46	10,853	1,231	59,150
40	1,515	39	10,061	1,083	51,533
41	1,298	34	9,229	942	44,391
42	1,101	29	8,378	811	37,775
43	922	24	7,526	689	31,711
44	762	20	6,690	578	26,232
45	621	16	5,863	477	21,342
46	497	13	5,047	388	17,065
47	391	10	4,248	309	13,392
48	301	7	3,486	242	10,308
49	226	5	2,787	185	7,758
50	165	4	2,159	138	5,684
51	117	3	1,609	99	4,029
52	79	2	1,144	69	2,741
53	50	1	761	45	1,752
54	28	1	452	25	999
55	12	1	203	11	426

TABLE 10

EFFECTS ON THE MODEL OFFICE OF CHANGE IN INTEREST RATES

	4%	5%	6%	7%	8%
	Model A				
Present value of net premium	$ 190,327	$ 175,725	$ 163,151	$ 152,238	$ 142,700
Present value of net payouts	$ 289,803	$ 252,491	$ 222,068	$ 197,004	$ 176,156
Gross premium reserve	$ 99,476	$ 76,766	$ 58,917	$ 44,766	$ 33,456
First moment net premium	$ 1,829,721	$ 1,595,248	$ 1,402,200	$ 1,241,761	$ 1,107,262
First moment net payout	$ 4,396,059	$ 3,604,945	$ 2,987,865	$ 2,501,205	$ 2,113,377
D_1	25.80	26.18	26.91	28.13	30.07
Second moment net premium	$ 28,745,031	$ 23,862,989	$20,001,900	$16,916,760	$14,427,933
Second moment net payout	$ 94,706,523	$ 74,219,075	$58,820,636	$47,116,289	$38,123,022
D_2	663.09	655.97	658.87	674.61	708.25
	Model B				
Present value of net premium	$ 214,800	$ 196,035	$ 180,180	$ 166,659	$ 155,030
Present value of net payout	$ 319,917	$ 273,597	$ 236,843	$ 207,323	$ 183,336
Gross premium reserve	$ 105,117	$ 77,562	$ 56,663	$ 40,664	$ 28,306
First moment net premium	$ 2,341,950	$ 2,002,150	$ 1,728,246	$ 1,505,243	$ 1,321,939
First moment net payout	$ 5,547,801	$ 4,425,418	$ 3,574,226	$ 2,921,284	$ 2,414,819
D_1	30.50	31.24	32.58	34.82	38.61
Second moment net premium	$ 41,755,592	$ 33,918,283	$27,836,855	$23,070,522	$19,298,298
Second moment net payout	$136,457,890	$103,989,870	$80,164,495	$62,292,982	$49,253,973
D_2	900.92	903.43	923.49	969.52	1,058.28
	Annuity ($1 Annual Income)				
Gross premium reserve	$ 10.76	$ 10.09	$ 9.34	$ 8.67	$ 8.08
D_1	8.57	8.37	8.02	7.69	7.39
D_2	112.12	101.77	100.17	93.08	86.61

officers realize that their actions are betting on a rise in rates. Only if we include the effects of equities is the model protected against a drop in interest rates below 4 per cent.

In Table 12 we illustrate the effects of changes in interest rates by using the relative values based on unity at 6 per cent for several asset portfolios and liability models. Examination of these values indicates that the asset model based on the life insurance industry parallels reasonably well the pattern of Model A plus annuities, and less well the pattern of the other liability models. However, we can easily see that the life industry asset

TABLE 11

SUMMARY OF D_I'S

	4%	5%	6%	7%	8%
Life industry assets	27.89	16.01	12.40	11.48	10.31
Assets except equity	6.58	8.67	8.24	8.87	8.42
Model A	25.80	26.18	26.91	28.13	30.07
Model B	30.50	31.24	32.58	34.82	38.61
Annuity	8.57	8.37	8.02	7.69	7.39
Model A plus annuity	20.02	19.41	18.87	18.42	18.05
Total reserves	$149,725	$123,886	$102,535	$85,255	$71,190
Model B plus annuity	23.47	22.60	21.90	21.28	20.77
Total reserves	$153,366	$124,682	$100,281	$81,153	$66,040

TABLE 12

RELATIVE PRESENT VALUES AT 6 PER CENT = 1 (AT CHANGING INTEREST RATES)

	4%	5%	6%	7%	8%
Life industry assets	1.383	1.144	1.000	0.892	0.804
Assets except equities	1.166	1.087	1.000	0.919	0.845
20-year 3% bond	1.321	1.147	1.000	0.877	0.773
50-year 3% bond	1.491	1.205	1.000	0.850	0.736
Half 20-year 3% bond and half Williams common ($D_I = 19.03$)	1.521	1.214	1.000	0.842	0.721
Model A	1.688	1.303	1.000	0.760	0.568
Model B	1.855	1.369	1.000	0.718	0.500
Annuity	1.152	1.080	1.000	0.928	0.865
Model A plus annuity	1.460	1.208	1.000	0.832	0.694
Model B plus annuity	1.549	1.243	1.000	0.809	0.659

model depends almost entirely on equities for its flexibility. I have argued elsewhere that this is a good reason for investing in equities. Although I believe in the equity models, they are surely less dependable than the debt models. Particularly because the reserves of a company are intended to be covered by debt securities, the effects of the short maturities of the fixed-dollar obligations are very unfortunate. This is especially obvious because examination of the table indicates that a twenty-year 3 per cent bond does twice as good a job as the industry model less equities, and the fifty-year 3 per cent bond does a better job even including equities and actually matches Model A with annuities. The last asset model is the

TABLE 13

D_1 AND D_2, ALTERNATE SALES ASSUMPTIONS

	Present Value of Premium	Present Value of Payout	First Moment Premium	First Moment Payout	D_1	Second Moment Premium	Second Moment Payout	D_2
Level sales	$ 2,721	$ 5,830	$ 22,460	$ 64,365	13.48	$ 312,834	$ 1,137,234	265.17
Sales increase 5% per year compounded	26,658	44,642	233,604	576,972	19.09	3,404,356	11,222,677	434.74

half twenty-year 3 per cent bonds and half Williams common model, with D_1 set slightly above that of Model A plus annuities. Obviously other combinations can be worked out which will match various liability models.

Effects of Other Sales Assumptions

Two calculations were made at 5 per cent using a level sales assumption and a 5 per cent annual increase for comparison (see Table 13). Comparing with previous values (Model A, 5 per cent, $D_1 = 26.18$, $D_2 = 655.97$), we can see that those high values of D_1 and D_2 are caused by the recent rapid actual growth in life insurance sales. Companies with more sedate sales records have smaller problems in the area of immunization. Companies with more rapid growth than the industry have greater problems.

Variants of Immunization Theory

If a company rejects reliance on equities to implement a program of matching, or simply has too many assets short and too many liabilities long to actually achieve an immunized position in the foreseeable future, can it still invest wisely without knowing future interest rates? I think so. At the very least, the company can concentrate new investments toward repairing the lack. If a company pursues a plan over a period of only a few years, substantial improvement in matching of D_1's is possible, and this automatically decreases the company's risks of changing interest rates later.

Variant approaches exist which have less stringent requirements and still reduce interest rate risks. In our previous examples only the gross premium reserve, present value of gross premiums less present value of claims and expenses, was invested. In American companies at least, the statutory net premium reserve, plus some surplus, is invested. This change is equivalent to using statutory accounting as opposed to a gross

TABLE 14

NET PREMIUM RESERVE IMMUNIZATION

	4%	5%	6%	7%	8%
Model A D_1	24.43	21.16	18.52	16.38	14.63
Per cent future premium released	4.84	18.17	30.51	41.99	52.72
Model B D_1	26.68	22.71	19.57	17.07	15.05
Per cent future premium released	11.83	27.02	40.99	53.92	65.94
Model A and annuity D_1	19.52	17.22	15.30	13.72	12.42
Per cent future premium released	3.98	19.11	33.56	47.32	60.34
Model B and annuity D_1	21.75	18.84	16.48	14.58	13.04
Per cent future premium released	11.07	27.77	43.76	58.79	72.95

premium valuation. Adjustments of earnings resulting from the use of current interest rates in valuing liabilities instead of statutory rates has the reverse effect of transferring profits from future years to the current year. Viewing this as a problem of asset matching, we obviously should not use a high rate and a lowered reserve unless we have been able to immunize our assets at that higher interest rate using the lower reserve. Table 14 shows the effect on D_1 of assuming that the cash value liability is invested and that only a portion of the future premiums is applied against future claims and expenses, the remainder being available in the future as profit. The per cent premium released is

$$\frac{\text{Present value of future benefits} - \text{Cash value liability}}{\text{Present value of future premiums}}.$$

When this portion of future premiums is set aside, the first moment of premiums is reduced, and the D_1 is decreased. This final table indicates that reduction in the D_1 of the liability side may be accomplished by dedicating a larger portion of the assets of the company to this purpose. This approach not only relates to investment practice but actually provides for differing emergence of profits.

In both this and the earlier cases,

$$\text{Present value of profit} = \text{Present value of premium} + \text{Investment reserve} - \text{Present value of costs}\,.$$

In the earlier form the profit was taken immediately by setting up only the gross premium reserve. In this later version the margin will emerge as a portion of premium in each future year, and the profit will vary depending upon future investment experience on that margin. In the earlier case we immunized premium benefits and profit and took this profit into our statement immediately, and in this last case we immunized the statutory reserve with premium less profit and costs. This is equivalent to considering profit another cost and deducting it before calculating the gross premium reserve. I would recommend this net premium reserve immunization as an approach which is practical for American companies and more consistent with our general approach toward valuation than the English systems.

Immunization for dividends for mutual companies has interesting possibilities:

1. Complete immunization can be attempted with or without anticipating new business. This is most desirable, and most difficult, in periods of higher interest rates.
2. Immunization can be ignored, and the risks and rewards of interest rate prediction and variation can be thrown completely on the policyholder. In periods of historically low interest rates this may be justified.
3. Some intermediate position can be accepted, either based on the extent of movement toward an immunized position, where immunizing includes or excludes the dividend, or utilizing paid-up immunization.

Bailey and Perks suggested immunization for $1 - GAP_x/GAP_{x+t}$ of a policy, where GAP is gross annual premium. This guarantees that, on a portion of his premium, an old policyholder would be treated as a new policyholder. This approach works well with a bonus system for distribution of surplus. Another reasonable alternative might be to immunize the reserve as a single premium for all benefits and expenses for however much insurance it would cover. The single premiums have D_1's around 14 or 15, so immunization by many discount bonds is possible. Quite a range of dividend formulas are possible, emphasizing that different aspects of equity exploration in this area are needed.

Among the variations in types of asset, in valuation rates, and in what is immunized, most companies should, in fact, be able to adjust their assets so that reliance upon future levels of interest rates can be minimized or completely eliminated.

INFERENCES AND CONCLUSION

In reviewing all these calculations, we can form some positive conclusions and draw some inferences on a wide range of topics. These topics have in common their reliance upon an assumed rate of interest available for investment into the distant future. These topics include (1) investment management; (2) product development and sales; (3) proper actuarial principles in assumption of future interest rates; and (4) surplus, earnings, adjusted earnings, and minimum valuation standards.

Investment Management

Since the obvious concern of most companies is to lengthen their assets, the sample tables of D_1 and D_2 give a clear guide to appropriate investment action. Sinking fund bonds should be avoided, and mortgages must be offset by longer-term assets. Bonds with short call features are to be avoided in periods of high interest rates. So-called discount, or low-coupon, bonds are especially attractive. Equities are very attractive, since they are the only long-term assets available in quantity in this country. Forward commitment of funds and borrowing short are also recommended. Issuance of short-term bonds in periods of high interest rates should be considered where legal. One large mutual company is apparently following this course, although not necessarily for these reasons.

Product Development and Sales

This, of course, is the other side of the coin from the assets, and the problem here is avoiding lengthening the payments. To avoid interest rate risks, we should provide guarantees or settlement options so low that maturities can be considered cash payouts and then allow excess interest on these funds, dependent upon the then current investment climate. Retirement income policies with liberal guarantees should be avoided, as should other than minimal guarantees in deferred annuities. Use of participating contracts in these areas in times of high interest rates also reduces the company's interest rate risk by transferring it to the policyholder. In general, long guarantees of high rates should be avoided and dividend projections used, so that the return to the policyholder is generous if interest rates are high. This is especially true because the public seems to be willing to pay very little for interest rate guarantees.

Immediate annuities should be encouraged on both a participating and a nonparticipating basis, and short-term life insurance contracts are also helpful.

Each company must decide the appropriate relation between its asset and sales philosophies, since they are complementary, but must always remember that this relationship exists.

Proper Actuarial Principles in Assumptions of Future Interest Rates

Neither accountants, actuaries, economists, politicians, insurance regulators, nor security analysts have any demonstrated competence in predicting long-term interest rates. We need only go back five years to establish that, when rates were about to increase drastically, no one foresaw it. Life actuaries have built their reputation on the use of mortality tables which do allow some prediction of the future, and these have had increasing safety margins. Our life clients want to live longer, and advances in medical science have caused substantial improvements in mortality. No actuary has such expertise in long-term interest rates, and there are tremendous political and social pressures against continuation of the current high level.

Several solutions exist:

1. The company may work with participating policies, only guaranteeing the historically reasonable rates of 3 or 3.5 per cent, and allowing the dividends to express assumptions of high rates in the future. This is a good approach for a mutual company.
2. The company may assume safe levels for the future in the 3 or 4 per cent area and content itself with selling term insurance until those interest levels return. This is actuarially sound but limits business too severely to be a practical solution.
3. The company may authorize the actuary to use a level 6 or $6\frac{1}{2}$ per cent rate or some series of rates which start at that level and decrease every few years, and hope that the business will lapse if interest rates fall below our assumption. Since, if interest rates fall, these policies will not lapse, this approach depends on the correctness of our prediction.
4. The company can examine the duration of its assets and its currently existing business and lengthen the D_1 of its assets beyond the D_1 of its projected flows from operating, so that, on the basis of reasonable sales assumptions, matching of the D_1's can take place one or two years in the future. We can assume that short-term assets will be acquired in the future and calculate what our current D_1 should be, so that the addition of these short-term assets will create the desired immunization.

Even if exact prospective matching cannot be achieved, we can still determine whether it is possible to come reasonably close, in which case the dangers to the company are minimized. If reasonable correspondence is not possible, the interest rate assumption should be reduced.

Under this system, an assumption of 7 per cent could be safe for one company and 4 per cent speculative for another, even though they have the same current earnings rate. If the analysis and matching of D_1's is not done, however, we are guaranteeing future interest earnings without assuring ourselves that they will be earned on schedule. If we do the analysis, we will know what we will earn under even unfavorable circumstances and may be able to adjust our portfolio so that our guarantee is warranted.

Gross Premium Valuation, Adjusted Earnings, and Minimum Valuation Standards

Since our basic approach has been a gross premium valuation, all the above topics fall together.

In the example of net premium reserve immunization, we invested the net premium reserve as opposed to the gross premium reserve. This had the effect of transferring earnings from the current year into future years, since a portion of the gross premium was allocated as annual profit.

The analyst's adjustment of earnings by revaluing the reserves may then overstate current earnings, because it is not established that the company can continue those earnings in the future. The analyst's approach actually attempts to go from a statutory rate to a current rate in the calculation of the increase in reserves, and that higher rate is implicitly assumed to be available for the indefinite future. Similarly, the AICPA method of using the premium assumption on interest rate in calculating reserves can be proper only if the assets are matched including these new sales as discussed above. Unless assets are immunized, both the analysts and the accountants are not only capitalizing future earnings but capitalizing a speculation on future interest rates. Even though the analysts and accountants approve current earnings adjustments, I do not believe that either will accept responsibility for the restatements of earnings and the losses that will result if interest rates move against us. A company can, however, establish that it is immunized, and under those conditions the analysts' or accountants' methods may be valid. This means, again, that only if we have properly arranged our assets can we assume a high interest rate.

Standards for interest rates used in statutory valuation follow also. Since the summary of D_1's does not make plausible the argument that the industry can match its liabilities at a rate higher than 4 per cent or even protect itself against a drop below 4 per cent (if equities are ignored), there seems no support for a general change in minimum valuation standards. In an individual case, if a company can demonstrate its ability to support such a change, the regulatory authorities should have discretionary power to approve it. Revision for immediate annuity valuation should, of course, take place to allow a valuation rate very close to the long-term interest rate if assets are appropriately matched.

Unresolved Problems

Serious problems remain:

1. The handling of equities is still open to question. I believe that the assumptions in my two equity examples are realistic and the mathematics correct, but I cannot quite believe all the results.
2. In the case of participating contracts, there are many different ways in which dividends could be immunized against future changes in interest rates. If a company has done this and interest rates are high, can the company advertise that the interest rate changes will not affect the dividends? What kind of disclosure is proper?
3. No effort has been made to relate this approach to statutory earnings. Some effects are apparent, but the implications are not all obvious.
4. Taxes have been ignored. This is partly because of the additional complexities which they would add but also because tax planning seems to be a company affair and many situations exist. Tax alternatives on capital gains and valuation bases could be quite important.
5. Pension funds may have entirely separate kinds of problems because of the varying kinds of guarantees that may exist, the extremely long durations, and the large unfunded liabilities. The discretion of the employer in payments toward the unfunded liability may require Mr. Benjamin's games theoretic approach in addition to the theory here expounded. Union welfare funds and group permanent have somewhat different but no less real problems.

Obviously, however sound the theory, its application involves a whole new host of problems.

Answers to Some Objections

The following are some objections, and answers to them, that have arisen with respect to the practice of immunization:

1. *Immunization will prevent profits as well as losses.*—This is true if you know which way interest rates are going to move and have correctly positioned yourself. (The industry, in general, seems poised to take advantage of interest rates jumping to the 10–15 per cent area.) However, most companies are invested too short, and without investigation and matching you do not know whether you are short or long. It does no good to know what is going to happen if you do not know which side you are, in fact, on. The mathe-

matical techniques presented are the only way to decide whether you are short or long, and speculation on interest rates is not possible without this knowledge.

2. *There is not one interest rate but many.*—This is true. Interest rates vary, depending upon quality, liquidity, term, size, and so on. This simply means that we have an investment department to pick out the best "buys" for our purposes. We are going to have to use one interest assumption, and problems in choosing it have nothing to do with matching theory but will always exist. Our industry and its continual liquidity at least avoid the term-structure problem because we can always deal with the "long-term rate." Interest rates varying with duration can also be used.
3. *Guaranteed cash values make the procedure impossible.*—This is a British objection; examination of the cash flow tables shows that cash payouts are not larger than death claims, so I do not know why they cannot be included. Mr. Sidney Benjamin has indicated that he no longer is sure that guaranteed cash values are an insurmountable obstacle.
4. *The theory will not work because lapses will not follow assumptions, and so on.*—This is, of course, correct; on the other hand, nothing ever works out exactly as planned. We are still obliged to use our best judgment to protect ourselves against the unforeseeable and uncontrollable.
5. *The investments required by the theory do not exist in sufficient quantity to make the theory usable by all companies.*—This is true, but if companies markedly prefer some merchandise to others, the required investments will become more available. If companies refuse to buy bonds with only five-year call protection, ten-, fifteen-, and twenty-year call protection will become available.
6. *Does it make sense for mortgages to have a low* D_1 *if bought by the company, while stock in a mortgage company has a high* D_1 *appropriate for equities?*—It probably does, if the mortgage company is run separately and is aimed at increasing earnings and dividends.
7. *It isn't worth the effort involved.*—Each company will have to make this judgment for itself. However, it should be remembered that failure to consider this subject properly has a possible penalty of 50 or 60 per cent of reserves—the possible variation in our gross premium reserves with changing interest rates.

EPILOGUE

In this paper I have attempted to describe the British system of immunization of assets and to adapt it to conditions in this country. Perhaps it was easier for the British to develop such theories, since they have had much more experience than we in the similar problem of matching assets and liabilities in varying currencies. Also, more actuaries are active in investment there than here, and perhaps that is related to the use of gross premium valuations without strict guarantees, in contrast to our system. Many British actuaries now believe that there can be no real valuation of liabilities without a corresponding integrated valuation of the assets. This is surely true but is not easily brought into our system of operation.

Despite these excuses, we have been negligent as a profession in examining the assets of our companies' total operation. Whereas we can go into and understand the investment functions, the investment people in general cannot understand ours. I hope that this paper will stimulate other actuaries to work in this area.

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DISCUSSION OF PRECEDING PAPER

RICHARD S. ROBERTSON:

Although this paper suggests many practical applications, the most significant to me is that referred to by the author in his Introduction—the implications with respect to the interest assumption for profit studies of nonparticipating life insurance. It is my impression that, while most actuaries are willing to recognize the current highly favorable yields which can be realized on new investments, they are reluctant to project those yields as realizable for the life of the policy, or even for the period of time over which profit tests are made. As a result, a typical aftertax interest assumption might be to project 5.5 per cent interest for the first five policy years, gradually reducing it to perhaps 4 per cent at the twentieth year.

Mr. Vanderhoof's paper demonstrates how an actuary can safely assume a higher rate of interest at the longer durations if the company lengthens the average maturity of its investment portfolio. Theoretically, if a company could manage its investments so that the positive cash flow from investments and from policies in their first fifteen contract years (using the assumptions of the hypothetical asset share presented in the paper) exactly offsets the negative cash flow from older policies, it would be entirely safe to assume that current new-money rates would last for the life of the insurance policy. On a far more practical basis, it may be possible, by a suitable management of his company's investment portfolio, for an actuary to substantially increase his long-term interest assumption for nonparticipation insurance.

The aspect of the paper which gives me the most difficulty, and I sense that it also gives the author difficulty, is the extraordinary attractiveness the theory ascribes to common stock investments. I would like to suggest that one possible difficulty in applying the theory to common stocks involves the basic objective of using the timing of the maturity of assets to reduce the uncertainty of investment yields on a specified portfolio. However, common stock investments carry an additional uncertainty related to the projected growth of a company or an industry, its long-term dividend prospects, and the prospects for capital gains, which may tend to offset some of the favorable aspects related to the improvement in immunization of assets. I do not know whether it would be possible to modify the theory to take into consideration this type of uncertainty, but it would appear to be proper justification for reducing the "D values" for common stock to a more reasonable level.

PAUL R. MILGROM:

Congratulations are due Mr. Vanderhoof for his very fine and thought-provoking paper, which provided me with an introduction to this subject. The underlying mathematical theory of the paper has some interesting aspects which deserve further critical attention.

The fundamental equation of the paper is

$$\Sigma A_t v^t - \Sigma B_t v^t = 0\ ,$$

where i is the reinvestment interest rate, not varying by duration. As Mr. Vanderhoof correctly remarks, the essential nature of the conclusions is not changed by allowing interest rates to vary by duration.

The left-hand side of the fundamental equation (which I will call the gross premium valuation margin) is then viewed as a function of i, so that, by differentiating this margin with respect to i, we can measure its sensitivity to changes in the level of interest rates. Dividing this derivative by $\Sigma A_t v^{t+1}$ gives $D_{1B} - D_{1A}$, which is an index of how large a change in the margin will result from a change in i. If, for example, reinvestment rates change by an amount Δi, the value of our gross premium reserve valuation margin changes by approximately $\Sigma A_t v^{t+1} \Delta i (D_{1B} - D_{1A})$. This suggests that the gross premium valuation margin will be immunized against a change in the level of interest rates whenever $D_{1A} = D_{1B}$.

This analysis, however, assumes that the reinvestment interest rates at the various durations move rigidly together; that is, if the interest rate at the first duration is $\frac{1}{2}$ per cent below the assumed rate, the interest rates at all other durations must also be $\frac{1}{2}$ per cent below their assumed rates. Since interest rates may realign themselves gradually and in any number of patterns, it would be worthwhile to examine the indexes of change that would be applicable under various other assumptions.

I offer three naïve examples here which are intended mostly as illustrations of the method which can be used and the variety of results which might be obtained. In each example, the index is a derivative of the gross premium valuation margin divided by $\Sigma A_t v^{t+1}$.

1. *Independently varying interest rates.*—Wherever v^t appears in the fundamental equation, replace it with $(1 + i_t)^{-t}$. Now we may take the derivative of the margin with respect to i_t:

$$\begin{aligned}\text{Index}_{1,\,t} &= \frac{1}{\Sigma A_S v^{S+1}} \frac{\partial}{\partial i_t} \Sigma (A_t - B_t)(1 + i_t)^{-t} \\ &= \frac{t v^t (B_t - A_t)}{\Sigma A_S v^S}\ .\end{aligned}$$

The result is as we would expect—that we can only immunize against independent fluctuations in interest rates by having exact matching in our cash flows from investment and insurance operations.

2. *Gradual change in interest rates.*—Let us represent a gradual change in interest rates by writing $i_t = i + t\alpha$ and taking the derivative of the margin with respect to α. The effect of this is that a change of $\frac{1}{8}$ per cent in the first year corresponds to $\frac{2}{8}$ per cent in the second year, $\frac{3}{8}$ per cent in the third year, and so on. The resulting index measures the sensitivity of the margin to a change in the first-year interest rate:

$$\text{Index}_2 = \frac{1}{\Sigma A_t v^{t+1}} \frac{d}{d\alpha} \Sigma(A_t - B_t)(1 + i + t\alpha)^{-t}$$

$$= \frac{\Sigma t^2 (B_t - A_t) v^t}{\Sigma A_t v^t} = D_{2B} - D_{2A} .$$

It seems that increasing D_{2A} relative to D_{2B}, besides helping to immunize against a change in the level of interest rates (as Mr. Vanderhoof's paper indicates), will also help to immunize against a gradual decline in interest rates.

3. *Special distortion in interest rates.*—Here suppose that a distortion afflicts this year's interest rates but will gradually fade out with the passage of time. This particular distortion may affect future interest rates as follows: $i_t = i + \alpha/t$. Is our margin immune to such a distortion?

$$\text{Index}_3 = \frac{1}{\Sigma A_t v^{t+1}} \frac{d}{d\alpha} \Sigma(A_t - B_t)(1 + i + \alpha/t)^{-t}$$

$$= \frac{1}{\Sigma A_t v^t} (\Sigma A_t v^t - \Sigma B_t v^t) = 0 .$$

Isn't it good to know that there are *some* things, at least, that you never have to worry about?

RALPH GARFIELD:

I would like to say at the outset that I enjoyed very much reading Mr. Vanderhoof's paper and feel that it is a valuable contribution to the actuarial literature. I relived some of the delights of studying Redington's classic 1952 paper. Incidentally, that paper is difficult but a masterpiece of exposition—it will handsomely repay careful study.

It has always been a mystery to me that actuaries in this country have never been more than superficially involved in the assets side of the balance sheet. But this attitude is almost certainly due to my British background, where actuaries are in the forefront of the investment world. It is indeed a pity that here we have lost the investment initiative to people who are certainly no more qualified (and in many cases are less qualified) than we are. Of course we do not have special insight into the Markovian characteristics of the stock market, but we do know a lot about our liabilities and when we expect them to fall due. Accordingly, I agree with the author that it follows that the actuary ought to have a big say in when the assets fall due. This, of course, brings us onto the immunization road.

To emphasize what the author says, no actuary recommends that immunization be followed slavishly. It is, however, an ideal toward which insurance and pension funds should reach. This is certainly the case with our British colleagues. I understand that there is little argument in Britain on the principle of immunization—it is pretty well accepted today as an investment objective (among others).

The author tells us that life insurance funds are, on the average, invested too short. This must certainly be the case with pension funds, the obligations of which may well be of sixty or seventy years' duration. Anything less than 100 per cent invested in irredeemables must be considered as being invested too short (always assuming that such undated investments are available—a situation which obtains in the United Kingdom and which presumably influences their acceptance of immunization). But how many pension funds have such a portfolio? With no statistics to back me up, my feeling is that most managers of pension fund portfolios do little more than recognize the long-term nature of the pension fund liabilities and accordingly have a portion (perhaps as much as 50–60 per cent) in common stocks. How many valuation reports, I wonder, include a projection of the benefits into the long-term future? And, in the case of those that do, how much cognizance is taken of this by the investment manager? The author has clearly made a sound case for such information to be available and to be used efficiently. I hope that he is listened to and that we regain some of that lost investment initiative.

BERNARD RABINOWITZ:

If this paper could be criticized, it would be because the explanation of immunization given may be too technical for nonactuaries, especially those responsible for investment management, to understand. I will therefore try to explain the investment aspects of immunization in nonmathematical terms.

Immunization theory deals with the total cash flow expected in the future from a defined block of policies and the assets corresponding thereto. Asset cash flow in any year is defined as the investment income and capital maturities in that year. Total cash flow in any year is defined as the asset cash flow in that year plus premium income less policy claims less policy surrenders less expenses in that year. Positive total cash flows are assumed to be invested at the market rate of interest at the time such positive cash flow occurs, whereas negative total cash flows assume liquidation of assets at market values corresponding to the market rate of interest at the time such negative cash flow occurs. The immunization model further assumes that the market value of the assets is always equal to the discounted value of the future asset cash flows at the market rate of interest. Immunization theory is concerned with obtaining protection against the investment risks resulting from unfavorable changes in interest rates in the future. These risks are, first, that positive total cash flows may have to be invested in the future at too low a rate of interest and, second, that negative cash flows may involve liquidation of assets in the future at depreciated values when interest rates are too high.

Put very simply, immunization theory shows that depreciation in the market value of the assets on account of a rise in the interest rate may be offset by the increase in the investment return on positive total cash flows to be invested in the future at this higher rate of interest. Conversely, on a fall in interest rates the appreciation in the market value of the assets may offset the lower rate of interest at which positive total cash flows are to be invested in the future. The immunization equations given in the paper attempt to solve for the spread of asset maturity dates required to give these offsets. In general, the later the asset maturity date (or, more precisely, the later the mean date of the future asset cash flow), the greater is the change in market value for a given change in the rate of interest.

Immunization theory will further dictate that, once there is a change in interest rates, the asset portfolio be rearranged with regard to the distribution of asset maturity dates. Common sense, however, should override the requirements of immunization. For example, on a sharp fall in interest rates, the capital gains should be realized and the proceeds reinvested in shorter-dated assets. On a sharp rise in interest rates, longer-dated assets would be preferred, so as to lock in the funds at these high interest rates.

Temporary depreciation in asset values due to a temporary rise in interest rates is real, even if the company does not liquidate assets when premium income is sufficient to pay current policy benefits to terminating policyholders. This becomes obvious if we consider that the same effect is achieved if the company liquidates assets at the depreciated current market value to pay these policy benefits and the current premium income of surviving policyholders is then used to purchase back these same assets at the same depreciated current market value. In other words, assets held in respect of terminating policies are in fact sold at depreciated current market values, but an actual sale on the market does not take place, since the purchasers are the surviving policyholders. I mention this point because we tend to think of fixed-income securities in terms of statutory values (i.e., book value for mortgages and amortized value for bonds), on the assumption that market value has significance only on liquidation.

Life insurance companies are unique savings institutions, in that they guarantee a rate of interest at which the policyholders' future savings are going to be invested. This is implicit in the assumptions underlying the premiums charged. The paper gives an elegant mathematical and numerical demonstration of the power of immunization theory in dealing with the problem of this guarantee. It is important, however, that immunization be viewed in its proper perspective as an investment objective and that it not be overrated because of its sophistication.

The primary objective of investment policy is that the life insurance company be able to meet its contractual liabilities to policyholders. The secondary objective is to obtain over the long term or over the lifetime of the block of business under review the highest investment yield that is commensurate with the investment risk undertaken. I do not know how immunization fits into the secondary objective, since immunization against losses implies immunization against profits. Possibly immunization theory may be used here as one of the criteria against which risks of investment action based on predictions of the trend of interest rates may be measured. The primary objective of investment policy demands that there be certainty of investment income and capital maturities as to both amount and date. Furthermore, the rate of interest earned over the lifetime of the block of business must not be less than that assumed in the premium rates. It is here, in the primary objective, that immunization as an investment policy is of importance. Immunization protects a company against unfavorable changes in the rate of interest, and to be concerned with immunization is to be concerned with solvency.

I therefore believe that one of the limiting factors of immunization is the requirement that the company be at all times able to meet its guaranteed cash surrender value and loan value commitments. This is crucial when assets depreciate on account of a sharp climb in interest rates, with a resultant increase in the amount of cash surrenders and policy loans to an extent not contemplated in the original premium rate assumptions.

This may explain why United States companies prefer shorter asset maturity dates than British companies that do not guarantee cash surrender values.

I also believe that equities have no place in immunization theory, which is primarily concerned with solvency. Immunization theory demands that, given a future interest rate, the market value of the asset be absolutely determinable, that is, that the market value be the discounted value of asset cash flow, where the asset cash flow is certain as to both amount and date. This is not true in the case of equities, where it is likely that the market rate of interest is one of the lesser items determining the market price. Equities do not fit into the primary objective of investment policy. Equities, in my opinion, enter into the secondary objective of investment policy, which is to maximize yield, and equities are bought with the expectation of a greater yield (commensurate with risk) than is provided by long-dated bonds. The results given in the paper showing the suitability of equities for the protection afforded by immunization are therefore misleading. These results merely reflect the hopes and expectations of the investor.

The author points out that there may not be assets of sufficiently long date to immunize a block of business. This problem tends to manifest itself when there is a very large influx of new business and contractual liabilities to the policyholders are pushed far into the future. The author rightfully suggests that a conservative interest rate assumption such as 4 per cent be used for determining premium rates for new business under these conditions. There is, however, one idea which may be worth exploring. A $5\frac{1}{2}$ per cent coupon bond maturing in twenty years at 100 with a current market value of 83.7 yields $7\frac{1}{2}$ per cent to maturity. If the company buys these bonds now with shareholders' additional surplus and sells these bonds after ten years to the survivors of today's new nonparticipating policyholders at a price to yield $5\frac{1}{2}$ per cent to maturity, then the following happens. The shareholders obtain a yield of 8 per cent over ten years, and the new policyholders are guaranteed a return of $5\frac{1}{2}$ per cent on moneys to be invested ten years hence. Both parties seem to be better off. This principle, if sound, may have startling implications where current market rates of interest are substantially in excess of that deemed reasonable for premium rate calculation purposes.

In conclusion, this paper is extremely valuable in showing that the actuaries' knowledge of both the nature of the contractual liabilities to policyholders and the effects of interest rate changes on the assets is an essential ingredient in the formulation of sound investment policy.

RICHARD W. ZIOCK:

Mr. Vanderhoof deserves a vote of gratitude from the Society of Actuaries for bringing our attention to the theory of immunization. He has thoroughly researched the ideas presented and covered the topic well. I thoroughly enjoyed his paper. We have, as he points out, neglected the asset side of the Annual Statement.

It is possible to approach this subject with either an active or a passive approach. An active approach would use the theory of immunization as a guide to investing the assets. A passive approach would involve looking at the maturity structure of the assets as a guide to the selection of the long-term interest rate assumption, which depends upon the degree to which investments have or have not been immunized. For example, if no immunization exists, then the interest rate forty years hence is completely uncertain. To the degree to which immunization is achieved, the interest rate is predictable. Obviously the passive approach is not nearly as ambitious in its scope as the active.

Should United States actuaries begin calculating the D's for assets and liabilities and advising the investment departments of their companies on the correct investments for matching? In short, should active application be made of immunization theory? My answer is no. All of the D_1 values the author has calculated, particularly those shown in Table 11, lead me to one conclusion: the D_1 for liabilities is usually greater than the D_1 for assets. The difference is sometimes large, particularly at the higher rates of interest. Furthermore, a visual examination of the D_1 values for various types of assets in Table 3 shows that no realistic shift in investments could enlarge the D_1 of the assets up to the D_1 of the liabilities. This assumes that the asset and liability models shown in the paper are realistic, and indeed they appear so to me.

If it is true that complete matching is impractical in the United States because of the unavailability of certain types of investments, then should matching be done to the maximum extent possible? Yes, it should be done when interest rates are high, and, to my knowledge, investment departments do expand forward commitments and invest in longer-term investments when interest rates are high and do the opposite when rates are low. I do not believe that matching should be sought when interest rates are low.

Even when greater matching is a current goal, it may sometimes have to take a back seat to other considerations. One example is when mortgage yields are high compared to bond yields.

A passive application of matching theory can help in the choice of the ultimate interest rate assumption, which is very crucial for premium and natural reserve calculations. If the theory could be modified to tell what percentage of assets is immunized at each future duration, then the long-term interest rate assumption would be equal to that on the immunized part plus the best estimate of the future rate (most likely the historical average rate), each weighted properly.

I would like now to switch to a critical examination of Mr. Vanderhoof's conclusions regarding common stock investments. He says that common stocks are the most effective type of investment to lengthen the average maturity of the assets and that when interest rates are high the average maturity of assets should be lengthened. This would seem to imply that when interest rates are high we should load up on common stocks. Yet doing this will cause investment in the stock market during periods of low bond prices (high yields), which goes contrary to the well-known investment observation that bond prices are low when stock prices are high, and vice versa. Quite obviously, it is ideal that common stock investments be made at the lowest possible price. Hence immunizing with common stocks may be very undesirable.

Policy loans are shown as having a weighted average maturity of one year in Table 3. My own investigations have revealed that total policy loan repayments run about 15 per cent of policy loans outstanding. This fact would argue for a longer weighted average maturity than that shown.

At about the same time that Mr. Vanderhoof was doing his work on this paper, I was doing some research into the closely related field of interest rates. I have written a paper on my research, which I hope to publish soon, but publication will be after the appearance of this discussion. Because of the difference in publication dates, I would like to tell here how some of my results relate to Mr. Vanderhoof's paper. The subject of my research was the pricing of life insurance in inflationary conditions like those of 1965–70. During this period interest rates were high (6–8 per cent), policy loans increased, and expenses inflated.

The major conclusion in my paper is that the increase in investment earnings of a typical life company is considerably dampened in periods of increased rates of interest, chiefly because of the increased federal income tax. Policy loans and concurrent inflation also play a part. Some indication of the degree of dampening is the marginal tax for a $100 increase in "fully taxable investment yield," which is between $39.82 and $47.38, depending upon the tax situation.[1]

[1] *The United States Federal Income Tax as It Applies to Life Insurance Companies* (Study Note of the Society of Actuaries, Part 10IE 2-1-63 [Chicago, Ill.: Society of Actuaries]), p. 57.

This dampening would seem to lessen the need for immunization. In addition, increases in policy loans during periods of high interest rates may make any attempt at immunization very difficult. Many companies experienced negative cash flows during 1965–70.

All of the above discussion is submitted in a spirit of hoping to add to a very excellent paper.

(AUTHOR'S REVIEW OF DISCUSSION)

IRWIN T. VANDERHOOF:

I greatly appreciate the interesting and perceptive comments involved in the various discussions. Since one of my major objectives in writing the paper was to try to interest members of the Society in investment work and investment actuarial co-ordination, the various comments are particularly rewarding to me.

I think that Mr. Robertson best expressed one of the problems inherent in the subject and best described its difficulty. I refer, of course, to the whole question of the handling of equity investments. From one point of view, there is clearly a problem of risk involved in common stock investments. Market values can vary tremendously from one period to the next, and the return, computed upon a basis which recognizes market value, can therefore be very inconsistent. On the other hand, there is little expectation that a company would be forced to sell common stocks at a time when market values were low. More probably, the proper expectation is that companies will continue to simply increase their assets and ownership of common stocks along with other types of assets. Under these circumstances common stocks would be sold only when the expectation for the particular security was inconsistent with the market price, that is, when the market price was too high in comparison with our opinion of the inherent worth of the security. I believe that the mathematical approach which I used is consistent with the idea of an irredeemable security with a slowly or moderately increasing payout. My current feeling as to the proper handling of equities would be to use one of the mathematical models described in the paper but arbitrarily penalize the equity 1 or 2 per cent yield to compensate the company for the annoyance of market value fluctuations and the risk of reduced returns as opposed to fixed guarantees.

Mr. Paul Milgrom provided a very interesting continuation of the mathematical development. His analysis of the effects of changes in the interest rate according to different assumptions as to the pattern of that change is a valuable extension of the whole theory. His analysis under

index_2, where he describes the effect of a gradual change in interest rates and the fact that setting the D_2's equal immunizes this whole pattern, is a new development as far as I know. This implies that, having properly set yourself up in advance, it is not even necessary to change your asset patterns if this kind of gradual change in interest rates is perceived for the future.

I naturally appreciate the comments from my friend Ralph Garfield, although his warning that we should not slavishly follow the recommendations of immunization formula may not be too necessary in this country, since there are, at the moment, very few people who are considering it at all. In any case, his comments are all pertinent and valuable.

Mr. Bernard Rabinowitz has presented an interesting nonmathematical explanation of the theory of immunization. Mr. Rabinowitz's point is well taken; however, his comments about allowing common sense to override immunization theory when there are sharp falls or a sharp temporary rise in interest rates disturbs me a little bit. The ability to determine that a rise in interest rates is in fact temporary presumes the ability to know what the long-range trend is in fact going to be. Until there is a demonstration of a greater degree of expertise in predicting long-term trends by our economists, my own feeling is that identifying something as a temporary rise in interest rates is more closely akin to feeling for the top or the bottom in long-term stock market moves than anything else.

Most losses of fortunes in the stock market have been caused by buying too early in a stock market decline in the opinion that it was a temporary drop. The question about using immunization in connection with high long-term yields relates again to the question of prediction of long-term rates. If, in fact, you can predict what the movement is going to be on long-term rates, you still have to know your position as far as immunization is concerned to take advantage of that knowledge, because, if interest rates are in fact going to drop, the knowledge is valueless if you are not at least trying to lengthen the position of your assets, and know what lengthening actually is in your particular case. I have mentioned before the question of the handling of equities in this theory. I do not agree with Mr. Rabinowitz that you can simply ignore them. They exist, they are investments of companies, and they are very good investments of companies. In general, common stocks have provided increasing returns over the years to their owners. If the model included in the paper is defective or overly optimistic, then a more suitable model can perhaps be found, but the question of equities cannot be ignored. Mr. Rabinowitz also points out the interesting situation of the interest assumption for new business if bonds can be bought which are somewhat longer in duration than the existing liabilities. This is, of course, one of my points in the paper, and I hope to expand and enlarge upon this possibility in the future.

Mr. Ziock brings up an interesting point in his discussion of passive versus active immunization. In disagreeing with the idea of active, complete immunization, he really is arguing against making immunization the only investment criterion. Other discussants have also mentioned this. Mr. Ziock's disagreement with the idea of active immunization does not extend to ignoring the immunized position of the company, and, even though he disclaims immunization as a practical possibility, he still feels that companies should try to lengthen the maturity of their assets during periods of high interest rates. I think that the problem with the current actions of the companies toward extending their maturity patterns is that the investment men in general do not realize how much further they should go than they are now going. The requirement of the life insurance industry for long maturities in investment portfolios is not generally recognized, and the great difference between the maturity structure for life insurance companies and that for banks, for instance, is not appreciated. I think that our investment people ought to be trained to think in longer terms than they now feel comfortable with.

The question that Mr. Ziock raises in connection with matching, of immunization taking a back seat when mortgage yields are high compared to bond yields, is not, to my recollection, a meaningful one. I do not remember the time when mortgage yields exceeded bond yields by a large enough margin, when bond yields were high, to make mortgages a better investment than bonds. At the present moment, the rate on FHA mortgages is about the same as the rate on high-grade discount bonds, and, unless a very substantial margin develops in favor of the mortgage investments, the bonds would be better.

The passive approach to matching is really the question of how we develop from our portfolio whatever interest assumptions we can make for future business. As I indicated, this is a subject that I hope to be able to develop in the near future. In connection with Mr. Ziock's comments on policy loans, I think that the point regarding policy loans having a one-year duration is simply that they can be paid against the company at any time. I agree that total policy loan repayments would be a small percentage now, but, if bank loan interest rates drop to 4 per cent, then we will see very substantial amounts of policy loans repaid in a very short period of time. The criterion then is not how long they will remain outstanding on the average but how long they will remain outstanding under adverse conditions. I certainly hope to see Mr. Ziock's paper on interest earnings and income tax in the near future, and I particularly thank him for his provocative comments.

I agree with Mr. Hickman when he says that immunization is a continuing process.

In conclusion, on this whole matter, I would like to point out certain areas in which questions and disagreements were not raised by any discussant, because perhaps these are more significant from my point of view than those in which questions were raised:

1. No one argued that actuaries should not become more active in investment matters and actuarial investment co-ordination.

2. No one argued that the actuaries in each company should not try to determine the position of the assets and the liabilities of the company as far as the immunization position is concerned.

3. No one argued that this calculation of the D_1's for liabilities and assets separately would be too expensive, too onerous, or too difficult for the actuaries of companies to do, at least at two- or three-year intervals.

4. No one argued that the asset structure of a life insurance company does not have a very important impact on the assumptions to be made in calculating adjusted earnings, future nonparticipating rates, or even valuation interest rates.

In conclusion, I would like to express my thanks to all of the people who provided encouragement and had an interest in this work.