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AN ALGORITHM FOR COMPUTING EXPECTED STOP-LOSS CLAIMS UNDER A GROUP LIFE CONTRACT

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ABSTRACT

This paper describes a method for computing the expected loss under a group life contract in excess of a stated limit. In accordance with a common assumption of collective risk theory, it is assumed that the number of claims under the contract is a Poisson-distributed random variable.

The possible aggregate amounts less than the stop-loss level which can occur are identified, and the probability of each such amount is calculated. From these calculations the probability of an aggregate claim less than the stop-loss level is determined, and also the expected value of such a claim.

The mean aggregate claim is then recognized to be the weighted average of the mean aggregate claim of each of two mutually exclusive subpopulations of aggregate claims—those with aggregate claims exceeding the stop-loss level and those with aggregate claims not exceeding the stop-loss level. From the information enumerated about the subpopulation of smaller aggregates, information is obtained about the subpopulation of larger aggregates—namely, the probability of occurrence of an aggregate claim exceeding the stop-loss level and the mean such claim. The expected stop-loss claim is then readily calculated.

The paper also shows how to compute the variance and standard deviation of the stop-loss claim and suggests how the method can be extended to group insurance benefits where the claim amount per certificate is not fixed.

INTRODUCTION

It is the objective of this paper to develop a method for computing the expected stop-loss claim under a group life insurance contract, where the stop-loss claim is defined as the excess of the aggregate claim in some interval over a specified stop-loss level.

Bartlett [1] reviewed a number of methods which had been proposed for generating this kind of information and went on to develop a method of fitting a gamma function to the frequency distribution of aggregate claims.

In our development we shall consider a group of lives each of which is insured for a fixed amount of death benefit and for each of which there is given the probability of dying during the year. The development assumes that lives who die are immediately replaced by lives of identical risk. This assumption is a simplification of what is likely to happen in practice, but it may not be less realistic than the assumption of no replacement at all. If we also assume that the force of mortality is a constant for each life during the year, then we have postulated an environment where the risk remains constant. This permits us to apply collective risk theory and assume that the number of claims during the year is a Poisson-distributed random variable and that we are dealing with a compound Poisson process as described by Bartlett.

We shall not attempt to find an expression for the cumulative distribution function of the aggregate claim but shall develop an algorithm using an enumerative approach for the computation of the expected stop-loss claim from (*a*) the expected aggregate claim, (*b*) the probability that the aggregate claim will not exceed the stop-loss level, and (*c*) the conditional expected aggregate claim given that the aggregate claim does not exceed the stop-loss level. The method will be first of all developed algebraically and then illustrated numerically.

EXPECTED AGGREGATE CLAIMS

Our first step will be to compute the expected value $\bar{Z}$ of the aggregate claim Z under the group. Let us assume that we have N certificates, with a_i the amount of insurance under certificate i and q_i the mortality rate applicable to certificate i.

Because we are considering certificate i to be immediately reissued to a new life in the event of a claim thereunder, the number of claims θ_i under certificate i is a Poisson-distributed random variable which can take on any integral value 0, 1, 2, ...

Let t_i be the expected number of claims under certificate i. From the Poisson function it follows that the probability of θ_i claims under the certificate is given by

$$\text{Probability of } \theta_i \text{ claims} = \frac{e^{-t_i} t_i^{\theta_i}}{\theta_i!}. \tag{1}$$

The probability of no claims is obtained from the above formula by substituting $\theta_i = 0$ and is given by e^{-t_i}. But the probability of no claims under certificate i is also given by $1 - q_i$. Therefore,

$$e^{-t_i} = 1 - q_i ,$$

and

$$t_i = E(\theta_i) = -\ln (1 - q_i) . \tag{2}$$

It may be of interest to note that t_i is also the force of mortality for certificate i.

The aggregate claim Z under the group is given by

$$Z = \sum_{i=1}^{N} a_i \theta_i ,$$

and, if we assume that claim experience under each certificate is independent, it follows that the expected value of the aggregate claims is

$$\bar{Z} = E(Z) = E(\Sigma a_i \theta_i) = \Sigma a_i E(\theta_i) ,$$

or

$$\bar{Z} = \Sigma a_i t_i . \tag{3}$$

The number of claims T under the group is given by

$$T = \Sigma \theta_i$$

and has an expected value $\bar{T} = E(T) = \Sigma E(\theta_i)$, or

$$\bar{T} = \Sigma t_i . \tag{4}$$

The number of claims T is also a Poisson-distributed random variable.

VARIANCE OF AGGREGATE CLAIMS

Our next step will be to compute the variance $V(Z)$ of the aggregate claims. This is given by

$$\begin{aligned} V(Z) &= E(Z - \bar{Z})^2 \\ &= E\Big[\sum_{i=1}^{N} a_i (\theta_i - t_i) \Big]^2 \\ &= E\Big[\sum_{i=1}^{N} a_i^2 (\theta_i - t_i)^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j (\theta_i - t_i)(\theta_j - t_j) \Big] \quad (j \neq i) \\ &= \sum_{i=1}^{N} a_i^2 E(\theta_i - t_i)^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j E(\theta_i - t_i)(\theta_j - t_j) \quad (j \neq i) \\ &= \sum_{i=1}^{N} a_i^2 V(\theta_i) + 0 . \end{aligned} \tag{5}$$

Therefore,

$$V(Z) = \Sigma a_i^2 t_i . \tag{6}$$

The simplification of the first term follows from the property of the Poisson distribution that its variance equals its mean. The vanishing of the second term is a consequence of the assumption that the numbers of claims on different certificates are independently distributed.

CALCULATION OF CONDITIONAL EXPECTED CLAIMS

The population of aggregate claims which could occur if the group could relive the year under study many times can be divided into two subpopulations according to whether or not the aggregate claim exceeds the stop-loss level S. The subpopulation of smaller aggregates can then be further subdivided according to the number of claims. Given below are some definitions which will be used.

Let $p(j)$ be the probability of occurrence of j claims.

Let $\bar{Z}_j$ be the expected aggregate claim conditional on the occurrence of exactly j claims.

Let L_j be the expected aggregate claim conditional on the aggregate claim's not exceeding S and conditional on the occurrence of exactly j claims.

Let f_j be the probability that the aggregate claim will not exceed S conditional on the occurrence of exactly j claims.

Let L be the expected aggregate claim conditional on the aggregate claim's not exceeding S.

Let H be the expected aggregate claim conditional on the aggregate claim's exceeding S.

Let f be the probability that the aggregate claim will not exceed S.

Let A_{jk} be the kth possible aggregate of j claims, the aggregates being ranked in ascending order. We will find it necessary to enumerate only the situations where the aggregate claim does not exceed the stop-loss level.

Let P_{jk} be the probability that the aggregate of j claims will equal A_{jk}.

Let n_j be the number of possible aggregates of j claims not exceeding the stop-loss level S.

Let us now obtain an expression for $\bar{Z}_1$, the expected aggregate claim given that only one claim occurs:

$$\bar{Z}_1 = \frac{\sum_{i=1}^{N} a_i \cdot \text{Probability}(\theta_i = 1) \prod_{j=1}^{N} \text{Probability}(\theta_j = 0)}{\sum_{i=1}^{N} \text{Probability}(\theta_i = 1) \prod_{j=1}^{N} \text{Probability}(\theta_j = 0)} \qquad (j \neq i)$$

$$= \frac{\sum_{i=1}^{N} a_i e^{-t_i} t_i \prod_{j=1}^{N} e^{-t_j}}{\sum_{i=1}^{N} e^{-t_i} t_i \prod_{j=1}^{N} e^{-t_j}}$$

$$= \frac{e^{-T}\Sigma a_i t_i}{e^{-T}\Sigma t_i},$$

or

$$\bar{Z}_1 = \frac{\bar{Z}}{\bar{T}}. \tag{7}$$

It is evident that the same expression represents the average-sized claim whenever it occurs during the year, and we conclude, therefore, that

$$\bar{Z}_j = j\bar{Z}_1. \tag{8}$$

Because the total number of claims under our model is a Poisson-distributed random variable, we can state the probability of exactly j claims to be

$$p(j) = \frac{e^{-\bar{T}}\bar{T}^j}{j!}. \tag{9}$$

As stated, the possible aggregates of a single claim are represented by A_{11}, A_{12}, and so on. The enumeration of these amounts is a straightforward process carried out by an examination of the certificate amounts a_i. The probability P_{1k} of occurrence of A_{1k} is given by

$$P_{1k} = \frac{\Sigma t_i \text{ (for those certificates where } a_i = A_{1k})}{\Sigma t_i \text{ (for all certificates)}}. \tag{10}$$

We then proceed to enumerate the A_{2k} values, the possible aggregates of two claims. This we can do systematically by sequentially associating each A_{1k} value with other A_{1k} values, but enumerating only those A_{2k} values which do not exceed the stop-loss level. Similarly, we can enumerate the A_{jk} values for any j by taking into account the $A_{j-1,k}$ and A_{1k} values. Enumeration stops for some $j = J$ if $J + 1$ occurrences of the smallest-sized claim would exceed the stop-loss level.

For each A_{jk} value we compute a probability P_{jk} from

$$P_{jk} = \Sigma P_{j-1,l} P_{1m}, \tag{11}$$

summed over all pairs such that $A_{j-1,l} + A_{1m} = A_{jk}$.

We now have the necessary data for computation of L_j, the expected aggregate of j claims conditional on the aggregate claim's not exceeding the stop-loss level.

$$L_j = \frac{\sum_{k=1}^{n_j} P_{jk} A_{jk}}{\sum_{k=1}^{n_j} P_{jk}} . \tag{12}$$

Also, the probability that the aggregate of the j claims will not exceed the stop-loss level is given by

$$f_j = \sum_{k=1}^{n_j} P_{jk} . \tag{13}$$

We can now compute L, the expected aggregate claim conditional on the aggregate's not exceeding the stop-loss level.

$$L = \frac{\sum_{j=0}^{J} p(j) f_j L_j}{\sum_{j=0}^{J} p(j) f_j} . \tag{14}$$

Also, the probability f that the aggregate claim will not exceed the stop-loss level is given by

$$f = \sum_{j=0}^{J} p(j) f_j . \tag{15}$$

We are now ready to compute H, the expected aggregate claim conditional on the aggregate claim's exceeding the stop-loss level. This we do by recognizing that the mean of the aggregate claims must be a weighted average of the means of the two mutually exclusive subpopulations into which we have decomposed the over-all population. That is,

$$\bar{Z} = fL + (1 - f)H . \tag{16}$$

Solving for the unknown value of H, the expected value of the aggregate claim conditional on the aggregate claim's exceeding the stop-loss level, we get

$$H = \frac{\bar{Z} - fL}{1 - f} . \tag{17}$$

EXPECTED STOP-LOSS CLAIM

We can now obtain an expression for the expected value of the stop-loss claim W, where

$$\begin{aligned} W &= T - S & (T \geq S) \\ &= 0 & (T < S) . \end{aligned}$$

We have

$$\bar{W} = E(W) = f\cdot 0 + (1-f)(H-S)\,,$$

or

$$\bar{W} = (1-f)(H-S)\,. \tag{18}$$

VARIANCE OF STOP-LOSS CLAIMS

Our next step will be to develop an expression for the variance of the stop-loss claim W.

Let us define LL_j as the expected sum of the squares of the claims conditional on there being exactly j claims whose aggregate does not exceed the stop-loss level S. It follows that

$$LL_j = \frac{\sum_{j=1}^{J} P_{jk}A_{jk}^2}{f_j}\,. \tag{19}$$

Also, let LL be the expected sum of the squares of the claims conditional on the aggregate claim not exceeding the stop-loss level:

$$LL = \frac{\Sigma p(j) f_j LL_j}{f}\,. \tag{20}$$

Now

$$V(W) = E(W-\bar{W})^2$$

$$= E(W^2) - \bar{W}^2\,,$$

and

$$E(W^2) = \sum_{j=0}^{\infty} p(j)\Big[\sum_{k=n_j+1}^{\infty} P_{jk}(A_{jk}-S)^2\Big]$$

$$= \sum_{j=0}^{\infty} p(j)\Big[\sum_{k=1}^{\infty} P_{jk}(A_{jk}-S)^2 - \sum_{k=1}^{n_j} P_{jk}(A_{jk}-S)^2\Big]$$

$$= \sum_{j=0}^{\infty} p(j)\Big(\sum_{k=1}^{\infty} P_{jk}A_{jk}^2 - 2S\sum_{k=1}^{\infty} P_{jk}A_{jk} + S^2\sum_{k=1}^{\infty} P_{jk}\Big)$$

$$- \sum_{j=0}^{\infty} p(j)\Big(\sum_{k=1}^{n_j} P_{jk}A_{jk}^2 - 2S\sum_{k=1}^{n_j} P_{jk}A_{jk} + S^2\sum_{k=1}^{n_j} P_{jk}\Big)$$

$$= \{[V(Z)+\bar{Z}^2] - 2S\bar{Z} + S^2\} - \{fLL - 2SfL + S^2 f\}\,.$$

Substituting, we have

$$V(W) = V(Z) + \bar{Z}^2 - 2S(\bar{Z} - fL) - fLL + S^2(1-f) - \bar{W}^2\,. \tag{21}$$

NUMERICAL EXAMPLE

Consider a group of fifty certificates with the characteristics outlined in Table 1, and let us assume a stop-loss level S of \$18,000. The results are shown in Tables 2–5.

TABLE 1

CERTIFICATE DETAILS

Certificate	Amount	q_i	t_i	Certificate	Amount	q_i	t_i
1	\$4,000	.001382	.001383	26	\$10,000	.001190	.001191
2	4,000	.001193	.001194	27	10,000	.002587	.002590
3	4,000	.001193	.001194	28	10,000	.002587	.002590
4	4,000	.001011	.001012	29	10,000	.005956	.005974
5	4,000	.000918	.000918	30	10,000	.011182	.011245
6	4,000	.000890	.000890	31	12,000	.002587	.002590
7	4,000	.001313	.001314	32	12,000	.003715	.003722
8	4,000	.004750	.004761	33	12,000	.004204	.004213
9	4,000	.008507	.008543	34	12,000	.004204	.004213
10	4,000	.013308	.013397	35	12,000	.006569	.006591
11	6,000	.001011	.001012	36	14,000	.002914	.002918
12	6,000	.000882	.000882	37	14,000	.003715	.003722
13	6,000	.000914	.000914	38	14,000	.004750	.004761
14	6,000	.000953	.000953	39	14,000	.004750	.004761
15	6,000	.001313	.001314	40	14,000	.008507	.008543
16	6,000	.001313	.001314	41	16,000	.001632	.001633
17	6,000	.001827	.001829	42	16,000	.002587	.002590
18	6,000	.002587	.002590	43	16,000	.004204	.004213
19	6,000	.003288	.003293	44	16,000	.004204	.004213
20	6,000	.003715	.003722	45	16,000	.009303	.009346
21	8,000	.000893	.000893	46	20,000	.003715	.003722
22	8,000	.001827	.001829	47	20,000	.005956	.005974
23	8,000	.003715	.003722	48	20,000	.009303	.009346
24	8,000	.006569	.006591	49	20,000	.021590	.021825
25	8,000	.012213	.012288	50	25,000	.015753	.015878

TABLE 2

EXPECTED VALUES

Expected aggregate claim, $\bar{Z} = \Sigma a_i t_i$	\$ 2,851.874
Expected number of claims, $\bar{T} = \Sigma t_i$	0.226116
Variance of expected claims, $V(Z) = \Sigma a_i^2 t_i$	44,989,822
Average-sized claim, $\bar{Z}_1 = \bar{Z}/\bar{T}$	\$12,612.44

$J = 4$, since 5 claims of the smallest size will exceed the stop-loss level.

TABLE 3

POSSIBLE AGGREGATES AND PROBABILITIES

j	k	A_{jk}	P_{jk}	j	k	A_{jk}	P_{jk}
1......	1	4,000	.15304534	2......	4	14,000	.04958834
1......	2	6,000	.07882237	2......	5	16,000	.05786145
1......	3	8,000	.11199119	2......	6	18,000	.07168055
1......	4	10,000	.10432698	3......	1	12,000	.00358476
1......	5	12,000	09432769	3......	2	14,000	.00553874
1......	6	14,000	.10925808	3......	3	16,000	.01072206
1......	7	16,000	.09727308	3......	4	18,000	.01592660
2......	1	8,000	.02342288	4......	1	16,000	.00054863
2......	2	10,000	.02412679	4......	2	18,000	.00113024
2......	3	12,000	.04049243				

TABLE 4

VALUES OF $p(j), f_j, L_j, LL_j$

j	$p(j)$	f_j	L_j	LL_j
0........	.79762608	1.00000000	0	0
1........	.18035602	0.74904473	9,668.620	110,522,297
2........	.02039069	0.26717244	14,315.960	215,213,058
3........	.00153689	0.03577216	16,179.935	265,761,537
4........	.00008688	0.00167887	17,346.429	301,778,860

TABLE 5

OTHER VALUES

f........	0.93822377	LL.....	17,179,357
L........	1,476.2604	$V(W)$...	4,089,333
H.......	23,743.92	$\sigma(W)$....	2,022.21
$\bar{W}$.......	354.84		

EXTENSIONS TO OTHER GROUP BENEFITS

The formulas that we have developed have assumed a group life insurance type of environment in which the amount of insurance per certificate is constant. The following discussion relates to group benefits in general, in which the above assumption may not be applicable.

Let us define the secondary distribution as Bartlett has done, to be the probability distribution of the size of a particular claim given that a claim has occurred, and let us assume that the first and second moments of the secondary distribution are available. For the group life situation described earlier the first moment μ_1^A is equivalent to $\bar{Z}_1$, and the second moment μ_2^A is equivalent to $V(Z)/\bar{T} - \bar{Z}_1^2$.

The secondary distribution itself is equivalent to the P_{1k} probabilities which we have previously defined. The method of this paper can then be used to evaluate L, f, H, LL, and hence $\bar{W}$, the expected value of the stop-loss claim, and $V(W)$, the variance of the stop-loss claim.

BIBLIOGRAPHY

1. Bartlett, Dwight K., III. "Excess Ratio Distribution in Risk Theory," *TSA*, XVII (1965), 435.

DISCUSSION OF PRECEDING PAPER

WILLIAM A. BAILEY:

The ingenuity of actuaries like Mr. Mereu, together with the speed and storage capacity of some modern computers, has made it possible to solve numerically problems in risk analysis which previously could be solved only approximately by somewhat deep theoretical mathematical techniques. This is especially encouraging because presumably one of the actuary's primary professional functions is the evaluation of risk, and versatile computer techniques like Mr. Mereu's permit more actuaries to perform this task on a scientific basis.

Mr. Mereu's method of calculating net stop-loss premiums for a group life contract involves the following:

1. Assuming a Poisson distribution of the number of deaths arising from each certificate.
2. Calculating the first part of a frequency distribution of aggregate claim by
   a) Enumerating each possible aggregate claim less than the stop-loss level.
   b) Determining the probability that each such aggregate claim will occur.
3. Making use of the fact that the expected aggregate claim $\bar{Z}$ is equal to $fL + (1 - f)H$ and the stop-loss premium $\overline{W}$ is equal to $(1 - f)(H - S)$.

The author measures the expected variability in the stop-loss claim by calculating the variance thereof.

To assess the efficacy of the Poisson assumption, I have used the "risk analyzer program" to calculate *two complete frequency distributions* of aggregate claim (together with the stop-loss premiums at each stop-loss level) for the group illustrated in Mr. Mereu's Table 1. Table 1 of this discussion assumes, as did Mr. Mereu, that the number of deaths under each certificate follows a Poisson distribution; this is equivalent to assuming that lives who die are immediately replaced by lives of identical risk. Table 2 assumes that only one death is possible under each certificate —that is, lives who die are assumed not to be replaced during the year. Of course, in practice a given life might be replaced by a life of different age (or risk).

The magnitude (absolute or relative) of the excess of the stop-loss premium based on the Poisson assumption over that based on the binomial assumption will depend on the distribution of lives by amount and age

TABLE 1

FREQUENCY DISTRIBUTION OF AGGREGATE CLAIM AND STOP-LOSS PREMIUM: EXAMPLE I

(Assumes Poisson Distribution for Each Certificate)

Amount	Frequency	Cumulative	Stop-Loss Premium
0	.79762557173	.79762557173	2,851.8740
4,000.0000	.02760263053	.82522820226	2,042.3763
6,000.0000	.01421608056	.83944428283	1,692.8327
8,000.0000	.02067588066	.86012016350	1,371.7213
10,000.0000	.01930794892	.87942811242	1,091.9616
12,000.0000	.01784373320	.89727184562	850.8178
14,000.0000	.02072499202	.91799683765	645.3615
16,000.0000	.01874013497	.93673697263	481.3552
18,000.0000	.00148619057	.93822316321	354.8291
20,000.0000	.03424170211	.97246486532	231.2754
22,000.0000	.00125970755	.97372457288	176.2052
24,000.0000	.00227776578	.97600233866	123.6543
25,000.0000	.01266469882	.98866703749	99.6567
26,000.0000	.00147878254	.99014582003	88.3237
28,000.0000	.00153145947	.99167727951	68.6153
29,000.0000	.00043827456	.99211555408	60.2926
30,000.0000	.00129059429	.99340614838	52.4082
31,000.0000	.00022572292	.99363187130	45.8143
32,000.0000	.00098707130	.99461894261	39.4462
33,000.0000	.00032829163	.99494723424	34.0651
. . .	. . .	. . .	. . .
43,000.0000	.00002359773	.99883452514	6.2914
44,000.0000	.00008168163	.99891620678	5.1260
. . .	. . .	. . .	. . .
63,000.0000	.00000153684	.99996431923	0.1896
64,000.0000	.00000439244	.99996871167	0.1540
. . .	. . .	. . .	. . .
83,000.0000	.00000006051	.99999914841	0.0045
84,000.0000	.00000014154	.99999928995	0.0036
. . .	. . .	. . .	. . .
103,000.0000	.00000000192	.99999998335	0.0001
104,000.0000	.00000000295	.99999998630	0.0001
. . .	. . .	. . .	. . .
123,000.0000	.00000000004	.99999999970	0.0000
124,000.0000	.00000000004	.99999999975	0.0000

Mean from table = 2,851.8739992; theoretical mean = 2,851.8700000.

Standard deviation from table = 6,707.4452621; theoretical standard deviation = 6,707.4452618.

Table variance = 44,989,821.9440; theoretical variance = 44,989,821.9400.

TABLE 2

Frequency Distribution of Aggregate Claim and Stop-Loss Premium: Example II

(Assumes Binomial Distribution for Each Certificate)

Amount	Frequency	Cumulative	Stop-Loss Premium
0	.79762119004	.79762119004	2,837.6710
4,000.0000	.02771677790	.82533796794	2,028.1558
6,000.0000	.01423335291	.83957132085	1,678.8317
8,000.0000	.02064997985	.86022130071	1,357.9743
10,000.0000	.01938145144	.87960275215	1,078.4169
12,000.0000	.01786950632	.87947225848	837.6224
14,000.0000	.02078721901	.91825947749	632.5670
16,000.0000	.01871234134	.93697181884	469.0859
18,000.0000	.00149309845	.93846491729	343.0296
20,000.0000	.03442248851	.97288740581	219.9594
22,000.0000	.00126262140	.97415002721	165.7342
24,000.0000	.00225176705	.97640179427	114.0343
25,000.0000	.01276602987	.98916782414	90.4360
26,000.0000	.00148557635	.99065340050	79.6039
28,000.0000	.00147870759	.99213210810	60.9107
29,000.0000	.00044361060	.99257571870	53.0428
30,000.0000	.00129843732	.99387415603	45.6185
31,000.0000	.00022780664	.99410196268	39.4927
32,000.0000	.00093554868	.99503751136	33.5946
33,000.0000	.00033050558	.99536801695	28.6321
⋮	⋮	⋮	⋮
43,000.0000	.00002389723	.99902239980	4.4227
44,000.0000	.00006955276	.99909195256	3.4451
⋮	⋮	⋮	⋮
63,000.0000	.00000150744	.99998367472	0.0750
64,000.0000	.00000105430	.99998472903	0.0587
⋮	⋮	⋮	⋮
83,000.0000	.00000003659	.99999984294	0.0008
84,000.0000	.00000000857	.99999985151	0.0006
⋮	⋮	⋮	⋮
103,000.0000	.00000000044	.99999999897	0.0000
104,000.0000	.00000000003	.99999999901	0.0000
⋮	⋮	⋮	⋮
110,000.0000	.00000000000	.99999999977	0.0000

Mean from table = 2,837.6709996; theoretical mean = 2,837.6700000.

Standard deviation from table = 6,650.2975704; theoretical standard deviation = 6,650.2975700.

Table variance = 44,226,457.7747; theoretical variance = 44,226,457.7700.

(or other underwriting characteristics) under the particular group life contract.

There is an overriding advantage in calculating the complete frequency distribution, in that the variability in the stop-loss claims is evident from the table, and, although the variance can be computed as a by-product, we do not need to rely on the variance to guess at the shape of the distribution. An enormous advantage is that other financial calculations can be made from the complete frequency distributions. For example, the aggregate claim amounts can be translated into profit (±) amounts, which would reflect premiums, expenses, dividends, and so on, as well as aggregate claim amounts. Thus trial or actual premium rates, credibility factors, and risk charges in the retrospective dividend formula can be evaluated on a frequency distribution basis. Convoluting the frequency distributions of profits (±) (one frequency distribution for each group contract) produces a frequency distribution of total profits (±) expected from the over-all group life portfolio.

A somewhat more complex situation occurs when the retrospective dividend is based both on the aggregate claim of the specific group and on the total of the aggregate claims for all groups in the portfolio. By convoluting the frequency distributions of aggregate claims (i.e., one frequency distribution for each group other than a selected group), we can then calculate a frequency distribution of profits (±) for the selected group, reflecting the frequency distribution of aggregate claims for all the other groups in the portfolio as well as the aggregate claim for the selected group. Thus we can evaluate premiums, credibility factors, and risk charges in the dividend formula for each group separately. However, the calculation of the frequency distribution of profits (±) from the over-all portfolio of group contracts is not as facile in this situation (i.e., where the dividend is based on both the experience of the specific group and the experience of the over-all portfolio of groups).

A different situation exists when we wish to determine the adequacy of contingency reserves or surplus for a portfolio of group life contracts where the prospective rate adjustment for each group is based in part on the previous year's premium and aggregate claim (loss ratio) for such group. This process can be treated as a specialized two-dimensional random walk,[1] where the pertinent variables for each group contract are (1) the accumulated profit (±) and (2) the prospective premium rate; that is, the second variable is required to continue the random walk from

[1] Equivalent to a Markov chain.

year to year, whereas the first constitutes primacy in the problem. By retaining the frequency distribution of accumulated profits (±) for each group at the end of each year in the random walk, we can convolute such frequency distributions (i.e., one for each group contract in the portfolio) for sums at the end of any specified number of years to produce a frequency distribution of accumulated profit (±) for the over-all portfolio of group contracts; then, translating these profits (±) into present values by discounting them at a suitable interest rate, we can attempt to evaluate the adequacy of various levels of contingency reserves.

A less complex but equally valuable use of complete frequency distributions of aggregate claim is in assessing the likelihood that the death claim experience (group by group or portfolio) is consistent with the mortality rates assumed in the pricing of the group life contract.

Mr. Mereu has presented his ideas clearly and succinctly. He has combined his knowledge of probability theory and computer science to produce a direct method for calculating stop-loss premiums for group life contracts, assuming that the Poisson distribution for the number of deaths applies. The extension of his method to group health contracts needs elaboration, but perhaps this furnishes a topic for a future paper.

WILLIAM J. TAYLOR:

Mr. Mereu refers to the paper "Excess Ratio Distributions in Risk Theory" by Dwight K. Bartlett, III (*TSA*, XVII, 435). Robert Tookey, in discussing Mr. Bartlett's paper, proposed a "voyage to the center of the earth." The vehicle he suggested for this voyage was the use of Monte Carlo techniques for the determination of excess risk measurement in specific group cases. Mr. Mereu is to be congratulated! His paper and the several discussions which it has inspired provide a choice of several vehicles for a "voyage to the center of the earth."

There are several points which I would like to make in my discussion, some of which will be elaborated upon. They are as follows:

1. Risk theory should be a rather fundamental subject for the life actuary, yet many of us have never studied the subject, since it was not added to the syllabus until 1964, and many people missed it when it was transferred in 1971 from Part 10 to Part 5 in the examinations; very few practitioners of the subject have emerged by virtue of its presence on the syllabus; and the few practitioners we have within our membership come almost exclusively from our better mathematicians.
2. The central idea of risk theory as it appears to a neophyte is the application of esoteric mathematical methods to obtain very rough approximations to

part or all of the probability distribution function for the total claims in a portfolio of risk, values which could be easily computed if such a function were available.

3. If risk theory is going to be utilized by a majority of actuaries, rather than a small minority, then we must have simple, easily understood, accurate, and efficient methods of calculating directly such probability distribution functions.

   Mr. Mereu's paper illustrates, under a simplifying assumption, how solutions can be found using only the probability and statistics from the preliminary examinations and a little imagination as to computational methods. My discussion will present two additional algorithms, as well as modifications to Mr. Mereu's model which require even less knowledge of probability and statistics.

   The emergence of FORTRAN and the general availability of computational power should be sufficient to spark the imagination of enough actuaries to bring about the application of risk theory throughout most of the life actuaries' work. There is, however, a computer language much more powerful than FORTRAN which facilitates the development of algorithms. In fact, the language was originally developed for the purpose of specifying algorithms rather than as a computer language. Its name is APL, which stands for "A Programming Language." All the algorithms in my discussion are precisely defined as working APL programs. Even though not everyone may be able to decipher them without some knowledge of the language, an understanding of what they accomplish and an examination of their brevity should be enough to whet one's appetite.

4. The assumption made in Mr. Mereu's paper is that any death which occurs on a case will be replaced by another life insured for the same amount, with probability of death for the remainder of the year equal to the force of mortality for the life it replaces. The justification for this assumption is presumably that some replacement will normally occur, and this assumption is as good as any. The motivation for the assumption is presumably to satisfy the requirements for the applicability of the Poisson probability distribution function.

   I would like to suggest that the composition of the group throughout the year will change for reasons other than replacement and that it is more appropriate to make the calculation without the assumption of replacement and relate the stop-loss premium to the premium on the closed group. As changes in the composition of the group occur, one can either recompute the stop-loss premium or, if this is either inconvenient or too expensive, maintain the stop-loss premium as the same percentage of the total premium.

5. In the section of this discussion headed "Mereu Model," Mr. Mereu's model is presented in APL, and an analysis is made of the changes necessary to eliminate the "replacement assumption."

6. In the section "Retention Convolution," an algorithm is presented which accurately computes the frequency distribution of the retained claim and, from this, the statistical parameters of the stop-loss claim.
7. In the section "Pull Convolution," an algorithm is suggested which computes the probability of each of the possible total claim amounts for the portfolio, discards those amounts for which the probability is less than some minimum value, and then produces a cumulative probability distribution function. Considering each of the possible claim amounts as a possible stop-loss level, the algorithm then goes on to compute the mean, variance, and standard deviation of the stop-loss claim as well as the mean, variance, and standard deviation for the related retention. The results are printed out in an abridged form to fit on an $8\frac{1}{2} \times 11$ page. The power of APL is illustrated by this algorithm, since all the calculations are specified in eight lines of program. The CPU time for both the calculations and the printing is about 12 seconds.

TABLE 1

SYMBOL EQUIVALENCE TABLE

APL	Paper	APL	Paper
A	a_i	PJK	P_{jk}
AJK	A_{jk}	$P1K$	P_{1k}
$A1K$	A_{1k}	Q	q_i
F	f	S	S
FJ	f_j	SDW	$\sigma(W)$
H	H	T	t_i
L	L	VW	$V(W)$
LJ	L_j	VZ	$V(Z)$
LL	LL	$\underline{T}$	$\bar{T}$
LLJ	LL_j	$\underline{W}$	$\bar{W}$
NJ	J	$\underline{Z}$	$\bar{Z}$
$NJ1$	$J-1$	$\underline{Z1}$	$\bar{Z}_1$
PJ	$p(j)$		

Mereu Model

Although no explanation of the APL code is given, a symbol equivalence table is given (Table 1), so that anyone familiar with both the paper and APL can easily read the program. Anyone familiar with the paper but not with APL can probably decipher most of the program simply with the additional knowledge that an APL program line is executed from right to left.

The following APL program *MEREU* performs all the calculations described by the author but does not print any of the results.

```
    ∇ A MEREU Q;J;T1;T2
[1]    T←-⍟1-Q
[2]    Z←+/A×T
[3]    T←+/T
[4]    VZ←+/A×A×T
[5]    Z1←Z÷T
[6]    NJ1←¯1+NJ←⌊S÷⌊/A
[7]    PJ←(*-T)×(T*J)÷!J←⍳1+NJ
[8]    AJK←A1K,[0](NJ1,⍴A1K←(T1≠0)/T1←T1×(T1←1000×⍳1+⌊0.001×S)∊A)
         ⍴0
[9]    PJK←P1K,[0](NJ1,⍴P1K←(+/((⍴T1)⍴T)×T1←A1K∘.=A)÷T)⍴0
[10]   J←1
[11]   L1:AJK[J;]←(⍴A1K)↑(((¯1↓T1)≠1↓T1),1)/T1←T2[⍋T2←,AJK[0;]∘.+
         AJK[J-1;]]
[12]   PJK[J;]←+/(AJK[J;]∘.=T2)×((⍴A1K),⍴T2)⍴,PJK[0;]∘.×PJK[J-1;]
[13]   →L1⌈⍳NJ1≥J←J+1
[14]   AJK←AJK×T1←AJK≤S
[15]   PJK←PJK×T1
[16]   LJ←(0,+/PJK×AJK)÷FJ←1,+/PJK
[17]   L←(+/PJ×FJ×LJ)÷F←+/PJ×FJ
[18]   H←(Z-F×L)÷1-F
[19]   W←(1-F)×H-S
[20]   LLJ←(0,NJ↑+/PJK×AJK*2)÷FJ
[21]   LL←(+/PJ×FJ×LLJ)÷F
[22]   SDW←(VW←VZ+(Z*2)+(S×S×1-F)-((2×S×Z-F×L)+(F×LL)+W
         *2))*0.5
    ∇
```

The following program *PJOHN* prints the results of the program *MEREU*.

```
    ∇ PJOHN;T1
[1]    HEAD3
[2]    SFL((-T2,0)↓T1),((T2←⌈0.5×⍴Q),0)↓T1←'I10,CI8,2F10.6' ΔFMT((1+
         ⍳⍴Q);A;Q;T)
[3]    LF;'Z=';Z;' T=';T;' VZ=';VZ;' Z1=';Z1
[4]    HEAD4
[5]    'BCI12' ΔFMT AJK
[6]    HEAD5
[7]    'BF12.8' ΔFMT PJK
[8]    HEAD6
[9]    'I2,2F14.8,CF14.3,CI15' ΔFMT((⍳⍴PJ);PJ;FJ;LJ;LLJ)
[10]   LF;'F=';F;' L=';L;' H=';H;' W=';W
[11]   LF,'LL=';LL;' VW=';VW;' SDW=';SDW
    ∇

    ∇ R←SFL P;T
[1]    R←((T+⌊(¯1+T←1↑⍴P)÷5)⍴1 1 1 1 1 0)⍀P
    ∇
```

The above program, as well as all other APL programs presented in this discussion, employs ΔFMT, which is the output formatting feature of APL-PLUS. This is the only feature employed that is not available under the standard APL. Table 2 illustrates the execution of both programs.

There are just five formulas in Mr. Mereu's model which are dependent upon the "replacement assumption." The necessary changes for four of them and their numeric effect on the example will now be set forth. Formula number references are to the paper; line-number references are to the above program *MEREU*. The first two changes are given in both the formulas and the APL code, whereas the latter two are in APL code only.

The first change is the expected number of claims under certificate i, formula (2), line 1: change to $t_i = q_i$ or $T \leftarrow Q$. This is the fundamental change—use of probabilities of death in one year as opposed to the average force of mortality.

The second change is the variance of aggregate claims, formula (6), line 4: change to

$$V(Z) = \Sigma\, a_i^2 t_i(1 - t_i) \qquad \text{or} \qquad VZ\leftarrow+/A\times A\times T\times 1-T\ .$$

The third change is the probability distribution function for the number of claims, formula (9), line 7: change to *PJ* ← *NJ CONVOLN T*, where *CONVOLN* is the following algorithm:

```
  ∇ R←J CONVOLN Q
[1]  R←+1,Jρ+0
[2] L1:R←(R×1−1↑Q)++0,(¯1↓R)×1↑Q
[3]  ←L1⌈ι0≠ρQ←1↓Q
  ∇
```

The above algorithm contains the basic computational idea employed in the calculation of all the probability distribution functions presented in this discussion. In line 1 the probability distribution function is initialized as a 1 followed by the correct number of zeros. This is the correct probability distribution function for zero risks. In line 2 the probability distribution function is modified to include one more risk in the group. In line 3 the looping through all risks is controlled.

The fourth change is the conditional probabilities P_{1k}, formula (10), line 9: change to

```
PJK←P1K,[0](NJ1,ρP1K←ONE)ρ0 ,
```

where *ONE* is the following algorithm:

```
  ∇ R←ONE;I;J;K
[1]  R←ιI←0
[2]  R←R,(÷/PJ[0 1])×+/J÷K←1−J←(A1K[I]=A)/Q
[3]  →2⌈ι(ρA1K)>I←I+1
  ∇
```

TABLE 2

A MEREU Q; PJOHN

Cert.	A	Q	T	Cert.	A	Q	T
1.......	4,000	0.001382	0.001383	26.....	10,000	0.001190	0.001191
2.......	4,000	0.001193	0.001194	27.....	10,000	0.002587	0.002590
3.......	4,000	0.001193	0.001194	28.....	10,000	0.002587	0.002590
4.......	4,000	0.001011	0.001012	29.....	10,000	0.005956	0.005974
5.......	4,000	0.000918	0.000918	30.....	10,000	0.011182	0.011245
6.......	4,000	0.000890	0.000890	31.....	12,000	0.002587	0.002590
7.......	4,000	0.001313	0.001314	32.....	12,000	0.003715	0.003722
8.......	4,000	0.004750	0.004761	33.....	12,000	0.004204	0.004213
9.......	4,000	0.008507	0.008543	34.....	12,000	0.004204	0.004213
10......	4,000	0.013308	0.013397	35.....	12,000	0.006569	0.006591
11......	6,000	0.001011	0.001012	36.....	14,000	0.002914	0.002918
12......	6,000	0.000882	0.000882	37.....	14,000	0.003715	0.003722
13......	6,000	0.000914	0.000914	38.....	14,000	0.004750	0.004761
14......	6,000	0.000953	0.000953	39.....	14,000	0.004750	0.004761
15......	6,000	0.001313	0.001314	40.....	14,000	0.008507	0.008543
16......	6,000	0.001313	0.001314	41.....	16,000	0.001632	0.001633
17......	6,000	0.001827	0.001829	42.....	16,000	0.002587	0.002590
18......	6,000	0.002587	0.002590	43.....	16,000	0.004204	0.004213
19......	6,000	0.003288	0.003293	44.....	16,000	0.004204	0.004213
20......	6,000	0.003715	0.003722	45.....	16,000	0.009303	0.009347
21......	8,000	0.000893	0.000893	46.....	20,000	0.003715	0.003722
22......	8,000	0.001827	0.001829	47.....	20,000	0.005956	0.005974
23......	8,000	0.003715	0.003722	48.....	20,000	0.009303	0.009347
24......	8,000	0.006569	0.006591	49.....	20,000	0.021590	0.021826
25......	8,000	0.012213	0.012288	50.....	25,000	0.015753	0.015878

$\underline{Z} = 2{,}851.955264$; $\underline{T} = 0.2261214934$; $VZ = 44{,}991{,}249.24$; $\underline{Z}1 = 12{,}612.49084$

AJK

4,000	6,000	8,000	10,000	12,000	14,000	16,000
8,000	10,000	12,000	14,000	16,000	18,000	
12,000	14,000	16,000	18,000			
16,000	18,000					

PJK

0.15304439	0.07882425	0.11198781	0.10432537	0.09432391	0.10926072	0.09727493
0.02342258	0.02412722	0.04049147	0.04958753	0.05785950	0.07167984	
0.00358470	0.00553880	0.01072185	0.01592633			
0.00054862	0.00113024					

J	PJ	FJ	LJ	LLJ
0.......	0.79762119	1.00000000	0.000	0
1.......	0.18035929	0.74904137	9,668.645	110,522,955
2.......	0.02039156	0.26716815	14,315.947	215,212,766
3.......	0.00153699	0.03577167	16,179.926	265,761,251
4.......	0.00008689	0.00167886	17,346.441	301,778,990

$F = 0.9382208646$; $L = 1{,}476.289731$; $H = 23{,}743.76765$; $W = 354.8449993$

$LL = 17{,}179{,}752.09$; $VW = 4{,}089{,}721.394$; $SDW = 2{,}022.30596$

The fifth necessary change is the remaining conditional probabilities, P_{jk} ($j \neq 1$), formula (11), lines 9–15. This becomes so involved that I did not derive the various formulas.

Table 3 shows the mean and standard deviation of the stop-loss claims from Mr. Mereu's model and the cumulative effect of making each of the four changes set forth above. The final line gives the theoretically correct results from the algorithm *CONVOLR*, described in the next section.

From the above, I would conclude that Mr. Mereu's model is also valuable as an approximate calculation using the "closed group" assumption. The accuracy of each of the above forms is greater than is generally obtained from Monte Carlo methods, and the error is probably less than the error of estimation in the various probabilities employed.

Retention Convolution

Table 4 illustrates the execution of the APL program *CONVOLR*, using the author's data. The claim probability distribution function should be

TABLE 3

Model	Stop-Loss		CPU Time in Seconds
	Mean	Standard Deviation	
MEREU.........	354.8450	2,022.31	1.23
Change line:			
1.............	351.7453	2,011.53	1.18
4.............	351.7453	1,880.99	1.27
7.............	345.9583	1,925.33	2.70
9.............	343.6949	1,931.20	2.82
CONVOLR.......	343.0296	1,933.26	3.48

TABLE 4

A CONVOLR Q

	Mean	Variance	Standard Deviation
Retention..........	2,494.641447	29,851,370.98	5,463.640817
Stop-loss...........	343.029553	3,737,494.39	1,933.260043

Amount	Probability	Cumulative	Amount	Probability	Cumulative
0.........	0.7976211900	0.7976211900	12,000...	0.0178695063	0.8974722585
4,000.....	0.0277167779	0.8253379679	14,000...	0.0207872190	0.9182594775
6,000.....	0.0142333529	0.8395713209	16,000...	0.0187123413	0.9369718188
8,000.....	0.0206499799	0.8602213007	18,000...	0.0014930985	0.9384649173
10,000....	0.0193814514	0.8796027522	18,000...	0.0615350827	1.0000000000

useful in explaining the need for risk charges to the policyholder. The first 18,000 refers to a total claim of exactly $18,000. The second 18,000 refers to a stop-loss claim which results in a retention of $18,000 of claim.

The author's model led to the development of this algorithm. The difference is that I compute the probability distribution function for the retained claim directly rather than through the use of conditional probabilities.

Although, for the example in the paper, this algorithm appears to take longer than the author's model, this would probably not be the case for very large cases with a more complete distribution of amounts and higher stop-loss coverage, especially if the algorithms are rewritten in assembler language or FORTRAN. The execution time for this algorithm is clearly directly proportional to both the number of risks and the number of amount units in the stop-loss level.

The following APL program *CONVOLR* employs an extension of the algorithm *CONVOLN* to compute the claims probability distribution function:

```
    ∇ A CONVOLR Q;T;I;B;R
[1]   ((⍴A)≠⍴Q)/'ERROR (⍴A)≠⍴Q',0/B←A,0/R←Q
[2]   A←⌊A÷1000,0/I←⍴FR←1,(⌊S÷1000)⍴0
[3]   L1:FR←(FR×1−1↑Q)+((T≥I)/+0),(T<I)/(T⍴0),((−T←1↑A)↓FR)×1
        ↑Q
[4]   →L1⌈⍳0≠⍴Q←1↓Q,0/A←1↓A
[5]   CFR←1 ∆PL FR←((T←FR>0)/FR),1−F←+/FR
[6]   CR←(1000×T/⍳I),S
[7]   SDR←(VR←(+/FR×CR*2)−(R←+/FR×CR)*2)*0.5
[8]   W←(Z←+/B×R)−R
[9]   SDW←(VW←(VZ←+/B×B×R×1−R)+(Z*2)+(S×S×1−F)+(2×S×+/
        ¯1↓CR×FR)−((2×S×Z)+(W×W)++/¯1↓CR×CR×FR))*0.5
[10]  HEAD1
[11]  '⍞RETENTION ⍞,CF16.6,CF16.2,CF16.6' ∆FMT(R;VR;SDR)
[12]  '⍞STOP-LOSS ⍞,CF16.6,CF16.2,CF16.6' ∆FMT(W;VW;SDW)
[13]  HEAD2
[14]  SFL 'CI9,2F16.10' ∆FMT(CR;FR;CFR)
    ∇
```

The above program employs a utility program ΔPL which computes subtotals. The following will illustrate the use of this program in both the above and the next section.

```
          ⍳5
0  1  2  3  4
          1 ∆PL ⍳5
0  1  3  6  10
          ¯1 ∆PL ⍳5
10  10  9  7  4
```

For completeness, the program ΔPL as well as ΔNS which it calls are listed. Some readers may be confused and bewildered by the complex logic used to accomplish such a simple function. APL is implemented only as an interpretive system, which means that the execution of loops is very inefficient. ΔPL accomplishes the summing process by looping only as many times as the smallest power of 2 which is greater than the number of elements in the array to be summed.

```
  ∇ Z←J ΔPL X;N
[1]  X←(×N)×(J←|J)=(~⎕IO)+⍳⍴N←−(×J)×⍴Z←ΔNS X
[2]  J←J−~⎕IO
[3]  Z←Z+N↑X↓Z
[4]  →3×(|N[J])>|(X←2×X)[J]
  ∇

  ∇ Z←ΔNS N
[1]  Z←((⍴N),(~⎕IO)+⍳0=⍴⍴N)⍴N
  ∇
```

Full Convolution

Table 5 is an illustration of the execution of the APL program *CONVOL* using Mr. Mereu's data. The accuracy of the calculations in Table 5 has been verified by independently calculating the stop-loss mean and standard deviation for a zero stop-loss level. The results agree precisely for the number of digits shown. All probabilities less than $1E-20$ have been discarded, and only every tenth value has been printed beyond 33,000. Obviously, two separate probability distribution functions for each of the retention and stop-loss claims can easily be constructed for any given stop-loss level from the figures shown in Table 5. The former would be useful in establishing the adequacy of contingency reserves for the case, and the latter, if convoluted across all group cases, could be used to evaluate the claim fluctuation risk for the stop-loss coverage. The following is a listing of the APL program *CONVOL* which produced Table 5.

```
  ∇ A CONVOL Q;T;I
[1]  ((⍴A)≠⍴Q)/'ERROR (⍴A)≠⍴Q'
[2]  →L̲2,0/F←(1−1↑Q),((¯1+1↑A)⍴0),1↑Q,0/A←⌊A÷1000
[3] L̲1:F←((F×1−1↑Q),T)+(T←(1↑A)⍴0),F×1↑Q
[4] L̲2:→L̲1⌈⍳0≠⍴Q←1↓Q,0/A←1↓A
[5]  CF←1 ΔPL F←(T←F>SCREEN)/F,0/I←⍴F
[6]  C←1000×T/⍳I
[7]  SDW←(VW←(¯1 ΔPL C×C×F)+(C×C×T)−(2×C×I)+(W̲←(I←¯1 ΔPL
       C×F)−C×T←¯1 ΔPL F)*2)*0.5
[8]  SDR←(VR←(1 ΔPL C×C×F)←(C×T)−(R̲←(1 ΔPL C×F)+T←C×1−CF)
       *2)*0.5
[9]  PC
  ∇

    SCREEN
1E¯20
```

TABLE 5

A CONVOL Q

Amount	Probability	Cumulative	Retention		Stop Loss	
			Mean	Standard Deviation	Mean	Standard Deviation
0..........	0.7976211900	0.7976211900	0.0000	0.0000	2,837.6710	6,650.2976
4,000......	0.0277167779	0.8253379679	809.5152	1,607.0924	2,028.1558	5,357.4352
6,000......	0.0142333529	0.8395713209	1,158.8393	2,321.2913	1,678.8317	4,752.1655
8,000......	0.0206499799	0.8602213007	1,479.6967	3,005.6285	1,357.9743	4,181.3689
10,000.....	0.0193814514	0.8796027522	1,759.2541	3,627.7215	1,078.4169	3,645.8435
12,000.....	0.0178695063	0.8974722585	2,000.0486	4,189.5850	837.6224	3,149.8362
14,000.....	0.0207872190	0.9182594775	2,205.1040	4,692.7365	632.5670	2,698.6218
16,000.....	0.0187123413	0.9369718188	2,368.5851	5,116.4925	469.0859	2,293.3279
18,000.....	0.0014930985	0.9384649173	2,494.6414	5,463.6408	343.0296	1,933.2600
20,000.....	0.0344224885	0.9728874058	2,617.7166	5,822.2727	219.9594	1,637.3160
22,000.....	0.0012626214	0.9741500272	2,671.9368	5,990.7833	165.7342	1,389.3634
24,000.....	0.0022517671	0.9764017943	2,723.6367	6,163.4999	114.0343	1,176.9698
25,000.....	0.0127660299	0.9891678241	2,747.2350	6,246.2742	90.4360	1,088.8585
26,000.....	0.0014855764	0.9906534005	2,758.0671	6,285.5984	79.6039	1,008.6698
28,000.....	0.0014787076	0.9921321081	2,776.7603	6,357.2570	60.9107	859.6582
29,000.....	0.0004436106	0.9925757187	2,784.6282	6,389.0085	53.0428	791.1734
30,000.....	0.0012984373	0.9938741560	2,792.0525	6,419.9735	45.6185	726.6544
31,000.....	0.0002278066	0.9941019627	2,798.1783	6,446.3549	39.4927	665.9105
32,000.....	0.0009355487	0.9950375114	2,804.0764	6,472.5594	33.5946	608.9176
33,000.....	0.0003305056	0.9953680170	2,809.0839	6,495.2854	28.6321	555.7542
43,000.....	0.0000238972	0.9990223998	2,833.2483	6,620.3005	4.4227	206.8578
53,000.....	0.0000236669	0.9998852526	2,837.0227	6,644.9451	0.6483	78.2050
63,000.....	0.0000015074	0.9999836747	2,837.5960	6,649.5687	0.0750	25.7760
73,000.....	0.0000006189	0.9999983340	2,837.6625	6,650.2020	0.0085	8.4720
83,000.....	0.0000000366	0.9999998429	2,837.6702	6,650.2879	0.0008	2.5185
93,000.....	0.0000000075	0.9999999856	2,837.6709	6,650.2966	0.0001	0.7197
103,000....	0.0000000004	0.9999999990	2,837.6710	6,650.2975	0.0000	0.1947
113,000....	0.0000000000	0.9999999999	2,837.6710	6,650.2976	0.0000	0.0489
123,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0119
133,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0027
143,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0006
153,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0001
163,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0000
173,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0000
185,000....	0.0000000000	1.0000000000	2,837.6710	6,650.2976	0.0000	0.0000

The left argument to the program A is a vector of amounts of benefit, and the right argument Q is a vector of the corresponding probabilities of claim. The first line merely checks to see that the two vectors are of identical lengths and prints an error message in the event that they are not.

The second line in the program converts the vector A into an integer number of thousands of coverage. It then initializes the probability distribution vector F so that it is correct for the first risk, that is, it is a vector one element longer than the amount of benefit for the first risk, with the cells representing the amount of claims in thousands from zero up to and including the amount for the first risk. The vector contains the probability of no claim in its first element and the probability of one claim in its last element. Control is then transferred to line 4, which decrements each of the vectors A and Q by throwing away one element, tests to see whether the Q vector is a null vector, and, if it is not, transfers control to line 3, where the impact of the next risk will be added to the probability frequency distribution vector. This is done by calculating separately the impact of a claim and no claim and adding the results.

The loop is continued until all the risks have been processed, at which point execution drops to line 5. Then all the probabilities which are less than the parameter *SCREEN* are discarded and the cumulative frequency distribution computed.

The above algorithm works well for small groups. It can be extended to work better for large groups by throwing away any contiguous string of insignificant probabilities at both ends of the distribution function inside the loop.

In line 6 the vector of corresponding claim amounts is determined and stored in the variable C. Line 7 computes the mean, variance, and standard deviation of the stop-loss claim and stores them in the vectors $\underline{W}$, VW, and SDW, respectively.

In line 8 the mean, variance, and standard deviation of the retention claim for all possible levels of stop loss are computed and stored in the vectors $\underline{R}$, VR, and SDR, respectively.

Line 9 calls a program denoted PC to print the results, the listing for which is as follows:

```
  ∇ PC
[1]  HEAD
[2]  SFL((20ρ1),((¯21+ρC)ρ(9ρ0),1),1)⍀'CI9,2F14.10,4CF12.4' ΔFMT(C;F;CF;R;
       SDR;W;SDW)
  ∇
```

GERALD J. RANKIN:

Mr. Mereu's paper is a welcome addition to the literature, since it provides a lucid exposition of the properties of the compound Poisson distribution, along with the stop-loss premiums associated with this distribution. This discussion will present an alternate method which uses a binomial distribution to calculate the frequency function of the aggregate amount of claims and the associated stop-loss premiums.

The risk theory literature has a plethora of statements about the presumed superiority of the Poisson distribution. However, for group life insurance and many other types of insurance, there is no practical difference in the two methods. The assumption that the deaths are replaced (Poisson) does not differ materially from the assumption that the deaths are not replaced (binomial), as long as the assumed claim rates are relatively low.

In my opinion, the binomial distribution is easier to work with, since the frequency function is obtained directly and it is not necessary to combine a conditional probability distribution, $P(j, k)$, with a frequency function, $p(j)$, for the number of claims. In addition, it is easier to calculate stop-loss premiums for all relevant values of aggregate claims after the frequency function has been determined.

The superiority of either method cannot be determined by a priori type arguments. Both models are of the "urn-wager" type and require empirical evidence and statistical testing to validate their use. As long as both models predict essentially the same claim distribution, I would suspect that the common statistical tests are too robust to differentiate between them.

The method of determining the frequency function of the aggregate claims using the binomial theorem is outlined below. Mr. Mereu's notation has been retained wherever possible.

A. *Basic Relationships*

Let $f(z, i)$ be the probability that the aggregate claims will be exactly z for a group of i lives. Then

$$f(z, i) = f(z - a_i, i - 1)t_i + f(z, i - 1)(1 - t_i) , \qquad (1)$$

where $f(z, i) = 0$, if $z < 0$. Since

$$\begin{aligned} f(z, 0) &= 1 , \quad z = 0 , \\ &= 0 , \quad z > 0 , \end{aligned}$$

formula (1) can be used recursively to calculate the frequency function for each life until all N lives have been considered.

For simplicity, let

$$f(z) = f(z, N)$$

and

$$F(z) = \text{c.d.f. of } f(z) .$$

From elementary statistics, the expected value and the variance of Z are as follows:

$$\bar{Z} = \Sigma a_i t_i ,$$

$$V(Z) = \Sigma a_i^2 t_i (1 - t_i) < \Sigma a_i^2 t_i .$$

The variance of the binomial model is, of course, less than the variance of the Poisson model. This results in smaller stop-loss premiums.

B. *Stop-Loss Premiums*

Let $\bar{W}$ be the expected value of the stop-loss claim for a deductible of s:

$$\bar{W} = \sum_{s}^{\infty} (z - s) f(z)$$

$$= \bar{Z} - \sum_{0}^{s-A} z f(z) - s[1 - F(s - A)] ,$$

where $(s - A)$ is the first nonzero value of f prior to s. $V(W)$ is the variance of the stop-loss claim:

$$V(W) = \sum_{z=0}^{s-A} (0 - \bar{W})^2 f(z) + \sum_{z=s}^{\infty} (z - \bar{W} - s)^2 f(z)$$

$$= \sigma^2 + (\bar{Z} - s)^2 - \sum_{z=0}^{s-A} z^2 f(z) + 2s \sum_{z=0}^{s-A} z f(z) - s^2 F(s - A) - \bar{W}^2 .$$

C. *Extension to Other Benefits*

Formula (1) can be extended to other types of benefits where there are more than two disjoint events, life or death, and different payoffs for each event. Let

$$a(i, k) = \text{Payoff for event } k \text{ for life } i,$$

$$t(i, k) = \text{Probability of event } k,$$

where $\Sigma t(i, k) = 1$ and $k = 1, 2, \ldots, M$. Then

$$f(z, i) = \sum_{k=1}^{M} f(z - a(i, k)) t(i, k) . \tag{2}$$

D. *Numerical Example*

Table 1 below shows the frequency function and the stop-loss premium for various claim amounts, using the group of fifty certificates outlined in Table 1 of Mr. Mereu's paper.

TABLE 1

NUMERICAL RESULTS

Amount of Claim z	Probability of Claim of Exact Amount z $f(z)$	Cumulative Distribution Function of $f(z)$ $F(z)$	Expected Value of Stop-Loss Claim $\bar{W}$	Standard Deviation of Stop-Loss Claim $\sigma(W)$
0	0.79684	0.79684	2,851.87	6,668.87
4,000	0.02780	0.82464	2,039.24	5,374.40
6,000	0.01424	0.83888	1,688.52	4,768.23
8,000	0.02072	0.85960	1,366.29	4,196.52
10,000	0.01944	0.87904	1,085.48	3,660.01
12,000	0.01790	0.89693	843.55	3,162.96
14,000	0.02083	0.91776	637.42	2,710.65
16,000	0.01875	0.93652	472.95	2,304.23
18,000	0.00150	0.93802	345.98	1,943.09
20,000	0.03465	0.97267	222.02	1,646.21
22,000	0.00127	0.97394	167.36	1,397.33
24,000	0.00227	0.97621	115.23	1,184.09
25,000	0.01286	0.98906	91.44	1,095.59
26,000	0.00149	0.99056	80.51	1,015.01
28,000	0.00149	0.99205	61.62	865.26
29,000	0.00045	0.99250	53.67	796.42
30,000	0.00131	0.99381	46.17	731.57

HANS U. GERBER* AND DONALD A. JONES:

For many years the determination of an adequate stop-loss premium has been a serious problem from a numerical point of view. This has led to a series of approximation formulas, some of which were really ingenious. The first merit of Mr. Mereu's paper is that it reminds us that in spite of all of the ingenious approximation formulas we should not forget the most natural way to determine a stop-loss premium, namely, to compute it explicitly. The use of a computer enables us to do so in many instances, and this paper shows how to do it economically—which is the paper's second merit.

The main idea of the paper is applied to a group life insurance portfolio under the assumptions of the collective risk model. While this model simplifies the numerical calculations, it is not essential to the main idea, as we shall illustrate by applying it under the assumptions of the indi-

* Dr. Gerber, not a member of the Society, is assistant professor of mathematics at the University of Michigan.

vidual risk model. Assume that N lives are to be covered with amounts at risk $z_1, z_2, \ldots, z_N$ and mortality rates $q_1, q_2, \ldots, q_N$. Introducing the random variables $Z_1, Z_2, \ldots, Z_N$, where $Z_i = 0$ if life i survives and $Z_i = z_i$ if life i dies within one year, we can express the aggregate claims of one year as $Z = Z_1 + Z_2 + \ldots + Z_N$, with expected value $E(Z) = z_1q_1 + z_2q_2 + \ldots + z_Nq_N$.

Since for each life there are two possible outcomes, the outcome space, say Ω, contains 2^N possible outcomes, which we will denote by ω. Mereu's idea is based on the practical consideration that most stop-loss covers will be set at a level, S, such that the number of outcomes which produce a stop-loss claim W is larger than the number of outcomes which produce no stop-loss claim, even though the probability of the second may exceed the probability of the first. Thus, instead of directly calculating $E(W)$, which is constant (i.e., zero) on the smaller set of outcomes and hence requires the numerical evaluation of the convolution formula on the larger set, he has used the identity $W = Z - (Z - W)$ and the fact that $Z - W$ is constant (i.e., S) over the larger set, so that he must evaluate the convolution formula only over the smaller set of outcomes. We observe that Mereu included the outcomes where $Z = S$ in the smaller set even though $Z - W = S$ on these outcomes also. We find it natural to partition Ω into

$$A = \{\omega | Z(\omega) < S\} ,$$

that is, the event that the retention, R, is less than S, and

$$B = \{\omega | Z(\omega) \geq S\} ,$$

that is, the event that the retention is equal to S. In this notation we have

$$\begin{array}{llll} \text{Stop-loss claim} & W(\omega) = 0 & \text{if} & \omega \in A \\ & \quad\;\;\;\;\; = Z(\omega) - S & \text{if} & \omega \in B , \\ \text{Retention} & R(\omega) = Z(\omega) & \text{if} & \omega \in A \\ & \quad\;\;\;\;\; = S & \text{if} & \omega \in B . \end{array}$$

The identity is now written

$$W(\omega) + R(\omega) = Z(\omega) .$$

Mereu's portfolio ($N = 50$, $S = 18{,}000$; see his Table 1) provides an impressive illustration of the size of these sets. Using elementary combinatorics, we find that A contains 1,951 outcomes and B contains 1, 125, 899, 906, 840, 673 outcomes! The number of outcomes where $Z = S$ is 2,170.

The net stop-loss premium, $E(W)$, may be calculated by

$$E(R) = \sum_{\omega \epsilon A} Z(\omega)P(\omega) + [1 - P(A)]S ,$$

and $E(W) = E(Z) - E(R)$.

The variance of W may also be calculated by using the probabilities of our smaller A. Since $W = Z - R$, we have, after squaring and using the properties of the linear operator E,

$$E[(Z - R)^2] = E(Z^2) - 2S\,E(Z) + S^2[1 - P(A)]$$

$$- \sum_{\omega \epsilon A} Z(\omega)^2 P(\omega) + 2S \sum_{\omega \epsilon A} Z(\omega)P(\omega) .$$

Up to this point we have not explicitly used the assumption of independent risks for the N lives. However, some assumption about the stochastic dependence of the risks would be necessary to calculate the $P(\omega)$'s. If we make the assumption of independent risks to calculate the P's, then we may also calculate $E(Z^2)$ in the last formula by

$$\sum_{i=1}^{N} p_i q_i z_i^2 + [E(Z)]^2 .$$

A second remark concerns the choice of the Poisson parameter t. Since Mereu considers an open portfolio, he sets

$$t = -\sum_{i=1}^{N} \ln\,(1 - q_i) .$$

If one is interested in the stop-loss premium for a closed portfolio (no replacements), the collective model with

$$t = \sum_{i=1}^{N} q_i$$

produces a stop-loss premium less than that for Mereu's collective model but greater than the true stop-loss premium for the closed portfolio. The first inequality follows from $q < \ln\,(1 - q)$. The proof of the second inequality is based on the following lemma:

If, in the closed portfolio described above, life number N is replaced by two independent risks, N and $N + 1$, with amounts at risk $\tilde{z}_N = \tilde{z}_{N+1} = z_N$ and mortality rates $\tilde{q}_N$, $\tilde{q}_{N+1}$ such that $\tilde{q}_N + \tilde{q}_{N+1} = q_N$, the stop-loss premium for this modified portfolio is at least as large as the one of the original portfolio. For the proof one considers a fixed outcome of $Z_1, Z_2, \ldots, Z_{N-1}$ and verifies that the conditional expectation of $\widetilde{W}$ is at least as large as that of W.

Now, by repeated application of the lemma, one obtains the result that the stop-loss premium in the limiting case, that is, the collective model for the portfolio with

$$t = \sum_{i=1}^{N} q_i ,$$

dominates the true stop-loss premium for the closed portfolio.

With the use of the University of Michigan computer, for which our acknowledgment is due, we have calculated for Mereu's portfolio the probability of the retention's being less than the stop-loss level and the means and variances of the total claims, the retention, and the stop-loss claim for closed portfolios (i.e., individual risk model assumptions) with mortality rates equal to the q_i's and to the t_i's in Mereu's Table 1. These

TABLE 1

	q_i	t_i
$P(A)$	0.9369	0.9365
$E(Z)$	2,837.6710	2,851.8740
$E(R)$	2,494.6414	2,505.8899
$E(W)$	343.0296	345.9841
Var (Z)	44,226,457	44,473,869
Var (R)	29,851,371	29,976,836
Var (W)	3,737,494	3,775,601

results, shown in Table 1 of this discussion, illustrate the above inequality for the net stop-loss premiums for the collective model and the individual model.

Our thanks to John Mereu for bringing practical considerations into the calculation of net stop-loss premiums and for stimulating our thinking in this area.

L. TIMOTHY GILES:

I have been using the Poisson distribution recently in a very approximate fashion to handle group life problems, so the precise method presented by Mr. Mereu is quite enlightening.

In addition to computing expected stop-loss claims, Mr. Mereu's algorithm has other applications. Because his numerical example has a stop-loss level that is rather large in relation to the net premium, I recalculated some values with $S = \$12,000$:

$$f = 0.89738 \quad \text{and} \quad H = \$20,273 .$$

The first additional application would be to determine a credibility factor. This could be done by assuming that claims in excess of the stop-loss limit would be adjusted to the stop-loss limit, that is, $Hz + \bar{Z}(1 - z) = S$:

$$20{,}273z + 2848.749(1 - z) = 12{,}000 ,$$

and z, the credibility factor, would be 0.5252. The stop-loss level would be set at the same value of f, say 0.90, for all groups, so that the resulting credibility values would vary appropriately by size.

A second additional application would be to establish a maximum coverage on any one life in relation to the average face amount for the group. This could be done by determining a stop-loss level such that the probability of exceeding it is equal to the probability of the death of the individual with the maximum amount. Typically, this would be an older individual, so that a probability of death of about 0.02 would be appropriate. In Mr. Mereu's example, \$18,000 will be exceeded $1 - f = 6.2$ per cent of the time, which indicates that a larger S with $f = 0.02$ should be determined. The average face amount in his example is \$10,100; hence a multiple in excess of 1.8 might be appropriate.

Finally, the algorithm involves a lengthy calculation, especially for the larger groups. The problem has been computerized, but it would be helpful if we had some procedure for approximate answers. For example, suppose that we were to attempt to find f with S = \$12,000, using only a table of the Poisson distribution with $\lambda = 0.2$ (the rounded value of $\bar{T}$). Dividing \$12,000 by $\bar{Z}$ (= 2,848.749) yields 4.21. Multiplying 4.21 by $\bar{T}$ yields 0.95, or approximately 1. The Poisson table tells us that the probability of zero occurrences (the only integral value less than 0.95) when $\lambda = 0.2$ is 0.8187, not terribly close to the precise answer of 0.89738. Better results would quite probably be obtained for larger groups. In any event, Mr. Mereu has shown us the extent of the approximation.

RICHARD S. HESTER, SR.:

At Philadelphia Life we have attacked the stop-loss problem on group life contracts in a different manner. A series of programs has been written which makes use of random number series generated by computer and a Monte Carlo simulation of expected claims.

This system requires a change from one of Mr. Mereu's basic assumptions, that of instantaneous replacement of dying members with identical new members. I am sure that we will all agree that either the assumption of replacement or the assumption of nonreplacement is equally valid, and the choice depends on the other features of the system chosen.

To illustrate the comparability of the two methods, I ran his fifty-life example through our system. To save time, I assigned an age to each certificate such that the mortality rate on the 1960 Basic Group Table (already stored in the computer) was approximately equal to the q in the example.

The computer first calculates the expected claims, in this case $3,034; this is somewhat higher than the $2,849 found in the example, the error being due to the slightly different q's. Next, a series of random numbers is matched against each q to simulate one year's experience, and a record is kept of number of deaths and amount of claims. This process is repeated n times, where n follows a rule of thumb based on number of lives, amount of insurance, and expected claims. The maximum value of n is 3,000 to satisfy the capacity of the hardware, and this limit was reached in this case.

To save time and paper, the computer only prints out the first ten simulated years plus the last year. In this case, only one claim occurred in the first ten years, a claim for $16,000 in year 7. Total claims over the 3,000 years amounted to $9,239,000 on 738 deaths for an average of $3,080 per year. This is only 1.5 per cent higher than the expected value of $3,034. The average claim in the simulations was $12,519, which is remarkably close to the $12,606 in the paper. The expected number of claims in a given year was 0.246, somewhat higher than the 0.226 in the paper, again presumably due to different q's.

Stop-loss premiums can be determined for any given level. The computer simply separates the years into those with and without claims in excess of the specified level. At $18,000, it found the split shown in the accompanying tabulation. The excess of $4,977,000 over the stop-loss

	No. of Years	No. of Claims	Claims
$18,000 or less......	2,791	455	$4,262,000
Over $18,000........	209	283	4,977,000

level of $18,000 times 209 years ($3,762,000) is $1,215,000. Spreading this over 3,000 years produces a stop-loss premium of $405. Once more, this is higher than the $354 found in the paper. However, it should be remembered that the purpose of our system is to determine whether the correct value is about $400, as opposed to $25 or $1,000, rather than to pinpoint a theoretical value to the nearest penny. The probability of not having a stop-loss claim is quite close to the paper's value of 0.938, since 2,791 divided by 3,000 is 0.930.

In actual practice, we do not calculate stop-loss premiums at such a relatively high level. (The stop-loss level of $18,000 is about six times the expected claims.) Instead, the computer solves for a stop-loss level, p, such that $p + S(p) = AM$, where $S(p)$ is the stop-loss premium at p and AM is the allowable mortality cost in our rate structure, including any previously established reserves. In this way each case has a risk charge assessed against it each year in lieu of actual claims above the stop-loss level. Therefore, it is never necessary to have any loss carry forward.

COURTLAND C. SMITH:

Mr. Mereu's interesting paper gives a method for obtaining the pure risk charge for a group life stop-loss cover using expected aggregate claims, the probability f that aggregate claims do not exceed the stop-loss point, and the average size L of these wholly self-insured claims. While his method was developed for a case with predetermined benefits on individual claims, the concept has general application.

The pure stop-loss premium may be expressed as the product of the probability $1 - f$ that aggregate claims exceed the stop-loss point S times the average size stop-loss claim $H - S$; but H or $H - S$ is often hard to determine withcut (1) making crude assumptions regarding the upper tail of the aggregate claim distribution or (2) doing extensive calculations or simulations on the computer. Working with the self-insured losses can be a great saving, but it has its own risks.

If we substitute Mr. Mereu's equation (16) in his equation (18), we obtain essentially the following expression for the pure stop-loss charge:

$$r(s, \infty) = E(z) - fL - (1 - f)S \,. \tag{1}$$

This expression is quite general and gives the expected value of stop-loss claims as the difference between the expected value of all claims and the expected value of self-insured claims, whether paid in full or in part by the insured. Equation (1) above also makes it clear that it is vital for the insurer *not to overestimate* $fL + (1 - f)S$.

In certain nonlife insurance lines an insurer or a reinsurer may be asked to provide an excess-of-loss coverage in which there are deductibles for each event and possibly also an aggregate deductible or self-insurance limit applicable to all events during a specified time period, say a year. Thus, in aviation hull insurance, we may be asked to cover a given airline fleet, and we may have accident frequency rates by type of equipment and class of carrier. In such cases the benefit payable on each claim is not predetermined but is itself a variable for which we may have claim severity distributions available. In many of these instances a variant of the method outlined by Mr. Mereu would apply.

For practical purposes the sorts of events we are concerned with range from (*a*) multicraft catastrophes, usually collisions, to (*b*) one-craft catastrophes to (*c*) occurrences which are relatively common but of low severity and cost. For convenience the three classes of events may be handled separately in pricing. The collisions may be so rare that their expected value can be taken to be an element in the catastrophe loading, and the minor occurrences may produce an aggregate claim cost that can be predicted within reasonable limits from separate studies of experience and cost trends. Hence the one-craft catastrophes may become the major element in aggregate costs and the major problem in premium determination. One simplifying assumption we can often make to reflect a low stop-loss point or a high minimum cost per catastrophe is that when two such catastrophes occur within the coverage period, the aggregate claims cost will necessarily exceed the stop-loss point. Therefore, the probability f that type b aggregate claims do not exceed S depends on the Poisson probabilities of exactly zero or one claim and on the probability f_1 that a single claim will not exceed S. If L_1 is the average size of a single-catastrophe claim which does not exceed S, then

$$fL = p(1)f_1L_1 , \tag{2}$$

where $f = p(0) + p(1)f_1$, and equation (1) can be used to find $r(S, \infty)$.

In the special case where the cost of a single catastrophe can be assumed always to exceed S, there are no wholly self-insured losses, $f_1 = 0 = fL, f = p(0)$, and

$$r(S, \infty) = E(z) - [1 - p(0)]S . \tag{3}$$

An interesting practical question arises when the problem is not to determine the pure stop-loss charge but rather to take the stop-loss premium as given and find the stop-loss point which the pure charge can "buy." In cases where the number of expected claims is less than about 0.3, the probability of zero claims, $p(0)$, becomes large, and L may be of the order of 0–10 per cent of S. We can then write

$$L = gS , \tag{4}$$

where g is small, and equation (1) becomes

$$r(S, \infty) = E(z) - [fg + (1 - f)]S ,$$

so that

$$S = \frac{E(z) - r(S, \infty)}{1 - f + fg} . \tag{5}$$

Three observations about equation (5) are worth noting for small numbers of expected claims:

1. S is very sensitive to changes in the denominator alone. The term fg may be disregarded by the insurer with safety, but doing so may in some instances lead to an uncompetitive estimate of S, because f is large (close to unity).
2. When expressed in absolute dollars or other monetary units, the value of S remains relatively unaffected by variations in the period of the coverage. Halving the period, for example, would reduce the volume of claims in the numerator and the probability of one or more claims in the denominator (and number of expected claims) to about the same degree. Therefore, S would be relatively stable and would depend largely on the average size of a claim.
3. The buyer of a stop-loss cover may request a stop-loss level quoted as a rate or as a ratio to expected claims. If the quote seems high, the buyer may then ask that the coverage period be reduced. Point 2 above indicates that if the coverage period is cut in half (say), then the quoted stop-loss level should be roughly doubled, or else the stop-loss charge should be appropriately increased.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. MEREU:

I appreciate very much the discussions of my paper and would like to thank the contributors for their observations.

Mr. Bailey has run my example through his famous risk analyzer program and has produced detailed tables of the frequency distributions of the aggregate claims under both the "Poisson replacement" and "binomial nonreplacement models." Advantages of developing the full distribution for a variety of problems are explained.

The probability of zero claims should be the same for each model. The slight difference in Mr. Bailey's results is attributable to the fact that he has used the rounded t_i values that I supplied him.

Mr. William Taylor is to be congratulated on his extensive expository discussion in which he illustrates and explains how APL can be used to program algorithms employing both the "binomial nonreplacement" and the "Poisson replacement" assumptions. The slight discrepancies between his Poisson results and those in the paper arise because the results in the paper were obtained by inputting t values which had been rounded, whereas Mr. Taylor inputs the q values and does not round them. His figures are consequently more accurate.

Mr. Rankin has paralleled the development in the paper using the binomial model. His recursive procedure involves bringing in one life at a time, whereas the recursive procedure described in the paper was connected to the claim amounts and number of claims. I believe that Mr.

Rankin has inputted my t_i values as probabilities. This would explain the difference between his results and Bailey's and Taylor's and his agreement with the second set of results provided by Dr. Jones and Dr. Gerber.

Dr. Jones and Dr. Gerber have helpfully explained my algorithm in terms of partitioned sets. They show correctly that in applying the algorithm it is proper to stop with the highest amount less than the stop-loss level. Nothing is gained by including claims exactly equal to the stop-loss level in the population of smaller aggregates.

Mr. Giles explores the possibility of using the algorithm for purposes other than the computation of expected stop-loss claims.

Mr. Hester validates the results in the paper with his simulation program.

Mr. Smith discusses the use of the algorithm for catastrophe coverages. His formula (1) expresses the stop-loss charge in a form explainable by general reasoning. In analyzing his formula (5), he appears to treat g as a constant independent of the stop-loss level. It would seem safer to have a program which produces a complete frequency distribution, from which it would be a simple matter to set a stop level corresponding to a charge.

In addition to thanking the discussers of the paper, I would like to acknowledge Mr. Ivan R. Taylor as a source of inspiration for the algorithm and Mr. David S. Patroch (not a member of the Society) from my company for programming the algorithm.