COST OF VESTING IN PRIVATE PENSION PLANS

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ABSTRACT

The continued interest in government circles in the mandatory vesting of private pension plans imposes upon the actuarial profession the obligation to answer as definitely and as clearly as possible the question continually raised—how much does vesting cost? This paper builds upon the previous work of Marples, McGinn, and McGill and only occasionally breaks new ground. Its basic contribution is in simplification rather than elaboration. It attempts to reduce the several variables to their bare minimum and to investigate how the cost of vesting depends upon those few that remain.

It is the contention of the author that the long-range cost of vesting, when expressed in relative rather than absolute terms, is basically a function of (1) the vesting conditions, (2) the rates of employee withdrawal *after* the earliest vesting age, and (3) the age at entry. With some important qualifications, the cost of vesting is essentially independent of nearly everything else.

All the difficulties in the cost-of-vesting problem are minor compared to the overriding problem of projecting high-age and long-service employee withdrawal rates. When and if predictive tools in this area are sharpened, the actuary has all the others needed to give good answers to the cost-of-vesting question.

INTRODUCTION

THE continued interest on the part of those in government circles in the mandatory vesting of private pension plans, as well as the natural interest among employers, unions, and pension technicians in the concept of vesting, imposes upon the actuarial profession the obligation to answer as definitely and as clearly as possible the question continually raised: How much does vesting cost?

Despite excellent work by William Marples and Daniel McGinn, culminating in two papers in Volume XVIII of these *Transactions*, the profession has not answered this basic question to the satisfaction of its public. The typical actuarial statement about the cost of vesting empha-

sizes that the cost depends upon interest, mortality, withdrawal rates, entry age, salary scales, the age-service distribution of the covered population, and the actuarial cost method, in addition to the terms of the vesting provisions themselves. Only rarely is a firm estimate made, and even then the answer can seldom be generalized to other situations. Under the circumstances it should not be surprising that there now appears to be a tendency to turn the problem over to the computer, before (rather than after) the basic thinking has been done.

The most recent attempt to get at the problem appears as a chapter on the "Cost of Vesting" in the volume entitled *Preservation of Pension Benefit Rights,* written by Professor Dan McGill of the Wharton School and published by the Pension Research Council. This chapter contains some excellent analysis, as well as the results of a computer simulation. It does not, however, clearly specify the variables upon which the cost of vesting essentially depends.

This paper is an attempt to reduce the several variables to their bare minimum and to investigate how the cost of vesting depends upon those few that remain. It is the contention of the author that the long-range cost of vesting, when expressed in relative rather than absolute terms, is basically a function of (1) the vesting conditions, (2) the rates of employee withdrawal *after* the earliest vesting age, and (3) the age at entry. With some important qualifications, the cost of vesting is essentially independent of (4) interest, (5) mortality, (6) the rates of employee withdrawal before the earliest vesting age, (7) the age-service distribution of the covered population, and (8) the actuarial cost method. It is hoped that this simplification of the problem will enable actuaries to concentrate on the fundamentals and thereby to give much better answers to the underlying question than they have previously been able to provide. It will be found that the fundamental difficulty in estimating the cost of vesting comes down to the problem of estimating rates of employee withdrawal (voluntary and involuntary) after the vesting conditions have been met.

This paper builds upon the previous work of Marples, McGinn, and McGill and only occasionally breaks new ground. Its basic contribution is in simplification rather than elaboration. As a tie to the past, McGinn's notation is employed, and the illustrations used are based on his.

RELATIVE VERSUS ABSOLUTE COST OF VESTING

The basic concept needed if the simplification attempted is to be successful is that the cost of vesting can be expressed as a fraction f of the cost of the basic nonvested plan. Only in this form, expressing costs of vesting relative to the costs of a nonvested plan, do the troublesome

variables cancel out. This view of the cost of vesting is in no way new, since all the authors previously noted have employed it. McGinn's "cost of vesting indices" are effectively $1 + f$, and Marples' "vesting cost ratios" are exactly f, as are McGill's "vesting cost percentages." The relative cost can, of course, be translated into an absolute or dollar cost by multiplication, using as the base whatever amount is being employed as illustrative of the cost of the basic nonvested plan.

SIMPLIFYING ASSUMPTIONS

For the initial development of the formula that we seek, it will be extremely helpful to make three simplifying assumptions. None of these is realistic. The complexities involved in the abandonment of these assumptions are not trivial and will not be ignored, but they will be deferred until later in this paper. Until then, we make the following assumptions:

1. That there are no employees or retirees at any age $x > z$, where z represents the earliest age at which vesting can take place. The age z is uniquely determined for a single entry age y by the terms of the vesting provisions, whether these are expressed in terms of age or of service or of a combination of the two.
2. That there is a single entry age y at which past employees were hired and at which future employees will be hired. If the plan has a waiting period before service is credited for pension plan purposes, y is perhaps more properly viewed as the age attained at the end of the waiting period.
3. That the plan requires no employee contribution.

BASIC FORMULA FOR f

The basic formula for f, subject to the simplifying assumptions, can be easily and directly derived by dividing (a) the present value of the benefits for those who withdraw at or after age z, and prior to retirement age r, by (b) the present value of a unit of pension to begin at age r. For any employee now aged x ($x \leq z$ by the first assumption),

$$f = \frac{v^{r-x} \sum_{k=z}^{r-1} t_k \; _{y} \; _{k-x}p_x^{(T)} q_k^{(ws)} \; _{r-k-1}p_{k+1}\bar{a}_r}{v^{r-x} \; _{r-x}p_x^{(T)}\bar{a}_r} \tag{1}$$

where

y = Entry age;

x = Attained age, $x \geq y$;

z = Earliest vesting age, $z \geq x$;

r = Retirement age, $r > z$;

$_tp_x$ = Probability that an employee aged x will remain alive for t years;

$_t p_x^{(T)}$ = Probability that an employee aged x will remain employed for t years;

$t_{k, v}$ = Fraction of the unit benefit payable at age r to the employee who withdraws between age k and age $k + 1$;

$q_k^{(ws)}$ = Probability that an employee aged k will terminate employment before attaining age $k + 1$ and will survive to age $k + 1$.

Formula (1) immediately simplifies to

$$f = \sum_{k=z}^{r-1} t_{k, v} q_k^{(ws)} \frac{_{r-k-1}p_{k+1}}{_{r-k}p_k^{(T)}} . \tag{2}$$

Formula (2) is essentially identical to McGinn's, that is,

$$f_v^{x, 65} = \sum_{k=z}^{64} \frac{k - y}{65 - y} \frac{q_k^{(ws)}}{_{65-k}p_k^{(T)}} \frac{_{64-k}p_{k+1}}{_{65-k}p_k^{(T)}} .$$

The minor difference comes from the more general form in which formula (2) is expressed. The retirement age remains unspecified, and $t_{k, v}$ is written generally, to allow for partial or nonproportional vesting. Formula (2) is also essentially the same as Marples' formula (10a) (modified to accommodate a slightly different definition of the probability of withdrawal) and the same as McGill's formulation (for the present value expected benefits method and attained ages $a < x < z$).

A close examination of formula (2) shows that f is particularly a function of the following variables:

1. The vesting conditions, represented in the formula by z, and the fractions $t_{k, v}$.
2. The withdrawal rates after vesting—that is, $q_k^{(ws)}$, $z \le k < r$. These enter the numerator directly and the denominator $_{r-k}p_k^{(T)}$ indirectly. The relationship is more than proportional, since doubling all the $q_k^{(ws)}$'s would double each numerator and at the same time substantially reduce the $_{r-k}p_k^{(T)}$ term in each denominator.
3. The entry age y. Although y does not enter formula (2) directly, both z and $t_{k, v}$ will be functions of y. In general, the higher y, the lower each $t_{k, v}$ and the higher the vesting age z, reducing the number of terms in the summation. Hence the higher the entry age y, the lower the resulting f. The term $q_k^{(ws)}$ for $k \ge z$ is a function of entry age only if the select period is longer than $z - y$ years.

On the other hand, formula (2) is absolutely independent of the following:

4. The interest rate. As noted by McGinn, all of the interest functions have canceled out.

5. The withdrawal rates prior to vesting—that is, $q_k^{(ws)}$, $k < z$.
6. The attained age x, which has also canceled out. It must be kept in mind that independence of f in formula (2) from the attained age x arises partly from the first simplifying assumption $x \leq z$ and cannot as yet be generalized to attained ages greater than z.

Finally, according to formula (2), f has the following characteristics:

7. It is absolutely independent of mortality rates for ages less than z or greater than r but is technically a function of q_k, $z \leq k < r$. McGinn shows that f is extremely insensitive to reasonable values of the death rates even within the range from z to r.
8. It is somewhat sensitive to the normal retirement age r. An increase in r lengthens the period from k to r, thereby adding to the number of terms in the summation and increasing each ratio $_{r-k-1}p_{k+1}/_{r-k}p_k^{(T)}$. This effect is likely to be slight, however, since $q_k^{(ws)}$ in the vicinity of r must be very small. More powerful is the lower value of each $t_{k,y}$ associated with a higher retirement age. An example of this latter effect can be found in the proportional vesting formula illustrated by McGinn, $t_{k,y} = (k - y)/(r - y)$. For constant values of k and y, $t_{k,y}$ is inversely related to r. All in all, f can probably be expected to decline slightly with increasing r, if everything else is held constant.
9. It is somewhat sensitive to wage or salary increase scales and to the benefit formula. Under a pension benefit formula based on some average of earnings, $t_{k,y}$ will depend upon wage or salary progressions as well as on the vesting provisions. Generally speaking, the steeper the salary scale, the lower the value of f.

ILLUSTRATION

At this point it will prove worthwhile to illustrate the effect on f of the three important variables—the vesting conditions, withdrawal rates after vesting, and the age at entry. McGinn has done this for us, in Table B1 of Appendix B, from which Table 1 of the present paper is obtained. Table 1 bears out the sensitivities indicated earlier. The variation between corresponding figures in the upper and lower sections of Table 1 illustrates the differences between two rather different vesting schedules, the second considerably more liberal than the first.

The horizontal variation illustrates the important sensitivity to postvesting withdrawal rates. Turnover Table IV has $q_k^{(ws)}$'s exactly double those of Table II, and the resulting f is everywhere more than doubled. Table IX crosses with Table IV, showing higher $q_k^{(ws)}$'s at age 43 and below, but lower values at age 44 and above. The crossing produces the higher Table IV results for the higher entry ages and clearly shows that the incidence of postvesting withdrawal rates, as well as their absolute

level, is important. The independence of f with respect to prevesting withdrawal is the reason why turnover tables similar beyond the first five years of service give identical results when no vesting occurs before five years.

The vertical variation indicates the important sensitivity to entry age. Variation by entry age is quite steep, with f cut roughly in half for each five-year increase in the entry age.

TABLE 1

VALUES OF f

ENTRY AGE y	VEST-ING AGE z	EMPLOYEE WITHDRAWAL TABLE					
		I	II or III	IV or V	VI or VII or VIII	IX	X
		100% Proportional Vesting after 15 Years					
20......	35	5.9%	13.3%	30.1%	79.0%	31.5%	62.7%
25......	40	1.9	7.9	17.0	39.8	12.9	36.3
30......	45	0.2	3.9	8.2	17.7	4.3	18.9
35......	50	0	1.2	2.5	5.1	0	7.7
		50% Proportional Vesting after 5 Years, Grading to 100% after 10 Years					
20......	25	13.5%	20.2%	50.6%	174.5%	110.9%	99.4%
25......	30	6.4	12.8	29.8	83.5	37.5	61.5
30......	35	2.4	7.8	17.0	42.0	14.8	36.4
35......	40	0.6	4.1	8.8	19.8	5.5	20.0

ACTUARIAL COST METHODS

General reasoning tells us that the object of our search, the percentage f that vesting adds to the cost of an unvested plan, is essentially independent of the actuarial cost method. If vesting costs f per cent more, this should be true no matter what plan is employed for the funding of the benefits. Obviously understanding will be greatly increased if f can also be considered invariant over time.

The derivation of f so far presented is a comparison based on the present value of benefits. It is independent of the actuarial cost method by which these present values are funded, and, if all the assumptions (including the two simplifying assumptions) are realized, f is invariant over time. Changing the comparison to base it on the "with supplemen-

tary liability" form of the projected benefit cost method gives identical results, whether the comparison is made on normal costs or on supplementary liabilities. McGinn's tables, which are based on the projected benefit cost method, give answers identical with those in this presentation for all attained ages less than z.

The principle that f is independent of the actuarial cost method and is invariant over time can easily be violated (even under the simplifying assumptions) unless we take great care in the comparison we draw. The application of some of the classic actuarial cost methods without careful attention to how vested benefits are handled can lead to f's clearly changing over time.

The accrued benefit cost method is an example of this. If we derive f_A, the ratio we seek, but in accordance with either the normal cost or the supplementary liability of the accrued benefit cost method, we find that $1 + f_A$ is simply the ratio of the two probabilities of survival from age z to age r:

$$1 + f_A = \frac{r-z p_z}{r-z p_z^{(T)}} .$$

$$f_A = \sum_{k=z}^{r-1} q_k^{(ws)} \frac{r-k-1 p_{k+1}}{r-k p_k^{(T)}} . \tag{3}$$

Formula (3) is identical with formula (2) previously derived from the present value of benefits, except that $t_{k, y}$ has been replaced by 1. Since $t_{k, y}$ in formula (2) is always ≤ 1, f_A is always greater than f. Thus the accrued benefit cost method appears to give a higher result (for all attained ages $\leq z$) than the comparison based on either the present value of benefits or the projected cost method. This seeming anomaly is explained by the fact that f_A declines with the passage of time, whereas f holds steady. This may become clearer as we abandon the first simplifying assumption and look at attained ages $> z$.

<center>ATTAINED AGES $> z$</center>

Up to this point this paper has ignored (via the first simplifying assumption) the probability that there are, or will be, pension plan members at attained ages $> z$. The purpose of this oversimplification is to handle in one place the various ramifications of the *delay* in vesting which arises from the "prospective only" nature of the vesting initially granted. It is not always appreciated that vesting at age z is a straightforward concept only for those at or below age z when the vesting provisions come into being. For those then aged $x > z$, vesting is delayed to

age x (except in the unlikely event that vested benefits are granted retroactively to those who have terminated in the past at ages $>z$). The fact that vested benefits have not always been granted has the effect of temporarily reducing the relative cost of vesting, the reduction gradually wearing off until it is eventually eliminated entirely. The value of f is therefore increasing, rather than invariant, over time, if initially there are plan members older than age z.

Formula (2) was derived in terms of active employees at ages $\leq z$. The present values of benefits for those who have previously withdrawn with vested benefits and for those who have already retired were properly ignored, since neither is associated with attained ages below z.

As we examine the situation with respect to the higher attained ages, the present value of the benefits for those who withdrew vested at some time in the past must become a part of the numerator, and the present value of the benefits of those retired from active employment must become a part of the denominator. It can easily be shown that formula (2) is also entirely valid for ages $>z$ *so long as there is a full complement of vested withdrawals* at all such ages, including those $>r$. This means essentially that the vesting provisions have been in existence for at least $x - z$ years, where x is the attained age of the oldest plan member, active or retired. Formula (2) is therefore a measure of the cost of vesting, entirely independent of attained age but appropriate for the ultimate situation after the effect of the initial vesting delay has worn off.

If we are calculating f when the vesting provisions are new, there are no vested withdrawals, and the correct formula for f at attained ages $x > z$ is obtained by substituting x for z in formula (2):

$$ f = \sum_{k=x}^{r-1} t_{k,v} q_k^{(ws)} \frac{_{r-k-1}p_{k+1}}{_{r-k}p_k^{(T)}} . \tag{4} $$

Formula (4) will always produce a lower result than formula (2), because of the missing terms in the summation, representing the missing vested withdrawals. For attained ages beyond which $q_k^{(ws)}$ is zero, f will be zero. The over-all f is then computed from formula (2) for $x \leq z$, and formula (4) for $x > z$, and will be smaller than that calculated from formula (2), the reduction depending upon the proportion of pension plan members at the higher attained ages.

In visualizing the differences between formulas (2) and (4), it helps to look at an example. Although McGinn's Table B1 was intended to illustrate the effect of varying z, Mr. Charles Farr, in his discussion of McGinn's paper, points out that it is equally appropriate for the variation

of x, $x > z$. The accompanying tabulation is based on Turnover Table IV and the "fully vested after fifteen years" case. The value of f goes to

Entry Age	100% Vesting at	f
25.........	$z=40$	17.0%
25.........	$x=45$	9.1
25.........	$x=50$	3.0
25.........	$x=55$	0

zero for those initially 55 or above because Table IV shows no withdrawal after that age.

The "prospective only" nature of the initial vesting provisions is indeed a complicating factor in estimating the cost of vesting. It can be handled by calculating both an initial and an ultimate f, with the indication of a gradual increase from one to the other.

The *initial* cost of vesting is the f calculated by formula (2) for attained ages z and below and by formula (4) for each attained age $> z$. The over-all initial cost of vesting is a weighted average of these results, depending heavily upon the attained-age distribution of the initial census. The correct weights are present value of prospective benefit factors and are not, unfortunately, independent of the various discounts (interest, mortality, prevesting withdrawal). In general, the higher any of these discounts, the higher the relative weight of the higher attained ages and the lower the over-all f.

The *ultimate* cost of vesting, toward which the initial cost must move, is clearly that obtained by formula (2)—which has been shown to be essentially independent of the various discounts and of the attained-age distribution.

Before leaving the general subject of the reduced initial cost of vesting, it may be well to indicate one point upon which the present development and that of McGinn appear to differ significantly. For attained age $x = 45$, entry age $y = 25$, and vesting age $z = 40$, the preceding table based on formula (4) indicates a 9.1 per cent additional cost of vesting, down from 17.0 per cent because of the five-year delay. This result is intended to apply to the present value of benefits, or to projected benefit normal cost and supplementary liabilities separately. McGinn, however, illustrates a 17.0 per cent increase in the normal cost (the result for vesting at $z = 40$) and a 7.3 per cent increase in supplementary liability. He has chosen to define his normal cost indices as those appropriate to the

ultimate situation where the vesting delay does not exist, and hence as independent of attained age. This higher result for normal cost (which does not reflect the vesting delay) produces a lower result for the supplementary liability (which does). This is the phenomenon which introduces negative supplementary liability adjustments in certain parts of McGinn's Table B2. With at least equal logic he might have computed his normal cost indices to recognize when vesting *in fact* commenced, in which case his normal cost and supplementary liability indices would have been equal, but both functions of attained age when $x > z$.

<div align="center">MULTIPLE ENTRY AGE</div>

We now inquire into the consequences of abandoning the second of the two simplifying assumptions—that is, we recognize that not all employees enter at a single age. We saw earlier that f is a sharply decreasing function of entry age. When more than one entry age is involved, but a composite f is needed, we run into an averaging problem similar to that encountered earlier. Under the approach of this paper, the proper weights would again be attained-age present value of benefit factors, which depend upon the various discounts and the attained-age distribution as well. Varying any of these will change the relative weights between the results of formula (2) for the several entry ages. This confounding cannot be of real importance, but it is largely the reason why results seem to differ with the interest rate and with the changing attained-age distributions in the McGill illustration, which is based on an entry age distribution rather than single entry age. The differences between normal cost and supplementary liability for the projected benefit method, and between either of these and the present value of benefit approach that McGill also illustrates, are due to a variation of this same averaging problem. Different weighting factors are employed in the three different situations.

<div align="center">CONTRIBUTORY PLANS</div>

The foregoing development has been confined to noncontributory plans, those in which the individual employee makes no contribution. It cannot be extended to contributory plans, without substantial modification, because pension practice in the United States requires that employee contributions be treated differently from those made by the employer in at least these two respects:

1. Employee contributions are essentially always vested. Withdrawal from employment prior to retirement almost invariably gives rise to a right to the return of employee contributions (usually with interest) in a lump sum.

2. The exercise of the cash withdrawal right usually negates any vesting provisions with respect to the employer-contributed portion of the pension benefit.

With some further complication it is possible to generalize to contributory plans the foregoing analysis of the relative cost of vesting. The first step is to separate the contributory plan into two parts: (1) the death, withdrawal, and pension benefits provided by the employee contributions on a money-purchase basis and (2) the withdrawal and pension benefits attributed to employer contributions that remain.

The cost of vesting of the employer-contributed second portion can then be analyzed in much the same way as for the noncontributory case. A lower f is likely to develop (even though it applies only to the employer contributions) than in the noncontributory situation, for two unrelated reasons:

1. The function indicated by the symbol $t_{k,y}$ will ordinarily be smaller. Under vesting commonly associated with contributory plans, proportionately less of the withdrawal benefit will be associated with the employer contribution than of the normal pension benefit.
2. The important $q_k^{(ws)}$ in the numerator of formula (2) must be reduced to eliminate those who divest themselves by withdrawing employee contributions. That a surprising amount of such divesting occurs has been observed by many.

While the relative cost of vesting of the employer-contributed portion of contributory plans is clearly less than for an otherwise similar noncontributory plan, it is not valid to consider the former to be zero. Clearly there is a continuum, depending largely on the degree to which the employee contributes. For plans with only nominal employee contributions, little error results from viewing the plans as noncontributory. For plans with very heavy reliance on employee contributions, little error results from viewing the cost of vesting of employer contributions as negligible.

ABANDONMENT OF SIMPLIFYING ASSUMPTIONS

The conclusions reached under the simplifying assumptions therefore need modification in three respects:

1. The attained-age distribution becomes important in measuring the temporary effect of delayed initial vesting, and hence the initial conclusion as to the independence of f with respect to the age distribution is not valid until the effect of the delay wears off.
2. During the initial period, and for all time if multiple entry ages are involved,

the independence claimed from variation in the three discounts, and from the actuarial cost method, is relative rather than absolute.
3. Contributory plans are a somewhat special situation, with lower costs of vesting.

Even with these qualifications the main thesis remains: the important variables upon which the cost of vesting depends, in addition to the vesting conditions themselves, are (1) the rate of withdrawal after vesting and (2) the entry age. The latter is relatively tractable, because credible assumptions as to future entry ages (more important than what has occurred in the past) come from examination of hiring policy. The difficulties in estimating the long-term cost of vesting come down essentially to a *single* difficulty, that of getting any firm fix on the rate of withdrawal after vesting. Because this particular issue is at the very heart of the problem, it deserves special attention.

RATES OF WITHDRAWAL AFTER VESTING

As the actuary directs his attention to rates of employee withdrawal after reasonable age and/or service conditions have been met, he finds himself in the realm of the social sciences. Economics, sociology, and psychology are involved in employee withdrawal rates at the higher ages and the longer periods of service.

Resignations, the usual form of voluntary terminations, are positively correlated with the willingness and the ability of employees to change employment, both of which are reduced at the higher ages. Layoffs or dismissals, sometimes termed involuntary terminations, are positively associated with job insecurity, which commonly decreases with length of service. These two effects work in the same direction, and both are strengthened by union emphasis on seniority. To these two concepts one could add a third—that any mismatching of employer and employee that will lead to employee termination will have been largely corrected before the vesting conditions are met. For all these reasons rates of employee termination at the higher ages and longer periods of service will be low in comparison with those at lower ages, but they are nonetheless high enough to have the most important effect on the problem of this paper. It is important to note that voluntary terminations will vary with general economic conditions and that involuntary terminations are a function of the economic forces on the particular employer.

Sociological factors affecting employee termination rates involve questions of family composition and attitudes toward working mothers. They apply particularly to female employees. Female workers have less

attachment to the work force, and the working wife can be affected in her job situation by a change in employment of her husband. On the other hand, female workers may have less opportunity or desire to advance via a job change. Termination rates of female workers are likely to be different from those of male workers of similar age and service, but the direction of difference (at the higher ages with which we are concerned here) is not entirely clear.

Some of the psychological factors are related to the presence or absence of a pension plan and to whether the plan is vested. It is often hypothesized that an environment of no pension plan or a vested one is neutral with respect to voluntary employee termination but that a nonvested plan will hold employees who might otherwise change jobs. A good theoretical case can be made for the hypothesis that labor mobility is restricted by a nonvested plan, and the argument would appear to be especially strong with respect to the high-age, long-service termination with which we are particularly concerned here. There is some empirical evidence to the contrary, particularly the observed tendency of vested employees to divest themselves by withdrawing employee contributions, a tendency not confined to low-age or short-service employees. It is a subtle but important point that the $q_k^{(ws)}$ $(k > z)$ that we need for the estimation of the cost of vesting is the termination rate associated with the nonvested situation. Any additional postvesting terminations brought about by a change from a nonvested to a vested plan are not properly chargeable to the cost of vesting as here developed, although the employer may consider them in another sense as undesirable (when he has his most productive employees in mind) or desirable (when he remembers that he has a certain number of the nonproductive).

There is yet another subtlety to the interaction between vesting provisions and withdrawal rates that has so far been ignored. The preceding mathematical development, which reached the conclusion that f is essentially independent of prevesting termination rates, implicitly assumed that $q_k^{(ws)}$ $(k < z)$ for a nonvested plan is unaffected by vesting granted at age z. Many would argue that vesting at age z would have the effect of reducing the voluntary component of $q_k^{(ws)}$ for at least those ages just lower than z. If so, the cost of vesting is understated by the f so far derived. The value of f', a cost of vesting adjusted for prevesting interaction between withdrawal rates and vesting provisions, can be shown to be

$$f' = (1 + f)\frac{_{z-v}p_v'^{(T)}}{_{z-v}p_y^{(T)}} - 1 ,$$

where the p' in the numerator represents the probability of survival on the lower withdrawal rates that vesting at age z might bring about. The cost f' is clearly a function of prevesting withdrawal rates or, more exactly, the change in prevesting withdrawal rates that vesting at age z might cause.

Because of the complications implied in the preceding paragraphs, historical withdrawal rates, even if tabulated in the age-service-sex-specific form needed, are likely to be of little value. Since $q_k^{(ws)}$ at the high ages is small, it would take a large employer indeed to have significant experience, and even statistically adequate experience is dubiously translated to another time period, to another employer, to a different situation with respect to a pension plan and its vesting, to a different mix of skills or of geography, or to a different economic climate. Good estimates of high-age or long-service employee withdrawal rates are not yet within the state of the art, and all published tables must be viewed as illustrative only.

<div align="center">SUMMARY</div>

All the difficulties in the cost-of-vesting problem are minor compared to the overriding problem of projecting high-age and long-service employee withdrawal rates. When and if predictive tools in this area are sharpened, the actuary has all the others needed to give good answers to the cost-of-vesting question. Formula (2) in itself is almost enough for noncontributory plans, although some correction is called for to recognize the delayed effect of prospective vesting, and careful attention must be given to entry age. Interest, mortality, attained age, and funding methods, so often the major concern of actuaries, are largely extraneous issues; but careful analysis, the main actuarial tool, is particularly required.

DISCUSSION OF PRECEDING PAPER

HOWARD WINKLEVOSS* AND ARNOLD SHAPIRO:

Mr. Trowbridge has done an admirable job of boiling down a complex vesting cost function to only a few important variables: (1) the vesting condition, (2) the termination rates *after* the earliest vesting age, and (3) the entry age. The authors feel, however, that this list excludes two potentially important factors, namely, the benefit function (and in particular the slope of this function) and the actuarial cost method. We will consider the benefit function first.

We have seen, for example, that a career average benefit formula in connection with a salary function that increases at a rate of 3 per cent per year may produce a vesting cost ratio that is one-third less than the corresponding ratio developed under a unit benefit formula. A 5 per cent rate of increase in the salary function has produced estimates that are approximately 50 per cent lower than the unit benefit vesting cost ratio. The effect of the salary slope, of course, becomes even more significant when, for example, a final five-year average salary benefit formula is used. Thus the benefit function—which necessarily involves the benefit formula and, when appropriate, the slope of the salary scale—is a factor which we feel must be added to the list of important variables affecting the relative cost of vesting.

As for the actuarial cost method, the author notes that the benefit fraction, $t_{k,y}$, in the general vesting cost ratio (eq. [2]) is eliminated for the corresponding ratio of the accrued benefit cost method, but no attempt is made to reason whether or not this will produce significantly different ratios. If one observes that $t_{k,y}$ is less than unity until retirement age—and for low entry ages it may be significantly less than unity over a rather extended portion of the vesting interval—it stands to reason that the vesting cost ratio of the accrued benefit cost method might be significantly greater than the general vesting cost ratio. For this reason we feel that this factor must also be placed on the list of important variables affecting the relative cost of vesting.

Our final comment relates to the author's observation regarding Mc-Ginn's paper (*TSA*, XVIII, 187). He states essentially that the negative

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accrued liability vesting cost ratio that occurred in McGinn's ratios at advanced ages is attributable to his not reflecting the vesting delay, that is, McGinn costed the entry-age cost of vesting rather than the employee's attained-age cost of vesting. It should be noted that, even if one were to cost only a participant's future vested benefits when his attained age is greater than z at the time vesting is introduced, the individual's supplemental liability ratio will still obtain a negative value prior to age r. The negative ratio occurs because the projected benefit cost method costs benefits on a level basis throughout one's employment period, while the vested benefit function (present value of vested benefits) decreases to zero by age r—and in McGinn's examples by age 55 for most of the termination assumptions. Clearly, if a vested benefit contribution is made at the time future vested benefits are zero, there must have developed a negative accrued liability prior to this time which the contribution is designed to eliminate. Actually, McGinn's ratios attain a negative value at advanced ages because they exclude the cost of vesting associated with vested withdrawals after age z, a point which Mr. Trowbridge develops earlier in his paper.

MAX BLOCH* AND HARRY S. PURNELL:

Again, as in 1966, two papers on the cost of vesting are appearing simultaneously. The paper by Howard E. Winklevoss and Arnold F. Shapiro, as well as that of Charles L. Trowbridge, acknowledges the pioneer work done by William S. Marples and Daniel F. McGinn. The new papers offer welcome additions and clarifications with regard to a subject that has lost none of its timeliness. A great portion of the discussions of 1966, as well as a basic article by Frank L. Griffin, Jr., and a whole book by Professor Dan M. McGill, was directed at the underlying philosophy of vesting. The new papers deal primarily with mathematical principles in a generalized, rarefied atmosphere. In our discussion we want to come a little more down to earth.

The most important general conclusion, surmised in the 1966 papers and confirmed in the two current papers, is the independence of vesting cost *ratios* of the interest and mortality assumptions used in pension liability calculations. At the end of his article Mr. Trowbridge points out that "good estimates of high-age or long-service employee withdrawal rates are not yet within the state of the art, and all published tables must be viewed as illustrative only." We tend to agree, but we feel (1) that the actuarial profession cannot wait until the art is perfected and must use

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"published" rates, at least in the computation of basic pension liabilities (as indeed it does); (2) that the present imperfections are no excuse for not computing vesting liabilities, which is still a widespread practice; and (3) that the "published" rates are in most instances within the realm of "conservative" actuarial assumptions and, as such, are not merely illustrative but are fully justifiable and acceptable tools.

How are these tools to be used? From any published or otherwise selected table of one-year withdrawal rates w_x it is possible to derive functions of the form

$$l_x = \prod_{k=0}^{k=r-x} (1 - w_{x+k}) ,\qquad (1)$$

which are probabilities of remaining in the covered group from age x to retirement age r. (If r' is the lowest age at which early retirement is permitted, and if the "conservative" table shows values of w_x greater than zero for age r' and higher ages, as the Winklevoss-Shapiro paper assumes, then the formation of l_x must break off at r'.) As everyone knows, a service table applicable to the covered group can be constructed from a life table by making

$$l_x^t = \frac{l_x}{l_x} \quad \text{and, consequently,} \quad D_x^t = \frac{D_x}{l_x} .\qquad (2)$$

It is just as well known that a deferred annuity which is "discounted for withdrawals" is connected with a deferred annuity which is based on mortality and interest discounts only by the relationship

$$_{r-x|}\ddot{a}_x^{(12)\,t} = {}_{r-x|}\ddot{a}_x^{(12)}l_x .\qquad (3)$$

It should be noted that this is a first example for the "independence of withdrawal rates."

We now form a new function, which is also based only on the withdrawal table,

$$c_x^v = \frac{w_x}{l_x} ,\qquad (4)$$

and look for its usefulness. Suppose that we have computed a liability, APL_x, for an accrued pension benefit AB_x, using deferred annuity rates which are discounted for withdrawals. Suppose, further, that we know that at age x the employee has met the requirements for 100 per cent vesting. What, then, is the additional liability to provide for full funding of the vested benefit in the event that this employee withdraws at that age? The probability of his doing so is w_x, and the liability is for a deferred annuity in the amount of AB_x but without a withdrawal discount.

Combining formulas (3) and (4), we see immediately that the required additional liability is

$$AB_{x\ \overline{r-x}|}\ddot{a}_x^{(12)t}c_x^v = APL_xc_x^v.\tag{5}$$

This is the "vesting term cost" at age x, to be multiplied by an additional factor g_x ($0 < g_x < 1$) if the vested benefit is less than 100 per cent of the accrued benefit.

The function c_x^v is a *ratio*, thus not an ordinary commutation function, and in order to emphasize the difference we are using a lower-case c. But what if we treat it as we would treat a "C" function? Then we would build a new function m_x^v by "summing c_x^v from the bottom up." First, we notice that there is a direct connection between m_x^v and t_x:[1]

$$m_x^v = \frac{1}{t_x} - 1.\tag{6}$$

Then we remember that the accrued benefit valuation method looks into the future with respect to decrements but not with respect to further benefit accruals. The accrued pension liability is of the same form as that used for the development of vesting term cost in equation (5). What the "accrued vesting liability" is depends on whether at age x the vesting requirements are or are not yet fulfilled. If they are (assuming 100 per cent vesting), a liability which is discounted for mortality and interest only must be set up, contingent on the employee's leaving the group at any time before retirement:

$$AVL_x = (1 - t_x)AB_{x\ \overline{r-x}|}\ddot{a}_x^{(12)} = AB_{x\ \overline{r-x}|}\ddot{a}_x^{(12)t}m_x^v = APL_xm_x^v.\tag{7}$$

If x is less than z, the minimum age for vesting, the "no withdrawals" liability is set up at age z, contingent upon the employee's remaining in the group until then; thus

$$AVL_x = \frac{D_z^t}{D_x^t}AB_{x\ \overline{r-x}|}\ddot{a}_z^{(12)}m_z^v = AB_{x\ \overline{r-x}|}\ddot{a}_x^{(12)t}m_z^v = APL_xm_z^v.\tag{8}$$

In general, the accrued vesting liability under this funding method is obtained by multiplying the accrued pension liability by the m^v function for the higher of x and z.

If the "vesting percentage" is less than 100 per cent, formula (7) must be modified by making $AVL_x = g_xAPL_xm_x^v$. A similar modification applies to formula (8), where the g remains at its lowest value up to age z, in

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[1] If relationship (6) holds, then $m_{x-1}^v - m_x^v = w_{x-1}/t_{x-1} = c_{x-1}^v$, which represents the rule of "summing up from the bottom."

keeping with the principle that the accrued benefit method disregards all changes in benefits after the valuation date.

The method envisages a "future service cost" for the year beginning at x. The formulas for 100 per cent vesting are analogous to formulas (7) and (8), except that the benefit expected to accrue for the year replaces the already accrued benefit. For lower vesting percentages there are several possible solutions. It might be best to include in the vesting future service cost $FSCV_x$ for the year beginning at x the "next" vesting percentage, as well as the "markup" of the accrued vesting liability. If $FSCP_x$ is the pension future service cost, then

$$FSCV_x = (g_{x+1}FSCP_x + \Delta g_{x+1}APL_x)m_x^v ,   \tag{9}$$

where $\Delta g_{x+1} = g_{x+1} - g_x$.

Many actuaries perform an annual experience analysis which demonstrates how the accrued liabilities develop from year to year. For the accrued vesting liability, we have to make a distinction with respect to the fulfillment of the vesting conditions:

$$(AVL_{x-1} + FSCV_{x-1})(1 + i) - g_x AB_x \, _{r-x+1}|\ddot{a}_{x-1}^{(12)}{}^t c_{x-1}^v(1 + i) \tag{10}$$
$$+ AVL_x \bar{q}_{x-1} = AVL_x .$$

This formula applies if x is higher than z; if x is equal to or lower than z, the negative middle term, the "vesting dropout," is missing. The $\bar{q}_{x-1}$ is the total probability of leaving the group between $x - 1$ and x and can be split into its components by well-known methods. It follows that for the experience analysis the pension and vesting liabilities and costs can be lumped together before the "expected release" factors are applied.

If we had a pension plan with immediate 100 per cent vesting, the sum of the accrued pension and vesting liabilities, using formulas (3) and (7), would be

$$AB_x[_{r-x}|\ddot{a}_x^{(12)}t_x + (1 - t_x) \, _{r-x}|\ddot{a}_x^{(12)}] = AB_x \, _{r-x}|\ddot{a}_x^{(12)} .   \tag{11}$$

This triviality is the formal proof of the often-heard statement that, if we had 100 per cent immediate vesting, any withdrawal discount would be redundant.

For use with projected benefit methods the new "family of commutation functions" can be expanded by developing r_x^v, the sum of m_x^v from the bottom up. For a simple "unit benefit" plan (a fixed amount for each year of service) and for the two possible cases of the employee's qualifying for 100 per cent vesting now or only later, we would have total vesting liabilities (y being the entry age, B the benefit per year of service)

$$TVL_x = B_{r-x}|\ddot{a}_x^{(12)}{}'[(x-y)m_x^v + r_{x+1}^v]\tag{12}$$

and

$$TVL_x = B_{r-x}|\ddot{a}_x^{(12)}{}'[(z-y)m_x^v + r_{x+1}^v]\,.\tag{13}$$

The case is still simple enough where a career average plan is valued under a projected benefit method, assuming no future salary increases:

$$TVL_x = {}_{r-x}|\ddot{a}_x^{(12)}{}'(AB_x m_x^v + FSB_x r_{x+1}^v)\tag{14}$$

and

$$TVL_x = {}_{r-x}|\ddot{a}_x^{(12)}{}'\{[AB_x + (z-x)FSB_x]m_x^v + FSB_x r_{x+1}^v\}\,.\tag{15}$$

The year-to-year development of formulas (13) and (15) offers nothing basically new. The "vesting dropout" which appeared in formula (7) becomes

$$(x-y-1)\,{}_{r-x+1}|\ddot{a}_{x-1}^{(12)}{}'c_{x-1}^v(1+i)\tag{16}$$

for formula (12) and

$$AB_{x-1}\,{}_{r-x+1}|\ddot{a}_{x-1}^{(12)}{}'c_{x-1}^v(1+i)\tag{17}$$

for formula (14).

In pension plans of the "final average pay" variety the "vested" benefit must be explicitly defined, and the computation of vesting liabilities is under the influence of such definitions. The difficulties are compounded if salary scales are used. It is perfectly possible to tailor special m_x^v and r_x^v functions for use with such plans, but valuations will be performed in all likelihood with the aid of high-speed computers, where higher commutation functions are generally much less important. The c_x^v function, however, will always be useful.

We have attempted to show the continued usefulness of the old concept of commutation functions and the applicability of this concept to ratios. There are actuaries who do not readily yield to the "tendency to turn the problem over to the computer before (rather than after) the basic thinking has been done," which Mr. Trowbridge seems to fear in the introduction to his paper, and who do try to translate important new developments of theory into simple tools for everyday use at their desks.

PAULETTE TINO:

This discussion is directed to the newcomer to the actuarial field who will have to perform the valuation of a plan with vesting benefits. Thanks to Mr. Trowbridge's paper and the paper by Dr. Winklevoss and Mr.

Shapiro, which completed former publications on the subject, he will have a good idea of what to expect as to the relative size of vesting liabilities and pension liabilities—quite a nice feeling, as we all know—and now his task begins.

1. The plan is contributory and provides for pensions at age 65, vesting benefits, and death benefits to age 65 in the form of return of employee contributions.
2. The liabilities are funded through the frozen initial liability method.
3. The liabilities are discounted for interest, mortality, and turnover.
4. The salaries are assumed to progress according to a salary scale.
5. For simplification we assume that there are no pensioners at time $t = 0, 1$ and no terminated vested employees at $t = 0$.

We shall explain in detail first how to express the vested liabilities in the contributory case and then how to develop them from one year to the next. This development serves as an illustration of the development of similar functions which will be introduced in the section dealing with the valuation proper. Given below is a list of the symbols not defined in that section.

$VE_{x+t} =$ A function equal to zero if the employee has not met the vesting requirement, and equal to one otherwise;

$D_x^t =$ Commutation symbol D_x with turnover included;

$j =$ Interest rate at which employee contributions grow;

$C_{x+t}^j =$ Accumulated employee contributions at time t;

$B_{x+t} =$ Vesting benefit at time t, inclusive of the percentage vesting if any (it can be a function of salary, service, or participation); salary dependency is implied in the text;

$i =$ Valuation interest rate;

$BR =$ Pension at retirement;

$\Sigma_{A_0} =$ Summation of an expression for all employees belonging to the population A_0;

$A_0 =$ Active employees at $t = 0$ $(A_0 = A_1 + D_1 + T_1)$;

$A_1 =$ Active employees at $t = 1$;

$D_1 =$ Decreased employees at $t = 1$ emerging from A_0;

$T_1 =$ Terminated employees at $t = 1$ emerging from A_0;

$VT_1 =$ Employees among T_1 entitled to vesting;

$UL_t =$ Unfunded liability at time t;

$F_t =$ Total fund at time t inclusive of the employee contributions;

$PSPyt =$ Past-service payment with interest, included in the employer contribution for the year;

$IG =$ Interest gain.

Since we assume $i \neq j$, the expression $1 - l_{65}/l_x$ is a symbolic way of writing the liability for returning with interest, on death from age x to age 65, a contribution of \$1 at age x. The exact expression is

$$\frac{1}{D_x} \sum_{t=0}^{64-x} C_x(1+j)^t q_{x+t} D_{x+t} \, .$$

The Vesting Liability

The liability to be set up at $t = 0$ for an employee then aged x in view of his eventual termination at time t is

If $VE_{x+t} = 0$:

$$\frac{1}{D_x^t} D_{x+t}^t w_{x+t} C_{x+t}^j \, , \tag{1}$$

the discounted value of the term cost in year t for returning the contributions;

If $VE_{x+t} = 1$:

$$\frac{1}{D_x^t} D_{x+t}^t w_{x+t} \left[B_{x+t} \frac{N_{65}^{(12)}}{D_{x+t}} + C_{x+t}^j \left(1 - \frac{l_{65}}{l_{x+t}} \right) \right] , \tag{2}$$

the discounted value of the term cost in year t for providing the deferred benefit and the death benefit.

Note that expression (1) is included in expression (2). As a consequence, we shall set up a liability for the benefit consisting of the return of the employee contribution on termination, without regard to the value of VE_{x+t}, and an *employer* vesting liability which can be looked at as a present value of future employer vesting term costs. An employer term cost in year t is considered when $VE_{x+t} = 1$.

The liability at time t for vesting and that for returning employee contributions on *death and termination* are therefore, respectively,

$$M_x^V = \frac{1}{D_x^t} \sum_{t=0}^{64-x} VE_{x+t} w_{x+t} D_{x+t}^t \left(B_{x+t} \frac{N_{65}^{(12)}}{D_{x+t}} - C_{x+t}^j \frac{l_{65}}{l_{x+t}} \right) ;$$

$$M_x^{RC} = \frac{1}{D_x^t} \sum_{t=0}^{64-x} (q_{x+t} + w_{x+t}) D_{x+t}^t C_{x+t}^j \, .$$

Development of the Vesting Liabilities

The development of M_x^V/D_x^t from one year to the next follows the standard pattern:

$$L_x(1+i) + (q_x + w_x)L_{x+1} - \text{Expected payments} = L_{x+1} \, . \tag{3}$$

This can be shown as follows:

$$\frac{M_x^V}{D_x^t} = VE_x w_x \left(B_x \frac{N_{65}^{(12)}}{D_x} - C_x^j \frac{l_{65}}{l_x} \right) + \frac{1}{D_{x+1}^t} \frac{D_{x+1}^t}{D_x^t} \sum_{t=0}^{64-(x+1)} (\dots) ,$$

$$\frac{M_x^V}{D_x^t} (1 + i) = VE_x w_x \left(B_x \frac{N_{65}^{(12)}}{D_x} - C_x^j \frac{l_{65}}{l_x} \right) (1 + i)$$

$$+ (1 - q_x - w_x) \frac{1}{D_{x+1}^t} \sum_{t=0}^{64-(x+1)} (\dots) ,$$

$$\frac{M_x^V}{D_x^t} (1 + i) + (q_x + w_x) \frac{M_{x+1}^V}{D_{x+1}^t} \tag{4}$$

$$- VE_x w_x \left(B_x \frac{N_{65}^{(12)}}{D_x} - C_x^j \frac{l_{65}}{l_x} \right) (1 + i) = \frac{M_{x+1}^V}{D_{x+1}^t} ,$$

which is the explicit form of equation (3) for M_x^V/D_x^t. The B_{x+t} terms in $\Sigma_t(\dots)$ are computed on the basis of salaries progressing according to the salary scale from the salary at age x, S_x.

We now sum over the appropriate populations and at the same time introduce two new elements: (a) on both sides of equality (4) the *employer* liability $\Sigma_{VT_1} VL_{x+1}$ actually set up at $t = 1$ for the new vested employees based on the expected salary $[VL_{x+1} = B_x(N_{65}^{(12)}/D_x) - C_x(l_{65}/l_x)]$ and (b) a corrective term on the left side of the equation to reflect the effect on the active and vested liabilities at time $t = 1$ of the deviation of salaries at that time from those expected, namely,

$$- \left(\sum_{A_1} \frac{M_{x+1}^V}{D_{x+1}^t} + \sum_{VT_1} VL_{x+1} - \sum_{A_1} \frac{^A M_{x+1}^V}{D_{x+1}^t} - \sum_{VT_1} {}^A VL_{x+1} \right).$$

The same correction on the right side of the equation has the effect of substituting the actual value for the expected ones. We obtain

$$\sum_{A_0} \frac{M_x^V}{D_x^t} (1 + i) - \left[\sum_{A_0} VE_x w_x \left(B_x \frac{N_{65}^{(12)}}{D_x} - C_x \frac{l_{65}}{l_x} \right) (1 + i) \right.$$

$$\left. - \sum_{VT_1} VL_{x+1} \right]$$

$$- \left(\sum_{T_1} \frac{M_{x+1}^V}{D_{x+1}^t} - \sum_{A_0} w_x \frac{M_{x+1}^V}{D_{x+1}^t} \right) - \left(\sum_{D_1} \frac{M_{x+1}^V}{D_{x+1}^t} - \sum_{A_0} q_x \frac{M_{x+1}^V}{D_{x+1}^t} \right) \tag{5}$$

$$- \left(\sum_{A_1} \frac{M_{x+1}^V}{D_{x+1}^t} + \sum_{VT_1} VL_{x+1} - \sum_{A_1} \frac{^A M_{x+1}^V}{D_{x+1}^t} - \sum_{VT_1} {}^A VL_{x+1} \right)$$

$$= \sum_{A_1} \frac{^A M_{x+1}^V}{D_{x+1}^t} + \sum_{VT_1} {}^A VL_{x+1} .$$

Note (1) that the above expression assumes that the expected time of establishment of vested liability is the beginning of the year and (2) that $[B_x(N_{65}^{(12)}/D_x) - C_x^j(l_{65}/l_x)]$ can be negative.

Equation (5) reads as follows: The liability at the beginning of the year, with interest, minus the gains from the different sources (liabilities for newly vested employees, turnover, death, and salary scale), is equal to the liability at the end of the year for the remaining active and newly vested employees.

The Valuation Proper

When a plan is contributory, the aim of the valuation is to find the *employer's* cost and liabilities. The employer normal cost expressed as a percentage of payroll U is equal to N/D, where N is the present value of future employer normal costs and D is the present value of future salaries.

$$D_0 = \sum \frac{1}{D_x^t} \sum_{t=0}^{64-x} S_{x+t} D_{x+t}^t = \sum_{A_0} \frac{M_x^{FS}}{D_x^t}.$$

Following the steps of the development given for M_x^V/D_x^t, we have

$$\sum_{A_0} \frac{M_x^{FS}}{D_x^t} (1 + i) - \sum_{A_0} S_x(1 + i) - \left(\sum_{T_1} \frac{M_{x+1}^{FS}}{D_{x+1}^t} - \sum_{A_0} w_x \frac{M_{x+1}^{FS}}{D_{x+1}^t} \right)$$

$$- \left(\sum_{D_1} \frac{M_{x+1}^{FS}}{D_{x+1}^t} - \sum_{A_0} q_x \frac{M_{x+1}^{FS}}{D_{x+1}^t} \right) - \left(\sum_{A_1} \frac{M_{x+1}^{FS}}{D_{x+1}^t} - \sum_{A_1} \frac{{}^A M_{x+1}^{FS}}{D_{x+1}^t} \right) \quad (6)$$

$$= D_1 = \sum_{A_1} \frac{{}^A M_{x+1}^{FS}}{D_{x+1}^t}.$$

In practice, N_0 is calculated in our case as

a) Present value of projected benefit at retirement: $\Sigma_{A_0} BR(N_{65}^{(12)}/D_x^t)$, plus

b) Present value of future employer vesting liabilities: $\Sigma_{A_0} (M_x^V/D_x^t)$, plus

c) Present value of the cost of returning employee contributions: $\Sigma_{A_0} (M_x^{RC}/D_x^t)$, minus

d) Present value of future employee contributions: $\Sigma_{A_0} (M_x^{EEC}/D_x^t)$, minus

e) Accumulated employee contributions to date: $\Sigma_{A_0} C_x^j$, minus

f) Unfunded liability: UL_0, minus

g) Employer assets: $F_0 - \Sigma_{A_0} C_x^j$.

The expression M_x^{EEC}/D_x^t is equal to

$$\frac{1}{D_x^t} \sum_{t=1}^{65-x} \gamma_{x+t} D_{x+t},$$

where γ_{x+t} is the employee contribution for the year $t - 1$, with t computed on the expected salary for that year. In further developments, $(a) + (b) + (c) - (d) - (e)$ will be denoted by ΣTL_x, an employer liability.

Let us develop N from $t = 0$ to $t = 1$, first working on each component separately:

(a) $\displaystyle\sum_{A_0} \left[BR \frac{N_{65}^{(12)}}{D_x^t} (1 + i) + (q_x + w_x) BR \frac{N_{65}^{(12)}}{D_{x+1}^t} \right] = \sum_{A_0} BR \frac{N_{65}^{(12)}}{D_{x+1}^t} \, ;$

(b) $\displaystyle\sum_{A_0} \left[\frac{M_x^V}{D_x^t} (1 + i) + (q_x + w_x) \frac{M_{x+1}^V}{D_{x+1}^t} \right.$

$$\left. - VE_x w_x \left(B_x \frac{N_{65}^{(12)}}{D_x} - C_x^j \frac{l_{65}}{l_x} \right) (1 + i) \right] = \sum_{A_0} \frac{M_{x+1}^V}{D_{x+1}^t} \, ;$$

(c) $\displaystyle\sum_{A_0} \left[\frac{M_x^{RC}}{D_x^t} (1 + i) + (q_x + w_x) \frac{M_{x+1}^{RC}}{D_{x+1}^t} \right.$

$$\left. - (q_x + w_x) C_x^j (1 + j) - (q_x + w_x) C_x^j (i - j) \right] = \sum_{A_0} \frac{M_{x+1}^{RC}}{D_{x+1}^t} \, ;$$

(d) $\displaystyle\sum_{A_0} \left[\frac{M_x^{EEC}}{D_x^t} (1 + i) + (q_x + w_x) \frac{M_{x+1}^{EEC}}{D_{x+1}^t} \right.$

$$\left. - (1 - q_x - w_x) \gamma_x \right] = \sum_{A_0} \frac{M_{x+1}^{EEC}}{D_{x+1}^t} \, ;$$

(e) $\displaystyle\sum_{A_0} [C_x^j (1 + i) + \gamma_x - C_x^j (i - j)] = \sum_{A_0} C_x^j \, ;$

(f) $UL_0 (1 + i) - PSPyt = UL_1 \, ;$

(g) $\displaystyle\left(F_0 - \sum_{A_0} C_x^j \right) (1 + i) + U_0 \sum_{A_0} S_x (1 + i)$

$$+ PSPyt + IG + \sum_{A_0} C_x (i - j) = F_1 - \sum_{A_0} C_{x+1}^j \, .$$

Using the technique demonstrated on M_x^V / D_x^t to go from equation (4) to equation (5), we write, remembering the definition of TL_x and N,

$$N_0(1 + i) - U_0 \sum_{A_0} S_x(1 + i) - \left(\sum_{T_1} TL_{x+1} - \sum_{A_0} q_x TL_{x+1} \right)$$

$$- \left(\sum_{D_1} TL_{x+1} - \sum_{A_0} w_x TL_{x+1} \right) - \sum_{A_0} (q_x + w_x) C_x^j (i - j) \quad (7)$$

$$- \left(\sum_{A_1} TL_{x+1} + \sum_{VT_1} VL_{x+1} - \sum_{A_1} {}^A TL_{x+1} - \sum_{VT_1} {}^A VL_{x+1} \right)$$

$$- IG = N_1 .$$

The ratio of expression (7) to expression (6) yields U_1, which is written below with NG and DG representing the numerator and denominator gains, respectively:

$$U_1 = \frac{N_1}{D_1} = \frac{N_0(1 + i) - U_0 \sum_{A_0} S_x(1 + i) - NG}{D_0(1 + i) - \sum_{A_0} S_x(1 + i) - DG} .$$

Remembering that $N_0 = U_0 D_0$, we have

$$U_1 = U_0 - \frac{NG - U_0 DG}{D_1} , \quad (8)$$

where $NG - U_0 DG$ is the dollar gain and D_1 is the spreading factor.

At this point we can make the link between the analysis of gain and loss under the frozen initial liability method and under the unit credit method. Take the mortality gain component of the dollar gain, which, as previously written, is

$$\left(\sum_{D_1} TL_{x+1} - \sum_{A_0} q_x TL_{x+1} \right) - U_0 \left(\sum_{D_1} \frac{M_{x+1}^{PS}}{D_{x+1}^t} - \sum_{A_0} q_x \frac{M_{x+1}^{PS}}{D_{x+1}^t} \right) .$$

Regrouping, we have

$$\sum_{D_1} \left(TL_{x+1} - U_0 \frac{M_{x+1}^{PS}}{D_{x+1}^t} \right) - \sum_{A_0} q_x \left(TL_{x+1} - U_0 \frac{M_{x+1}^{PS}}{D_{x+1}^t} \right) . \quad (9)$$

The expression $TL_{x+1} - U_0(M_{x+1}^{PS}/D_{x+1}^t)$ is the accrued liability under the frozen initial liability method, and with this definition expression (9) does not differ in any way from what would have been written under the unit cost method. Note that all the functions in expression (9) are obtained from the salaries expected according to the salary scale starting from S_x.

It now remains to introduce the new entrants. Let us write $_{A_1,VT_1}U_1$ for the U_1 given by equation (8), thus enumerating the population to which it applies. For the new entrants we have

$$_{NE}U_1 = \frac{\displaystyle\sum_{NE} TL_{x+1}}{\displaystyle\sum_{NE} (M_{x+1}^{PS}/D_{x+1}^t)} = \frac{_{NE}N_1}{_{NE}D_1} = U_0 - \frac{U_0\,_{NE}D_1 - _{NE}N_1}{_{NE}D_1} \tag{10}$$

$$= U_0 - \frac{NEG}{_{NE}D_1}.$$

With equations (8) and (10), we write

$$U_0 = _{A_1.VT_1}U_1 + \frac{NG - U_0 DG}{_{A_1}D_1} = _{NE}U_1 + \frac{NEG}{_{NE}D_1}$$

$$= \frac{_{A_1.VT_1}N_1 + NG - U_0 DG + _{NE}N_1 + NEG}{_{A_1.NE}D_1}$$

$$= _{A_1.VT_1\,NE}U_1 + \frac{NG - U_0 DG + NEG}{_{A_1.NE}D_1}.$$

Then

$$_{A_1\,VT_1.NE}U_1 = U_0 - \frac{\text{Total dollar gain}}{_{A_1.NE}D_1}.$$

In practice, the liabilities do not have to be obtained by using commutation functions. This discussion is essentially an exercise in reconciliation. The usefulness of the reconciliation is not limited to the analysis of gain and loss. The insight it gives into the progression of liabilities can lead to powerful shortcuts in projecting benefits and liabilities over a series of years.

DONALD S. GRUBBS, JR.:

These two papers add substantially to the published knowledge of techniques for determining the cost of vesting, and each of the authors is to be commended.

My study of the cost of mandatory vesting provisions, prepared for the Senate Subcommittee on Labor, also involved such techniques, and these differed in some respects from those of the authors. Copies of my report are available upon request.

Nonactuaries beseech us to report the average increase in cost for vesting provisions. One thing upon which all actuaries agree is that any such average cost is unknown and even if known would have no significance for any particular employer, whose costs might be far different from the average. Both papers emphasize this variety, which needs to be communicated to the layman.

The papers deal with the ratios of costs for plans with certain vesting provisions to costs for plans with none. Any of the techniques described can easily be adapted to determine the ratio of cost increase for plans with some current vesting provision to change to a more liberal vesting requirement. Since 77 per cent of plan members are now covered under

plans with some vesting, the cost increase for this group should be presented as clearly as the cost increase for the 23 per cent with no vesting.

Both papers develop ratios of increase in cost to present plan cost. These ratios need to be translated into the increases in cost, as percentages of pay and as numbers of dollars, as they affect the actual contributions employers must make to plans. For this purpose, increases in the normal cost and in the accrued liability have direct application, while ratios of increase in the present value of future benefits cannot be related as appropriately.

The level of present plan funding has an important effect upon vesting cost ratios, as indicated by very different ratios shown in my report for plans which are now fully funded and those which are unfunded. Suppose that two companies have identical plans and identical distributions of employees, but Company A has a fully funded plan and Company B is completely unfunded. Company A may contribute the annual normal cost of 5 per cent of pay, while Company B contributes 10 per cent of pay to meet the normal cost and past-service payment. If vesting increases the cost by 0.5 per cent of pay, this is a 10 per cent increase for Company A and a 5 per cent increase for Company B.

Mr. Trowbridge points out the difference in vesting cost ratios for wage-related pensions and those not wage-related, and the effects of future salary increases. These effects are dealt with more specifically in my report, as well as the similar effects of plan amendments which increase benefits for active employees under plans which are not wage-related.

The vesting cost ratio for new employees is different from that for current employees, particularly since many current employees are past the point of initial vesting. Techniques which do not take account of the attained age of current employees properly reflect only the ultimate cost of vesting for new employees after all current employees have retired. We must also be able to determine what the increase in cost will be during the next thirty years, and show this difference and the transition. The cost increase for present employees is somewhat more predictable, since the distribution of these employees is known.

Both papers state that prevesting termination rates do not affect the vesting cost ratio. While this is correct for employees of a particular entry age, it is not correct for the plan as a whole. The prevesting termination rates vary significantly by both age and duration, thus affecting the proportion of employees at different entry ages who will eventually become eligible for either vesting or retirement. This has a very significant effect on the vesting cost ratios for the plan as a whole.

Both papers deal with techniques which can be used to determine

vesting costs. Specific figures are presented to illustrate the techniques and do not claim to represent any actual costs. But the use of termination rates of zero at ages over 55 draws attention to the importance of disability and early retirement provisions in the pension plan and to rates of disablement. In a plan with no special disability benefit, any disablement prior to early retirement is just another termination of employment, subject to the plan's vesting provision. Such disablements may form a significant portion of terminations at older ages, and, of course, their post-termination mortality differs from that for other employees.

Mr. Trowbridge states that future ages at hire and future rates of termination of employment may be quite different from those of the past. This is correct, but the future trends are unknown, and the recent past at least gives us the best guide to our starting position. But we have very little information on past experience. For my study I sought select and ultimate termination rates from actuaries and pension plans throughout the United States, but a relative handful had such data available. Table 1

TABLE 1

ANNUAL RATES OF TERMINATION OF EMPLOYMENT PER 1,000 EMPLOYEES, INCLUDING DISABLEMENTS BUT EXCLUDING DEATHS, FOR ILLUSTRATIVE AGE AND SERVICE COMBINATIONS

COMPLETED YEARS OF SERVICE	GROUP 1		GROUP 2		GROUP 3	GROUP 4	GROUP 5	GROUP 6	GROUP 7	
	Males	Females	Males	Females	Males	Males	Males	Males	Males	Females
Age at hire	22	22	20	20	22	22	20	22	22	22
0	168	366	294	292	557	421	331	381	313	401
1	99	310	187	250	428	251	208	161	240	397
5	22	163	87	192	176	86	128	51	92	211
10	11	81	22	105	65	30	96	31	70	116
15	8	40	10	54	38	15	45	23	40	79
20	6	20	6	24	29	10	38	20	30	54
25	5	10	5	11	24	7	30	13	25	54
30	8	6	6	17	20	8	12	8	21	26
35	14	10	11	17	18	14	12	14	14	10
Age at hire	32	32	30	30	32	32	30	32	32	32
0	156	248	217	287	489	268	291	371	237	306
1	81	189	149	212	269	159	193	151	180	266
5	25	68	49	79	111	55	114	63	40	79
10	13	36	8	22	41	19	72	33	30	54
15	8	23	7	10	24	9	35	21	25	54
20	8	18	6	12	20	8	14	10	21	26
25	14	13	11	14	18	14	14	14	14	10
Age at hire	42	42	40	40	42	42	40	42	42	42
0	136	197	198	204	450	178	262	343	162	257
1	67	119	111	149	214	106	153	148	174	139
5	24	50	60	49	88	36	91	57	25	54
10	15	35	12	15	33	12	60	15	21	26
15	14	29	11	14	19	14	28	15	14	10
Age at hire	52	52	52	52	47	52	50	47	52	52
0	136	197	221	188	323	72	206	330	162	130
1	67	119	145	136	168	43	147	150	174	178
5	24	50	80	92	69	15	98	60	14	10

shows termination rates, including disablements but excluding deaths, at illustrative age and service combinations, based on the actual experience of seven pension plans in the United States. These plans include considerable variety in industry and other characteristics, as discussed in my report.

Both papers state vesting cost as a percentage of present plan cost, as I do also. But we should understand that ultimately the cost of a pension plan is the cost of benefits for members who receive benefits. When vesting

TABLE 2

RANGE OF INCREASE IN PENSION PLAN COSTS
FOR MANDATORY VESTING PROVISIONS

	PRESENT VESTING			ALL PLANS
	None	Moderate	Liberal	
Percentage of pension plan members covered under such plans.	23%	56%	21%	100%
Range of present plan cost as a percentage of payroll	1.8%–10.4%	2.2%–11.8%	2.2%–11.9%	1.8%–11.9%
Range of increase in cost as a percentage of payroll:				
1. 30% at 8 years, graded, no past service vested	0.2 – 0.6	0.0 – 0.2	0.0 – 0.0	0.0 – 0.6
2. 30% at 8 years, graded, all past service vested	0.2 – 1.4	0.1 – 0.3	0.0 – 0.0	0.0 – 1.4
3. 30% at 8 years graded, past service vested for members age 45 and over .	0.2 – 1.2	0.1 – 0.2	0.0 – 0.0	0.0 – 1.2
4. Rule of 50, no past service vested . .	0.2 – 0.7	0.0 – 0.3	0.0 – 0.2	0.0 – 0.7
Range of increase in cost as a percentage of present plan cost:				
1. 30% at 8 years, graded, no past service vested	3–25	0– 6	0– 1	0–25
2. 30% at 8 years, graded, all past service vested	5–53	1– 8	0– 1	0–53
3. 30% at 8 years, graded, past service vested for members age 45 and over .	5–44	1– 6	0– 1	0–44
4. Rule of 50, no past service vested . .	3–28	0–12	0– 5	0–28

is added, it does not increase the cost for the man who would have received a pension anyway; rather, it adds a cost for the man who previously would have had no benefit and no cost.

A summary of the pension plan costs in my report is given in Table 2, but this should be considered in the context of the report itself.

CLAUDE Y. PAQUIN:

Mr. Trowbridge is to be thanked for his lucid presentation on the relative cost of vesting. His highly readable paper greatly facilitates the understanding of this complex subject, and his gently disguised admonition to do one's basic thinking before turning the problem over to the computer is well taken.

Vesting has always appeared to me as some mysterious legalistic

animal of dubious likability. It seems to produce, in a pension plan, the equivalent of what one's "equity" is in a mortgaged home, or perhaps of a nonforfeiture value in a whole life insurance contract. A few thoughts on the concept might perhaps be appropriate.

A careful distinction should be made between mentally expected benefits, mathematically expected benefits, funded benefits, vested benefits, and real benefits. The portion of a pension actually collected by the pensioner is a real benefit, and everything else about his pension is, to him, a mental expectation. To the actuary, a pension not yet payable is a mathematical expectation depending generally, at the least, upon survival. Funding, it would appear, gives a real foundation to this expectation, while vesting gives it a legal foundation based upon contract law. If the foundation is real without being legal (as where funding precedes vesting, the usual case), or if it is legal without being real (as where vesting precedes funding), there can be good cause for insecurity. The question which I feel is foremost in the consumerist public's mind is why funding and vesting should not be forced to go hand in hand. There cannot be pension security, which is in the nature of a firm mental expectation, without both. As a pauper can break his contracts with impunity, so can a pension plan provide generous early vesting and, for lack of funding, leave the pensioner stranded. Likewise, as moral claims can be enforced only in heaven, a pension plan can promise all sorts of "moral" benefits which are but empty expectations until they become legally enforceable through vesting.

Vesting, as it is considered in this paper and as it is normally conceived of, consists of an accrual schedule for the enforceability of pension claims; this accrual schedule is usually different from that for the cost of these pension claims. Except with the accrued benefit cost method, where a year-by-year matching of benefits and costs is usually implied, the accrual of the benefits is also different from that of the costs. So we often have benefits, vesting, and costs accruing in their separate ways, although obviously in some interrelated ways.

By and large, vesting can be viewed as a method which retards the enforceability of mentally expected pension benefits and thus cuts down on the ultimate cost of pensions. If, as the paper's approach infers, vesting adds to the cost of a pension presumed to be nonvested (i.e., unenforceable) until retirement age, then it is true, as it demonstrates, that the extra cost from vesting is absolutely independent of the withdrawal rates prior to the time when some vesting (or enforceability of benefits) first occurs. But a strong argument can be made, I feel, for describing "vesting" as a method for reducing the cost of pension plans. If one views

pensions in that way, then vesting becomes an instrument of effecting forfeitures, and the withdrawal rates prior to vesting do matter considerably.

If one refers to the equation of equivalence between prospective and retrospective reserves in life insurance, one should say that Mr. Trowbridge's view of the cost of vesting in this paper is essentially prospective. I think that a truly realistic view of vesting requires equal consideration of it on a retrospective basis as a means of forfeiture of mental pension expectations and of redistribution of the mathematical pension expectations of the participants. Somehow, I cannot but feel that would-be pensioners still should worry about employer-induced terminations before vesting. Mr. Trowbridge's approach looks at those who would collect a pension in any event, without vesting (except at retirement), and adds the cost of some preretirement vesting. The public looks at those who begin work, qualify for the pension in a nonvested way, or even in a vested way, and then fall by the wayside, and counts their losses.

There is something to be said for the retrospective view on the question of vesting if years of service are to be more than a mere factor in a basic pension benefit formula. Such a method might serve us well in exposing the losses suffered by those who "terminated" early, their "reserve released by termination." My regret is that the time available for the preparation of a discussion on such an interesting and important paper is too short to allow for a mathematical exposition of the retrospective side of the story and of how it relates to the side ably presented here by Mr. Trowbridge.

(AUTHOR'S REVIEW OF DISCUSSION)

C. L. TROWBRIDGE:

The appearance of two papers in the same publication at the same time and on the same general subject must be somewhat confusing. It might help to let the membership know that these two papers are not entirely independent, both having arisen as a result of efforts by the Pension Research Council at the Wharton School to add to knowledge in a particularly confusing area.

Pressure of other activity has prevented me from writing a formal discussion of the Winklevoss-Shapiro paper, for which I have a high degree of admiration. These two authors from the academic world have made a significant contribution to actuarial knowledge. Their discussion of my paper, which I should have reciprocated, gives me an opening for comment on theirs, to make up in part for what I should have done earlier.

First, I entirely agree with the Winklevoss-Shapiro comment that the

benefit formula, and the slope of the salary scale when benefits are based on salary, have an important bearing on the relative cost of vesting. I did not intend to leave these variables off the list of important determiners of the cost of vesting, and I consider them to be included within the concept of vesting conditions, the first of the three important variables enumerated. By "vesting conditions" I meant the entirety of all those factors which determine when vesting occurs and in what amount. More specifically, I had in mind the function $t_{k,y}$, the proportion of the benefit at retirement vested in event of withdrawal in year of age k to $k + 1$. This function reflects any grading of vesting percentages and the ratio of the accrued benefit, however defined, to the ultimate pension benefit. In the latter role it clearly is a function of the benefit function and of the slope of the salary scale. This is easily seen if my $t_{k,y}$ is rewritten in the identical Winklevoss-Shapiro form:

$$t_{k,y} = {}_{k-z}g_z \frac{k-aB_a}{r-aB_a}.$$

My presentation could have been clearer on this point. In any event, I have no disagreement with my academic friends, unless they feel that my somewhat broadened concept of vesting conditions is not valid.

There is possibly some disagreement between the authors of the two papers on the role of the actuarial cost method in determining the relative cost of vesting. Dr. Winklevoss and Mr. Shapiro point out that the relative cost of vesting, as measured by the accrued benefit cost method, is likely to produce results significantly different from those obtained if other measures are employed, and they conclude that the actuarial cost method is therefore one of the important variables affecting the cost of vesting. Mr. Grubbs makes a somewhat similar point when he suggests that the relative cost of vesting is affected by the degree of funding. My view is that the relative cost of vesting is by definition independent of both the funding method employed and the degree of funding accomplished and that if results different from those of formulas (2) and (4) are obtained, then the methods employed are misleading. While I grant that the accrued benefit cost method will indicate a relative cost of vesting different from that obtained by the formulas of the paper, I consider this method unsatisfactory for measuring the relative cost of vesting.

Perhaps my point can be made clearer by a plebeian example. Suppose that an automobile without "extras" costs $3,000; with a certain set of extra equipment it costs $3,600. My view is that the extras add 20 per cent to the cost of the basic automobile no matter what the financing arrangements may be and no matter where in the payment process we make the

comparison. If the basic automobile and the extras are similarly financed, it is unlikely that any reasonable comparison will give an answer different from 20 per cent. Suppose, however, that the \$600 down payment is considered as against the extras and that the \$3,000 basic automobile is financed by equal instalments over three years. If a calculation of the relative cost of extras is based on the first-year payments under this peculiar funding method, clearly the relative cost will be badly overstated; but for the second and third years the relative cost will be reduced to zero. I view any actuarial cost method which does not produce relative costs of vesting (or of extras) that are essentially invariant over time as unsatisfactory for this particular purpose; I attribute the results obtained to the nonhomogeneity of the funding of the two kinds of benefits.

Dr. Winklevoss and Mr. Shapiro are to be commended on picking up one point that escaped me entirely. My paper clearly gives the impression that the relative cost of vesting increases if withdrawal rates after vesting increase. Formulas (2) and (4) clearly indicate that this must be true if *all* the $q_k^{(ws)}$'s for age z and later are increased, but it is not necessarily true if some (with low weight) are increased and others (with higher weight) are decreased. The weights are $l_{k,y}(_{r-k-1}p_{k+1}/_{r-k}p_k)$, the first factor of which increases rather sharply with attained age, while the second decreases with age. It would be interesting to see examples of situations where withdrawal after vesting increases but the relative cost of vesting decreases. Where this occurs, it is the shape of the $q_k^{(ws)}$ function, rather than its level, that becomes important.

There is one important point of notation in which the two papers disagree. I have followed McGinn and have used $q_k^{(ws)}$ to represent the probability than an employee aged k will terminate employment before age $k + 1$ but will be alive at age $k + 1$. Dr. Winklevoss and Mr. Shapiro have used p_k^{aw} to represent exactly the same concept. Probably their notation, using p to emphasize that the individual survives, is preferable to mine (and McGinn's), which employs q to emphasize that the individual withdraws. Except for this and other notational differences, Winklevoss and Shapiro's formulas (10b) and (10c) are identical with formulas (2) and (4) of my paper, except that they have been more specific in separating high-age withdrawal from early retirement by cutting off the summation at age r'. I now like their formulation on this point better than my own. On the other hand, they have missed one simplification,

$$\frac{_{k-z}p_z^{aa}}{_{r-z}p_z^{aa}} = \frac{1}{_{r-k}p_k^{aa}} \, ,$$

that I feel helps.

In their last paragraph Dr. Winklevoss and Mr. Shapiro seem to disagree with me as to the reason for the negative sign on some of McGinn's vesting cost ratios. I was suggesting not that McGinn might have used an attained-age approach to the cost of vesting but rather that the entry age factor includes withdrawal benefits beginning at age x (the age at which vesting provisions are first introduced) rather than at age z, if $x > z$. This will make the normal cost and accrued liability vesting factors identical initially and will avoid a negative sign on the latter. Of course, I agree that later on, for attained ages above the original age x, the relative cost of vesting as to the accrued liability for active employees only may become negative, simply because the liability for the vested withdrawals at the same attained age has been (inappropriately) ignored.

There are possibly some problems in reconciling the algebra when we look into the discussions by Messrs. Bloch and Purnell and by Mrs. Tino.

Bloch and Purnell have devised an imaginative and interesting vesting algebra based on probabilities of withdrawal. For those who wonder whether their algebra is consistent with mine, I submit the following reconciliation between Bloch and Purnell's formula (8) and my formula (3). Both are intended to define the relative cost of vesting under the accrued benefit cost method. Bloch and Purnell's formula (8) is

$$AVL_x = APL_x m_z^v, \quad x < z,$$

or

$$\frac{AVL_x}{APL_x} = m_z^v, \quad x < z;$$

but AVL_x/APL_x is the ratio that I define as f_A. Hence

$$f_A = m_z^v = \sum_{k=z}^{r} c_k^v = \sum_{k=z}^{r} \frac{w_k}{t_k}$$

by Bloch and Purnell's definitions. But w_k in Bloch and Purnell's notation is $q_k^{(ws)}/p_k$ in my notation, and $1/t_k$ in Bloch and Purnell's notation is $_{r-k}p_k/_{r-k}p_k^t$ in my notation. Therefore,

$$f_A = \sum_{k=z}^{r} q_k^{(ws)} \frac{_{r-k-1}p_{k+1}}{_{r-k}p_k^T},$$

which is my formula (3).

For the projected benefit methods, and to introduce graded vesting, Bloch and Purnell would expand their family of commutation columns, the result of which is to introduce $t_{k,y}$ into the summation. Although their discussion does not follow this process all the way to my formulas (2) and (4), I feel confident that the Bloch and Purnell algebra gives the same

results as my own and hence the same as that of Dr. Winklevoss and Mr. Shapiro.

I must confess my inability to follow Mrs. Tino's development entirely. She seems to be more interested in the absolute cost of vesting, and in the gain and loss analysis associated with it, than in the relative cost of vesting emphasized in the paper. Moreover, she develops the more complicated contributory case rather than the simpler noncontributory situation toward which most of the paper is directed. It would be my presumption that there is nothing irreconcilable between her algebra and mine, although I am not entirely sure.

Mr. Grubbs very correctly states that the techniques of these papers can be adapted to determine the ratio of cost increase for plans that currently have some vesting and are considering more liberal vesting; he recommends that comparisons be drawn between two degrees of vesting instead of measuring always from the zero-vesting situation. It would perhaps help the reader to show how easy the adaptation is. If, for a particular vesting condition v_1, the relative cost compared to zero vesting is $1 + f_1$, and for a second vesting condition v_2 it is $1 + f_2$, then the relative cost of change from v_1 to v_2 is $(1 + f_2)/(1 + f_1)$. From Table 1 of my paper, a change from 100 per cent proportional vesting after fifteen years to 50 per cent after five years, grading to 100 per cent after ten years, would add (entry age 25, Withdrawal Table IV or V) $(1.298/1.170) - 1 = 10.9$ per cent for all new employees and for those in the initial census who have not yet reached age 30.

Mr. Grubbs makes me wonder whether he really had an opportunity to read the two papers closely. In particular, his paragraph dealing with current employees past the point of initial vesting, and the immediately following paragraph relating to interaction of prevesting rates and multiple entry ages, strike me as introducing points well covered not only in one of the papers but in both.

I have no real problem with Mr. Grubbs's point that the lack of good information on postvesting termination should not be an excuse for lack of effort. Nonetheless, I view his seven tables as illustrative of recent past experience in seven plans rather than as tables that can be generalized to other plans and other time frames. He gives us little idea as to the exposure in the seven cases or as to the vesting provisions at the time the experience is drawn.

For those who have a feeling that employee termination rates can be reasonably well predicted from experience in the past, I suggest that terminations due to forces beyond the employee's control, particularly terminations relating to plant closings, relocations, mergers, and the like,

are certainly unpredictable even if the more usual voluntary termination rate can be shown to be relatively stable. It helps us keep our perspective if we realize that the relative cost of vesting is literally infinite if the employer's business terminates before anyone retires but after some employees have met the vesting conditions. Another unpredictable situation can occur in a small firm where the pension of one individual is so much larger than the rest that it dominates the plan. Under these circumstances the withdrawal experience with respect to one man is all that really counts.

Let me encourage Mr. Paquin to develop more fully his retrospective view of the cost of vesting. I agree completely that my view has been prospective and that a retrospective look may have good possibilities for further insight. One approach to the cost of vesting has always been an inventory of the vested withdrawals (real, or as if certain provisions were in effect) and a valuation of the benefits these individuals enjoy. At least the retrospective look substitutes facts for conjecture and, if the past is long enough, gains thereby in credibility.

I have one difficulty in following Mr. Paquin's discussion. It has to do with the meaning of the word "vesting." In the sense used in the paper, "vesting" is simply the elimination of *one* of the circumstances (employee withdrawal) under which an expected benefit might disappear. Other influences can frustrate pension expectations, particularly plan termination without adequate funding. A pension benefit is never completely vested in the absolute certainty sense unless there are *no* circumstances under which it can disappear. I believe that it is in this larger sense that Mr. Paquin uses the term, as is consistent with the way the same word is used in certain nonpension connections.

Finally, let me thank the seven writers of the five discussions for the thoughts they have added. Discussion makes any paper more valuable, and both of the papers on the cost of vesting are the better for it.

[Dr. Winklevoss and Mr. Shapiro did not prepare a review of the discussions.]