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ON SOME ACTUARIAL INEQUALITIES—ACTUARIAL NOTE

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INTRODUCTION

One can prove by a simple mathematical exercise that the value of an immediate life annuity of 1 payable at age x is less than the value of an annuity-certain payable for a period equal to the curtate expectation of life, and the expression

$$a_x < a_{\overline{e_x|}}\,, \tag{1}$$

is well known to actuaries. Apparently, however, very little has been written about the attendant inequalities involving insurances, term periods, and stationary population concepts. The purpose of this paper is to derive and discuss both continuous and curtate inequalities involving such attendant inequalities and to show how the proofs of the continuous inequalities are of an entirely different nature from the usual proofs for the above curtate inequality. In this presentation a number of varied techniques familiar to students of life contingencies will be found to be useful.

We will make a brief derivation of the above inequality, in order to recall and observe the pattern which will be required later. The usual demonstration is based on the fact that the arithmetic mean of n distinct positive quantities (not all equal and all > 0) is always greater than their geometric mean.

Consider l_x quantities consisting of d_x such quantities each equal to v, d_{x+1} each equal to v^2, and so on. The arithmetic mean of these l_x quantities is given on the left-hand side of inequality (2) below, and the geometric mean on the right-hand side. Then, using the principle stated above, we may say that

$$\frac{vd_x + v^2d_{x+1} + \dots + v^{\omega-x}d_{\omega-1}}{l_x} > [(v)^{d_x}(v^2)^{d_{x+1}}(v^3)^{d_{x+2}} \dots (v^{\omega-x})^{d_{\omega-1}}]^{1/l_x}\,, \tag{2}$$

or

$$A_x > v^{1+e_x}\,. \tag{3}$$

This last inequality can be transformed routinely into inequality (1), as is done in reference [4], but we will tend to concentrate our attention on inequality (3).

It might be well to comment at the outset that (*a*) all the basic inequalities derived herein can be given verbal interpretations; (*b*) the left-hand member of an inequality can be and occasionally is used as a rough approximation to the right-hand member (or vice versa), since one member rarely overstates or understates the other member to a marked degree; and (*c*) essentially any inequality herein could be, in the same way as inequality (1), a source of monetary mischief, confusion, and/or incorrect calculation for the layman (and perhaps the courts). It should also be noted that, given a mortality table, any inequality shown (unless otherwise indicated) purports to hold for all integral ages x and for all interest rates (except possibly $i = 0$ and $i \rightarrow \infty$ and except for the single premiums in the last year of life, where $x = \omega - 1$ and $q_x = 1$, where these and certain other combinations may produce a degenerate equality).

We know that inequality (3) is strictly obtainable from inequality (1); by adding unity to each side of (1), we also can obtain

$$\ddot{a}_x < \ddot{a}_{\overline{1+e_x}|}\,. \tag{4}$$

It should first be mentioned that it will usually be convenient to think of the l_x persons shown in the mortality table as representing l_x "starters" on their xth birthday, where all were born at the same instant; then a different subscript, as in the symbol l_{x+5}, in a given context, will designate the reduced cohort of those same people five years later. Moreover, any l_x should be thought of as a relative representation of such a large quantity (in theory, infinite) that the deaths occur continuously; our concept of the mortality table (which is merely a reiteration of the usual concept) will not be related to the stationary population model until so noted; our l_x curve will be a continuous curve, then, about which we have information *only* at discrete points (unless we know the law of mortality upon which it is based). Through the inequalities involving continuous functions, to be derived later, it is interesting that we will be making very definitive statements about curves where we know an infinitesimally small amount, relatively speaking, about their ordinates, except for the very general routine requirements for an l_x curve.

Returning to inequality (3), we notice that $1 + e_x = (l_x + l_{x+1} + \ldots)/l_x$, where the *numerator* is the total number of future life-years to be enjoyed by our cohort of l_x starters, giving a whole year's credit for those entering upon any year of age; that is, we are effectively assuming that deaths occur at the end of the year of age. Division by l_x produces the average number of years to be enjoyed after age x by each of the l_x

persons at exact age x, who also constitute all the eventual deaths in the cohort.

We now want to derive the analogous term-period version of inequality (3). To do this, we consider, as before, the arithmetic mean of d_x quantities each equal to v, d_{x+1} each equal to $v^2, \ldots$, and finally d_{x+n-1} each equal to v^n. The number of quantities to be averaged is the sum of the d's, which is $l_x - l_{x+n}$; the sum of the quantities to be averaged divided by the number of quantities is then

$$\frac{vd_x + v^2d_{x+1} + \ldots + v^nd_{x+n-1}}{l_x - l_{x+n}} = \frac{l_x}{l_x - l_{x+n}} A^{\,1}_{x:\overline{n}|} = \frac{1}{{}_nq_x} A^{\,1}_{x:\overline{n}|}\,. \tag{5}$$

Now the geometric mean of these same $l_x - l_{x+n}$ quantities is

$$\begin{aligned}&[(v)^{d_x}(v^2)^{d_{x+1}}(v^3)^{d_{x+2}} \ldots (v^n)^{d_{x+n-1}}]^{1/(l_x-l_{x+n})}\\ &\qquad = \left(v^{d_x+2d_{x+1}+3d_{x+2}+\ldots+nd_{x+n-1}}\right)^{1/(l_x-l_{x+n})}\,.\end{aligned} \tag{6}$$

Although at this point we could interpret verbally the exponents on the v, it will be well to simplify into more compact symbols. Dealing only with the sum involving the various d_x's, it is not entirely straightforward to turn this into l_x's, and it may be refreshing, if not to our advantage, to use finite integration by parts. It will be recalled that the formula can be written

$$\sum_{t=1}^{n} V_t \Delta U_t = V_t U_t \Big]_1^{n+1} - \sum_{t=1}^{n} U_{t+1} \Delta V_t\,,$$

where the interval of differencing is unity and t is the variable. In our case, since $\Delta l_y = -d_y$, we have

$$\begin{aligned}\sum_{t=1}^{n} t d_{x+t-1} &= -\sum_{t=1}^{n} t \Delta l_{x+t-1}\\ &= -\left(t l_{x+t-1}\Big]_1^{n+1} - \sum_{t=1}^{n} l_{x+t}\right)\\ &= -(n+1) l_{x+n} + 1 \cdot l_x + \sum_{t=1}^{n} l_{x+t}\,.\end{aligned}$$

The entire exponent associated with v in expression (6) is then

$$\begin{aligned}\frac{l_x - l_{x+n} - nl_{x+n} + \sum_{t=1}^{n} l_{x+t}}{l_x - l_{x+n}} &= 1 + \frac{l_x}{l_x - l_{x+n}} \frac{\sum_{t=1}^{n} l_{x+t} - nl_{x+n}}{l_x}\\ &= 1 + \frac{1}{{}_nq_x}\left(e_{x:\overline{n}|} - n\ {}_np_x\right)\,.\end{aligned} \tag{7}$$

Since the arithmetic mean of our $(l_x - l_{x+n})$ quantities is greater than their geometric mean, we can state that expression (5) is greater than expression (6), and, multiplying both sides of this inequality by ${}_nq_x$, we obtain, using equation (7),

$$A^{\,1}_{x:\overline{n}|} > {}_nq_x v^{\eta} , \tag{8}$$

where $\eta = 1 + (e_{x:\overline{n}|} - n \ {}_np_x)/{}_nq_x$; this is one form of the term insurance inequality that is strictly analogous to inequality (3).

By using the first expression in equation (7) and slightly simplifying the numerator, we can readily write the same inequality in a different form:

$$A^{\,1}_{x:\overline{n}|} > {}_nq_x v^{\theta_x} , \tag{9}$$

where

$$\theta_x = \frac{\sum_{t=0}^{n-1} l_{x+t} - n l_{x+n}}{l_x - l_{x+n}} .$$

Here θ_x represents the same type of special average as did $1 + e_x$: θ_x is the average number of years since age x, for each of the $(l_x - l_{x+n})$ deaths (which occur in the n years following the xth birthday), that shall have been enjoyed by those deaths, considering all deaths as occurring at the end of the year of age. As a concrete observation, θ_x frequently will be near $n/2$ for many tables, ages, and durations n.

Inequality (9) then states that the single premium for an n-year term insurance for \$1 payable at the end of the year of death is greater than the present value (at interest only) of a single payment-certain of $\$({}_nq_x)$ deferred θ_x years, where θ_x could be described very loosely as "the (special) average time until a claim occurs." The right-hand member involves a present value *not* of the face amount of insurance on the left (which is \$1) but rather of the pro rata portion of the unit, or of the life, "expected" to become a claim, according to the mortality table used; this, of course, is ${}_nq_x$.

CONTINUOUS INEQUALITIES

The basic inequality involving continuous functions is not presented in Jordan's text [4], nor does it seem to be mentioned in Spurgeon [8]. In fact, the student who attempts to turn inequality (3) into a reasonably analogous continuous function version might extract inequality (3) from Jordan's derivation of inequality (1) and multiply both sides by $(1 + i)^{1/2}$, obtaining $(1 + i)^{1/2}A_x > v^{1/2 + e_x}$ or, using two standard approximations, $\bar{A}_x \mathrel{\dot{>}} v^{\mathring{e}_x}$; the two dots signify a very tentative relationship, since we have replaced both members of the first (exact) inequality by approximations, and we have apparently arrived at an inexact, or "approximate," inequality—whatever *that* means! Because of this devious derivation, the student might conclude that the elegant relationship

$$\bar{A}_x > v^{\mathring{e}_x} \tag{10a}$$

or, what is algebraically the same,

$$\bar{a}_x < \bar{a}_{\overline{\mathring{e}_x}|} \tag{10b}$$

might not always hold for all x and i in a given mortality table. A little reflection would show him, however, that the approximate substitutions were at best inconsistent, since the right-hand substitution, $\mathring{e}_x \doteq e_x + \frac{1}{2}$, is always exact under the assumption of uniform distribution of deaths, for example, whereas the left-hand replacement is ***not*** exact under that assumption, the correct substitution being $\bar{A}_x \doteq (i/\delta)A_x$. As a result, no conclusion could be drawn as to the general validity of inequality (10a) without further investigation.

A proof of inequality (10b) (and hence of [10a]) is given in reference [3], using the method to be utilized by us presently; we will not derive that basic inequality by this method, for it will emerge later in another connection.

The following definite integrals will be found to be useful hereafter, and they are presented here as a convenience in verifying, mentally or otherwise, some of the integrations that will be needed later.

$$\int_0^\infty t\ {}_tp_x dt = \frac{Y_x}{l_x}; \tag{11a}$$

$$\int_0^\infty t\ {}_tp_x\mu_{x+t}dt = \frac{T_x}{l_x} = \mathring{e}_x = \int_0^\infty {}_tp_x dt; \tag{11b}$$

$$\int_0^n t\ {}_tp_x\mu_{x+t}dt = \frac{T_x - T_{x+n} - nl_{x+n}}{l_x} = \mathring{e}_{x:\overline{n}|} - n\ {}_np_x; \tag{11c}$$

$$\int_0^\infty t^2\ {}_tp_x\mu_{x+t}dt = \frac{2Y_x}{l_x}. \tag{11d}$$

We wish to derive now the continuous version of inequality (8) or (9), using Taylor's theorem, which we shall apply informally. It will be recalled that, for suitable functions $f(t)$, which we will assume, the value of the function at some arbitrary point t can be expressed in terms of its value at a fixed point r and the derivatives of $f(t)$ evaluated at that point r. For our purposes we will state this simply as

$$f(t) = f(r) + (t - r)f'(r) + \frac{(t - r)^2}{2!}f''(t_1), \tag{12}$$

where we have truncated the basic series after the second term and brought in the "remainder term" (Lagrange's form), in which t_1 represents a numerical value of the abscissa. This value will lie somewhere between r and the t that we may be concentrating on at the moment; hence t_1 will generally be different for different t's that one may work with, although we will always keep r rigidly constant. Letting $f(t) = v^t$ and considering the expression for a single-premium n-year term insurance payable at the moment of death,

$$\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n v^t \; {}_tp_x\mu_{x+t}dt \,, \tag{13}$$

we may replace v^t by its form as given by equation (12), without deciding on the value of the constant r as yet, and we have

$$\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n \left[v^r + (t-r)(\ln v)v^r + \frac{(t-r)^2}{2!}(\ln v)^2 v^{t_1} \right] {}_tp_x\mu_{x+t}dt \,, \tag{14}$$

or

$$\bar{A}^{\,1}_{x:\overline{n}|} = v^r({}_nq_x) - \delta v^r \left(\frac{T_x - T_{x+n} - nl_{x+n}}{l_x} - r \; {}_nq_x \right) + R \,, \tag{15}$$

where the integration has been performed routinely with the aid of equations (11) but the integration involving the last term in the brackets in expression (14) has been called R. Since, in order to integrate, we effectively used equation (12) an infinite number of times (namely, for all values of t from zero to n), t_1 was hardly constant; regardless of this, however, it is not difficult to see that the result for R will *always* be a positive quantity. We may now choose r to be the value which makes the second term in expression (15) vanish, and this produces

$$r = \frac{T_x - T_{x+n} - nl_{x+n}}{l_x} \frac{1}{{}_nq_x} = \frac{T_x - T_{x+n} - nl_{x+n}}{l_x - l_{x+n}} \,. \tag{16}$$

If we omit the positive quantity R in equation (15), the right-hand member clearly will be smaller than $\bar{A}^{\,1}_{x:\overline{n}|}$; then, by using expression (16) for r, we may state that

$$\bar{A}^{\,1}_{x:\overline{n}|} > {}_nq_x v^{(T_x - T_{x+n} - nl_{x+n})/(l_x - l_{x+n})} \,. \tag{17}$$

This is the continuous analogue of inequality (9); the price, or payment, on the right-hand side is not unity but ${}_nq_x$, as before. This payment is deferred for a period of time given by the exponent on v, which will be seen to be the exact average number of years lived in the n years, by each person who dies in that period, and is comparable to the exponent in inequality (9). In fact, the exponents in both (9) and (17) give the "average time until claim," but the former exponent gives this mean value under the assumption of deaths at the end of the year of age.

Inequality (17) also may be written using the temporary complete expectation of life; in this case we have

$$\bar{A}^{1}_{x:\overline{n|}} > {}_nq_x v^{\xi} , \tag{18}$$

where $\xi = (\mathring{e}_{x:\overline{n|}} - n\ {}_np_x)/{}_nq_x$; this is analogous to inequality (8).

If r is defined as either of the two equal exponents on the v's in formulas (17) and (18), we may transform the above inequalities into annuity inequalities. However, inequalities involving insurances, especially the continuous forms, have somewhat more compact "general reasoning" forms, as may be seen from the annuity forms which we will now derive.

Adding $v^n\ {}_np_x$ to both sides of inequality (17) or (18), we obtain inequalities involving the endowment insurance:

$$\bar{A}_{x:\overline{n|}} > {}_nq_x v^r + v^n\ {}_np_x = {}_nq_x(v^r - v^n) + v^n \tag{19}$$

and

$$1 - \delta \bar{a}_{x:\overline{n|}} > {}_nq_x[(1 - v^n) - (1 - v^r)] + v^n . \tag{20}$$

Changing signs, subtracting unity, dividing by δ, and introducing $\bar{a}_{\overline{n|}}$ for $(1 - v^n)/\delta$, and so on, we obtain

$$\bar{a}_{x:\overline{n|}} < \bar{a}_{\overline{n|}} - {}_nq_x(\bar{a}_{\overline{n|}} - \bar{a}_{\overline{r|}}) , \tag{21}$$

or

$$\bar{a}_{x:\overline{n|}} < {}_np_x \bar{a}_{\overline{n|}} + {}_nq_x \bar{a}_{\overline{r|}} . \tag{22}$$

Again, from formula (21), we can write, remembering that $r \leq n$,

$$\bar{a}_{x:\overline{n|}} < \bar{a}_{\overline{n|}} - {}_nq_x v^r \bar{a}_{\overline{n-r|}} . \tag{23}$$

Inequality (22) is one that also could be arrived at automatically by using Taylor's theorem, just as before, but in this instance starting with the identity

$$\bar{a}_{x:\overline{n|}} = \int_0^n {}_tp_x \mu_{x+t} \bar{a}_{\overline{t|}} dt + {}_np_x \bar{a}_{\overline{n|}} , \tag{24}$$

and expanding the $\bar{a}_{\overline{t|}}$ inside the integral sign by Taylor's theorem along the same lines as used for v^t in equation (14).

Each of the above inequalities has a verbal interpretation; for example, inequality (22) states that the continuous n-year temporary life annuity is smaller than (but generally not far from) the full n-year annuity-certain payable to the portion of the group that survives the n years, plus an annuity-certain to the portion that does not survive, the latter annuity being for the exact average amount of time enjoyed by each of them. Somewhat similar verbal statements can be given for inequalities (21) and (23).

In connection with the use of the identity (24) for proving inequality (22), we might inquire about applying Taylor's theorem to the more usual formula

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x dt \,.$$

Using the expansion for v^t from formula (14), we have

$$\begin{aligned} \bar{a}_x &= \int_0^\infty \left[v^r + (t - r)(\ln v)v^r + \frac{(t-r)^2}{2!} (\ln v)^2 v^{t_1} \right] {}_tp_x dt \\ &= v^r \mathring{e}_x + \left(-\delta v^r \frac{Y_x}{l_x} + r v^r \delta \mathring{e}_x \right) + R \,, \end{aligned} \tag{25}$$

where R is always positive. In this case, r, chosen so as to make the term in parentheses on the right-hand side vanish, would be $(Y_x/l_x) \div \mathring{e}_x$, or $r = Y_x/T_x$, and we have, dropping the R as before, the interesting result

$$\bar{a}_x > v^{Y_x/T_x} \mathring{e}_x \,, \quad \text{or} \quad \frac{\bar{a}_x}{\mathring{e}_x} > v^{Y_x/T_x} \,. \tag{26}$$

Viewing the mortality table strictly as a stationary population, $\bar{a}_x/\mathring{e}_x$ may not be immediately familiar, but it is the solution to the problems in texts and examinations which in effect ask for the net single premium that should be paid immediately by each of the T_x persons aged x and over who are *now living*, to provide for a unit payable at the death of each of them. This is a contract on a closed group, and all obligation of the hypothetical company would cease in exactly $\omega - x$ years from now. The right-hand member, by comparison, represents a payment of unity, as in the right-hand member of inequality (10a), but here the deferral period (i.e., the exponent), Y_x/T_x, will be remembered as the total future life-years (measured from now) that will be enjoyed by the group in the aggregate, averaged over all T_x of them; that is, it is the future lifetime, on the average, for each one of the T_x people involved in the contract; or, again, it is the exact average time until death (i.e., a claim) occurs, measured from the present time, for each of the T_x people. This inequality, despite the appearance of the left-hand member, deals with insurances. Transforming inequality (26) in the usual way, so that the right-hand member becomes an annuity-certain, we obtain

$$\frac{\mathring{e}_x - \bar{a}_x}{\delta \mathring{e}_x} < \bar{a}_{\overline{Y_x/T_x|}} \,, \tag{27}$$

and it can be verified that the left-hand member of this inequality, as one might suspect, is the single premium payable by each of the T_x people to provide a continuous life annuity on each of them.

Returning to the cohort concept of l_x starters, there is an appealing relationship between (*a*) the remainder term in the Taylor expansion (as we have used it) of

$$\bar{a}_x = \int_0^\infty {}_tp_x\mu_{x+t}\bar{a}_{\overline{t}|}\,dt\,,$$

(*b*) the variance of the random variable whose mean is $\mathring{e}_x$ (to be clarified later), and (*c*) the expression for an approximate "corrective subtraction" from the right-hand member in inequality (10b), which correction will draw the magnitude of the right-hand member down to near-equality with that of the left-hand member. The resulting basic correction, derived in a different form by a single, but different, method in various sources [8, 9], is thought to be worth noting in connection with the demonstrations given so far, since the resulting approximate formula can be of great value in calculating a network of life annuity values (and hence other functions) at several unavailable interest rates, with essentially no more labor than for one interest rate; the approximation requires only the l_x column and its associated sums, which may be of special advantage. Undoubtedly the statement has been made before that the advent of computers, and the attendant opportunities to use more sophisticated formulas and methods for various purposes, may require the actuary to re-examine and use older techniques and approximations in order to help determine initially the choices of tables, interest rates, and other assumptions which will be used ultimately in the computer programs for premiums, dividends, asset shares, and similar calculations.

The derivation that has been given in other sources deals with the curtate life annuity and determines $c > 0$ such that $a_x \doteq a_{\overline{e_x - c}|}$. We shall determine, however, a $C > 0$ such that $\bar{a}_x \doteq \bar{a}_{\overline{\mathring{e}_x}|} - C$. If we expand $f(t) = \bar{a}_{\overline{t}|}$ in the integral for $\bar{a}_x$ above, exactly as we did v^t in equation (14), we obtain, using $f'(t) = [d(1 - v^t)/dt]/\delta = v^t$, and $f''(t) = -\delta v^t$,

$$\bar{a}_x = \int_0^\infty \left[\bar{a}_{\overline{r}|} + (t - r)v^r + \frac{(t - r)^2}{2!}(-\delta v^{t_1})\right] {}_tp_x\mu_{x+t}dt\,. \quad (28)$$

Integrating the first two terms in the brackets, using equation (11b) as an aid, we find that r should be set equal to $\mathring{e}_x$ to make the second such term vanish, and we have

$$\bar{a}_x = \bar{a}_{\overline{\mathring{e}_x}|} - \frac{\delta v^{\xi}}{2!}\int_0^\infty (t - \mathring{e}_x)^2\ {}_tp_x\mu_{x+t}dt\,, \quad (29)$$

where $\hat{t}$ represents some average or composite value for the various values that t_1 took on in the integration process and is effectively the result of application of a mean-value theorem. If we omit the subtraction of the integral term on the right (which is always positive), we have incidentally derived the basic continuous inequality (10b), in the same way that inequality (17) was obtained. For each value of t in the integration process $(0 \leq t \leq \omega - x)$, t_1 took on some specific value between that t and $r = \mathring{e}_x$ (say, halfway between), and it is not unreasonable to say that $\hat{t} \doteq \mathring{e}_x$, especially since $\mathring{e}_x$ generally is about halfway to the end of the table and the correction term is comparatively small.

Now ${}_tp_x\mu_{x+t}$ can be regarded as a probability density function of a random variable t, if only because $\int_0^\infty {}_tp_x\mu_{x+t}dt = 1$; but, further, it also measures the probability of enjoying exactly t years after age x and then dying at age $x + t$. The mean of this random variable t, the "number of life-years that are enjoyed after age x by a random starter," should be the first moment of the probability distribution, or

$$\int_0^\omega t \; {}_tp_x\mu_{x+t}dt = \mathring{e}_x$$

(see eq. [11b]), which is also obvious from the definition of $\mathring{e}_x$, since $\mathring{e}_x$ is the mean of the variable indicated in the quotation marks above; another, shorter description of the random variable t whose mean is $\mathring{e}_x$ is simply "duration of life after attainment of age x." It is now easily seen that the integral in expression (29) is indeed the *second* moment about the mean of the probability distribution of t and hence is the variance of t, "time left until death," a third and even more terse description of what is, perhaps, the random variable with more significance in real life than just about any other imaginable. Further discussions of allied random variables are found in reference [1]. Evaluating the integral in expression (29), either by using equations (11b) and (11d) or otherwise, and replacing $\hat{t}$ by $\mathring{e}_x$ as discussed above, we obtain

$$\bar{a}_x \doteq \bar{a}_{\overline{\mathring{e}_x}|} - \delta v^{\mathring{e}_x} \frac{1}{2}\left[\frac{2Y_x}{l_x} - (\mathring{e}_x)^2\right]. \tag{30}$$

The factor in brackets is the variance of our random variable t. The negative correction above adjusts by the present value of a small payment due in $\mathring{e}_x$ years, in the amount of "interest" on one-half of the variance of the random variable t, "duration of life after attainment of age x."

The above formula can be quite practical in connection with various computations in which approximate values of functions are desired at several unavailable interest rates. One need not have several "starting" commutation columns but only a table of l_x, T_x, and Y_x, which are standard actuarial functions independent of interest.

As regards net annual premiums, we may easily say that

$$\frac{A_x}{\ddot{a}_x} > \frac{v^{1+e_x}}{\ddot{a}_{\overline{1+e_x}|}}, \tag{31}$$

where we have combined, so to speak, inequalities (3) and (4); the numerator of the left-hand side is seen to be larger than that of the right, and vice versa for the denominators, thus making the inequality always valid. Defining $v^n/\ddot{a}_{\overline{n}|} = P_{\overline{n}|}$ as in reference [2], where the symbol on the right-hand side can be called "the annual sinking fund premium which provides for the accumulation of a unit in n years," we can rewrite inequality (31) as

$$P_x > P_{\overline{1+e_x}|}\,. \tag{32}$$

A continuous version similarly can be derived by merging inequality (10a) with (10b), from which we obtain

$$\bar{P}(\bar{A}_x) > \bar{P}_{\overline{\mathring{e}_x}|}\,. \tag{33}$$

The above net annual premium represents a contract where both benefits and premiums are payable continuously; where only benefits are payable at the moment of death, and premiums are payable annually, we can merge inequality (10a) with (4) and obtain

$$P(\bar{A}_x) > P(A_{\overline{e_x}|})\,,$$

where P on the right-hand side is based on $1 + a_{\overline{e_x}|}$.

It is hoped that this actuarial note will be of interest to students and members alike; the author is aware that there are many other related extensions and demonstrations of similar inequalities, branching out into other areas of life contingencies (even pensions), and perhaps some discussants may wish to present discoveries which they make.

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DISCUSSION OF PRECEDING PAPER

HARRY M. SARASON:

Inequalities and approximations are an important part of actuarial expertise, and they require more care and explanation than do exact calculations. Actuarial approximations have been legalized into actuarial exactness in policy calculations but hardly anywhere else.

My "Layman's Explanation of the Expectancy Annuity," which is listed in the bibliography of Mr. Olson's paper, has had some interesting reactions. After I had laboriously written a presumably perfect and very concise explanation of the expectancy annuity fallacy, I showed the result to another actuary, Mr. Henry E. Belden. Mr. Belden took a look at my explanation and said, half-a-dozen times, "You don't mean that; you mean this." He was obviously correct in every single case!

The following explanation, due largely to Ronald Kobrine, is better.

> The expectation of life is the average number of years a number of people will live, according to a mortality table. The number of payments in the expectancy annuity is for that number of years and is the same as the number of payments according to the mortality table. But, *with interest discount*, the mortality table effect of spreading the payments all the way up to at least age 100 or 110 makes the life annuity value less than the value of the payments all concentrated within the life expectancy period—and the life expectancy is a *very* approximate estimate of the future life of an *individual*.

Quantitative comparisons are always in order, and the comparisons shown in Table 1 of this discussion are an outgrowth of my paper.

United States population mortality tables for white females are not too different from the 1971 Group Annuity Mortality Table—Males (another "actuarial inequality"!). Correspondingly, population tables for males with the age set back five or six years in the male *table*, are not too "unequal" to the female table—as we use "actuarial inequalities" here.

The expectation of life at birth is quite important in various conversational circles. The life expectancy at birth is significantly lower than the expectancy at age 1, and the percentages shown in Table 1 at age 5 are quite low because the death rates are less than one in a thousand up to age 30; thus we have almost an annuity certain for a long period even in valuing the annuity by the life table for age 5. On the female population mortality table, at 5 per cent interest, the excess is about 4.0 per cent, and on the male table the excess is about 4.7 per cent, at birth, compared to the 1.47 per cent at age 5 shown in Table 1.

TABLE 1

COMPARISON OF IMMEDIATE ANNUITY VALUES ON THE 1971 GROUP ANNUITY MORTALITY TABLE—MALES

AGE	FUTURE LIFE EXPECTANCY (YEARS)	VALUE OF 1 PER YEAR FOR LIFE AT INTEREST RATES			EXCESS OF EXPECTANCY ANNUITY VALUE OVER LIFE ANNUITY VALUE		
		3%	5%	8%	3%	5%	8%
5.....	69.86	28.574	19.059	12.335	1.86	1.47%	0.87%
15....	60.13	27.072	18.569	12.224	2.31	1.98	1.26
25....	50.40	25.073	17.788	11.994	2.97	2.82	2.06
35....	40.76	22.457	16.577	11.551	3.94	4.13	3.51
45....	31.36	19.121	14.762	10.730	5.33	6.14	6.06
55....	22.71	15.256	12.359	9.448	6.82	8.38	9.25
65....	15.11	11.052	9.402	7.601	8.63	10.93	13.04
75....	9.24	7.184	6.389	5.453	10.87	13.55	16.60
85....	5.34	4.230	3.897	3.479	14.99	17.60	20.92

I expect to follow through on Mr. Olson's suggestion concerning approximations in pension plan calculations—sometime, somewhere.

HANS U. GERBER AND DONALD A. JONES:

The author reminds us that, in general, the expected value of a function of a random variable differs from the function evaluated at the expected value of the random variable. A general result, Jensen's inequality, gives the relationship between these values for the case of a convex function. The reader may find a lengthy discussion of the result in the Steffensen paper listed in Mr. Olson's references and also in some beginning probability and statistics books [1, 2]. Steffensen puts it as follows: "Dr. Jensen's theorem may be expressed very simply in the language of probabilities, . . . thus: *A continuous convex function of a mathematical expectation is not greater than the mathematical expectation of the function*" (italics his). His equation (61), using EX for the expected value of X and Ψ for the convex function, is

$$\Psi(EX) \leq E\Psi(X) .$$

The majority of the author's inequalities follow immediately from this one, with the appropriate choices for Ψ and X. We will summarize this in Table 1 of our discussion, using the following random variables:

T = Exact time until death for a life now aged x;
Y = Integral part of T, i.e., number of complete years;
W = Exact age of an individual selected at random from among those with age greater than x in a population proportional to the life table.

Steffensen included a discrete version of an inequality more general than (26), which may be obtained by Jensen's inequality as follows. If, for the random variable W, the population is proportional to the commutation function D for some interest rate j, then

$$E[W] = (\bar{I}\bar{a})_x/\bar{a}_x \quad \text{at } j\,,$$

and $E[v^W] = \bar{a}'_x/\bar{a}_x$, where v is at i and $\bar{a}'_x$ is at the rate $(1+i)(1+j)-1$. Then Jensen's inequality provides a generalized version of inequality (26) that reduces to (26) for $j = 0$.

TABLE 1

Random Variable X	Convex Function Ψ	Expected Value of X	Expected Value of $\Psi(X)$	Author's Inequality Number
Y........	$-a_{\overline{t}\rvert}$	e_x	$-a_x$	(1)
$Y+1$....	v^t	e_x+1	A_x	(3)
$Y+1$....	$-\ddot{a}_{\overline{t}\rvert}$	e_x+1	$-\ddot{a}_x$	(4)
T........	v^t	$\mathring{e}_x$	$\bar{A}_x$	(10a)
T........	$-\bar{a}_{\overline{t}\rvert}$	$\mathring{e}_x$	$-\bar{a}_x$	(10b)
$W-x$...	v^t	Y_x/T_x	$\bar{a}_x/\mathring{e}_x$	(26)
$Y+1$*...	v^t	$\eta,\ \theta_x$	$A^{1}_{x:\overline{n}\rvert}/{}_nq_x$	(8), (9)
T*.......	v^t	$\dfrac{T_x - T_{x+n} - nl_{x+n}}{l_x - l_{x+n}}$	$\bar{A}^{1}_{x:\overline{n}\rvert}/{}_nq_x$	(17), (18)

* For the last two the expectations are conditional, given that T, or Y, is less than n.

Another statement from Steffensen's paper is an appropriate conclusion here: "examples . . . in the table suffice to prove the practical utility of Dr. Jensen's inequality which has not yet, amongst actuaries, received the popularity it deserves" (p. 292).

REFERENCES

1. Mood, A. M., Graybill, F. A., and Boes, D. C. *Introduction to the Theory of Statistics*. 3d ed. New York: McGraw-Hill Book Co., Inc., 1974.
2. Parzen, E. *Modern Probability Theory and Its Applications*. New York: John Wiley & Sons, Inc., 1960.

ARNOLD F. SHAPIRO:

The inquiry which Mr. Olson has undertaken certainly should be of interest to students, since various of his conclusions have found their way into the Society's examinations. For example, a question relating to an extension of inequality (1) to multilife statuses can be found at least as far back as 1945, and his inequality (31) provides the solution to one of the questions of the 1948 examination. However, one is tempted to inquire into the usefulness of his results as approximations, other than for solving examination problems. Most of the results are given in the form of inequalities; one wonders how sensitive these inequalities are. The purpose of this discussion is to investigate empirically some of the results given in the paper. Only the curtate functions are considered.

Turning first to Table 1 of this discussion, which compares the ratio of $a_{\overline{e_x}|}$ to a_x for interest rates of 0.02, 0.03, and 0.04, decennial issue ages, and rates of mortality based upon the 1958 Commissioners Standard Ordinary Table, one notices that the disparity between $a_{\overline{e_x}|}$ and a_x is

TABLE 1

$a_{\overline{e_x}|}/a_x$

x	$i=0.02$	$i=0.03$	$i=0.04$
0	1.02950	1.03383	1.03447
10	1.02652	1.03081	1.03166
20	1.02828	1.03413	1.03642
30	1.03060	1.03833	1.04248
40	1.03523	1.04603	1.05327
50	1.04000	1.05431	1.06542
60	1.04231	1.05931	1.07381
70	1.04014	1.05766	1.07359
80	1.03277	1.04789	1.06219
90	1.02074	1.03070	1.04038

small, the relative error nowhere exceeding 0.075. The ratio is parabolic in nature, with maxima at ages 61, 63, and 65 for interest rates of 0.02, 0.03, and 0.04, respectively. It would appear that for moderate interest rates $a_{\overline{e_x}|}$ provides a reasonable approximation to a_x, particularly for the younger or older ages.

Table 2 provides a comparison of the ratio of A_x and v^{1+e_x}, based upon the same data. As one would expect, this ratio is a decreasing function of age and an increasing function of the interest rate. For lower ages and relatively high interest rates A_x is considerably larger than v^{1+e_x}. However, it is worth noting that for higher ages and/or lower interest rates v^{1+e_x} appears to provide a reasonable approximation to A_x.

Table 3 investigates the ratio $P_x/P_{\overline{1+e_x}|}$. This ratio has essentially the same characteristics as the ratio A_x/v^{1+e_x}, so that for lower interest rates or older ages $P_{\overline{1+e_x}|}$ appears to provide a reasonable approximation to P_x.

Table 4 investigates the ratio $A^{1}_{x:\overline{n}|}/{}_nq_xv^{\eta}$, where η is defined in formula

TABLE 2

A_x / v^{1+e_x}

x	$i=0.02$	$i=0.03$	$i=0.04$
0.	1.08105	1.21005	1.44253
10.	1.05740	1.14149	1.28068
20.	1.04634	1.11113	1.21333
30.	1.03686	1.08622	1.16075
40.	1.02970	1.06824	1.12462
50.	1.02234	1.05055	1.09071
60.	1.01471	1.03284	1.05809
70.	1.00810	1.01794	1.03143
80.	1.00355	1.00783	1.01368
90.	1.00106	1.00234	1.00410

TABLE 3

$P_x / P_{\overline{1+e_x}|}$

x	$i=0.02$	$i=0.03$	$i=0.04$
0.	1.11207	1.24958	1.49013
10.	1.08463	1.17540	1.31944
20.	1.07499	1.14758	1.25550
30.	1.06745	1.12608	1.20762
40.	1.06444	1.11499	1.18118
50.	1.06105	1.10418	1.15731
60.	1.05452	1.08925	1.12950
70.	1.04424	1.06997	1.09821
80.	1.03083	1.04756	1.06517
90.	1.01576	1.02395	1.03236

(8), for issue ages 20, 40, and 60 and quinquennial terms. As indicated in the table, this ratio is an increasing function of both the term of the policy and the interest rate. For short terms and/or low interest rates ${}_nq_xv^n$ appears to be a reasonable approximation to $A^{1}_{x:\overline{n}|}$. It is interesting to note that for terms of less than twenty-five years, the relative error of using ${}_nq_xv^n$ in lieu of $A^{1}_{x:\overline{n}|}$ is less than 0.045.

On the basis of the foregoing it might be concluded that the inequalities mentioned by Mr. Olson may, under proper circumstances, be used as approximations. However, it should be mentioned that applications of this type are limited to those instances in which only isolated values are required. When a complete table of values is required, it is just as simple and essentially as fast to generate the exact values. This latter observation is particularly appropriate with respect to the excellent approximation given by equation (30) of the paper.

TABLE 4

$A^{1}_{x:\overline{n}|}/{}_nq_xv^n$

n	x=20			x=40			x=60		
	$i=0.02$	$i=0.03$	$i=0.04$	$i=0.02$	$i=0.03$	$i=0.04$	$i=0.02$	$i=0.03$	$i=0.04$
5	1.00039	1.00087	1.00154	1.00039	1.00087	1.00153	1.00039	1.00087	1.00153
10	1.00162	1.00361	1.00637	1.00158	1.00353	1.00623	1.00158	1.00354	1.00625
15	1.00370	1.00828	1.01461	1.00345	1.00775	1.01374	1.00348	1.00779	1.01378
20	1.00676	1.01515	1.02684	1.00587	1.01323	1.02360	1.00600	1.01345	1.02386
25	1.01077	1.02428	1.04334	1.00869	1.01971	1.03538	1.00899	1.02017	1.03580
30	1.01515	1.03449	1.06216	1.01185	1.02702	1.04880	1.01185	1.02655	1.04711
35	1.01939	1.04461	1.08135	1.01525	1.03497	1.06353	1.01387	1.03102	1.05493
40	1.02313	1.05383	1.09936	1.01895	1.04361	1.07957	1.01471	1.03284	1.05809
45	1.02639	1.06206	1.11593	1.02290	1.05277	1.09653			
50	1.02936	1.06969	1.13158	1.02642	1.06086	1.11136			
55	1.03233	1.07728	1.14717	1.02877	1.06618	1.12096			
60	1.03566	1.08562	1.16399	1.02970	1.06824	1.12462			
65	1.03943	1.09484	1.18222						
70	1.04296	1.10328	1.19856						
75	1.04537	1.10892	1.20925						
80	1.04634	1.11113	1.21333						

(REVIEW OF DISCUSSION)

JOHN A. SCHUTZ:

[EDITOR'S NOTE.—Mr. Schutz, a friend of the late Mr. Olson, kindly consented to review the discussions and prepare a response to them.]

It is indeed an honor to have a response from Mr. Sarason, affectionately known in the industry as "Uncle Harry." His insight is a welcome addition to the actuarial note. The verbal analysis will be particularly helpful to students, and the quantitative comparisons will be helpful to all readers.

Mr. Shapiro very ably measures the consequences of treating several of the inequalities as approximations when 1958 CSO mortality is employed. As he points out, exact computations of entire tables often can be made with relative ease. In many such situations it is advisable for the actuary to establish check values in order to validate the tables produced. By the use of inequalities it is possible to determine upper or lower bounds which aid in applying tests of reasonableness.

The discussion prepared by Messrs. Gerber and Jones adds a new dimension to the entire topic of actuarial inequalities. It brings forth a simple, powerful mathematical technique which should be brought to the attention of every actuary and every actuarial student. Hopefully, Dr. Jensen's inequality will achieve "the popularity it deserves."