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SALARY-SCALE RETROACTIVITY UNDER RETIREMENT PLANS

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ABSTRACT

This paper gives a mathematical proof of the frequently expressed concept that under the entry age normal method of valuing a retirement plan the salary scale applies retroactively. That is, it demonstrates that the valuation of a plan assumes that all past contributions to the plan were based on salaries which, when projected by the salary scale, reproduce the present level of income. The general approach to proving this concept is to actually apply the salary scale retroactively and obtain the accumulated fund on this basis. The accumulated fund plus the present value of future payments is equated to the present value of future benefits, and the equation is solved for the normal cost factor. Since the normal cost factor obtained in this manner is equivalent to the normal cost factor obtained at date of entry into the plan, it was concluded that the concept of retroactivity was established.

The concept of retroactivity is also applied to a situation where there is a constant offset such as the social security benefit. In addition, the paper discusses the effect on this liability resulting from an increase in salary in excess of the increase provided by the salary scale and the effect on the liability of certain types of salary-scale changes.

THE purpose of this paper is to give a mathematical development of some frequently encountered concepts in the valuation of retirement plans using a salary scale. These concepts probably are so familiar to the actuary who spends most of his time in the pension field that he will find a mathematical development of this type unnecessary. He will probably be able to arrive at these results by general reasoning. On the other hand, I have not been able to find in the actuarial literature the mathematical development of these concepts which might be helpful to students or to actuaries who spend only a limited amount of time in the pension field.

It has been stated frequently that, in application of the normal cost factor to current salaries under the entry age normal method of funding, the salary scale applies retroactively. For retirement plans valued by the entry age normal method, there is a significant increase in the liability when there is an increase in salaries that exceeds the increase provided for by the salary scale. Increases of this type have not been uncommon in recent years where salaries of plan participants have been adjusted upward by cost-of-living increases.

Although it is evident that cost-of-living increases will increase future costs of the plan, it is not so evident that past-service costs or accrued liabilities will also be increased.

Joseph C. Noback mentions this in the discussion (*TSA*, II, 350) of his paper "The Valuation of Self-insured Retirement Plans" (*TSA*, II, 49). He states: "If the change takes the form of a fixed percentage increase in actual salaries and if it may be assumed that the salary scale is not affected, then the value of $C^{Er}_{(x)}$ (the normal cost accrual rate) is, in general, not changed. This conclusion applies to both average salary plans and final salary plans. Such a development, however, would tend to create a deficit in the fund because additional benefits will accrue to present contributors for which an adequate charge is not made. In a final salary plan, the deficit would tend to be greater than in an average salary plan. Furthermore, the deficit would tend to be greater where the salary is flatter." Mr. Noback evidently arrived at this conclusion on the basis of general reasoning.

Mr. Noback commented on the effects of other changes in salary scale, but his conclusions were arrived at by general reasoning. He suggested that additional work could be done in this area. Changes in salary scale were discussed in Mr. William F. Marples' paper "Salary Scales" (*TSA*, XIV, 1), but the change was not directly related to the funding of a plan by the entry age normal method.

RETROACTIVITY OF SALARY SCALE

It will be shown that when the normal cost factor (NCF) for a plan funded by the entry age normal method is applied to the salary at any attained age, the salary scale applies retroactively.

Consider a plan which is not integrated with social security and which provides f per cent of final annual wage at age 65 for each year of service. The normal cost factor for such plan for entry age x is calculated as follows:

$$(NCF)_x(AS)_x\frac{{}^sN_x}{{}^sD_x} = (AS)_x(65-x)f\frac{{}^sD_{65}}{{}^sD_x}\,a_{65}^{(12)}\,, \qquad (1)$$

where

x = Entry age;

$x + t$ = Attained age;

$(NCF)_x$ = Normal cost factor at entry age x;

$(AS)_x$ = Annual salary at entry age x;

f = Percentage credit for each year of service;

D and N = Commutation columns from service table, where

$$N_x = \sum_{t=0}^{65-x-1} D_{x+t} ;$$

${}^sD$ and ${}^sN$ = Commutation columns from service table with salary scale, so that ${}^sD_x = s_x D_x$ and

$$ {}^sN_x = \sum_{t=0}^{65-x-1} {}^sD_{x+t} ;$$

$a_x^{(12)}$ = Life annuity value for \$1 per year payable monthly beginning at age x.

From equation (1) the normal cost factor is

$$(NCF)_x = \frac{(65 - x)f \, {}^sD_{65} a_{65}^{(12)}}{{}^sN_x} . \tag{2}$$

If this factor is to be applied to salaries from age $x + t$ to retirement, that is, $(NCF)_x(AS)_{[x]+t}$, $(NCF)_x(AS)_{[x]+t+1}$, . . . , then it must have been applied to $(AS)_x$, where $(AS)_x = (AS)_{x+t}\, s_x/s_{x+t}$, in order for the accumulated normal cost plus the present value of future normal cost to equal the present value of future benefits. This can be demonstrated in the following way.

The present value of benefits at the attained age is

$$(AS)_{x+t}(65 - x)f \frac{{}^sD_{65}}{{}^sD_{x+t}} a_{65}^{(12)} , \tag{3}$$

and the present value of future normal costs is

$$(NCF)_x(AS)_{x+t} \frac{{}^sN_{x+t}}{{}^sD_{x+t}} . \tag{4}$$

Applying the salary scale retroactively, we derive all salaries from the salary at the attained age, and the accumulated value of the normal cost is

$$(NCF)_x \sum_{n=1}^{t} (AS)_{x+t} \frac{s_{x+n-1}}{s_{x+t}} \frac{D_{x+n-1}}{D_{x+t}} .$$

In the preceding formula the value of $(AS)_{x+t}s_{x+n-1}/s_{x+t}$ can be replaced by $(AS)_{x+n-1}$, giving

$$(NCF)_x \sum_{n=1}^{t} (AS)_{x+n-1} \frac{D_{x+n-1}}{D_{x+t}} ,$$

and, since

$$(AS)_{x+n-1} = (AS)_x \frac{s_{x+n-1}}{s_x} ,$$

the expression becomes

$$\frac{(NCF)_x}{D_{x+t}} \sum_{n=1}^{t} (AS)_x \frac{s_{x+n-1}}{s_x} D_{x+n-1} = \frac{(NCF)_x}{s_x D_{x+t}} (AS)_x \sum_{n=1}^{t} s_{x+n-1} D_{x+n-1}$$
$$= \frac{(NCF)_x}{s_x D_{x+t}} (AS)_x ({}^sN_x - {}^sN_{x+t}) . \tag{5}$$

Adding this to the present value of future normal cost at the attained age and equating the sum to the present value of benefits at the attained age give

$$\frac{(NCF)_x (AS)_x}{s_x D_{x+t}} ({}^sN_x - {}^sN_{x+t}) + (NCF)_x (AS)_{x+t} \frac{{}^sN_{x+t}}{{}^sD_{x+t}}$$
$$= (AS)_{x+t}(65 - x) f \frac{{}^sD_{65}}{{}^sD_{x+t}} a_{65}^{(12)} . \tag{6}$$

If both numerator and denominator of the first term are multiplied by s_{x+t}/s_x, the result is

$$(NCF)_x \frac{s_{x+t}}{s_x} (AS)_x \frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}} + (NCF)_x (AS)_{x+t} \frac{{}^sN_{x+t}}{{}^sD_{x+t}}$$
$$= (AS)_{x+t}(65 - x) f \frac{{}^sD_{65}}{{}^sD_{x+t}} a_{65}^{(12)} . \tag{7}$$

Combining terms and simplifying, we obtain

$$(NCF)_x (AS)_{x+t} \left(\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}} + \frac{{}^sN_{x+t}}{{}^sD_{x+t}} \right)$$
$$= (AS)_{x+t}(65 - x) f \frac{{}^sD_{65}}{{}^sD_{x+t}} a_{65}^{(12)} ,$$

which reduces to

$$(NCF)_x (AS)_{x+t} \frac{{}^sN_x}{{}^sD_{x+t}} = (AS)_{x+t}(65 - x) f \frac{{}^sD_{65}}{{}^sD_{x+t}} a_{65}^{(12)} ;$$

solving for $(NCF)_x$ gives

$$(NCF)_x = \frac{(65 - x) f \ {}^sD_{65} a_{65}^{(12)}}{{}^sN_x} , \tag{8}$$

which is the same as the normal cost factor calculated at the entry age. If the salary scale had not been applied retroactively, a different value of $(NCF)_x$ would have resulted.

INCREASE IN UNFUNDED LIABILITY RESULTING FROM INCREASE IN SALARIES

After it has been established that the salary scale applies retroactively, it can be shown without much difficulty that an increase in salaries in excess of the increase provided for by the salary scale will increase the unfunded liability. This can be demonstrated by an analysis of the retrospective formula for the accrued liability (AL).

The retrospective formula is

$$\begin{aligned}(AL)_{x+t} &= \frac{(NCF)_x(AS)_x}{s_x D_{x+t}}({}^sN_x - {}^sN_{x+t}) \\ &= \frac{(NCF)_x(AS)_x(s_{x+t}/s_x)({}^sN_x - {}^sN_{x+t})}{s_x(s_{x+t}/s_x)\,D_{x+t}} \\ &= (NCF)_x(AS)_{x+t}\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}}\,.\end{aligned} \tag{9}$$

If the actual salaries in formula (9) to which the normal cost factor is applied are less than the salaries that would be found by applying the salary scale retroactively, the accumulated fund would be less than the accrued liability from formula (9), with a resulting increase in the unfunded liability.

RETROACTIVITY OF SALARY SCALE WHEN FORMULA CONTAINS A FACTOR INDEPENDENT OF SALARY

It might be helpful to determine what the result would be if the benefit formula contained a constant factor that did not vary by salary, such as the social security benefit.

Consider a plan providing f per cent of the final annual wage at age 65 for each year of service but having the total reduced by a constant benefit (CB). In this case the formula for the normal cost factor is

$$(NCF)_x(AS)_x\frac{{}^sN_x}{{}^sD_x} = (AS)_x(65-x)f\frac{{}^sD_{65}}{{}^sD_x}a_{65}^{(12)} - (CB)\frac{D_{65}}{D_x}a_{65}^{(12)}\,. \tag{10}$$

Solving for the normal cost factor, we obtain

$$(NCF)_x = \frac{(65-x)f\,{}^sD_{65}a_{65}^{(12)}}{{}^sN_x} - \frac{(CB)(D_{65}/D_x)a_{65}^{(12)}}{(AS)_x\,{}^sN_x/{}^sD_x}\,. \tag{11}$$

At the attained age the present value of the benefit is

$$(AS)_{x+t}(65-x)f\frac{{}^sD_{65}}{{}^sD_{x+t}}a_{65}^{(12)}-(CB)\frac{D_{65}}{D_{x+t}}a_{65}^{(12)}$$

and the present value of future normal costs is

$$(NCF)_x(AS)_{x+t}\frac{{}^sN_{x+t}}{{}^sD_{x+t}}.$$

The accumulated amount of past normal costs assuming that the salary scale has been applied retroactively is

$$\frac{(NCF)_x(AS)_x}{s_xD_{x+t}}({}^sN_x-{}^sN_{x+t})=(NCF)_x(AS)_{x+t}\frac{{}^sN_x-{}^sN_{x+t}}{{}^sD_{x+t}}.$$

Equating the benefit to the sum of the accumulated normal cost and the present value future normal cost at the attained ages gives

$$(NCF)_x(AS)_{x+t}\frac{{}^sN_x-{}^sN_{x+t}}{{}^sD_{x+t}}+(NCF)_x(AS)_{x+t}\frac{{}^sN_{x+t}}{{}^sD_{x+t}}$$

$$=(AS)_{x+t}(65-x)f\frac{{}^sD_{65}}{{}^sD_{x+t}}a_{65}^{(12)}-(CB)\frac{D_{65}a_{65}^{(12)}}{D_{x+t}}.$$

Solving for the normal cost factor, we obtain

$$(NCF)_x=\frac{(65-x)f\ {}^sD_{65}a_{65}^{(12)}}{{}^sN_x}-\frac{(CB)(D_{65}/D_{x+t})a_{65}^{(12)}}{(AS)_{x+t}\ {}^sN_x/{}^sD_{x+t}}.$$

Substituting $(AS)_xs_{x+t}/s_x$ for $(AS)_{x+t}$ in the last term and substituting $s_{x+t}D_{x+t}$ for ${}^sD_{x+t}$, we have

$$\frac{(CB)(D_{65}/D_{x+t})a_{65}^{(12)}}{(AS)_x(s_{x+t}/s_x)\ {}^sN_x/s_{x+t}D_{x+t}}=\frac{(CB)(D_{65}/D_x)a_{65}^{(12)}}{(AS)_x\ {}^sN_x/{}^sD_x},$$

so that

$$(NCF)_x=\frac{(65-x)f\ {}^sD_{65}a_{65}^{(12)}}{{}^sN_x}-\frac{(CB)(D_{65}/D_x)a_{65}^{(12)}}{(AS)_x\ {}^sN_x/{}^sD_x},\qquad(12)$$

which is the same as the normal cost factor in expression (11) calculated at the entry age.

In the preceding formula the first term of the normal cost factor is dependent upon entry age only and is independent of salary and attained age. The second term is not independent of salary, and the salary at the entry age must be known to make the calculation. Since the salary at entry age is generally unknown, the concept that the salary scale applies retroactively can be used to calculate the normal cost factor. The salary at entry age x can be determined by applying the salary scale retroactively to $(AS)_{x+t}$. This is done by multiplying $(AS)_{x+t}$ by s_x/s_{x+t}. Substituting this for $(AS)_x$ in expression (12) gives

$$(NCF)_x = \frac{(65 - x)f\ {}^sD_{65}a_{65}^{(12)}}{{}^sN_x} - \frac{(CB)(D_{65}/D_x)a_{65}^{(12)}}{(AS)_{x+t}(s_x/s_{x+t})\ {}^sN_x/{}^sD_x}\ .$$

The above calculation can be carried out for each employee after the value of (CB) for each employee has been determined.

EFFECT OF SALARY-SCALE CHANGES

The effect of a change in the salary scale on the accrued liability differs from the analysis in the second preceding section, where it was assumed that the salary scale did not change but the salaries increased in a given year by an amount in excess of the increase provided by the salary scale.

In this case it is assumed the salary scale is changed and that the salary at age $x + t$ was arrived at by the application of the new salary scale. The analysis will be made for a participant aged $x + t$ who entered at age x. As before, consider a plan that is not integrated with social security and is funded by the entry age normal method providing f per cent of the final annual wage at age 65 for each year of service.

Consider the situation where the salary scale is increased from 4 to 5 per cent. This means that originally the value of s_{x+1} is $1.04s_x$, or $s_{x+1}/s_x = 1.04$. The change in scale means that $s'_{x+1}/s'_x = 1.05$. Also,

$$\frac{s'_{x+1}}{s'_x} = \frac{s_{x+1}}{s_x}\left(\frac{1.05}{1.04}\right) = \frac{s_{x+1}}{s_x}\frac{(1.05/1.04)^{x+1}}{(1.05/1.04)^x}\ .$$

For convenience let

$$(1.05/1.04)^x = r_x\ .$$

The new normal cost factor is

$$(NCF)'_x = [(65 - x)fs_{65}r_{65}D_{65}a_{65}^{(12)}]\Big/\left(\sum_{t=0}^{65-x-1} r_{x+t}s_{x+t}D_{x+t}\right).$$

This can be written as

$$(NCF)'_x = (NCF)_x r_{65}\ {}^sN_x\Big/\left(\sum_{t=0}^{65-x-1} s_{x+t}r_{x+t}D_{x+t}\right).$$

Since $r_x = (1.05/1.04)^x > 1$, the values of $r_{65}/r_{x+t} = (1.05/1.04)^{65-x-t} > 1$ for all values of t from $t = 0$ to $t = 65 - x - 1$.

Since $r_{65}/r_{x+t} > 1$, then $r_{65} > r_{x+t}$, so that

$$r_{65}\,{}^{s}N_x > \sum_{t=0}^{65-x-1} r_{x+t}s_{x+t}D_{x+t}$$

and

$$r_{65}\,{}^{s}N_x \Big/ \left(\sum_{t=0}^{65-x-1} r_{x+t}s_{x+t}D_{x+t} \right) > 1 \,;$$

therefore, the normal cost factor is increased.

To determine the effect of the change in salary scale on the funding, we must consider two situations, as follows: (1) where actual past salaries conformed to the new salary scale, that is, increased 5 per cent per year, and (2) where actual past salaries did not conform to the new scale but followed the old scale and where the new scale applies to only future increases in salary.

In the first situation, where the actual past salaries conformed to the new scale, the accumulated normal cost would be

$$(NCF)_x(AS)_x \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{s'_x D_{x+t}} \,,$$

where the superscript s' indicates that the new salary scale applies. On the basis of the new normal cost factor $(NCF)'$, the accumulated normal cost should have been

$$(NCF)'_x(AS)_x \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{s'_x D_{x+t}} \,.$$

The amount of underfunding is, therefore, the accumulation of the difference between (NCF) and $(NCF)'$.

The difference between (NCF) and $(NCF)'$ can be determined from the two formulas

$$(NCF)'_x = [(65 - x)fs_{65}r_{65}D_{65}a_{65}^{(12)}] \Big/ \left(\sum_{t=0}^{65-x-1} s_{x+t}r_{x+t}D_{x+t} \right)$$

and

$$(NCF)_x = [(65 - x)fs_{65}D_{65}a_{65}^{(12)}] \Big/ \left(\sum_{t=0}^{65-x-1} s_{x+t}D_{x+t} \right) .$$

From these formulas, and as before,

$$(NCF)'_x = (NCF)_x r_{65}\,{}^{s}N_x \Big/ \left(\sum_{t=0}^{65-x-1} r_{x+t}s_{x+t}D_{x+t} \right) .$$

The amount of the underfunding is, therefore, the accumulation of the following:

$$(NCF)'_x - (NCF)_x = (NCF)_x\left[r_{65}\,{}^sN_x \Big/ \left(\sum_{t=0}^{65-x-1} r_{x+t}s_{x+t}D_{x+t}\right) - 1\right].$$

In the second situation the actual past salaries did not conform to the new salary scale but, instead, followed the old scale, so that the new scale would apply only to future increases in salary. Since the salary scale applies retroactively and assumes that salaries in the past followed the new scale, an increase in scale would imply smaller salaries in the past in order to reach the salary at age $x + t$ using the increased salary scale. These assumed salaries would be lower than the actual salaries. To these lower salaries would be applied a larger normal cost factor. It cannot be stated that the plan will be underfunded or overfunded because of the two offsetting factors, that is, the lower assumed salaries and the larger normal cost factor. Whether the plan is underfunded or overfunded will depend upon the particular case and the duration at which the change in salary scale is made.

This section has shown the effect on the funding of a plan of one type of change in salary scale. A similar approach can be used to analyze the effects of other types of changes in salary scale.

DISCUSSION OF PRECEDING PAPER

CLAUDE Y. PAQUIN:

Mr. Kemper's paper presents an interesting demonstration of pension mathematics, but what the paper seeks to prove would be a bit difficult to show to pension clients in that form. Hence this discussion will explore in words, rather than in formulas, concepts touched on in the paper.

First, the following syllogism might be helpful: In pension plans, costs depend upon benefits; in some plans, benefits depend upon salaries; hence, in the latter, costs depend upon salaries. This is true a fortiori when costs are expressed as a function of salaries, as was done in the paper. Costs, then, have two reasons to depend upon salaries.

Second, it might be noted that the paper confines its demonstration to the entry age normal cost method, which, *by definition*, calculates normal costs upon the assumption that the pension plan has always existed in its present form; this is an inherently retroactive approach, and all the factors in the formula (including not only salaries but mortality, withdrawals, and interest as well) are retroactive. There is no denying, of course, that the actuary may devise such formula modifications as he deems appropriate, and he often does.

It might be fair to state that an actuary's costs depend on assumptions and a client's costs on realities. When, as is alluded to in the paper, an actuary's assumptions seek to catch up with realities (such as cost-of-living increases not anticipated in assumed salary scales), the result is but a change in assumptions. The assumptive cost change which follows theoretically can be handled in numerous ways, and there is no supreme authority which mandates that "past-service costs or accrued liabilities . . . be increased." That the pension formula selected by the author has this property is ably demonstrated here; nevertheless it should be remembered that actuarial formulas are the actuary's servants, not his masters. The determination of pension costs is in large measure a matter of cost allocation, a process which may become inflexible only *after* one has imposed a specific formula upon oneself.

It is good to know of the properties of the entry age normal cost method. But it is also important to remember that professionally the actuary is expected to decide on the properties he wants his formula to have before he picks the formula. This informative paper provides him with knowledge which, in this respect, may help him make a more enlightened decision.

PAULETTE TINO:

The discussion will cover three items: (1) the validity of the demonstration given of the retroactivity of the salary scale; (2) the mechanism of the maintenance of the equality between the accrued liability and the accumulated funds, once attained, and its meaning; and (3) the relevance of attributing any role to actual past salaries in connection with accrued liabilities.

I. *Assumptions*

The plan under consideration covers one employee. The benefits are funded under the entry age normal method. The liabilities are discounted for interest and mortality. In covering points 2 and 3, in order to link accrued liabilities and accumulated funds as closely as possible, we shall assume that funding started from age x. The contributions will be such that, at each valuation, the expected unfunded liability is zero.

II. *Retroactivity of the Salary Scale*

The proof given in the paper consists in solving for the normal cost factor the equality established at time t between (a) the sum of the present value of past and future normal costs and (b) the present value of future benefits and comparing the result with that given with equation (1) or equation (10) of the text. Let us note that, whatever the benefit under the plan, the right-hand side of the equality at entry age will be of the form $Numerator/D_x$. The numerator for equation (10) is

$$(AS)_x \frac{s_{65}}{s_x}(65-x)fD_{65}a_{65}^{(12)} - (CB)D_{65}a_{65}^{(12)} .$$

If the valuation is performed at time t, $(AS)_x$ is equal to $(AS)_{x+t}s_x/s_{x+t}$, and we write

$$(NCF)_x(AS)_{x+t}\frac{s_x}{s_{x+t}}\frac{{}^sN_x}{s_xD_x} = \frac{\text{Numerator}}{D_x} . \qquad (1)$$

What is the meaning of this equation? The right-hand side expresses the present value at age x of the plan benefits. The left-hand side represents the present value of future normal cost contributions under the assumption that the future salaries will progress from $(AS)_x$ in accordance with the salary scale.

Now, if we want the equality of present values of normal costs and benefits at attained age $x+t$, we substitute D_{x+t} for D_x in equation (1), and in order to distinguish accrued liabilities and present value of future normal costs we break ${}^sN_x$ into two parts and rearrange the terms. We then have

$$(NCF)_x(AS)_{x+t}\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}} + (NCF)_x(AS)_{x+t}\frac{{}^sN_{x+t}}{{}^sD_{x+t}} = \frac{\text{Numerator}}{D_{x+t}} . \tag{2}$$

Equation (2) is nothing other than equation (1) read at time t. In other words, simplifying equation (1) by multiplying both sides by D_x and simplifying equation (2) by multiplying both sides by D_{x+t} produce identical equalities. Therefore, both equations are bound to give the same result in solving for $(NCF)_x$.

III. *Accrued Liability and Accumulated Fund*

In the section entitled "Increase in Unfunded Liability Resulting from Increase in Salaries" the author restricts his analysis to the case where the benefit at retirement is solely a function of the final salary and the normal cost percentage therefore is independent of salary.

At time t the accrued liability based on the expected salary $(AS)_{x+t}$ is

$$(AL)_{x+t} = (NCF)_x(AS)_{x+t}\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}} . \tag{3}$$

The actual accrued liability based on the actual salary $(AS)'_{x+t}$ is

$$(AL)'_{x+t} = (NCF)_x(AS)'_{x+t}\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}} . \tag{4}$$

Here my point of difference with the author is his referring to expression (3) as the accumulated fund without qualification. This would be the case, for example, if the initial unfunded liability had been fully amortized at some time, and payments of the normal cost as adjusted for salary and mortality experience dutifully made, including the payment of the mortality loss in the last year. If payments started from age x and the funding proceeded as explained in section II of this discussion, an unfunded liability never existed.

Let us see how the plan is kept fully funded by going through the determination and analysis of the gain (or loss) incurred in the year ending at time t. In order to simplify the language, we assume that the plan incurs over-all losses. The unfunded liability at time $t - 1$ was equal to the loss of the previous year. For the sake of generality we assume that the normal cost factor is a function of salary.

At time $t - 1$ the recommended contribution was equal to the normal cost plus the loss of the previous year payable immediately, or

$$(NCF)_x(AS)_{x+t-1} + L_{t-1} = ENC + L_{t-1} .$$

As stated before, the expected unfunded liability U_t^E is zero:

$$U_t^E = (U_{t-1}^A + ENC)(1+i) - (ENC + L_{t-1})(1+i) = 0\,.$$

The actual accrued liability is

$$(AL)'_{x+t} = (NCF)'_x\,(AS)'_{x+t}\,\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}}\,,$$

where $(NCF)'_x$ is determined by solving equation (1) with a salary at time t equal to $(AS)'_{x+t}$.

The fund is

$$\begin{aligned} F_t &= F_{t-1}(1+i) + ANC + L_{t-1}(1+i) + IG \\ &= (AL)_{x+t-1}(1+i) + ANC + IG, \end{aligned} \tag{5}$$

where IG is the interest gain and ANC the actual normal cost paid, with interest. The actual unfunded liability is $U_t^A = (AL)'_{x+t} - F_t$ and is equal to the over-all loss L_t incurred during the year, since $U_t^E = 0$.

It remains to develop $(AL)'_{x+t}$ from $(AL)_{x+t-1}$. If the salary had progressed as expected, the accrued liability would have been

$$(AL)_{x+t} = (NCF)_x(AS)_{x+t}\,\frac{{}^sN_x - {}^sN_{x+t}}{{}^sD_{x+t}}\,.$$

The loss due to the actual salary increase over that expected is equal to $(AL)'_{x+t} - (AL)_{x+t} = SL$. But we have

$$(AL)_{x+t} = [(AL)_{x+t-1} + ENC](1+i) + EM\,,$$

where EM is the expected mortality release. This can be seen as follows:

$$\begin{aligned} (AL)_{x+t} &= (NCF)_x(AS)_{x+t}\,\frac{{}^sN_x - {}^sN_{x+t-1}}{{}^sD_{x+t}} + (NCF)_x(AS)_{x+t}\,\frac{{}^sD_{x+t-1}}{{}^sD_{x+t}} \\ &= \left[(NCF)_x(AS)_{x+t-1}\,\frac{{}^sN_x - N_{x+t-1}}{{}^sD_{x+t-1}} + (NCF)_x(AS)_{x+t-1}\right]\frac{D_{x+t-1}}{D_{x+t}}\,, \end{aligned}$$

from which we obtain

$$\begin{aligned} (AL)_{x+t} &= [(AL)_{x+t-1} + ENC](1+i) + q_{x+t-1}(AL)_{x+} \\ &= [(AL)_{x+t-1} + ENC](1+i) + EM\,, \end{aligned} \tag{6}$$

$$(AL)'_{x+t} = [(AL)_{x+t-1} + ENC](1+i) + EM + SL\,.$$

Therefore, the loss, broken into its components, is given by

$$L_t = [ENC(1+i) - ANC] - IG + EM + SL\,, \tag{7}$$

where $ENC(1 + i) - ANC$ is a gain if ANC exceeds the expected normal cost with interest.

The payment of L_t will force the funding to get in line with the actuarial assumptions. The fund is made equal to the accumulation with the assumed interest and mortality of the normal costs computed as the product of $(NCF)'_x$ and the past salaries derived from $(AS)'_{x+t}$ by the application of the salary scale. The work of actuarial science is to substitute appearance for facts.

IV. *More on Past Salaries*

In the section of the paper entitled "Effect of Salary-Scale Changes" the assumption is made that the salaries actually progressed from $(AS)_x$ according to the salary scale s'_{x+t}. At time t the actual salary is $(AS)'_{x+t} = (AS)_x\, s'_{x+t}/s'_x$. The normal cost percentage is here independent of salary but varies with the salary scale. On the basis of the proposed salary scale s'_{x+t}, the accrued liability is

$$(AL)'_{x+t} = (NCF)'_x (AS)'_{x+t} \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{{}^{s'}D_{x+t}}\,. \tag{8}$$

On the basis of the valuation salary scale s_{x+t}, the accrued liability is

$$(AL)_{x+t} = (NCF)_x (AS)'_{x+t} \frac{{}^{s}N_x - {}^{s}N_{x+t}}{{}^{s}D_{x+t}}\,. \tag{9}$$

The additional liability created by the change in salary scale is equal to expression (8) minus expression (9). This result differs from that given by Mr. Kemper. He states that, if the salaries actually progress from $(AS)_x$ according to the salary scale, the fund at time t will be equal to

$$F'_t = (NCF)_x (AS)'_{x+t} \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{{}^{s'}D_{x+t}}\,, \tag{10}$$

and that the difference in accrued liability or the amount of underfunding is equal to expression (8) minus expression (10).

We have seen in Section III that after payment of the loss L_t the fund F_t will be equal to $(AL)_{x+t}$, differing from F'_t by the underlying salaries. Therefore, only a fund built over t years, in the absence of interim valuations, by contributions equal to the product of $(NCF)_x$ by the actual salaries $(AS)_{x+z} = (AS)_x\, s'_{x+z}/s'_x$ would approximate F'_t. (It will be lagging by the accumulation of mortality and interest adjustments not made.) However, even in that particular funding situation the difference in accrued liabilities is still equal to expression (8) minus expression (9). It is the actual unfunded liability at time t which is equal to expression (8) minus expression (10).

(AUTHOR'S REVIEW OF DISCUSSION)

LEE H. KEMPER:

I wish to thank Mr. Paquin and Mrs. Tino for taking the time to prepare discussions of this paper.

I agree with the point made by Mr. Paquin that the actuary is expected to decide on the properties of the formula before he picks it. I hope that, as Mr. Paquin has suggested, the paper provides additional knowledge in making a more enlightened decision. Mr. Paquin's discussion, although brief, deserves a great deal of thoughtful consideration.

Mrs. Tino suggests that the demonstration I have presented is somewhat impractical. This is perhaps true, but the purpose of choosing such an example was to demonstrate the retroactivity of salary scales under a pension plan funded by the entry age normal method. It is not inconceivable to me that an actuary may wish to change the salary scale under a retirement plan and in so doing would like to know the impact of the change in salary scale on the valuation of the plan.

Mrs. Tino has further suggested that I have proved nothing by my demonstration in the first section of the paper. However, she assumes that the salary scale applies retroactively when she states that $(AS)_x = (AS)_{x+t}\, s_x/s_{x+t}$. Using this assumption, she goes on to prove that, after substitution of $(AS)_{x+t}\, s_x/s_{x+t}$ for $(AS)_x$, equation (1) is equivalent to equation (2). By assuming $(AS)_{x+t}\, s_x/s_{x+t} = (AS)_x$ in the first place, she has, in fact, proved my original premise that the salary scale applies retroactively.

In Section III Mrs. Tino develops a very interesting practical situation, in which she demonstrates the manner in which the payment of L_t brings the fund into actuarial balance. Many employers would allow the unfunded liability to increase by the amount of the loss. My formula (9) is based on the assumption that such adjustments for losses were not made.

In Section IV Mrs. Tino uncovers an error in my two formulas for the accumulated normal cost (p. 8). The first formula should be

$$(NCF)_x(AS)_x \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{s'_x D_{x+t}},$$

and the second formula should be

$$(NCF)'_x(AS)_x \frac{{}^{s'}N_x - {}^{s'}N_{x+t}}{s'_x D_{x+t}}$$

However, this does not alter the conclusion that the difference in the accumulated fund is the accumulated difference between (NCF) and $(NCF)'$. These corrections were made in the final printing of the paper.