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VARIABLE LIFE INSURANCE ASSET SHARES UNDER THE NEW YORK LIFE DESIGN

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ABSTRACT

Variable life insurance asset shares involve, in general, both the separate account and the general account. This paper develops for both accounts the annual asset share equations of equilibrium appropriate to the daily basis New York Life variable life insurance design. As part of that development the paper derives the equations of equilibrium for the separate account asset share, for the general account asset share, for the basic reserve, and for the surplus and also derives the individual expressions for the interest and mortality contributions to the surplus.

INTRODUCTION

THE basic theory underlying New York Life's variable whole life insurance policy was presented in the paper "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance," by John C. Fraser, Walter N. Miller, and Charles M. Sternhell (*TSA*, XXI, 343). The formulas necessary to adapt that design to the daily basis contemplated by New York Life were derived in my paper "The New York Life Variable Life Insurance Design on a Daily Basis" (*TSA*, XXIII, 367). The current paper derives the asset share equations of equilibrium appropriate to the New York Life policy.

For the separate account the paper derives the equations of equilibrium for the asset share (eq. [6]), for the basic reserve (eq. [7]), and for the surplus (eq. [14]) and also derives the expressions for the interest contribution (eq. [15]) and for the mortality contribution (eq. [16]) to the surplus.

For the general account the paper specifies the amount of the transfer from the separate account (eq. [17]), consisting of the interest and mortality contributions to the surplus, and the equation of equilibrium for the asset share (eq. [18]).

BASIC ASSUMPTIONS

In contrast to the situation under a fixed benefit policy, the variable life insurance environment involves interaction between the general account and the separate account, and it is important to make explicit how such transactions will be handled.

The specific assumptions underlying the results developed in this paper are as follows:

1. The gross annual premium is credited to the general account at the beginning of each year, and the daily net premium element (DNPE—the daily equivalent of the net annual premium) is transferred from the general account to the separate account each day.
2. All expenses, other than separate account investment expenses that are netted out when determining the separate account net investment rate, are charged to the general account, as are also pro rata premium refunds at death.
3. Deaths are assumed to occur uniformly throughout the year, and withdrawals are assumed to occur at the end of the year,
4. Upon death, the separate account is charged with the applicable variable face amount, which is exclusive of any portion of the death benefit payable that is attributable to the minimum death benefit guarantee (MDBG).
5. Upon withdrawal, the general account is charged with the applicable cash value, $F_t({}_tCV)$, and the corresponding reserve, $F_t({}_t\bar{V})$, is transferred from the separate account to the general account.
6. At the end of each year a transfer is made from the separate account to the general account, consisting of the mortality and interest contributions to surplus developed during the year in the separate account. The separate account is thus effectively netted of surplus each year.
7. A charge (E_3 in eq. [18]) in the nature of a premium is made to the general account at the beginning of each year to finance the MDBG.

It should be noted that the New York Life policy provides for a pro rata premium refund at death and also for a MDBG, neither of which affects the operations of the separate account. As may be inferred from equation (18), any surplus resulting from the MDBG is considered to be handled separately from the surplus developed in the equations in this paper.

INTEREST RATES

Three different interest rates are involved in the various equations in the paper, and their definitions should be kept clearly in mind.

j = Net interest rate earned in the general account;

i'' = Net interest rate earned in the separate account after investment expenses have been deducted;

i' = Net interest rate earned in the separate account after investment expenses have been deducted and after the mortality and expense risk charge has been deducted.

The interest rate i' is the separate account interest rate used in the calculation of variable face amounts. The difference between i'' and i', that is, the mortality and expense risk charge, forms the basis for the interest contribution to surplus, which is available (in whole or in part) for distribution as part of the dividend.

In addition to the above three interest rates, the AIR (assumed interest rate for calculation of net premiums and reserves), although not entering explicitly in any of the equations, nonetheless is a fourth interest rate that must be clearly distinguished, since it enters the equations indirectly via the reserve factors, premiums, and variable face amounts.

VARIABLE FACE AMOUNTS

In my earlier paper, cited in the introduction, formulas for the New York Life theoretical daily face amounts and for the New York Life actual daily face amounts that would be used in practice were developed, the latter resulting from the use of an approximation to the theoretical daily reserve factors. Both of these face-amount concepts appear in the present paper, the first denoted by the symbol F^{Th}, the second by the symbol F^{NYL}.

SEPARATE ACCOUNT

The equation of equilibrium connecting the fund at the beginning of the tth year with the fund at the end of the tth year just before the transfer is

$$F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1+i'')+P_t''=q_{t-1}'(DB_t'')+q_{t-1}'^{(w)}F_t^{\mathrm{NYL}}({}_t\bar{V})+p_{t-1}'F_t^{\mathrm{NYL}}({}_t\bar{V}+S_t)\,, \tag{1}$$

where

F_t^{NYL} = New York Life face amount at end of tth year;

${}_t\bar{V}$ = ${}_t\bar{V}(\bar{A})$, the tth-year terminal reserve per \$1 of New York Life face amount, based on continuous functions;

i'' = Net interest rate earned in separate account after investment expenses have been deducted;

P_t'' = Value in separate account at end of year t of daily net premium elements transferred to separate account during year t, based on interest rate i'';

$q_{t-1}'(DB_t'')$ = Value in separate account at end of year t of death benefits, based on New York Life face amounts, paid during year t, based on interest rate i'';

q_{t-1}' = Experience probability of death during year t;

$q_{t-1}'^{(w)}$ = Experience probability of withdrawal during year t (assumed to occur at end of year t);

p'_{t-1} = Experience probability of surviving year t in force;
S_t = Surplus per \$1 of New York Life face amount at end of tth year.

Note that $p'_{t-1} + q'_{t-1} + q'^{(w)}_{t-1} = 1$. If we substitute $1 - p'_{t-1} - q'_{t-1}$ for $q'^{(w)}_{t-1}$, we have, after rearranging terms,

$$F^{\mathrm{NYL}}_{t-1}({}_{t-1}\bar{V})(1 + i'') + P''_t - q'_{t-1}[DB''_t - F^{\mathrm{NYL}}_t({}_t\bar{V})] - F^{\mathrm{NYL}}_t({}_t\bar{V}) = p'_{t-1}F^{\mathrm{NYL}}_t(S_t) \,. \tag{2}$$

Similarly, the basic equation of equilibrium connecting successive terminal reserves is

$$F^{\mathrm{Th}}_{t-1}({}_{t-1}\bar{V})(1 + i') + P'_t = q_{t-1}({}^{\mathrm{Th}}DB'_t) + p_{t-1}F^{\mathrm{Th}}_t({}_t\bar{V}) \,, \tag{3}$$

where

F^{Th}_t = Theoretical face amount at end of tth year;
i' = Net interest rate earned in separate account after investment expenses and the mortality and expense risk charge have been deducted;
P'_t = Value in separate account at end of year t of daily net premium elements transferred to separate account during year t, based on interest rate i';
$q_{t-1}({}^{\mathrm{Th}}DB'_t)$ = Value in separate account at end of year t of death benefits, based on theoretical face amounts, paid during year t, based on interest rate i';
q_{t-1} = Probability of death during year t, based on standard table;
p_{t-1} = Probability of surviving year t, based on standard table.

Note that $p_{t-1} + q_{t-1} = 1$. If we substitute $1 - q_{t-1}$ for p_{t-1}, we have, after rearranging terms,

$$F^{\mathrm{Th}}_{t-1}({}_{t-1}\bar{V})(1 + i') + P'_t - q_{t-1}[{}^{\mathrm{Th}}DB'_t - F^{\mathrm{Th}}_t({}_t\bar{V})] - F^{\mathrm{Th}}_t({}_t\bar{V}) = 0 \,. \tag{4}$$

If we substitute in equation (2) the value of P''_t based on Appendix I and the value of DB''_t based on Appendix II, we obtain

$$F^{\mathrm{NYL}}_{t-1}({}_{t-1}\bar{V})(1 + i'') + 365DNPE\left[\frac{i''}{d''^{(365)}} - q'_{t-1}\left(\frac{i'' - i''^{(365)}}{i''^{(365)}d''^{(365)}}\right)\right]$$
$$- q'_{t-1}\left\{\frac{i''^{(365)}}{\delta''}(1 + i'')\left[F^{\mathrm{NYL}}_{t-1}\left(\frac{d''^{(365)} - d''}{i''^{(365)}d''^{(365)}}\right) + F^{\mathrm{NYL}}_t\left(\frac{d'' - v''d''^{(365)}}{i''^{(365)}d''^{(365)}}\right)\right]\right.$$
$$\left. - F^{\mathrm{NYL}}_t({}_t\bar{V})\right\} - F^{\mathrm{NYL}}_t({}_t\bar{V}) = p'_{t-1}F^{\mathrm{NYL}}_t(S_t) \,. \tag{5}$$

To simplify equation (5) and later equations, let us define the following five compound interest functions:

$$K_1 = \frac{i}{d^{(365)}}, \quad K_2 = \frac{i - i^{(365)}}{i^{(365)}d^{(365)}}, \quad K_3 = \frac{i^{(365)}}{\delta}(1+i),$$

$$K_4 = \frac{d^{(365)} - d}{i^{(365)}d^{(365)}}, \quad K_5 = \frac{d - vd^{(365)}}{i^{(365)}d^{(365)}}.$$

(The values of the above functions for $i = 0$ per cent should be taken as follows: $K_1^0 = 1$, $K_2^0 = \frac{1}{2} \times 364/365$, $K_3^0 = 1$, $K_4^0 = \frac{1}{2} \times 364/365$, $K_5^0 = 1 - \frac{1}{2} \times 364/365$.)

Using single and double primes to indicate the interest rates at which the K functions are evaluated and substituting in equation (5), we obtain

$$\begin{aligned} &F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1+i'') + 365DNPE[K_1'' - q_{t-1}'(K_2'')] \\ &- q_{t-1}'\{K_3''[F_{t-1}^{\mathrm{NYL}}(K_4'') + F_t^{\mathrm{NYL}}(K_5'')] - F_t^{\mathrm{NYL}}({}_t\bar{V})\} \\ &\qquad - F_t^{\mathrm{NYL}}({}_t\bar{V}) = p_{t-1}'F_t^{\mathrm{NYL}}(S_t). \end{aligned} \tag{6}$$

Equation (6) is the equation of equilibrium for the separate account asset share, connecting the fund at the beginning of the tth year with the fund at the end of the tth year immediately before the transfer.

Similarly, if we substitute in equation (4) the value of P_t' based on Appendix I and the value of ${}^{\mathrm{Th}}DB_t'$ based on Appendix II, we have

$$\begin{aligned} &F_{t-1}^{\mathrm{Th}}({}_{t-1}\bar{V})(1+i') + 365DNPE[K_1' - q_{t-1}(K_2')] \\ &\quad - q_{t-1}\{K_3'[F_{t-1}^{\mathrm{Th}}(K_4') + F_t^{\mathrm{Th}}(K_5')] - F_t^{\mathrm{Th}}({}_t\bar{V})\} - F_t^{\mathrm{Th}}({}_t\bar{V}) = 0. \end{aligned} \tag{7}$$

Equation (7) is the basic equation of equilibrium for the reserve in the separate account.

Subtracting equation (7) from equation (6) and rearranging, we have

$$\begin{aligned} &F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1+i'') - F_{t-1}^{\mathrm{Th}}({}_{t-1}\bar{V})(1+i') + 365DNPE(K_1'' - K_1') \\ &+ q_{t-1}\{365DNPE(K_2') + K_3'[F_{t-1}^{\mathrm{Th}}(K_4') + F_t^{\mathrm{Th}}(K_5')] - F_t^{\mathrm{Th}}({}_t\bar{V})\} \\ &- q_{t-1}'\{365DNPE(K_2'') + K_3''[F_{t-1}^{\mathrm{NYL}}(K_4'') + F_t^{\mathrm{NYL}}(K_5'')] - F_t^{\mathrm{NYL}}({}_t\bar{V})\} \\ &\qquad + F_t^{\mathrm{Th}}({}_t\bar{V}) - F_t^{\mathrm{NYL}}({}_t\bar{V}) = p_{t-1}'F_t^{\mathrm{NYL}}(S_t). \end{aligned} \tag{8}$$

Equation (8) represents the equation of equilibrium for the surplus, relating the surplus at the beginning of year t, which is zero, to the surplus at the end of year t, S_t.

Equation (8) is, generally speaking, in the desired form, indicated by equation (9) below.

$$I_t + M_t = p'_{t-1}F_t^{\mathrm{NYL}}(S_t) , \tag{9}$$

where I_t is the interest contribution in year t per M and M_t is the mortality contribution in year t per M. (Note: M refers to \$1,000 of initial face amount of insurance.)

However, the interest and mortality contributions are not readily separable and identifiable in equation (8). To make these items more readily distinguishable, let us add and subtract from the left-hand side of equation (8) the quantity

$$F_{t-1}^{\mathrm{Th}}({}_{t-1}\bar{V})(1 + i') - F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1 + i') .$$

After suitable rearrangement, we have

$$\begin{aligned} &q_{t-1}\{365DNPE(K_2') + K_3'[F_{t-1}^{\mathrm{Th}}(K_4') + F_t^{\mathrm{Th}}(K_5')] - F_t^{\mathrm{Th}}({}_t\bar{V})\} \\ &- q'_{t-1}\{365DNPE(K_2'') + K_3''[F_{t-1}^{\mathrm{NYL}}(K_4'') + F_t^{\mathrm{NYL}}(K_5'')] - F_t^{\mathrm{NYL}}({}_t\bar{V})\} \\ &+ [F_t^{\mathrm{Th}}({}_t\bar{V}) - F_t^{\mathrm{NYL}}({}_t\bar{V})] - [F_{t-1}^{\mathrm{Th}}({}_{t-1}\bar{V})(1 + i') \\ &\qquad - F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1 + i')] \\ &+ 365DNPE(K_1'' - K_1') + (i'' - i')F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V}) = p'_{t-1}F_t^{\mathrm{NYL}}(S_t) . \end{aligned} \tag{10}$$

In equation (10) the terms $365DNPE(K_1'' - K_1')$ and $(i'' - i')F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})$ clearly will be part of I_t. It should be noted at this point that because the New York Life face amounts differ (although only slightly) from the theoretical face amounts, some amount in the nature of a contribution to surplus will be present each year even if $i'' = i'$ and $q' = q$. Such amount will be very small, since it is a function of the difference between the corresponding New York Life and theoretical face amounts. Nonetheless it must be considered if the entire surplus built up in the separate account each year is to be accounted for. In equation (10) the terms

$$[F_t^{\mathrm{Th}}({}_t\bar{V}) - F_t^{\mathrm{NYL}}({}_t\bar{V})] - [F_{t-1}^{\mathrm{Th}}({}_{t-1}\bar{V})(1 + i') - F_{t-1}^{\mathrm{NYL}}({}_{t-1}\bar{V})(1 + i')]$$

are of this nature.

In equation (10) the terms $q_{t-1}\{\ \} - q'_{t-1}\{\ \}$ remain to be analyzed into their I_t and M_t components. Let us regard the expression in braces multiplying q_{t-1} as a function of i', say $f(i')$, and let us regard the expression in braces multiplying q'_{t-1} as a different function of i'', say $h(i'')$. The distinction between the two functions is that f involves theoretical face amounts, while h involves New York Life face amounts. Then we have

$$q_{t-1}\{\ \} - q'_{t-1}\{\ \} = q_{t-1}f(i') - q'_{t-1}h(i'') . \tag{11}$$

The right-hand side of equation (11) may in turn be expanded as follows:

$$\begin{aligned} q_{t-1}f(i') - q'_{t-1}h(i'') = q_{t-1}f(i') - q_{t-1}h(i') + (q_{t-1} - q'_{t-1})h(i') \\ - q'_{t-1}[h(i'') - h(i')] . \end{aligned} \tag{12}$$

Substituting in equation (11), we have

$$\begin{aligned} q_{t-1}\{\ \} - q'_{t-1}\{\ \} = q_{t-1}f(i') - q_{t-1}h(i') + (q_{t-1} - q'_{t-1})h(i') \\ - q'_{t-1}[h(i'') - h(i')] . \end{aligned} \tag{13}$$

In this form it may be seen that the first three terms of the right-hand side of equation (13) will be part of M_t, while the last term will be part of I_t.

After rearrangement, the final form of the equation of equilibrium for the surplus is

$$\begin{aligned} &(i'' - i')F_{t-1}^{\text{NYL}}({}_{t-1}\bar{V}) + 365DNPE(K_1'' - K_1') \\ &- q'_{t-1}[365DNPE(K_2'' - K_2') + F_{t-1}^{\text{NYL}}(K_3''K_4'' - K_3'K_4') \\ &\qquad + F_t^{\text{NYL}}(K_3''K_5'' - K_3'K_5')] \\ &+ (q_{t-1} - q'_{t-1})\{365DNPE(K_2') + K_3'[F_{t-1}^{\text{NYL}}(K_4') + F_t^{\text{NYL}}(K_5')] \\ &\qquad - F_t^{\text{NYL}}({}_t\bar{V})\} \\ &+ q_{t-1}\{K_3'[K_4'(F_{t-1}^{\text{Th}} - F_{t-1}^{\text{NYL}}) + K_5'(F_t^{\text{Th}} - F_t^{\text{NYL}})] \\ &\qquad - [F_t^{\text{Th}}({}_t\bar{V}) - F_t^{\text{NYL}}({}_t\bar{V})]\} \\ &+ [F_t^{\text{Th}}({}_t\bar{V}) - F_t^{\text{NYL}}({}_t\bar{V})] - [F_{t-1}^{\text{Th}}({}_{t-1}\bar{V})(1 + i') \\ &\qquad - F_{t-1}^{\text{NYL}}({}_{t-1}\bar{V})(1 + i')] = p'_{t-1}F_t^{\text{NYL}}(S_t) . \end{aligned} \tag{14}$$

From equation (14), the equation for the interest contribution to the surplus is

$$\begin{aligned} I_t = {} &(i'' - i')F_{t-1}^{\text{NYL}}({}_{t-1}\bar{V}) + 365DNPE(K_1'' - K_1') \\ &- q'_{t-1}[365DNPE(K_2'' - K_2') + F_{t-1}^{\text{NYL}}(K_3''K_4'' - K_3'K_4') \\ &\qquad + F_t^{\text{NYL}}(K_3''K_5'' - K_3'K_5')] , \end{aligned} \tag{15}$$

and the equation for the mortality contribution to the surplus is

$$
\begin{aligned}
M_t = {} & (q_{t-1} - q'_{t-1})\{365DNPE(K'_2) \\
& \qquad + K'_3[F_{t-1}^{\text{NYL}}(K'_4) + F_t^{\text{NYL}}(K'_5)] - F_t^{\text{NYL}}({}_t\bar{V})\} \\
& + q_{t-1}\{K'_3[K'_4(F_{t-1}^{\text{Th}} - F_{t-1}^{\text{NYL}}) + K'_5(F_t^{\text{Th}} - F_t^{\text{NYL}})] \\
& \qquad - [F_t^{\text{Th}}({}_t\bar{V}) - F_t^{\text{NYL}}({}_t\bar{V})]\} \\
& + [F_t^{\text{Th}}({}_t\bar{V}) - F_t^{\text{NYL}}({}_t\bar{V})] - [F_{t-1}^{\text{Th}}({}_{t-1}\bar{V})(1 + i') \\
& \qquad - F_{t-1}^{\text{NYL}}({}_{t-1}\bar{V})(1 + i')] \, .
\end{aligned}
\tag{16}
$$

GENERAL ACCOUNT

The transfer from the separate account to the general account at the end of each year equals the sum of the interest and mortality contributions for that year. Thus we have

$$T_t = I_t + M_t \, , \tag{17}$$

where T_t is the transfer per M at the end of the tth year. The equation of equilibrium for the general account asset share then becomes

$$
\begin{aligned}
& {}^{G}AS_{t-1}(1 + j) + [GP(1 - E_1) - E_2 - E_3](1 + j) \\
& \qquad - 365DNPE[K_1^j - q'_{t-1}(K_2^j)] \\
& \qquad - q'_{t-1}(\tfrac{11}{24}GP + E_4 + TD_{t-1} + fD_t)\left(1 + \frac{j}{2}\right) \\
& - q'^{(w)}_{t-1}(E_5 + TD_t + D_t) + q'^{(w)}_{t-1} F_t^{\text{NYL}}({}_t\bar{V} - {}_tCV) + T_t \\
& \qquad = p'_{t-1}({}^{G}AS_t + D_t) \, ,
\end{aligned}
\tag{18}
$$

where

${}^{G}AS_t$ = General account asset share per M at end of tth year, just after the transfer has been made;
j = Net interest rate earned in general account;
GP = Gross annual premium per M;
E_1 = Expenses per \$1 of premium;
E_2 = Regular per M expenses, including per policy expenses expressed as expenses per M;
E_3 = Charge per M for MDBG;
E_4 = Claim expenses per M;
E_5 = Withdrawal expenses per M;

$_tCV$ = tth-year cash value per M;
TD_t = Termination dividend payable at end of tth year;
D_t = Annual dividend payable at end of tth year;
f = Average fraction of annual dividend payable on death;
K_1^j and K_2^j = Previously defined K functions evaluated at interest rate j.

Expressions for the end-of-the-year value of the daily net premium elements transferred during the year differ between equations (6) and (18), since in equation (6) they are being evaluated at the separate account interest rate i'', while in equation (18) they are being evaluated at the general account interest rate j.

APPENDIX I

VALUE AT END OF YEAR OF DAILY NET PREMIUM ELEMENTS TRANSFERRED DAILY DURING THE YEAR

From first principles, P_t, the value at the end of the tth year of daily net premium elements ($DNPE$) transferred daily from the general account to the separate account, is

$$P_t = DNPE\sum_{s=0}^{364} {}_{s/365}p_{t-1}(1+i)^{1-s/365}\,, \tag{19}$$

where ${}_{s/365}p_{t-1}$ is the probability that a life aged $t-1$ survives s days. If we factor out $(1+i)$ and substitute $v^{s/365}$ for $(1+i)^{-s/365}$, we obtain

$$P_t = (1+i)DNPE\sum_{s=0}^{364} {}_{s/365}p_{t-1}v^{s/365} \tag{20}$$

$$= (1+i)365DNPE\ddot{a}^{(365)}_{t-1:\overline{1}|}\,. \tag{21}$$

If we assume a uniform distribution of deaths over the year, we have

$$\ddot{a}^{(365)}_{t-1:\overline{1}|} \doteq \frac{d}{d^{(365)}} - q_{t-1}v\left(\frac{i-i^{(365)}}{i^{(365)}d^{(365)}}\right). \tag{22}$$

(This relationship is derived as eq. [A10] in my earlier paper, cited in the introduction.)

Substituting the above value for $\ddot{a}^{(365)}_{t-1:\overline{1}|}$ in equation (21) and simplifying, we obtain

$$P_t \doteq 365DNPE\left[\frac{i}{d^{(365)}} - q_{t-1}\left(\frac{i-i^{(365)}}{i^{(365)}d^{(365)}}\right)\right]. \tag{23}$$

APPENDIX II

VALUE AT END OF YEAR OF DEATH BENEFITS PAID AT MOMENT OF DEATH DURING THE YEAR

The amount $q_{t-1}(DB_t)$, the value at the end of the tth year of death benefits paid at moment of death during the year, is, from first principles,

$$q_{t-1}(DB_t) = \sum_{s=0}^{364} {}_{s/365|1/365}q_{t-1}F_{t-1+(s+1)/365}(1+i)^{1-(s+1)/365}\left(\frac{i^{(365)}}{\delta}\right), \tag{24}$$

where ${}_{s/365|1/365}q_{t-1}$ is the probability that a life aged $t-1$ dies during the $(s+1)$st day of the tth year and $F_{t-1+(s+1)/365}$ is the face amount at the end of the $(s+1)$st day of the tth year. (The factor $i^{(365)}/\delta$ in eq. [24] is the daily analogue of the familiar factor i/δ in the annual case, and is necessary, for the same reason, to recognize that death benefits are assumed to be paid at moment of death. For the details of the derivation of $i^{(365)}/\delta$ see eqs. [4] to [16] in my earlier paper, cited in the introduction.)

Assuming a uniform distribution of deaths over the year, we have

$${}_{s/365|1/365}q_{t-1} \doteq \frac{q_{t-1}}{365}. \tag{25}$$

Using linear interpolation on the face-amount function, we have

$$F_{t-1+s/365} \doteq \left(\frac{365-s}{365}\right)F_{t-1} + \left(\frac{s}{365}\right)F_t. \tag{26}$$

Substituting the values from equations (25) and (26) in equation (24) and noting that $v^{s/365} = (1+i)^{-s/365}$, we obtain, after simplifying and appropriately adjusting the limits of summation,

$$q_{t-1}(DB_t) \doteq \frac{q_{t-1}}{365}\left(\frac{i^{(365)}}{\delta}\right)(1+i)\sum_{s=1}^{365} v^{s/365}\left[\left(\frac{365-s}{365}\right)F_{t-1} + \left(\frac{s}{365}\right)F_t\right]. \tag{27}$$

In order to evaluate the above, we need expressions for

$$\sum_{s=1}^{365} v^{s/365} \quad \text{and} \quad \sum_{s=1}^{365} s v^{s/365}.$$

From compound-interest theory, we have

$$\sum_{s=1}^{365} v^{s/365} = 365\left(\frac{d}{i^{(365)}}\right) \tag{28}$$

and

$$\sum_{s=1}^{365} sv^{s/365} = 365\left(\frac{d/d^{(365)} - v}{i^{(365)}/365}\right). \tag{29}$$

Substituting the values from equations (28) and (29) in equation (27), we have

$$\begin{aligned} q_{t-1}(DB_t) \doteq \frac{q_{t-1}}{365}\left(\frac{i^{(365)}}{\delta}\right)(1+i)\Big\{F_{t-1}\Big[365\left(\frac{d}{i^{(365)}}\right) \\ - \left(\frac{d/d^{(365)} - v}{i^{(365)}/365}\right)\Big] + F_t\left(\frac{d/d^{(365)} - v}{i^{(365)}/365}\right)\Big\}. \end{aligned} \tag{30}$$

After simplification, we have, finally,

$$\begin{aligned} q_{t-1}(DB_t) \doteq q_{t-1}\left(\frac{i^{(365)}}{\delta}\right)(1+i)\Big[F_{t-1}\left(\frac{d^{(365)} - d}{i^{(365)}d^{(365)}}\right) \\ + F_t\left(\frac{d - vd^{(365)}}{i^{(365)}d^{(365)}}\right)\Big]. \end{aligned} \tag{31}$$

And, therefore,

$$DB_t \doteq \left(\frac{i^{(365)}}{\delta}\right)(1+i)\left[F_{t-1}\left(\frac{d^{(365)} - d}{i^{(365)}d^{(365)}}\right) + F_t\left(\frac{d - vd^{(365)}}{i^{(365)}d^{(365)}}\right)\right]. \tag{32}$$

DISCUSSION OF PRECEDING PAPER

DAVID G. ADAMS:

Mr. Scher presents the equation of equilibrium for the separate account asset share, the equation of equilibrium for the reserve in the separate account, and the equation of equilibrium for the surplus arising during the year in the separate account.

In order to apply these formulas in developing variable life asset shares, values of F_t^{Th} (and F_t^{NYL}) must be generated, using the formulas presented in Mr. Scher's earlier paper.

In an effort to avoid the voluminous additional calculations required to produce the values of F_t, we found that a single equation could serve as both the generator of the values of F_t and the equation of equilibrium for the reserve in the separate account. From the theoretical recursive formula for F_t^{Th} on a daily basis, we made some general observations: (1) The reserve held at the end of the year t is approximately $F_t({}_t\bar{V})$. (2) The net premium transferred to the separate account in year t is approximately $\bar{P}(1 - \frac{1}{2}q_{t-1})$, and this transfer occurs approximately at the middle of the year. (3) The death benefit in year t is approximately $\frac{1}{2}(F_{t-1} + F_t)$, and this benefit is transferred from the separate account (approximately) at the middle of the year.

These observations suggested replacing F_t^{Th} with F_t^* and using the following formula as the equation of equilibrium for the reserve in the separate account:

$$F^*_{t-1}({}_{t-1}\bar{V})(1 + i') + \bar{P}(1 - \tfrac{1}{2}q_{t-1})(1 + i')^{1/2} - \tfrac{1}{2}(F^*_{t-1} + F^*_t)q_{t-1}(1 + i')^{1/2} - (1 - q_{t-1})F^*_t({}_t\bar{V}) = 0 . \tag{1}$$

Here F_t^* is defined not as F_t^{Th} or F_t^{NYL} but rather by the equation of equilibrium itself, and the remaining terms follow Mr. Scher's definitions. It should be noted that this approach is workable only if the values of F_t^* are reasonably close to F_t^{Th}. To test the fit, we solved the equation for F_t^* and compared the results with the corresponding values of F_t^{Th} for selected values of i'. We found surprisingly good results, as illustrated by Table 1, which is based on a \$1,000 variable whole life policy issued to a male aged 35.

Equation (1) can easily be expanded to serve as the equation of equilibrium separate account asset share:

$$
\begin{aligned}
&F^*_{t-1}({}_{t-1}\bar{V})(1+i'') + \bar{P}(1-\tfrac{1}{2}q'_{t-1})(1+i'')^{1/2} \\
&\qquad - \tfrac{1}{2}(F^*_{t-1}+F^*_t)\,q'_{t-1}(1+i'')^{1/2} - q'^{(w)}_{t-1}F_t({}_t\bar{V}) \qquad (2)\\
&\qquad\qquad - (1-q'_{t-1}-q'^{(w)}_{t-1})F^*_t({}_t\bar{V}+S_t) = 0\,.
\end{aligned}
$$

TABLE 1

Policy Year (t)	$i'=0\%$		$i'=3\%$		$i'=6\%$	
	F^*_t	F^{Th}_t	F^*_t	F^{Th}_t	F^*_t	F^{Th}_t
1	$986.47	$986.41	$999.98	$1,000	$1,013.26	$1,013.45
2	973.02	972.99	999.95	1,000	1,027.02	1,027.22
3	959.75	959.71	999.97	1,000	1,041.18	1,041.37
4	946.62	946.57	999.97	1,000	1,055.70	1,055.89
5	933.64	933.58	999.98	1,000	1,070.60	1,070.79
6	920.81	920.75	999.98	1,000	1,085.88	1,086.07
7	908.14	908.08	999.98	1,000	1,101.54	1,101.73
8	895.62	895.57	999.98	1,000	1,117.59	1,117.79
9	883.27	883.22	999.98	1,000	1,134.04	1,134.24
10	871.08	871.03	999.98	1,000	1,150.89	1,151.10

Following the author's approach and subtracting equation (1) from equation (2) gives the equation of equilibrium for the surplus generated in the separate account during the year:

$$
\begin{aligned}
&(i''-i')F^*_{t-1}({}_{t-1}\bar{V}) \\
&\qquad + \bar{P}[(1-\tfrac{1}{2}q'_{t-1})(1+i'')^{1/2} - (1-\tfrac{1}{2}q_{t-1})(1+i')^{1/2}] \\
&\qquad - \tfrac{1}{2}(F^*_{t-1}+F^*_t)\,[q'_{t-1}(1+i'')^{1/2} - q_{t-1}(1+i')^{1/2}] \qquad (3)\\
&\qquad\qquad - F^*_t({}_t\bar{V})(q-q') = p'_{t-1}F^*_tS_t\,.
\end{aligned}
$$

Separating $p'_{t-1}F^*_tS_t$ into the interest and mortality contributions such that $p'_{t-1}F^*_tS_t = I_t + M_t$, we have

$$
\begin{aligned}
I_t &= (i''-i')F_{t-1}({}_{t-1}\bar{V}) \\
&\quad + [(1+i'')^{1/2} - (1+i')^{1/2}][\bar{P}(1-\tfrac{1}{2}q'_{t-1}) - \tfrac{1}{2}(F_{t-1}+F_t)q'_{t-1}] \qquad (4)
\end{aligned}
$$

and

$$M_t = (q-q')\{[\tfrac{1}{2}(F_{t-1}+F_t)(1+i')^{1/2} - F_t({}_t\bar{V})] + \tfrac{1}{2}\bar{P}(1+i')^{1/2}\}\,. \qquad (5)$$

Equations (2)–(5) reflect the assumption that contributions to surplus are transferred from the separate account to the general account at the end of each year. They can be modified to reflect other assumptions, such as the assumption that interest and mortality contributions are transferred to the general account daily and thus on the average at the middle of the year.

J. ROSS HANSON:

Ever since the landmark paper was presented to the Society in 1969 by Messrs. Sternhell, Fraser, and Miller, I have been personally convinced that variable life insurance will become a most important part of the life insurance business.

We are indebted to the New York Life actuaries and the discussants of their paper for their illumination of this subject, and we continue to be indebted to the New York Life staff for their willingness to share the fruits of their research with the rest of us. Mr. Scher's current paper, like his previous one, is essentially a treatise on the way the design works. And for that reason, it will prove immensely valuable.

This discussion is not an analysis of Mr. Scher's presentation, since I can add very little to that. But I felt that this would be an appropriate place to describe briefly one special aspect of asset share calculations for variable life insurance which I find particularly interesting.

A major factor affecting the profitability of variable life insurance is the investment result in the separate account. However, the actual investment result that will be experienced over any period of time is unpredictable. Mortality assumptions may be made fairly safely because the large number of insured lives gives us some assurance that our experience will be close to the mean mortality rate. This is not true for the investment result. Therefore, it is necessary to measure the effect on profit of various possible investment results. There are some very sophisticated applications of risk theory to this problem. We have used a simple Monte Carlo technique which I think is probably adequate to give the actuary and the company's management a basis for judgment of the risk involved.

The basic statistical premise is that the annual yield of the Standard and Poor's composite index (500 common stocks) is a random variable. The monthly annualized rate of return from 1916 to 1966 supplies the data on which the statistical distribution of this variable is based. The investment result for some specific period in the future, say twenty years, can be simulated by making twenty random entries into a cumulative probability density table calculated from this distribution. The numerical data we have used were derived from Mr. Turner's paper "Asset Value

Guarantees under Equity-based Products" (*TSA* XXI, 459). (Of course, the underlying data can be modified or extended, or, in fact, other data can be used, provided that the condition of randomness is still present.)

Each choice of twenty values represents a trial, and we usually make 1,000 trials for each sample. This number of trials is more than sufficient to give us 99 per cent confidence that the entire potential range of the variable has been used. We are thus confident that the statistical characteristics of the sample are very close to those of the underlying distribution.

TABLE 1

STATISTICS FOR PRESENT VALUE AT ISSUE OF TWENTIETH-YEAR SURPLUS SHARE

Issue Age 25 Percentile Table		Issue Age 45 Percentile Table	
Probability	Amount	Probability	Amount
1%	$ 0.24	1%	$ 33.30
2	1.71	2	37.68
5	3.59	5	42.56
10	5.29	10	47.70
20	8.08	20	55.82
30	10.10	30	61.62
40	12.42	40	68.09
50	14.65	50	74.57
60	17.08	60	81.34
70	20.17	70	89.62
80	23.59	80	99.94
90	29.67	90	116.47
95	36.03	95	133.10
98	43.04	98	154.02
99	48.03	99	167.62
Mean	$16.48	Mean	$ 79.39
Median	$14.65	Median	$ 74.57
Std. dev	$10.03	Std. dev	$ 28.31

Applying this technique in an asset share model, we construct the distribution function for several monetary functions, including the asset share, the mean reserve, and the amount of insurance (for any benefit design).

In examining the results, it is very important to realize that the mean is not necessarily likely to be our experience; there is no law of large numbers operating here as there is with the mortality risk. Over many twenty-year periods the experience will approach the mean; we can only state the probability that for any twenty-year period the monetary function will take its mean value. Therefore, we must decide not only which monetary results are satisfactory but also what degree of probability we feel should be associated with its occurrence.

As an example of the technique, I have included Table 1, which shows the distribution of the present value at issue of the contribution to surplus in the first twenty policy years. The results shown in the table, of course, are unique to the particular actuarial, design, premium, and expense assumptions employed.

The table shows the probability that the present value will be less than the amount shown. For example, at issue age 25, there is a 20 per cent chance that the amount will be less than \$8.08 per \$1,000 of original face amount issued; alternatively, at issue age 45, there is a 30 per cent chance that the amount will exceed \$89.62.

BRUCE E. NICKERSON:

This extension of asset shares to variable life insurance is a logical sequel to Mr. Scher's 1971 paper presenting the theory of the New York Life design on a daily basis. We are all indebted to Mr. Scher and his associates for their continuing contributions to the development of actuarial theory and practice for variable life insurance.

The basic assumption underlying this paper is that the separate account will be adjusted annually by a lump-sum transfer to the general account of the difference between the funds actually accumulated and the reserve. Thus, no "surplus" is allowed to accumulate in the separate account, and the separate account "asset share" is, by definition, equal to the reserve at the beginning of each policy year. Equation (2) specifies the amount of this transfer for any policy year.

The general account "asset share" corresponds to the "surplus share" of a fixed benefit policy. That is, the result at the end of any policy year equals the excess (positive or negative) of the asset share over the reserve. This is the critical result for a company writing nonparticipating business. The actuary interested in pricing or determining profitability of a nonparticipating policy might well wish to go directly from equation (2) to a version of equation (18), such as the following:

$$
\begin{aligned}
&{}^{G}AS_{t-1}(1+j) + [GP(1-E_1) - E_2 - E_3](1+j) - P_t^j \\
&\quad - q'_{t-1}(\tfrac{11}{24}GP + E_4)(1+j/2) + q'^{(w)}_{t-1}[F_t^{\mathrm{NYL}}({}_t\bar{V} - CV_t) - E_5] \\
&\quad + p'_{t-1}F_t^{\mathrm{NYL}}S_t = p'_{t-1}\,{}^{G}AS_t\,,
\end{aligned}
$$

where P_t^j is the value at end of year t of daily net premium elements transferred to the separate account during year t, based on interest rate j.

The conclusion that the first three terms of equation (13) are the mortality element of equation (11) and the fourth term is the interest element seems arbitrary. A different expansion of the right-hand side of equation (11) could lead, for example, to the conclusion that

$$q_{t-1}\{\ \} - q'_{t-1}\{\ \} = q'_{t-1}f(i'') - q'_{t-1}h(i'') + (q_{t-1} - q'_{t-1})f(i'') - q_{t-1}[f(i'') - f(i')] .$$

It can be argued similarly that the first three terms of the right-hand side of this equation are the mortality element and the fourth term is the interest element.

Since the mortality gain could in theory (although not in practice, of course) be transferred to the general account as it accrues, an alternative approach would be to identify the mortality element of the separate account surplus as the difference between $q_{t-1}({}^jDB_t'')$ and $q'_{t-1}({}^jDB_t'')$, where

$$q_{t-1}({}^jDB_t'') = \sum_{s=0}^{364} {}_{S/365|1/365}q_{t-1}F^{\mathrm{NYL}}_{t-1+(S+1)/365}(1+j)^{1-(S+1)/365}\left(\frac{i^{(365)}}{\delta}\right) .$$

Thus the mortality gain would be the accumulation, at the general account interest rate, of the difference between the mortality costs based on the standard and the experience mortality tables, based on the death benefits actually payable.

Similarly, the benefit formula gain (from the difference between the theoretical and the actual death benefits and reserves) would be the sum of (a) the accumulation, at interest rate j, of the differences between the daily claim costs based on the experience mortality table and the theoretical face amounts and the daily claim costs based on the experience mortality table and the New York Life face amounts plus (b) the accumulation, at interest rate j, of the differences between the theoretical and the New York Life reserves released on lapses during the year plus (c) the difference between the theoretical and New York Life reserves on the survivors at the end of the year. Assuming, as in the paper, that all lapses occur at the end of the year, this element would be symbolized as the difference between $q'_{t-1}({}^jDB_t'')$ and $q'_{t-1}({}^jDB_t')$, plus $p'_{t-1}[F_t^{\mathrm{NYL}}({}_t\bar{V}) - F_t^{\mathrm{Th}}({}_t\bar{V})]$.

The interest element would then be the excess of the total surplus over the mortality and benefit formula gains. In principle, this "interest" element might be subdivided into an asset charge element (equal to the accumulation, at interest rate j, of the daily charges for mortality and expense risks based on what the fund assets would have been if the mortality, benefit formula, and asset charge gains were all transferred to the general account as they accrued) and an investment gain (which is the profit, or loss, to the company resulting from deferring the transfers, thereby earning the separate account investment return instead of the general account return).

One effect of the annual surplus transfer is to compensate for the difference between the New York Life and the theoretical face amounts. As a result, it appears that the two face amounts must be exactly equal at the beginning of each policy year. If so, then some of the formulas in the paper could be simplified. For example, the fourth major term in equation (10) would always be zero.

The linear interpolation used to obtain equation (26) is recognized in the paper to be only an approximation. It would be helpful if Mr. Scher could indicate the extent of the error introduced for various levels of difference between the actual and the assumed investment return.

I offer an apology for the belated nature of my final remark, since it applies both to this paper and to Mr. Scher's 1971 paper. In working with annual interest functions for fixed benefit life insurance, we have often found the assumption that death claims are paid at the moment of death to be a useful abstraction. A daily basis variable life policy requires extensive use of daily interest functions, however, and the amount of the death benefit is determined as of the end of the day of death. For these policies, the assumption of payment at the moment of death and the consequent introduction of the factor $i^{(365)}/\delta$ would appear to decrease, rather than increase, the correspondence of the formulas to the operation of the policy.

(AUTHOR'S REVIEW OF DISCUSSION)

EDWARD SCHER:

I would like to thank Messrs. Adams, Hanson, and Nickerson for their discussions of the paper.

From a desire to simplify some of the equations of the paper, Mr. Adams has made some assumptions relative to the flow of premiums into the separate account and the flow of death benefits out of the separate account. On the basis of these assumptions he has developed equations for the reserve, asset share, surplus, and mortality and interest contributions, corresponding to analogous equations developed in the paper. If one makes a term-by-term comparison of, for example, his equation (4), for the interest contribution, with equation (15) of the paper, one obtains some insight into the effect of his assumptions upon the deviation of the resulting equations from those in the paper.

As indicated by the figures in his table, the face amounts produced by his equation (1) are certainly very close to the theoretical face amounts for the particular age, interest rates, and durations shown. It would be of interest, however, to see a similar comparison at higher interest rates and longer durations, since these situations generally represent a much more severe test of the acceptability of an approximation.

This raises a basic problem relative to the use of approximations in variable life insurance. Whereas in fixed benefit life insurance one can generally feel reasonably secure about the impact of an approximation, in variable life insurance the effect of a particular assumption may not be so easy to gauge on the basis of a priori general reasoning. Thus, although it may often be desirable, or even necessary in practice, to use some particular approximation, one may nevertheless find it expedient to make extensive tests under various separate account assumptions in order to feel reassured about the potential effect of such an approximation. And the user may feel such testing to be overly burdensome.

Mr. Hanson's interesting discussion concerns the application of a simple Monte Carlo technique to the problem of estimating the value of various monetary functions, such as surplus, asset shares, and benefit amounts, at some future time.

Essentially what he has done is to perform, by means of random sampling, a large number of "trials," each "trial" consisting of "predicting" the investment rates to be earned in each of the next twenty years. The results of such "trials" are tabulated, and a cumulative distribution function is thereby determined for, say, the twentieth-year surplus. From such a distribution, one is able to make probability statements about the likelihood of the twentieth-year surplus being equal to or greater than some given amount.

The use of some risk-theoretic method, such as the one used by Mr. Hanson, is a valuable means of obtaining such probabilities as the above. I think it is particularly important in variable life insurance to carry out some analysis of this sort, as opposed to simply calculating future results based on various assumed fixed net earned interest rates. Even the sort of retroactive analysis in which issues are assumed to have been made in certain past years and exact historical investment experience is used to determine what "would have happened" must be appraised with great caution. It is very easy for such results to be quite misleading if one interprets them as a direct guide for future performance.

Mr. Nickerson raises a number of interesting points in his discussion. He suggests that the paper's analysis of equation (11) into its interest and mortality contribution components is arbitrary, and he sets forth a different set of such components into which equation (11) can be separated. While it is true that equation (11) can be analyzed in various ways, and Mr. Nickerson's set of components is certainly one of the sets which is mathematically equivalent to that developed in the paper, the word "arbitrary" is a little too strong.

The reason I chose the particular analysis shown was so that the form of the resulting interest and mortality contributions would parallel conceptually the analogous expressions in fixed benefit life insurance. Thus, in fixed benefit life insurance, the interest contribution is a function of the difference between the actual net earned interest rate and the assumed interest rate (AIR), and the mortality contribution is a function of (in addition to the mortality differential, of course) the AIR only. In variable life insurance, for the purpose of surplus determination, the interest rate i', the rate upon which the variable face amount is based, may be thought of as playing a role parallel to that of the AIR in fixed benefit life insurance.

In any event, it probably should be emphasized that any reasonable definition of the mortality contribution results in an expression which is a function of the separate account net earned interest rate, whether i' or i''. This is a very different situation from that in fixed benefit life insurance, where the mortality contribution is independent of the actual net earned interest rate.

Mr. Nickerson also suggests a somewhat different analysis of surplus, in which he identifies not only a mortality gain and an interest gain but also a benefit formula gain (the gain due to the difference between the theoretical and the actual death benefits and reserves). This last gain will be very small, as I have indicated, which is why I included it as part of the mortality gain in the paper.

Mr. Nickerson's mortality gain, as well as some other elements in his analysis of surplus, is based on an accumulation throughout the year at the general account interest rate instead of the separate account interest rate, the assumption being that such gains are transferred to the general account as they accrue. This idea is of interest, and its pertinence depends on the frequency with which a company actually intends to transfer surplus in practice.

It should be noted that Mr. Nickerson's mortality gain involves the full mortality cost rather than the mortality cost based on the net amount at risk which is customary. The reason for this change is unclear. Also, the mortality gain should include that part which arises from the daily premium payments during the year.

In the definition of the first portion of the benefit formula gain (component a), it will be found on analysis that it is necessary to accumulate differences of the daily claim costs based on the standard mortality table rather than on the experience mortality table as Mr. Nickerson specifies. Otherwise the interest gain, which he has defined essentially as a balancing item, will include a gain that will be nonzero even if $i'' = i'$.

Also, note that the factor multiplying the difference in reserves in his expression for the benefit formula gain should be $1 - q'_{t-1}$ and not p'_{t-1}, which is defined in the paper, and also used in one of his previous formulas, as the probability of surviving *in force*.

Mr. Nickerson has the impression that as a result of the annual transfer of surplus, the New York Life and the theoretical face amounts are exactly equal at the beginning of each policy year. This is incorrect. The annual transfer of surplus is a transfer of the excess of the fund in the separate account over the reserve. While the amount of such surplus is related to the difference between the New York Life and the theoretical face amounts, the two face amounts are distinct entities and the formula for each face amount is independent of the amount of surplus or its disposition.

In order to obtain an expression for the face amount on any day during the year, linear interpolation between the face amounts at the beginning and at the end of the year was used, as noted by Mr. Nickerson, to develop equation (26). Tests have shown that, under a constant separate account interest rate, the progression of the actual daily face amounts throughout the year is very smooth and gradual. Hence the use of linearly interpolated face amounts in place of the actual formula face amounts introduces only an insignificant error in equation (26).

With regard to Mr. Nickerson's final remark, the assumption of payment of death benefits at the moment of death is used in the current paper to be consistent with its use in the derivation of the daily net premium in my prior paper. While it is true that this is an abstraction, as noted by Mr. Nickerson, this is of course a very common assumption, and premiums and reserves are routinely computed on this basis in fixed benefit life insurance.

The frequency with which the variable face amount changes is normally set equal to the frequency with which the net premium is assumed to be paid. Mr. Nickerson appears to confuse this frequency with the frequency with which death benefits will be paid. These two frequencies need not be identical. While a daily net premium could have been developed under such an assumption of identical frequencies, it was desired to have a daily premium which would develop reserves at the end of the year as close as possible to fully continuous reserves, without introducing any extraneous considerations, such as a return of premium benefit. The closeness of the reserves developed by the daily net premium as I have defined it to the fully continuous reserves was noted by Mr. Adams.