TRANSACTIONS OF SOCIETY OF ACTUARIES
1975 VOL. 27

ON UNDERSTANDING THE EFFECTS OF GAAP RESERVE ASSUMPTIONS

PAUL R. MILGROM

ABSTRACT

This paper deals with the effects of GAAP reserve assumptions on the resulting net premiums and reserves.

The developments of the first two sections of the paper revolve around a basic theorem relating tabular costs to net premium and reserve levels. Viewed in one particular way, the theorem predicts the effects of margins for adverse deviations on net premium and reserve levels.

Also developed and proved in the second section is a corollary of the basic theorem that is a sort of double decrement Lidstone's theorem. The two theorems are applied to certain hypothetical situations where the results stand in marked contrast to those one would expect in similar situations involving statutory reserves.

The third and final substantive section examines the feasibility of developing reserves for policies paying premiums other than annually while continuing to employ an annual premium reserve formula. The tentative conclusion with regard to terminal reserve formulas is that good results can be obtained if the cash values in early policy years are modified to reflect the extra costs developing on early surrenders. With regard to interim (or "intermediate" or "mean") reserve formulas, an approach is suggested which resembles the traditional statutory mean reserve formula but contains a parameter to adjust for the effect of off-anniversary lapses. When the parameter is set to zero (the annual premium case), the resulting formula closely resembles other published formulas.

I. INTRODUCTION

THE role of the actuary in preparing financial statements for life insurance companies in accordance with generally accepted accounting principles (GAAP) is one fraught with potentially conflicting objectives and difficult questions of judgment. Our basic models are submitted to the scrutiny of accountants, whose embarrassing questions often show us just how limited our understanding is. Consider the following problems:

1. After carefully explaining the computation of terminal reserves to the auditing C.P.A., you are asked: "But is this approach justified when premiums are not paid annually?"

2. Despite your expressed belief that provisions for adverse deviations should be selected to release profits in proper relation to risk, management and the auditors view the provisions as "conservatism," that is, as a process which necessarily results in the deferral of earnings. Are these two concepts the same in practice? Under what circumstances are differences likely to arise?

3. You have a deadline to meet, and the adjusted reserve factors available for your use are based on the wrong set of cash values. You must decide quickly whether the resulting reserves will be misstated materially or whether the error can be estimated and an appropriate adjustment made. What tools are available to you in the face of this task?

The key to the solution in each of these situations is not some textbook formula but good judgment and a deep understanding of the actuarial formulas involved. This paper attempts to develop some insights into the operation of GAAP reserve assumptions within the reserve formulas.

Section II of this paper deals with what we may call the "explicit assumptions," that is, the assumptions concerning rates of interest, mortality, withdrawal, and expense. Section III deals with one important "implicit assumption," namely, the assumption that withdrawals occur at the ends of policy years. Section III also contains some ideas for recognizing the effects of premium mode variations within an annual premium-type model.

A. *Notation and Approach*

In order to facilitate comparisons with earlier papers by other authors, the notation adopted here (detailed in Table 1) follows closely that used by Richard G. Horn in his paper "Life Insurance Earnings and the Release from Risk Policy Reserve System" (*TSA*, XXIII, 391). Although the concepts developed herein have broad applicability, this paper will be confined for practical reasons to limited payment, level premium, nonparticipating life insurance. Section II is limited further to the annual premium case, with withdrawals at the ends of policy years.

Again, following Mr. Horn's example, we shall (1) assume that all expenses are incurred at the beginnings of policy years and (2) use probabilities rather than rates of withdrawal, since these devices simplify the presentation. Finally, the presentation in this paper deals primarily with the total natural reserve, without separate reference to the benefit and expense components. The formulas lose no generality from this approach, since the benefit reserve, for example, may be viewed simply as the total natural reserve assuming zero acquisition expenses.

II. THE EXPLICIT ASSUMPTIONS

A. *Theory*

In order to develop an approach which is broad enough for the purposes of Section III of this paper, we shall allow both the death benefits and the cash values to vary between the reference assumptions (unprimed symbols) and the alternate assumptions (primed symbols). To understand the need for this generality, consider the computation for statutory accounting purposes of the reserve for the nondeduction of deferred premiums. The computation, in effect, considers that there is an additional death benefit during the premium-paying period which is equal to the average outstanding net deferred premium. A similar reserve may be calculated for GAAP purposes by a direct modification of the assumed death benefit. Of more interest, however, is the possibility that the addi-

TABLE 1

Symbol	Meaning
$q_{[x]+n-1}$	Rate of mortality for policy year n, issue age $[x]$
$wq_{[x]+n-1}$	Probability of withdrawal for policy year n, issue age $[x]$
i_n	Rate of interest for policy year n
$l_{[x]}$	Radix of the select mortality table
$l_{[x]+n}$	$l_{[x]+n-1}(1 - q_{[x]+n-1} - wq_{[x]+n-1})$
$D_{[x]+n}$	$l_{[x]+n}\prod_{k=1}^{n}(1+i_k)^{-1}$
$N_{[x]+n}$	$\sum_{k=n}^{w-x} D_{[x]+n}$
E_n^A	Expenses per unit incurred in policy year n
$E_n^{\%}$	Expenses incurred as a percentage of premiums for policy year n
${}_nCV_{[x]}$	Cash value at the end of policy year n, issue age $[x]$
${}_nDB_{[x]}$	Average death benefit during policy year n, issue age $[x]$
$G_{[x]}$	Gross premium for issue age $[x]$
$P_{[x]}$	Valuation net premium for issue age $[x]$
${}_nV_{[x]}$	Terminal reserve for issue age $[x]$, policy year n
a	The premium-paying period, a positive integer
b	The benefit period, a positive integer

NOTE.—When unprimed, these symbols denote the "reference assumptions"; when primed, the "alternate assumptions."

tional reserve for withdrawals at fractional durations may be computed by modifying the cash value used in the reserve computation. This is treated in more detail in Part III.

1. THE POLICY-YEAR GAIN AND LOSS FORMULA[1]

Based on the foregoing assumptions, the year-to-year, retrospective, reserve accumulation formula during the premium-paying period is

$$\begin{aligned}{}_nV_{[x]} = [&({}_{n-1}V_{[x]} + P_{[x]} - E_n^A - E_n^{\%}G_{[x]})(1+i_n) \\ &-q_{[x]+n-1}\ {}_nDB_{[x]}(1+i_n)^{1/2} - wq_{[x]+n-1}\ {}_nCV_{[x]}]/ \\ &(1 - q_{[x]+n-1} - wq_{[x]+n-1})\ .\end{aligned} \tag{1}$$

Formula (1) may be rearranged into a form which expresses ${}_nV_{[x]}$ as the preceding initial reserve plus interest, less two terms which might be called the "tabular cost of mortality" and the "tabular cost of withdrawals," as follows:

$$\begin{aligned}{}_nV_{[x]} = &({}_{n-1}V_{[x]} + P_{[x]} - E_n^A - E_n^{\%}G_{[x]})(1+i_n) \\ &- q_{[x]+n-1}[{}_nDB_{[x]}(1+i_n)^{1/2} - {}_nV_{[x]}] \\ &- wq_{[x]+n-1}({}_nCV_{[x]} - {}_nV_{[x]})\ .\end{aligned} \tag{2}$$

If actual experience follows the alternate assumptions, then the gain for the year (expressed at the end of year n per unit of insurance beginning the year) is given (during the premium-paying period) by

$$\begin{aligned}{}_n\text{Gain}_{[x]} &= ({}_{n-1}V_{[x]} + G_{[x]} - E_n^{A\prime} - E_n^{\%\prime}G_{[x]})(1+i_n') \\ &\quad - q'_{[x]+n-1}\ {}_nDB'_{[x]}(1+i_n')^{1/2} \\ &\quad - wq'_{[x]+n-1}\ {}_nCV'_{[x]} \\ &\quad - (1 - q'_{[x]+n-1} - wq'_{[x]+n-1})\ {}_nV_{[x]} \\ &= ({}_{n-1}V_{[x]} + G_{[x]} - E_n^{A\prime} - E_n^{\%\prime}G_{[x]})(1+i_n') \\ &\quad - q'_{[x]+n-1}[{}_nDB'_{[x]}(1+i_n')^{1/2} - {}_nV_{[x]}] \\ &\quad - wq'_{[x]+n-1}({}_nCV'_{[x]} - {}_nV_{[x]}) - {}_nV_{[x]}\ .\end{aligned} \tag{3}$$

[1] To analyze a company's operating results, gain and loss analysis must be done on a calendar-year basis. As with the policy-year gain and loss analysis, the key to any calendar-year formula is to express the increase in reserve factors in terms of tabular premiums, tabular interest, tabular cost of mortality, tabular cost of withdrawals, and tabular expenses. It follows that the mean reserve formula used will have an effect on the gain and loss formula and that a mechanistic approach to the problem would be ill-advised.

Substitution of the right-hand side of formula (2) for the final term of formula (3) and rearranging produces a policy-year gain and loss formula:[2]

$$
\begin{aligned}
{}_n\mathrm{Gain}_{[x]} = {} & (G_{[x]} - P_{[x]})(1 + i'_n) \quad \text{(Gain from loading)} \\
& + (i'_n - i_n)({}_{n-1}V_{[x]} + P_{[x]} - E^A_n - E^{\%}_n G_{[x]}) \quad \text{(Gain from interest)} \\
& + \{q_{[x]+n-1}[(1 + i_n)^{1/2}\ {}_nDB_{[x]} - {}_nV_{[x]}] \\
& \quad - q'_{[x]+n-1}[(1 + i'_n)^{1/2}\ {}_nDB'_{[x]} - {}_nV_{[x]}]\} \quad \text{(Gain from mortality)} \\
& + [wq_{[x]+n-1}({}_nCV_{[x]} - {}_nV_{[x]}) \\
& \quad - wq'_{[x]+n-1}({}_nCV'_{[x]} - {}_nV_{[x]})] \quad \text{(Gain from withdrawals)} \\
& + [E^A_n - E^{A\prime}_n + (E^{\%}_n - E^{\%\prime}_n)G_{[x]}](1 + i'_n) \quad \text{(Gain from expense)}\,.
\end{aligned} \tag{4}
$$

One may verify readily that formula (4) essentially is Mr. Horn's formula by making two changes: (1) ignore the variation in cash values by setting ${}_nCV'_{[x]} = {}_nCV_{[x]}$; (2) ignore the variation in death benefits, and the partial year's interest thereon, by setting

$$
{}_nDB_{[x]}(1 + i_n)^{1/2} = {}_nDB'_{[x]}(1 + i'_n)^{1/2} = 1{,}000\,.
$$

The resulting equation is

$$
\begin{aligned}
{}_n\mathrm{Gain}_{[x]} = {} & (G_{[x]} - P_{[x]})(1 + i'_n) \quad \text{(Gain from loading)} \\
& + (i'_n - i_n)({}_{n-1}V_{[x]} + P_{[x]} - E^A_n - E^{\%}_n G_{[x]}) \quad \text{(Gain from interest)} \\
& + (q_{[x]+n-1} - q'_{[x]+n-1})(1{,}000 - {}_nV_{[x]}) \quad \text{(Gain from mortality)} \\
& + (wq_{[x]+n-1} - wq'_{[x]+n-1})({}_nCV_{[x]} - {}_nV_{[x]}) \quad \text{(Gain from withdrawal)} \\
& + [E^A_n - E^{A\prime}_n + (E^{\%}_n - E^{\%\prime}_n)G_{[x]}](1 + i'_n) \quad \text{(Gain from expense)}\,.
\end{aligned} \tag{4a}
$$

[2] We say "*a* formula," since there is judgment involved in the arrangement of the terms. Other actuaries may prefer a different arrangement.

The reader should note that the "gain" portrayed in formula (4) is not synonymous with "earnings," since the latter, as generally used, includes interest earned on accumulated surplus. This difference is important because, although any two reserve bases must result in the same total "earnings" over the lifetime of a policy, generally they will not result in the same total "gains." Similarly, the present value of "gains" generated over the lifetime of a contract (discounted at experience rates of interest) is independent of the reserve basis, but the present value of reported "earnings" is not.

A similar formula, without the loading element, applies after the premium-paying period.[3]

2. BASIS OF A BASIC THEOREM

Introduce $P'_{[x]}$, the natural premium based on the alternate assumptions, and ${}_n\Delta_{[x]}$, which might be called "the experience gain per starter," expressed at the beginning of the year and defined as follows during the premium period:[4]

$$ {}_n\Delta_{[x]} = {}_n\mathrm{Gain}_{[x]}/(1+i'_n) - (G_{[x]} - P_{[x]}) \ . \qquad (5) $$

Both sides of equation (6) below represent the accumulated value of the profits generated by the contract in question, assuming an initial

[3] The analogues of formulas (1)–(4), after the premium-paying period, are given by

$$ \begin{aligned} {}_nV_{[x]} = [&({}_{n-1}V_{[x]} - E_n^A)(1+i_n) \\ &- q_{[x]+n-1}\ {}_nDB_{[x]}(1+i_n)^{1/2} - wq_{[x]+n-1}\ {}_nCV_{[x]}]/ \\ &\qquad (1 - q_{[x]+n-1} - wq_{[x]+n-1}) \ ; \end{aligned} \qquad (1a) $$

$$ \begin{aligned} {}_nV_{[x]} = &({}_{n-1}V_{[x]} - E_n^A)(1+i_n) \\ &- q_{[x]+n-1}[{}_nDB_{[x]}(1+i_n)^{1/2} - {}_nV_{[x]}] \\ &\qquad - wq_{[x]+n-1}({}_nCV_{[x]} - {}_nV_{[x]}) \ ; \end{aligned} \qquad (2a) $$

$$ \begin{aligned} {}_n\mathrm{Gain}_{[x]} = &({}_{n-1}V_{[x]} - E_n^{A\prime})(1+i'_n) \\ &- q'_{[x]+n-1}[{}_nDB'_{[x]}(1+i'_n)^{1/2} - {}_nV_{[x]}] \\ &- wq'_{[x]+n-1}\ {}_nCV'_{[x]} \\ &- (1 - q'_{[x]+n-1} - wq'_{[x]+n-1})\ {}_nV_{[x]} \\ = &({}_{n-1}V_{[x]} - E_n^{A\prime})(1+i'_n) \\ &- q'_{[x]+n-1}[{}_nDB'_{[x]}(1+i'_n)^{1/2} - {}_nV_{[x]}] \\ &\qquad - wq'_{[x]+n-1}({}_nCV'_{[x]} - {}_nV_{[x]}) - {}_nV_{[x]} \ ; \end{aligned} \qquad (3a) $$

$$ \begin{aligned} {}_n\mathrm{Gain}_{[x]} = &(i'_n - i_n)({}_{n-1}V_{[x]} - E_n^A) \qquad \text{(Gain from interest)} \\ &+ \{q_{[x]+n-1}[(1+i_n)^{1/2}\ {}_nDB_{[x]} - {}_nV_{[x]}] \\ &\quad - q'_{[x]+n-1}[(1+i'_n)^{1/2}\ {}_nDB'_{[x]} - {}_nV_{[x]}]\} \qquad \text{(Gain from mortality)} \\ &+ [wq_{[x]+n-1}({}_nCV_{[x]} - {}_nV_{[x]}) \\ &\quad - wq'_{[x]+n-1}({}_nCV'_{[x]} - {}_nV_{[x]})] \qquad \text{(Gain from withdrawals)} \\ &\qquad + (E_n^A - E_n^{A\prime})(1+i'_n) \qquad \text{(Gain from expense)} \ . \end{aligned} \qquad (4b) $$

[4] After the premium-paying period, the definition becomes

$$ {}_n\Delta_{[x]} = {}_n\mathrm{Gain}_{[x]}/(1+i'_n) \ . \qquad (5a) $$

number $l_{[x]}$ and assuming that experience follows the alternate assumptions.

$$\frac{(G_{[x]} - P'_{[x]})(N'_{[x]} - N'_{[x]+a})}{\prod_{k=1}^{b}(1+i'_k)^{-1}} = \frac{\sum_{n=1}^{b}[{}_n\mathrm{Gain}_{[x]}D'_{[x]+n-1}/(1+i'_n)]}{\prod_{k=1}^{b}(1+i'_k)^{-1}} . \tag{6}$$

Rearrangement of equation (6) produces

$$(P_{[x]} - P'_{[x]})(N'_{[x]} - N'_{[x]+a}) = \sum_{n=1}^{b} {}_n\Delta_{[x]} D'_{[x]+n-1} . \tag{7}$$

BASIC THEOREM. *Let* $P^{\Delta}_{[x]} = P'_{[x]} - P_{[x]}$, *and let* ${}_nV^{\Delta}_{[x]} = {}_nV'_{[x]} - {}_nV_{[x]}$. *Then*

$$P^{\Delta}_{[x]} = -\left(\sum_{n=1}^{b} {}_n\Delta_{[x]} D'_{[x]+n-1}\right) \Big/ (N'_{[x]} - N'_{[x]+a}) . \tag{8a}$$

For $n < a$,

$$\begin{aligned} {}_nV^{\Delta}_x &= \sum_{k=1}^{n} (P^{\Delta}_{[x]} + {}_k\Delta_{[x]}) \frac{D'_{[x]+k-1}}{D'_{[x]+n}} \qquad \text{(Retrospective form)} \\ &= \frac{1}{D'_{[x]+n}}\left[-\left(\sum_{k=n+1}^{b} {}_k\Delta_{[x]} D'_{[x]+k-1}\right) - P^{\Delta}_{[x]}(N'_{[x]+n} - N'_{[x]+a})\right] \\ &\qquad \text{(Prospective form)} . \end{aligned} \tag{8b}$$

For $n \geq a$,

$$\begin{aligned} {}_nV^{\Delta}_{[x]} &= \frac{1}{D'_{[x]+n}}\left[P^{\Delta}_{[x]}(N'_{[x]} - N'_{[x]+a}) + \sum_{k=1}^{n} {}_k\Delta_{[x]} D'_{[x]+k-1}\right] \\ &\qquad \text{(Retrospective form)} \\ &= -\sum_{k=n+1}^{b} {}_k\Delta_{[x]} \frac{D'_{[x]+k-1}}{D'_{[x]+n}} \qquad \text{(Prospective form)} . \end{aligned} \tag{8c}$$

Proof: Equation (8a) follows directly from formula (7). Suppose $n < a$. Then at the end of policy year n the accumulated assets per survivor may be expressed as either the left- or right-hand side of formula (9). Each side expresses the assets as reserve plus accumulated surplus. The left-hand side uses reserves based on the reference assumptions, while the right-hand side uses reserves based on the alternate assumptions.

$$\begin{aligned} &{}_nV_{[x]} + \sum_{k=1}^{n} [{}_k\mathrm{Gain}_{[x]}/(1+i'_k)] D'_{[x]+k}/D'_{[x]+n} \\ &\qquad = {}_nV'_{[x]} + (G_{[x]} - P'_{[x]})(N'_{[x]} - N'_{[x]+n})/D'_{[x]+n} . \end{aligned} \tag{9}$$

Rearranging terms,

$$
\begin{aligned}
{}_nV_x^{\Delta} &= {}_nV'_{[x]} - {}_nV_{[x]} \\
&= \frac{1}{D'_{[x]+n}} \sum_{k=1}^{n} \left[\frac{{}_k\text{Gain}_{[x]}}{1+i'_k} - (G_{[x]} - P_{[x]}) + (P'_{[x]} - P_{[x]}) \right] D'_{[x]+k} \\
&= \sum_{k=1}^{n} ({}_k\Delta_{[x]} + P^{\Delta}_{[x]}) \frac{D'_{[x]+k}}{D'_{[x]+n}} ,
\end{aligned}
\tag{10}
$$

which produces the retrospective form.

Rearranging equation (8a), we have

$$
\begin{aligned}
&\sum_{k=1}^{n} ({}_k\Delta_{[x]} + P^{\Delta}_{[x]}) \frac{D'_{[x]+k}}{D'_{[x]+n}} \\
&= \frac{1}{D'_{[x]+n}} \left[-\left(\sum_{k=n+1}^{b} {}_k\Delta_{[x]} D'_{[x]+k-1} \right) - P^{\Delta}_{[x]} (N'_{[x]+n} - N'_{[x]+a}) \right] ,
\end{aligned}
\tag{11}
$$

which establishes the equivalence of the prospective and retrospective forms. The case for $n \geq a$ is similar. Q.E.D.

The basic theorem can be used to prove a form of Lidstone's theorem that is generalized in the sense that it applies to both single and double decrement policies and to both limited and full payment policies.

COROLLARY (LIDSTONE'S THEOREM). *Let C_n (the "critical function") be defined by*

$$
\begin{aligned}
C_n &= {}_n\Delta_{[x]} + P^{\Delta}_{[x]} && \text{during premium period} \\
&= {}_n\Delta_{[x]} && \text{after premium period .}
\end{aligned}
$$

(i) *If C_n is nondecreasing in n, then, for all k,*

$$ {}_kV'_{[x]} \leq {}_kV_{[x]} . $$

(ii) *If C_n is nonincreasing in n, then, for all k,*

$$ {}_kV'_{[x]} \geq {}_kV_{[x]} . $$

Proof: The basic theorem may be written as follows in terms of C_n:

$$ \sum_{k=1}^{b} C_n D'_{[x]+k-1} / D'_{[x]} = 0 ; \tag{12a} $$

$$ {}_nV'_{[x]} - {}_nV_{[x]} = \sum_{k=1}^{n} C_k D'_{[x]+k-1} / D'_{[x]+n} \tag{12b} $$

$$_nV'_{[x]} - {_nV_{[x]}} = -\sum_{k=n+1}^{b} C_k D'_{[x]+k-1}/D'_{[x]+n} \,. \tag{12c}$$

Formula (12a) restates (8a); formula (12b) restates the retrospective form of (8b) and (8c); and formula (12c) restates the prospective form of (8b) and (8c).

Suppose C_n is a nondecreasing function of n. If C_n is constant, then, by (12a), $C_n = 0$. In that case, for all n, $_nV'_{[x]} - {_nV_{[x]}} = 0$, and the theorem is proved.

If C_n is not constant, then, by (12a), it has both negative and positive terms. Since C_n is nondecreasing, and has both negative and positive terms, $C_1 < 0$ and $C_b > 0$. Also, since C_n is nondecreasing, there is some integer m with the property that for $n < m$, $C_n < 0$ and for $n \geq m$, $C_n \geq 0$. It follows that, for $0 < n < m$, the right-hand side of (12b) is negative. Similarly, for $m \leq n < b$, the right-hand side of (12c) is negative. So, for $0 < n < b$,

$$_nV'_{[x]} - {_nV_{[x]}} < 0 \,.$$

This establishes the first part of the theorem. Proof of the second part is similar.

B. *Practical Applications*

1. SIGN

In practical situations, Lidstone's theorem alone is of limited value. The special conditions it requires are often not present, and, even where they are, the theorem gives only the sign—not the magnitude—of the change in reserves resulting from a change in assumptions. Despite these limitations, or perhaps even because of them, Lidstone's theorem is a useful starting point for understanding the effects of reserve assumptions.

Consider two examples where the special conditions required by Lidstone's theorem are present.

Example 1

Reserves have been calculated for a full-pay, straight-line decreasing term plan using the 1955–60 Basic Select and Ultimate Mortality Tables. Wishing to test the effect of using $(1 + k)$ times the Basic Tables, we find that the critical function is

$$\begin{aligned} C_n &= P^{\Delta}_{[x]} + {_n\Delta_{[x]}} \\ &= P^{\Delta}_{[x]} + (q_{[x]+n-1} - q'_{[x]+n-1})[(1+i_n)^{-1/2}\, {_nDB_{[x]}} - {_nV_{[x]}}/(1+i_n)] \\ &= P^{\Delta}_{[x]} + kq_{[x]+n-1}[(1+i_n)^{-1/2}\, {_nDB_{[x]}} - {_nV_{[x]}}/(1+i_n)] \,. \end{aligned} \tag{13}$$

Two things about formula (13) are noteworthy. First, since the policy is full pay, the critical function varies from year to year only as ${}_n\Delta_{[x]}$ varies. Second, if the expression

$$q_{[x]+n-1}[(1+i_n)^{-1/2}\ {}_nDB_{[x]} - {}_nV_{[x]}/(1+i_n)]$$

is a decreasing function of n, any positive value of k results in a decreasing critical function and, consequently, a reduction in reserves.

Example 2

Reserves have been calculated on a mass-marketed whole life plan which has demonstrated relatively high acquisition costs and poor persistency. The assumed interest rates are as follows: for years 1–10, $6\frac{1}{2}$ per cent; years 11–20, $5\frac{1}{2}$ per cent; years 21 and over, $4\frac{1}{2}$ per cent. As a result of high costs and poor persistency, the valuation premiums on the basis tested exceed the gross premiums at the important ages. It has been suggested that in light of the high yields currently available, the assumed interest for the first ten years could be increased to 8 per cent. You are asked the question, "What will this do to the reserves?"

You observe that, at the important ages, the natural reserves are negative in year 1, increasing gradually to zero around year 10. The critical function, therefore, is (omitting the interest on claims)

$$\begin{aligned} C_n &= \frac{0.08-0.065}{1.08}({}_{n-1}V_{[x]} + P_{[x]} - E_n^A - E_n^{\%}G_{[x]}) + P_{[x]}^{\Delta} \\ &\qquad\qquad \text{for} \quad n \le 10 \qquad (14) \\ &= P_{[x]}^{\Delta} \qquad \text{for} \quad n > 10\,, \end{aligned}$$

which is nondecreasing. The net effect will be to reduce reserves.

Comments on Examples 1 and 2

Examples 1 and 2 contain many important points. In Example 1 the assumption of additional mortality causes a reduction in reserves. The same effect might arise at some ages on endowment plans if the amount at risk (death benefit less natural reserve) declines proportionately faster than mortality rates rise.

In Example 2 an important question was not asked because it is not covered by Lidstone's theorem. The question is, "Will the proposed change of assumptions remedy the premium deficiency?" Recalling that, under the conditions described, ${}_n\Delta_{[x]}$ is negative for the first ten years and zero thereafter, formula (8a) gives the answer: "No; it will worsen the deficiency."

All the old rules change under natural reserves. An interest assumption which grades down to an interest rate of i can result in higher net premiums, and lower reserves, than an assumption of a level rate i. This is the chief lesson of Example 2.

Consider next an example involving lapse rates.

Example 3

Company A sells a large volume of limited payment life and endowment plans. "What effect," the actuary wonders, "will the assumption of some lapses after the premium-paying period have on the reserves?" Reserves already have been calculated on the assumption of no lapses after the premium-paying period.

Suppose that a glance at a table of paid-up cash values and paid-up GAAP reserves reveals that at attained ages below 80, GAAP reserves are markedly less than the cash values. The difference then decreases rapidly toward zero at attained age 100.

In this case,

$$\begin{aligned} {}_n\Delta_{[x]} &= (wq_{[x]+n-1} - wq'_{[x]+n-1})({}_nCV_{[x]} - {}_nV_{[x]})/(1+i_n) \\ &= -wq'_{[x]+n-1}({}_nCV_{[x]} - {}_nV_{[x]})/(1+i_n) \qquad \text{for} \quad n > a\,. \end{aligned} \tag{15}$$

Application of the basic theorem previously stated leads to the following conclusions: First, the valuation premium will be increased (formula [8a]) by the higher lapse rates. Second, during the premium-paying period, the new reserve will exceed the old reserve by an amount equal to the accumulated excess valuation premium (formula [8b], restropective). Third, after the premium-paying period, the extra reserve will be equal to the present value of the remaining ${}_n\Delta_{[x]}$'s as expressed in formula (15) (formula [8c], prospective).

2. MAGNITUDE

Our review thus far has been focused on the sign (+ or −) of effects caused by changes in valuation assumptions. We have asked, "Does the change increase or decrease the valuation premium?" and "Does it increase or decrease the natural reserve?" This information is useful, but the magnitude of the increase or decrease also is important. Our study of magnitudes also will shed some light on the interplay among the various assumptions.

Let us focus on one very common notion regarding valuation assumptions, namely, that the interest rate used in the late policy years has the most important effect on the valuation premium and the reserves. To this end, suppose the alternate assumptions differ from the reference assumptions only in the assumed rates of interest. Then, letting $\Delta i_n = i'_n - i_n$, we have

$$\begin{aligned} {}_n\Delta_{[x]} &= (i'_n - i_n)({}_{n-1}V_{[x]} + P_{[x]} - E^A_n - E^{\%}_n G_{[x]})/(1+i'_n) \\ &\quad + q_{[x]+n-1}\ {}_nDB_{[x]}[(1+i_n)^{1/2} - (1+i'_n)^{1/2}]/(1+i'_n) \\ &\doteq \Delta i_n({}_{n-1}V_{[x]} + P_{[x]} - E^A_n - E^{\%}_n G_{[x]} \\ &\qquad - \tfrac{1}{2} q_{[x]+n-1}\ {}_nDB_{[x]})/(1+i'_n)\ . \end{aligned} \tag{16}$$

We may view the first parenthesized term of the approximate expression as being a kind of initial reserve adjusted for half the claim payout. It represents the average balance on which interest is being earned. Then ${}_n\Delta_{[x]}$ is the value, at the start of the policy year, of the excess interest to be earned if the alternate assumptions were realized, measured per unit beginning the year.

The numerator of the right-hand side of formula (8a) suggests that a given change in interest rates, Δi_n, has its greatest effect on the valuation premium in the year that ${}_n\Delta_{[x]}D'_{[x]+n-1}$ is maximized. So the importance of the interest rate for a given year appears to depend upon the size of the total natural reserve fund to which it applies and upon the time elapsed since issue.

This result confirms intuition and suggests a relationship between assumed persistency and assumed interest. The interest rate assumed for later policy years diminishes in its effect on the valuation premium as the assumed rates of withdrawal increase. Similarly, an analysis of formulas (8b) and (8c) shows that the interest rate assumed for later policy years diminishes in its effect on the earlier years' reserves as the early-year rates of withdrawal increase.

Pursuing the sort of analysis demonstrated above confirms that similar properties hold for mortality, withdrawals, and expense.

1. Variations in the individual mortality rates tend to have their maximum effects on net premiums and reserves in those years when the amount at risk discounted to issue with interest and survivorship is maximized, that is, when

$$\frac{{}_nDB_{[x]}(1+i_n)^{1/2} - {}_nV_{[x]}}{1+i_n}\ \frac{D_{[x]+n-1}}{D_{[x]}}$$

is maximized. When the above term is zero, reserves and net premiums are independent of the corresponding mortality rate.

2. Variations in individual withdrawal rates tend to have their maximum effects in those years when

$$\frac{{}_nCV_{[x]} - {}_nV_{[x]}}{1+i_n}\ \frac{D_{[x]+n-1}}{D_{[x]}}$$

is maximized. When this term is zero, variations in the assumed withdrawal rate have no net effect.[5]

3. Variations in the assumed expenses have their maximum effect on reserves and premiums in the first year. Changes in assumed expenses for later years diminish in importance in proportion to $D_{[x]+n-1}$.

C. *Using the Theorem*

The theorem that has been presented here is not offered as a practical approach to calculating the difference between reserves on different bases; rather, it is offered as a solid theoretical foundation against which to test GAAP intuition and as a starting point for making educated guesses.

In this writer's opinion, provisions for adverse deviation must be chosen with an eye to their earnings effects. Intelligent choices are more likely to be made when the actuary in charge has a deep understanding of the workings of the explicit reserve assumptions.

III. THE IMPLICIT ANNUAL PREMIUM ASSUMPTION

Most published GAAP reserve formulas employ the time-honored assumption that premiums are collected and lapses occur on policy anniversaries. For policies paying premiums more frequently than annually, our standard model continues to apply simply by focusing on a period shorter than one year. For example, we could assume monthly interest, mortality, lapse, and expense rates and continue to use the same model. In practice, of course, this is not done, for several reasons: (1) the inaccuracies in assumptions do not justify such refinement; (2) the costs involved are too great; (3) the results are well approximated by the annual premium formulas.

It may be desirable to explore just how much error is involved. For any particular plan, the degree of error can be tested by direct calculations. Rather than focus on such an inductive approach, let us take the deductive approach here.

A. *Terminal Reserve Formulas*

The key to testing the effect of a change in the assumed timing of withdrawals is to view it as a modification of the reference assumptions—not as a new model. With an annual premium assumption, premium col-

[5] In each of these cases we are referring to the natural reserve. It is possible for the benefit reserve and acquisition cost asset to vary without their difference varying. Indeed, when ${}_nCV_{[x]} = {}_nV_{[x]}$ and the acquisition cost reserve factor is not zero, a change in the nth withdrawal rate will have this result.

lections and related expense payouts occur at the beginnings of years, while withdrawals occur at the ends.

To build a formula for nonannual premiums, we need to reflect premiums lost and expense savings on account of deferred premiums outstanding at time of death and the effects of early withdrawals on (1) premiums and expenses, (2) exposure to mortality, (3) interest earnings, and (4) cash values paid. It is known that the effect of the nondeduction of deferred fractional premiums at death can be estimated by treating the average outstanding deferred premiums, less related expenses, as an adjustment to the death benefit. We shall investigate whether the remaining items might be treated as an adjustment to the cash value.

To avoid some of the notational complexities which might otherwise arise, let I_n represent the assumed interest earned in policy year n under the annual premium assumption,[6] and let $P'_{[x]}$ be the annualized net premium under the new assumptions. Expressing it differently, let

$$P'_{[x]} = P^{(m)}_{[x]}\ddot{a}^{(m)}_{\overline{1}|} ; \tag{17}$$

$$I_n = i_n({}_{n-1}V_{[x]} + P'_{[x]} - E^A_n - E^{\%}_n G_{[x]} - \tfrac{1}{2}q_{[x]+n-1}\ {}_nDB'_{[x]}) ; \tag{18}$$

$${}_nDB'_{[x]} = {}_nDB_{[x]} + \frac{m-1}{2m} P'_{[x]} . \tag{19}$$

Finally, define an adjusted cash value as follows:

$$\begin{aligned} {}_n\overline{CV}_{[x]} = \Bigg[\!\!\Bigg[\sum_{k=0}^{m-1} {}_{k/m|1/m}wq_{[x]+n-1} \Big\{ {}_{n-1+(k+1)/m}CV_{[x]} + \left(1 - \frac{k+1}{m}\right) \\ \times [P'_{[x]} + I_n - E^{\%}_n G_{[x]} + q_{[x]+n-1}({}_nDB_{[x]} - {}_nV'_{[x]})] \Big\} \Bigg]\!\!\Bigg] \Big/ \\ wq_{[x]+n-1} . \end{aligned} \tag{20}$$

Then, approximately,

$$\begin{aligned} {}_nV'_{[x]} &= ({}_{n-1}V'_{[x]} + P'_{[x]} + I_n - q_{[x]+n-1}\ {}_nDB'_{[x]} \\ &\quad - wq_{[x]+n-1}\ {}_n\overline{CV}_{[x]})/(1 - q_{[x]+n-1} - wq_{[x]+n-1}) \\ &= {}_{n-1}V'_{[x]} + P'_{[x]} + I_n - q_{[x]+n-1}({}_nDB'_{[x]} - {}_nV'_{[x]}) \\ &\quad - wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV'_{[x]}) . \end{aligned} \tag{21}$$

Formula (21) forces the nonannual premium case into the annual premium mold. It demonstrates that, if ${}_nDB'_{[x]}$ and ${}_n\overline{CV}_{[x]}$ be chosen appro-

[6] The author asks the reader to forgive the flawed allocation of interest between time periods which mars this presentation. Unfortunately, the use of more exact formulas would bury the terms we seek to highlight in a mass of minor adjustments.

priately, very good results may be obtained from annual premium formulas.

In practice, if one knows the mix of business among premium modes, one can estimate ${}_nDB'_{[x]}$ with comparative ease. First estimate r_x, the ratio of net to gross for the plan and issue age. Then compute $\bar{m}$, the weighted average frequency of premium payments anticipated for the plan and issue age. Then

$$\begin{aligned} {}_nDB'_{[x]} &= r_x[(\bar{m}-1)/2\bar{m}]G_{[x]} + {}_nDB_{[x]} \qquad (n \le a) \\ &= {}_nDB_{[x]} \qquad (n > a)\,. \end{aligned} \tag{22}$$

Calculating ${}_n\overline{CV}_{[x]}$ may be more difficult. Assume ${}_{n-1+k/m}CV_{[x]}$ to be linear in k, and let

$$c_n = \sum_{k=0}^{m-1}\left(\frac{1+k}{m}\right)\frac{{}_{k/m|1/m}wq_{[x]+n-1}}{wq_{[x]+n-1}}\,. \tag{23}$$

Then formula (20) becomes[7]

$$\begin{aligned} {}_n\overline{CV}_{[x]} &= c_n\ {}_nCV_{[x]} + (1-c_n)[{}_{n-1}CV_{[x]} \\ &\quad + P'_{[x]} + I_n - E_n^{\%}G_{[x]} + q_{[x]+n-1}({}_nDB'_{[x]} - {}_nV'_{[x]})] \\ &= c_n\ {}_nCV_{[x]} + (1-c_n)\{{}_{n-1}CV_{[x]} \\ &\quad + [{}_nV'_{[x]} + wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV'_{[x]})] - ({}_{n-1}V'_{[x]} - E_n^A)\} \\ &= {}_nCV_{[x]} + [c_n\ {}_nCV_{[x]} + (1-c_n)\ {}_{n-1}CV_{[x]}] \\ &\quad - \{c_n[{}_nV'_{[x]} + wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV'_{[x]})] \\ &\qquad + (1-c_n)({}_{n-1}V_{[x]} - E_n^A)\} \\ &\quad - \{{}_nCV_{[x]} - [{}_nV'_{[x]} + wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV_{[x]})]\}\,. \end{aligned} \tag{24}$$

The various forms of formula (24) have interesting general reasoning explanations, but the final form is the most significant. Its various terms may be interpreted as follows:

Term	Interpretation
(a) $c_n\ {}_nCV_{[x]} + (1-c_n)\ {}_{n-1}CV_{[x]}$	The interpolated cash value at the average time of lapse

[7] The second equivalence is established by use of formula (2).

Term	Interpretation
(b) $c_n[{}_nV'_{[x]} + wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV'_{[x]})] + (1 - c_n)({}_nV'_{[x]} - E^A_n)$	The interpolated midterminal reserve at the average time of lapse
(c) (a) − (b)	The interpolated average "amount at risk" for midyear lapses
(d) ${}_nCV_{[x]} - [{}_nV'_{[x]} + wq_{[x]+n-1}({}_n\overline{CV}_{[x]} - {}_nV'_{[x]})]$	The "amount at risk" for end-of-the-year lapses
(e) (c) − (d)	The increase in the average "amount at risk" resulting from the assumption of midyear lapses

Thus the adjusted cash value exceeds the actual cash value by an amount equal to the excess of (*a*) the average "amount at risk" on assumed withdrawals during the policy year over (*b*) the corresponding "amount at risk" just prior to the end of the policy year.

It follows that when the increase in the cash value during the year parallels that of the natural reserve (so that the "amount at risk" throughout the year is essentially constant), the adjusted cash value is equal to the actual year-end policy cash value. This is important because, for practical purposes, the foregoing condition is often satisfied for years when the cash value is nonzero.

Substituting an adjusted cash value for the actual cash value in the first few policy years of an annual premium formula with no other changes may often give good practical results for nonannual premium plans.

B. *Interim Reserve Formulas*

Once the appropriate terminal reserves are calculated, the problem of determining appropriate interim reserves arises. Many possible formulas exist,[8] all incorporating some adjustment to recognize that, because of

[8] Two such formulas are presented in *TSA*, Vol. XXV, by Claude Y. Paquin in "The Development of Mean Natural Reserve Factors and Methods of Amortizing Acquisition Expenses in Adjusting Life Insurance Company Earnings" (p. 459) and Melvin L. Gold and Paul L. Weichert in "GAAP in Practice" (p. 599).

the timing of lapses, reserve increases should not progress uniformly through the policy year.

One possible formula which includes a parameter ($\alpha_{[x]+n}$) to adjust for the timing of lapses is given by

$$
\begin{aligned}
{}_nM_{[x]} &= \tfrac{1}{2}({}_{n-1}V_{[x]} + P_{[x]} + {}_nV_{[x]}) \\
&\quad + (\tfrac{1}{2} - \alpha_{[x]+n})({}_n\overline{CV}_{[x]} - {}_nV_{[x]}) \,.
\end{aligned}
\tag{25}
$$

The term $\alpha_{[x]+n}$ represents the fraction of the tabular cost of withdrawal which, on the average, is incurred during the first part of policy year n. For example, in the annual premium case $\alpha_{[x]+n} = 0$. The resulting version of formula (25) resembles closely the other annual premium interim reserve formulas in general use. In the case where the cost of lapse is spread uniformly through the year, $\alpha_{[x]+n} = \frac{1}{2}$. The resulting version of formula (25) is the traditional mean reserve formula, a result which intuition confirms as correct.

The determination of $\alpha_{[x]+n}$ may be carried out in extensive calculations or using simple algebraic formulas. As an example of the latter, when it is reasonable to assume that in some policy years the cost of lapse falls equally on each premium due date, then the $\alpha_{[x]+n}$ factor arising from mode (m) in those years is $(m - 1)/2m$. For the entire block of business (all modes combined) these factors may be weighted by the prevalence of the mode to obtain one over-all factor.

IV. CONCLUSION

The complex interactions of GAAP reserve assumptions are not mysterious or inscrutable. We have seen that the practicing actuary can learn to understand and predict effectively the effects of his assumptions.

The author believes that the same fundamental principles that govern the effects of actuarial assumptions on premiums and reserves—principles which have only been touched upon here—can be applied effectively to such varied problems as gross premium determinations and pension valuations and in interpreting more accurately pension gain and loss analyses. These tasks, however, must remain for others.

V. ACKNOWLEDGMENTS

The author gratefully acknowledges the contributions of John Stedman, F.S.A., whose probing questions, steady encouragement, and careful readings made this paper possible.

DISCUSSION OF PRECEDING PAPER

GOTTFRIED O. BERGER:*

To my knowledge, this is the first rigorous treatment of the two closely related problems of GAAP reserve variations due to alternate assumptions and the effect on earnings if actual experience departs from the reserve assumptions. Paul Milgrom has tackled the rather intricate subject in a most refreshing manner. His paper is very readable and should trigger a lively discussion. The following remarks refer to only a very few of the many questions which are raised or answered in the paper.

Any author, in dealing with a subject of such complexity, is faced with the dilemma that (in slight variation of Milgrom's footnote 6), the more "exact" a formula, the greater the danger that essentials get buried in a mass of details. This dilemma is compounded by the inflexibility of our "generally accepted actuarial notation." Perhaps we should tolerate notations which are especially designed to deal with particular problems, subject to easy communication and convenient translation into computer instructions.

In essence, a reserve accumulation is controlled by a sequence of events which depend on survival or departure. Thus we could introduce for the survival function the notation

$$\nabla l_k = l_k - l_{k-1} \quad (l_0 = 1) .$$

Here k indicates not policy years but decrements. The reserve accumulation ${}_kV$ may now be defined by the recursive formula

$${}_kV = {}_{k-1}V(1 + i_k) + E_k l_k + D_k \nabla l_k ,$$

where

i_k = "Interest" from step $k - 1$ to step k;

E_k = Balance of endowment transactions at step k, including premiums (positive sign) and expenses (negative sign);

D_k = Departure payments at step k (positive sign), including death benefits and cash surrender values.

This notation may serve to illustrate two questions which are raised in the paper, namely, the impact of different premium payment modes and

* Dr. Berger, not a member of the Society, is president of Cologne Life Reinsurance Company.

mean reserves versus terminal reserves. To this end, we introduce the following model for the first policy year:

Symbol	Refers to Units That
l_0	Enter the first policy year
∇l_1	Lapse in the first half of the policy year
l_1	Pay a full annual premium
∇l_2	Die in the first half of the policy year
l_2	Carry mean reserves
∇l_3	Die in the second half of the policy year
l_3	Are eligible for dividends (if any)
∇l_4	Lapse in the second half of the policy year
l_4	Enter the second policy year (and carry terminal reserves)

This sequence is repeated for each policy year. Obviously the model is nothing new; it merely expresses customary computer programs in algebraic terms. The reader will recognize that the suggested notation is very flexible. For instance, we could say that the model chosen in Paul Milgrom's paper represents the special case $\nabla l_1 - \nabla l_2 = 0$, while Claude Paquin (*TSA*, Vol. XXV) has selected the special case $\nabla l_1 = 0$ to demonstrate his theory.

There is no doubt that GAAP assumptions include the *rate* of departure, that is, the total of expected deaths and withdrawals within a policy year. The question is whether GAAP assumptions should include in addition the *timing* of departures or (what comes to the same thing) an allocation ∇l_k as suggested in the model presented above.

Any assumption that we make with respect to the timing of departures affects

1. The expected amount of collected premiums. In this model, l_1 represents the number of full annual premiums collected from l_0 units which enter the policy year. Clearly, l_1 depends upon the premium mode. Thus we could turn the procedure around and estimate the relation l_1/l_0 as the primary variable. This determines the allocation of total lapses ($\nabla l_1 + \nabla l_4$) into "early" lapses ∇l_1 and "late" lapses ∇l_4.
2. The mean reserve formula. If we make assumptions as to the timing of departures, then mean reserves and terminal reserves are related by accumulation formulas, as Paquin has demonstrated.

My final remark refers to the definitions of "profits," "gains," and "earnings." For the purpose of his paper, Paul Milgrom makes the following definition in his footnote 2:

> The reader should note that the "gain" portrayed in formula (4) is not synonymous with "earnings," since the latter, as generally used, includes interest earned on accumulated surplus. This difference is important because,

although any two reserve bases must result in the same total "earnings" over the lifetime of a policy, generally they will not result in the same total "gains." Similarly, the present value of "gains" generated over the lifetime of a contract (discounted at experience rates of interest) is independent of the reserve basis, but the present value of reported "earnings" is not.

Unfortunately, the terminology used in the papers on GAAP is not uniform. The authors shown at the left in the list below use the indicated terms (each of the two columns at the right represents the same entity):

Pharr (1971):	Earnings	Earnings plus interest
Paquin (1973):	Earnings	
Sondergeld (1974):	Profit	Earnings
Milgrom (1975):	Gains	Earnings

Personally, I prefer the terminology of Sondergeld, since "gain" is too reminiscent of "gains from operations," which is synonymous to "earnings" as defined by Milgrom.

In Paul Milgrom's paper, "gains" are expressed per unit of insurance beginning the year. Thus, if both sides of the first equation (3) are multiplied by l'_{n-1}, we obtain

$$l'_{n-1}\ {}_n\text{Gain} = l'_{n-1}\ {}_{n-1}V(1+i'_n) + l'_{n-1}H'_n - l'_n\ {}_nV\ . \qquad (3)$$

Here the index $[x]$ is omitted, and H'_n combines all terms which do not depend on reserves ${}_nV$.

Let $v'(n)$ denote the discount factors derived from interest rates i'_n. Then the present values of both sides of equation (3) are

$$\sum_{n=1}^{\omega} l'_{n-1}\ {}_n\text{Gain}\ v'(n) = \sum_{n=1}^{\omega} l'_{n-1}H'_n v'(n) + [l'_0\ {}_0V - l'_\omega\ {}_\omega V v'(\omega)]\ .$$

The right-hand side depends on the limiting reserves ${}_0V$ and ${}_\omega V$. However, we may impose the conditions that generally ${}_0V = 0$ and $l'_\omega = 0$.

This confirms Milgrom's statement that the present values (discounted at experience rate of interest) of "emerging gains" $l'_{n-1}\ {}_n\text{Gain}$ are independent of the reserves ${}_nV$.

We might add that the total of "earnings" $l'_{n-1}\ {}_n\text{Earnings}$ is also independent of the reserves ${}_nV$. Indeed, let

$$l'_{n-1}\ {}_n\text{Earnings} = l'_{n-1}(H'_n + {}_{n-1}Ai'_n) - (l'_n\ {}_nV - l'_{n-1}\ {}_{n-1}V)\ .$$

The right-hand side of this equation displays the difference between the increase of assets and the increase of reserves. Thereby, "asset shares" ${}_nA$ are defined by the recursive formula

$$l'_n\ {}_nA = l'_{n-1}\ {}_{n-1}A(1+i'_n) + H'_n\ .$$

By summation, we find, as expected,

$$\sum_{n=1}^{\omega} l'_{n-1}\ {}_{n}\text{Earnings} = \sum_{n=1}^{\omega} l'_{n-1}(H'_n + {}_{n-1}Ai'_n) + (l'_0\ {}_0V - l'_\omega\ {}_\omega V)\ .$$

(AUTHOR'S REVIEW OF DISCUSSION)

PAUL MILGROM:

Dr. Berger's interesting and useful discussion contains two major points. In his second and third paragraphs, he illustrates a general multiple decrement approach to reserve calculations. This approach offers the advantage of allowing the computation of mean reserves directly from the sorts of simple timing assumptions which actuaries have traditionally used. I believe that reserve formulas developed on the basis of this approach will be more understandable and will involve fewer computations than the formulas developed in my paper.

The second major point in Dr. Berger's discussion deals with the differences in terminology used by various authors and the relationships between what I have called "gains" and "earnings." Dr. Berger's discussion complements my treatment of these relationships by proving two of my assertions and by adding and proving a third assertion. I predict that the regrettable problem of inconsistent terminology will be resolved as younger actuaries adopt whichever terms are used in the study notes for the actuarial examinations. I hope that the author of the relevant study note will, as Dr. Berger has done, select his terms to avoid ambiguities.