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SOME PRACTICAL CONSIDERATIONS IN CONNECTION WITH THE CALCULATION OF STOP-LOSS PREMIUMS*

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ABSTRACT

For the evaluation of a compound Poisson distribution it is an advantage if the claim amount distribution is arithmetic, that is, if all claim amounts are multiples of a common span. Furthermore, the larger the span, the fewer the calculations. This idea is the keystone in the development of upper and lower bounds for stop-loss premiums. An upper bound is obtained when a stop-loss premium is calculated for a claim amount distribution constructed by dispersing the mass of the original claim amount distribution to uniformly spaced points. A lower bound is obtained when the "new" distribution is constructed by a truncation procedure. The advantages of this method are that it reduces round-off errors and saves computing time.

I. WHY THE EXPONENTIAL PRINCIPLE

A PREMIUM calculation principle is a rule that assigns a premium, say P, to any risk, say S. Mathematically, a risk is a random variable, given by its (supposedly sufficiently regular) distribution. The following four examples illustrate this concept.

a) The net premium principle:

$$P = E[S].$$

b) The exponential principle:

$$P = \frac{1}{a}\ln E[e^{aS}]\,, \qquad a > 0\,.$$

c) The variance principle:

$$P = E[S] + b\operatorname{Var}[S], \qquad b > 0.$$

d) The standard deviation principle:

$$P = E[S] + c\sqrt{(\operatorname{Var}[S])}, \qquad c > 0.$$

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In example a there is no loading; in examples c and d the loading is proportional to the variance and to the standard deviation, respectively. The exponential principle involves the evaluation of the moment generating function of S at the argument a.

The following two properties are highly desirable for a principle of premium calculation:

$$(P_1): \quad P \geq E[S],$$
$$(P_2): \quad P \leq \text{Max}\,[S],$$

for any risk S, where Max $[S]$ denotes the right-hand endpoint of the range of S. The first property means that the expected gain, $P - E[S]$, is nonnegative for any risk S. The second property guarantees that the premium for any risk is not unreasonably high: if $P >$ Max $[S]$, the premium exceeds the maximal possible benefit, and nobody in the world would buy such a policy.

Obviously, the principles a, c, and d above satisfy property (P_1). But so does the exponential principle: Jensen's inequality tells us that

$$E[e^{aS}] \geq e^{aE[S]}. \tag{1}$$

Now we take logarithms, divide by a, and recognize that (P_1) is satisfied for the exponential principle.

Principles a and b satisfy (P_2). The latter assertion follows from the inequality

$$E[e^{aS}] \leq e^{a\,\text{Max}\,[S]}. \tag{2}$$

Unfortunately, neither the variance principle nor the standard deviation principle satisfies property (P_2), as is seen from the following examples.

Example 1: Let $S = 0$ or Z, each with probability $\frac{1}{2}$ $(Z > 0)$. Thus Max $[S] = Z$. If the variance principle is applied, we find that

$$P = Z/2 + bZ^2/4. \tag{3}$$

But this means that $P >$ Max $[S]$, whenever $Z > 2/b$.

Example 2: Let $S = 1$ (with probability p) or $S = 0$ (with probability $q = 1 - p$). Thus Max $[S] = 1$. If we apply the standard deviation principle, we obtain

$$P = p + c\sqrt{[p(1-p)]}. \tag{4}$$

By examining P as a function of p, we find that $P >$ Max $[S]$ whenever $(1 + c^2)^{-1} < p < 1$. Worse than that, if p is close to 1, P is a decreasing function of p!

The net premium principle and the exponential principle are *additive* as well as *iterative*; these concepts are explained in [1] (pp. 87 and 91). Under a mild continuity condition, these two principles can be characterized by these properties (see [3]). Also, the exponential principle fits into the framework of the collective theory of risk: The parameter a plays essentially the role of an adjustment coefficient.

For these reasons we shall adopt the exponential principle. Of course net premiums are always of interest; in fact, they may be obtained as a limiting case ($a \to 0$) from exponential premiums.

II. MEREU'S FORMULA, OR WHY THE TAIL IS NOT A PROBLEM

In the following we consider a stop-loss coverage (deductible α) for aggregate claims X. Let $F(x)$ denote the cumulative distribution function of X. We assume that $F(x) = 0$ for $x < 0$ (no negative claims). The "risk" in question is now

$$\begin{aligned} S = (X - \alpha)+ &= 0 \qquad &\text{if } X \leq \alpha \\ &= X - \alpha \qquad &\text{if } X > \alpha. \end{aligned} \tag{5}$$

We denote the stop-loss premiums (based on principles a and b above) by $P(F, \alpha)$, $P(F, \alpha, a)$, respectively. Thus

$$P(F, \alpha) = E[(X - \alpha)+] = \int_\alpha^\infty (x - \alpha) dF(x) \tag{6}$$

and

$$\begin{aligned} P(F, \alpha, a) &= \frac{1}{a} \ln E[e^{a(x-\alpha)+}] \\ &= \frac{1}{a} \ln \left[F(\alpha) + \int_\alpha^\infty e^{a(x-\alpha)} dF(x) \right]. \end{aligned} \tag{7}$$

These formulas entail integration over the tail of $F(x)$, which may cause computational problems. Fortunately, these difficulties can be avoided.

Mereu's idea [5] was to take expectations in the identity

$$(X - \alpha)+ = (X - \alpha) + (\alpha - X)+, \tag{8}$$

to obtain

$$\begin{aligned} P(F, \alpha) &= E[X] - \alpha + E[(\alpha - X)+] \\ &= E[X] - \alpha + \int_0^\alpha (\alpha - x) dF(x) . \end{aligned} \tag{9}$$

Thus, if $E[X]$ is obtainable otherwise, this formula requires the knowledge of $F(x)$ only for $0 < x < \alpha$.

In analogy to Mereu's idea, let us now consider the identity

$$e^{a(X-\alpha)+} = e^{a(X-\alpha)} + (1 - e^{a(X-\alpha)})I_{[X<\alpha]}. \tag{10}$$

Then

$$\begin{aligned} E[e^{a(X-\alpha)+}] &= E[e^{a(X-\alpha)}] + E[(1 - e^{a(X-\alpha)})I_{[X<\alpha]}] \\ &= e^{-\alpha a}E[e^{aX}] + \int_0^\alpha (1 - e^{a(x-\alpha)})dF(x) . \end{aligned} \tag{11}$$

Thus

$$P(F, \alpha, a) = \frac{1}{a} \ln \left\{ e^{-\alpha a}E[e^{aX}] + \int_0^\alpha (1 - e^{a(x-\alpha)})dF(x) \right\} . \tag{12}$$

Assuming that the moment generating function of X is available otherwise, this formula requires the knowledge of $F(x)$ for $0 < x < \alpha$ only, as before. Therefore formulas (9) and (12) are preferable to the original formulas (6) and (7).

Remark

By differentiating formula (11) k times, and setting $a = 0$, one obtains an expression for the kth absolute moment of $(X - \alpha)+$ in terms of the first k moments of X and the values of $F(x)$ for $0 < x < \alpha$. In the case $k = 1$, this brings us back to formula (9). For $k = 2$, it leads to Mereu's formula (21) (see [5]).

III. THE COMPOUND POISSON DISTRIBUTION

In the following we assume that the aggregate claims X have a compound Poisson distribution, say with Poisson parameter λ (= expected number of claims) and jump amount distribution $H(x)$ (= cdf of the individual claim amounts). Thus

$$F(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H^{*k}(x) . \tag{13}$$

It is assumed that $H(0) = 0$, which implies that $F(x) = 0$ for $x < 0$. In a certain sense, the assumption of a compound Poisson distribution is conservative (see the last paragraph in [2]).

If $\mu = \int xdH(x)$ denotes the average claim size and $\phi(t) = \int e^{tx}dH(x)$ is the moment generating function of the claim amounts,

$$E[X] = \lambda\mu; \quad E[e^{aX}] = e^{\lambda[\phi(a)-1]}. \tag{14}$$

These expressions should be substituted in formulas (9) and (12). It remains to calculate $F(x)$ for $0 < x < \alpha$.

The latter problem is greatly simplified if $H(x)$ is an arithmetic distribution, say with a span $d > 0$. This means that all possible claim amounts are multiples of the span d. Let h_i denote the probability that a given claim amount equals id ($i = 1, 2, \ldots$), and let $f_i = P[X = id]$. Since

$$h_i^{*k} = 0 \quad \text{for} \quad k > i \,, \tag{15}$$

we see from formula (13) that

$$f_i = e^{-\lambda} \sum_{k=0}^{i} \frac{\lambda^k}{k!} h_i^{*k} \,. \tag{16}$$

For the evaluation of formulas (9) and (12) we have to compute h_i^{*k} and f_i for $i = 0, 1, \ldots, [\alpha/d]$. Thus the larger the span d, or the smaller the deductible α, the fewer calculations have to be made.

Remark

It is possible to calculate f_i by an algorithm that is somewhat different from the approach that results from formula (16). Let N_j denote the number of claims of amount jd. Obviously,

$$X = dN_1 + 2dN_2 + 3dN_3 + \ldots \,. \tag{17}$$

Also, it is well known that N_j has a Poisson distribution with parameter λh_j, and that the random variables $N_1, N_2, \ldots$ are mutually independent. From this it follows that

$$f_i = (p^{(1)} * p^{(2)} * p^{(3)} * \ldots)_i \,, \tag{18}$$

where $p_i^{(j)} = P[jN_j = i]$, or

$$\begin{aligned} p_i^{(j)} &= e^{-\lambda h_j} \frac{(\lambda h_j)^m}{m!} && \text{if } m = \frac{i}{j} \text{ is an integer} \\ &= 0 && \text{if } m = \frac{i}{j} \text{ is not an integer} \,. \end{aligned} \tag{19}$$

For the calculation of f_i ($i = 0, 1, \ldots, r = [\alpha/d]$), one needs $p_i^{(j)}$ for $i = 0, 1, \ldots, [r/j]$. Thus, if $j > r$, we need only

$$p_0^{(j)} = e^{-\lambda h_j} \,. \tag{20}$$

Therefore, formula (18) can be rewritten as follows:

$$f_i = (p^{(1)} * \ldots * p^{(r)})_i \exp\left(-\lambda \sum_{r+1}^{\infty} h_j\right) . \tag{21}$$

An algorithm similar to this has been developed very recently [4]. However, the underlying idea is not new (see the lemma on p. 120 of [6]).

IV. THE METHOD OF DISPERSAL

In the last section we saw that the exact calculation of a stop-loss premium is feasible if the claim amount distribution is arithmetic with a sufficiently large span. If F is an arbitrary compound Poisson distribution (nonarithmetic or arithmetic with a small span), the exact calculation of the stop-loss premium may be an extensive procedure and may lead to considerable round-off errors. Instead, we suggest a procedure that is outlined in this and the following section. The idea is to replace the original distribution F by arithmetic compound Poisson distributions F^u, F^l, respectively, calculating the stop-loss premiums for these, thereby getting upper and lower bounds for $P(F, \alpha, a)$.

We pick a span $d > 0$. Then we construct a compound Poisson distribution F^u, given by its Poisson parameter λ^u and the jump amount distribution $H^u(x)$, as follows: We set $\lambda^u = \lambda$, and H^u is arithmetic with span d, so that

$$h_i^u = \int_{(i-1)d}^{(i+1)d} \left(1 - \left|\frac{x}{d} - i\right|\right) dH(x) \tag{22}$$

for $i = 0, 1, 2, \ldots$. This simply means that the probability mass of $H(x)$ between id and $(i + 1)d$ is dispersed to the endpoints id and $(i + 1)d$, so that the conditional mean remains unchanged (for $i = 0, 1, 2, \ldots$). Consequently, the mean of F^u equals the mean of F.

THEOREM 1. *$P(F, \alpha, a) \leq P(F^u, \alpha, a)$ for all $a \geq 0, \alpha$.*

Thus the method of dispersal leads (for each d) to an upper bound for the stop-loss premium. Theorem 1 can be proved by using (in this order)

1. The second part of Example 2 in [2], with $a = id$, $b = (i + 1)d$, applied to the conditional distribution of $H(x)$, $a \leq x < b$.
2. Theorem 2 in [2].
3. Lemma 1 in [2] to show that $H(x) < H^u(x)$.
4. Lemma 3 and Lemma 1 in [2] to show that $F(x) < F^u(x)$.
5. Theorem 1 in [2] to complete the proof.

Here our concern is with the content of the theorem but not with its proof.

V. THE METHOD OF TRUNCATION

As before, we first pick a span $d > 0$. Then we associate a compound Poisson distribution F^l (Poisson parameter λ^l, jump amount distribution H^l arithmetic with span d) to the original distribution F (given by λ and

H) as follows:

$$\lambda^l h_i^l = \lambda \int_{id}^{(i+1)d} \frac{x}{id}\, dH(x) \tag{23}$$

for $i = 1, 2, \ldots$. The value of λ^l is determined by the condition that $h_1^l + h_2^l + \ldots = 1$. The underlying idea is the following: if $id \leq x < (i+1)d$, a claim of amount x is replaced by one of amount id. In order to keep expected values unchanged, we compensate by increasing the Poisson frequency proportionately. (A similar construction was suggested in [6], p. 123, with the symbol F^- instead of F^l.) In this truncation process, the claims between 0 and d are ignored. Thus

$$\int_0^\infty x dF(x) - \int_0^\infty x dF^l(x) = \lambda \int_0^d x dH(x)\ . \tag{24}$$

Truncation leads to a lower bound for the stop-loss premium:

THEOREM 2. *$P(F, \alpha, a) \geq P(F^l, \alpha, a)$ for all $a \geq 0, \alpha$.*

The proof is similar to the one for Theorem 1, with the following modification: In the first step, part 1 of Example 2 in [2] is applied, with $a = 0$ and $b = (i+1)d$, to the mixture of the degenerate distribution concentrated at zero and the conditional distribution of $H(x)$ given $id \leq x < (i+1)d$, where the weights are chosen so that the mean of this mixture equals id ($i = 1, 2, \ldots$).

VI. AN ILLUSTRATIVE EXAMPLE

To illustrate the dispersal and truncation methods, let us consider the sample portfolio of five policies that is defined in Table 1. For the compound Poisson distribution F we have $\lambda = 1.4$, and the claim amount distribution is concentrated at the arguments 1.7, 2.3, etc., with probabilities 0.2/1.4, 0.3/1.4, etc. Thus the original claim amount distribution is arithmetic with a span $d = 0.1$.

The methods of dispersal and truncation are best applied policy by policy. Table 2 shows this procedure for $d = 1$. The data on the bottom line are used to calculate $P(F^u, \alpha, a)$ and $P(F^l, \alpha, a)$, which are shown in Tables 5 and 6. For $d = 2$ the calculations are summarized in Table 3. However, in the case of a large number of policies, it is more economical to obtain the bottom line of Table 3 directly from the bottom line of Table 2. The calculations are displayed in Table 4.

In general, this shortcut works if the second span is a multiple of the first span. From this observation, and from Theorems 1 and 2, respectively, it follows that, if the original span is replaced by a multiple of it, the upper bound is increased while the lower bound is decreased. In practice one uses this in the opposite direction: if for a given span the upper bound differs from the lower bound by too much, one may want to replace this span by one nth of it.

In Tables 5 and 6 the numerical results are shown for our sample portfolio for a span $d = 1$. The first column shows the "amounts." These should be interpreted as the arguments of the frequency function of the

TABLE 1

A SAMPLE PORTFOLIO

Policy	Amount at Risk	Mortality Rate
A	$1.7	0.2
B	2.3	0.3
C	3.4	0.3
D	3.6	0.4
E	5.0	0.2
Total	....	1.4

aggregate claims (second column) and their cumulative distribution function (third column), on the one hand, and as deductibles for the net stop-loss premiums (fourth column) and the exponential stop-loss premiums (fifth column), on the other hand. The latter are based on $a = 0.1$. Tables 7 and 8 display the results for $d = 2$. Of course our sample portfolio is small enough so that the exact distribution of aggregate claims and the exact stop-loss premiums can be calculated. Table 9 shows these values calculated as lower bounds for $d = 0.1$ which are exact.

For the convenience of the reader, Tables 10 and 11 provide comparisons of the bounds for the premiums from Tables 5–8 with the exact values from Table 9. Tables 10 and 11 show the relative errors of the bounds increasing as the deductible increases—but remember that all are going to zero.

TABLE 2

DISPERSAL FOR $d = 1$

POLICY	CONTRIBUTION TO									
	$\lambda^u h_1^u$	$\lambda^u h_2^u$	$\lambda^u h_3^u$	$\lambda^u h_4^u$	$\lambda^u h_5^u$	$\lambda^l h_1^l$	$\lambda^l h_2^l$	$\lambda^l h_3^l$	$\lambda^l h_4^l$	$\lambda^l h_5^l$
A	0.06	0.14	......	......	......	0.34	......	......	......	......
B	......	0.21	0.09	......	......	......	0.345	......	......	......
C	......	......	0.18	0.12	......	......	......	0.34	......	......
D	......	......	0.16	0.24	......	......	......	0.48	......	......
E	......	......	......	......	0.20	......	......	......	......	0.20
Total	0.06	0.35	0.43	0.36	0.20	0.34	0.345	0.82	......	0.20

TABLE 3

DISPERSAL FOR $d = 2$

POLICY	CONTRIBUTION TO					
	$\lambda^u h_0^u$	$\lambda^u h_1^u$	$\lambda^u h_2^u$	$\lambda^u h_3^u$	$\lambda^l h_1^l$	$\lambda^l h_2^l$
A	0.03	0.17	....	....	....	....
B	....	0.255	0.045	....	0.345	....
C	....	0.09	0.21	....	0.51	....
D	....	0.08	0.32	....	0.72	....
E	....	....	0.10	0.10	....	0.25
Total	0.03	0.595	0.675	0.10	1.575	0.25

TABLE 4

TRUNCATION

AMOUNT AT RISK	$\lambda^u h_i^u$		$\lambda^l h_i^l$	
	$d=1$	$d=2$	$d=1$	$d=2$
\$0		0.03		dropped
1	0.06		0.34	
2	0.35	0.35+0.03+0.215	0.345	$0.345+\frac{3}{2}(0.82)$
3	0.43		0.82	
4	0.36	0.36+0.215+0.10		$\frac{5}{4}(0.20)$
5	0.20		0.20	
6		0.10		

TABLE 5

UPPER BOUNDS (d=1.00) FOR NET AND EXPONENTIAL (a=0.10) STOP-LOSS PREMIUMS

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
0.0	0.246597	0.246597	4.490000	5.410417
1.00	0.014796	0.261393	3.736597	4.560266
2.00	0.086753	0.348146	2.997990	3.733002
3.00	0.111224	0.459370	2.346135	2.981955
4.00	0.110397	0.569766	1.805505	2.334229
5.00	0.092859	0.662625	1.375271	1.797797
6.00	0.061008	0.723633	1.037897	1.363697
7.00	0.065427	0.789060	0.761530	1.006613
8.00	0.054577	0.843637	0.550590	0.730186
9.00	0.041321	0.884958	0.394228	0.522717
10.00	0.030579	0.915537	0.279186	0.369178
11.00	0.023308	0.938845	0.194723	0.256592
12.00	0.018344	0.957189	0.133568	0.175434
13.00	0.013149	0.970338	0.090757	0.118692
14.00	0.009218	0.979556	0.061096	0.079494
15.00	0.006504	0.986061	0.040652	0.052622
16.00	0.004596	0.990656	0.026713	0.034416
17.00	0.003176	0.993833	0.017369	0.022278
18.00	0.002123	0.995956	0.011202	0.014301
19.00	0.001414	0.997370	0.007158	0.009097
20.00	0.000940	0.998309	0.004528	0.005731
21.00	0.000617	0.998926	0.002837	0.003577
22.00	0.000398	0.999324	0.001764	0.002216
23.00	0.000253	0.999578	0.001088	0.001362
24.00	0.000161	0.999738	0.000666	0.000830
25.00	0.000101	0.999839	0.000404	0.000502
26.00	0.000063	0.999902	0.000243	0.000301
27.00	0.000039	0.999941	0.000145	0.000180
28.00	0.000024	0.999965	0.000086	0.000106
29.00	0.000014	0.999979	0.000051	0.000063
30.00	0.000009	0.999988	0.000030	0.000037
31.00	0.000005	0.999993	0.000017	0.000021
32.00	0.000003	0.999996	0.000010	0.000012
33.00	0.000002	0.999998	0.000006	0.000007
34.00	0.000001	0.999999	0.000003	0.000004
35.00	0.000001	0.999999	0.000002	0.000002
36.00	0.000000	1.000000	0.000001	0.000001

TABLE 6

LOWER BOUNDS ($d = 1.00$) FOR NET AND EXPONENTIAL ($a = 0.10$) STOP-LOSS PREMIUMS

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
0.0	0.181772	0.181772	4.490000	5.287705
1.00	0.061803	0.243575	3.671772	4.399739
2.00	0.073218	0.316793	2.915347	3.563379
3.00	0.171566	0.488359	2.232140	2.794000
4.00	0.065222	0.553581	1.720499	2.175059
5.00	0.100489	0.654070	1.274080	1.632818
. . .	. . .	. . .	. . .	. . .
10.00	0.027452	0.924600	0.227178	0.293951
. . .	. . .	. . .	. . .	. . .
15.00	0.004900	0.989611	0.027959	0.035414
. . .	. . .	. . .	. . .	. . .
20.00	0.000600	0.998973	0.002564	0.003181
. . .	. . .	. . .	. . .	. . .
25.00	0.000054	0.999922	0.000185	0.000226
. . .	. . .	. . .	. . .	. . .
30.00	0.000004	0.999995	0.000011	0.000013
. . .	. . .	. . .	. . .	. . .
34.00	0.000000	1.000000	0.000001	0.000001
35.00	0.000000	1.000000	0.000001	0.000001
36.00	0.000000	1.000000	0.000000	0.000000

TABLE 7

UPPER BOUNDS ($d=2.00$) FOR NET AND EXPONENTIAL ($a=0.10$) STOP-LOSS PREMIUMS

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
0.0	0.254107	0.254107	4.490000	5.459282
1.00	0.0	0.254107	3.744107	4.612913
2.00	0.151194	0.405301	2.998214	3.780000
3.00	0.0	0.405301	2.403515	3.067901
4.00	0.216502	0.621803	1.808815	2.376726
5.00	0.0	0.621803	1.430618	1.879491
6.00	0.136387	0.758190	1.052421	1.407223
⋮	⋮	⋮	⋮	⋮
12.00	0.036552	0.961712	0.144897	0.194409
⋮	⋮	⋮	⋮	⋮
18.00	0.004818	0.996095	0.013509	0.017659
⋮	⋮	⋮	⋮	⋮
24.00	0.000419	0.999716	0.000923	0.001181
⋮	⋮	⋮	⋮	⋮
30.00	0.000027	0.999984	0.000049	0.000062
·	·	·	·	·
32.00	0.000010	0.999994	0.000018	0.000022
33.00	0.0	0.999994	0.000012	0.000014
34.00	0.000004	0.999998	0.000006	0.000008
35.00	0.0	0.999998	0.000004	0.000005
36.00	0.000001	0.999999	0.000002	0.000003

TABLE 8

LOWER BOUNDS ($d=2.00$) FOR NET AND EXPONENTIAL ($a=0.10$) STOP-LOSS PREMIUMS

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
0.0	0.161218	0.161218	4.150000	4.716655
1.00	0.0	0.161218	3.311218	3.821895
2.00	0.253918	0.415135	2.472435	2.936929
3.00	0.0	0.415135	1.887571	2.257233
4.00	0.240265	0.655400	1.302706	1.599683
5.00	0.0	0.655400	0.958106	1.170472
6.00	0.168459	0.823859	0.613506	0.765562
⋮	⋮	⋮	⋮	⋮
12.00	0.020420	0.987843	0.036514	0.045071
⋮	⋮	⋮	⋮	⋮
18.00	0.000940	0.999594	0.001126	0.001360
⋮	⋮	⋮	⋮	⋮
24.00	0.000023	0.999992	0.000021	0.000025
⋮	⋮	⋮	⋮	⋮
30.00	0.000000	1.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮
32.00	0.000000	1.000000	0.000000	0.000000
33.00	0.0	1.000000	0.000000	0.000000
34.00	0.000000	1.000000	0.000000	0.000000
35.00	0.0	1.000000	0.000000	0.000000
36.00	0.000000	1.000000	0.000000	0.000000

TABLE 9

LOWER BOUNDS ($d=0.10$) FOR NET AND EXPONENTIAL ($a=0.10$) STOP-LOSS PREMIUMS

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
0.0	0.246597	0.246597	4.490000	5.392013
0.10	0.0	0.246597	4.414660	5.306456
⋮	⋮	⋮	⋮	⋮
1.00	0.0	0.246597	3.736597	4.542136
⋮	⋮	⋮	⋮	⋮
1.60	0.0	0.246597	3.284555	4.038491
1.70	0.049319	0.295916	3.209215	3.955027
1.80	0.0	0.295916	3.138806	3.875032
⋮	⋮	⋮	⋮	⋮
2.20	0.0	0.295916	2.857173	3.556667
2.30	0.073979	0.369895	2.786765	3.477485
2.40	0.0	0.369895	2.723754	3.403706
⋮	⋮	⋮	⋮	⋮
3.40	0.078911	0.448806	2.093650	2.676737
3.50	0.0	0.448806	2.038530	2.611190
3.60	0.098639	0.547445	1.983411	2.545870
3.70	0.0	0.547445	1.938155	2.488432
3.80	0.0	0.547445	1.892900	2.431239
3.90	0.0	0.547445	1.847645	2.374291
4.00	0.014796	0.562241	1.802389	2.317588
4.10	0.0	0.562241	1.758613	2.262305
⋮	⋮	⋮	⋮	⋮
4.60	0.011097	0.573338	1.539734	1.989605
4.70	0.0	0.573338	1.497067	1.936719
4.80	0.0	0.573338	1.454401	1.884083
4.90	0.0	0.573338	1.411735	1.831696
5.00	0.049319	0.622657	1.369069	1.779558
5.10	0.015125	0.637782	1.331335	1.731798
5.20	0.0	0.637782	1.295113	1.685559
5.30	0.019728	0.657510	1.258891	1.639569
5.40	0.0	0.657510	1.224642	1.595500
⋮	⋮	⋮	⋮	⋮
6.00	0.0	0.710775	1.029209	1.344943
⋮	⋮	⋮	⋮	⋮
7.00	0.031564	0.768125	0.747126	0.984301
⋮	⋮	⋮	⋮	⋮
8.00	0.003551	0.816655	0.546480	0.718940
⋮	⋮	⋮	⋮	⋮

TABLE 9—*Continued*

Amount	Frequency	Cumulative	Net Premium	Exponential Premium
9.00	0.002959	0.872099	0.387504	0.510486
⋮	⋮	⋮	⋮	⋮
10.00	0.004932	0.900067	0.273838	0.359412
⋮	⋮	⋮	⋮	⋮
11.00	0.001815	0.936334	0.189156	0.247744
⋮	⋮	⋮	⋮	⋮
12.00	0.006381	0.951186	0.128682	0.168073
⋮	⋮	⋮	⋮	⋮
14.00	0.001303	0.976464	0.058388	0.075471
⋮	⋮	⋮	⋮	⋮
16.00	0.000367	0.989989	0.025239	0.032298
⋮	⋮	⋮	⋮	⋮
18.00	0.000148	0.995527	0.010488	0.013286
⋮	⋮	⋮	⋮	⋮
20.00	0.000063	0.998051	0.004197	0.005265
⋮	⋮	⋮	⋮	⋮
24.00	0.000024	0.999703	0.000594	0.000735
⋮	⋮	⋮	⋮	⋮
28.00	0.000002	0.999962	0.000074	0.000091
⋮	⋮	⋮	⋮	⋮
32.00	0.000000	0.999995	0.000008	0.000010
⋮	⋮	⋮	⋮	⋮
34.00	0.000000	0.999998	0.000003	0.000003
⋮	⋮	⋮	⋮	⋮
35.00	0.000000	0.999999	0.000002	0.000002
⋮	⋮	⋮	⋮	⋮
35.90	0.000000	0.999999	0.000001	0.000001
36.00	0.000000	1.000000	0.000001	0.000001

TABLE 10

NET PREMIUMS

α	$P(F^l, \alpha)/P(F, \alpha)$		$P(F^u, \alpha)/P(F, \alpha)$	
	$d=2$	$d=1$	$d=1$	$d=2$
0	0.924	1.000	1.000	1.000
1	0.886	0.983	1.000	1.002
2	0.825	0.972	1.000	1.000
3	0.805	0.952	1.000	1.025
4	0.723	0.955	1.002	1.004
5	0.700	0.931	1.005	1.045
6	0.596	0.902	1.008	1.023
7	0.585	0.905	1.019	1.085
8	0.478	0.863	1.008	1.041
9	0.468	0.852	1.017	1.114
10	0.371	0.830	1.020	1.076
11	0.365	0.802	1.029	1.162
12	0.284	0.794	1.038	1.126
13	0.277	0.760	1.032	1.212
14	0.209	0.741	1.046	1.170
15	0.206	0.718	1.044	1.275
16	0.151	0.696	1.058	1.226
17	0.151	0.677	1.063	1.361
18	0.107	0.655	1.068	1.288
19	0.109	0.633	1.081	1.451
20	0.075	0.611	1.079	1.358

TABLE 11

EXPONENTIAL PREMIUMS

α	$P(F^l, \alpha, 0.1)/P(F, \alpha, 0.1)$		$P(F^u, \alpha, 0.1)/P(F, \alpha, 0.1)$	
	$d=2$	$d=1$	$d=1$	$d=2$
0	0.875	0.981	1.003	1.012
1	0.842	0.969	1.004	1.016
2	0.790	0.959	1.005	1.017
3	0.761	0.942	1.006	1.035
4	0.690	0.939	1.007	1.026
5	0.658	0.918	1.010	1.056
6	0.569	0.893	1.014	1.046
7	0.546	0.889	1.023	1.094
8	0.454	0.854	1.016	1.069
9	0.435	0.839	1.024	1.127
10	0.352	0.818	1.027	1.106
11	0.338	0.792	1.036	1.176
12	0.268	0.781	1.044	1.157
13	0.257	0.750	1.040	1.229
14	0.198	0.731	1.053	1.205
15	0.191	0.709	1.053	1.293
16	0.144	0.687	1.066	1.264
17	0.140	0.668	1.071	1.379
18	0.102	0.646	1.076	1.329
19	0.101	0.625	1.089	1.468
20	0.072	0.604	1.089	1.404

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DISCUSSION OF PRECEDING PAPER

B. J. J. ALTING VON GEUSAU:*

I read the above paper with great interest and was quite surprised and impressed by the elegant solution the authors found for a problem as complicated as the rating in stop-loss contracts. However, I have a few questions, and I would also like to make some remarks.

1. In the first part of the article the question "Why the exponential principle?" is answered. A score of reasons is provided why the variance and the standard deviation principle should be rejected, but formally speaking I cannot find any good reason why the net premium principle should be overruled.
2. Is there any reason to restrict the discussion to the Poisson model for the number of claims concerned? The nontruncated Poisson distribution does not fit too well in the described situation (as can be proved), so why not try the same technique on the truncated Poisson density or the negative binomial distribution as a model for the number of claims?
3. Immediately preceding equation (15), the probabilities h_i are introduced. Did the authors try to fit some a priori defined probability distribution on h_i?
4. From formula (17) the expectation and variance of the random variable X can be calculated. Would it not be a good idea to use these parameters with the central limit theorem in cases where the span d is too small to make the proposed calculations executable?
5. If there are objections to the use of the central limit theorem in the above question, why not use the Edgeworth series approximation as described by Cramér?

HARRY H. PANJER:

Professors Gerber and Jones have shown very neatly the theoretical pitfalls of some of the premium principles to which the exponential principle is not subject. Given that the exponential (and the net premium, as a special case) principle satisfies both properties P_1 and P_2, the natural question to ask is whether this is the only principle satisfying P_1 and P_2. The answer is clearly no. The exponential principle is of the form

$$P = f^{-1}E[f(s)] \, .$$

* Mr. von Geusau, not a member of the Society, is actuary, Nederlandse Reassurantie Groep.

If $f(s)$ is any convex function, then by Jensen's inequality

$$E[f(s)] \geq fE(s) ,$$

and P_1 is satisfied. Also, if $f(s)$ is a strictly monotonic (increasing or decreasing) continuous function that takes on finite values over the range $0 < s < \text{Max}\,[s]$, then both P_1 and P_2 are satisfied. Thus, the entire class of continuous monotonic convex functions that take on finite values over the range $0 < s < \text{Max}\,[s]$ satisfies the properties P_1 and P_2. This class does not contain all the functions satisfying P_1 and P_2 but serves only as an example. One could then construct premiums consistent with P_1 and P_2 by using such functions as

$$P = [E(s^a)]^{1/a} , \qquad a \geq 1 , \tag{1}$$

$$P = [E(b + s)^a]^{1/a} - b , \qquad a \geq 1, b \geq 0 , \tag{2}$$

and

$$P = \frac{2}{\pi} \text{Max}\,[s] \tan^{-1} \left\{ E\left[\tan\left(\frac{s}{\text{Max}\,[s]} \frac{\pi}{2} \right) \right] \right\} , \qquad 0 < s < \text{Max}\,[s] . \tag{3}$$

The exponential principle is just one example of functions in this class. The key to reducing the class of desirable functions is to impose additional properties that must be satisfied by the premium principle. The requirement that premium principles be iterative does not reduce the class. Any premium of the form $P = f^{-1}E[f(s)]$ is iterative, as is shown in the author's reference [3], page 165. The requirement that premiums be additive does, however, reduce the class significantly. Gerber[1] shows that, if the utility function has certain desirable properties consistent with economic theory, then only the net premium principle and the exponential principle produce premiums that are additive. It is important to stress the effect of this additivity requirement on the choice of premium principle. It is this property that makes the exponential principle so attractive.

In Section II of the paper, following the derivation of equation (12), analogous to Mereu's formula (9), the authors state that, if the moment generating function is available, determination of $P(F, \alpha, a)$ requires the knowledge of the distribution only in the left-hand tail region. However, it is known[2] that the moment generating function completely specifies

[1] Hans U. Gerber, "On Additive Premium Calculation Principles," *ASTIN*, VII (1974), 217.

[2] Samuel S. Wilks, *Mathematical Statistics* (New York: John Wiley & Sons, 1962), p. 125.

the distribution, and therefore the knowledge of the entire distribution is implicit in the knowledge of the moment generating function.

I thank the authors for another stimulating paper in this area.

(AUTHORS' REVIEW OF DISCUSSION)

HANS U. GERBER AND DONALD A. JONES:

We appreciate the comments of the two discussants.

In reply to Mr. von Geusau's remarks (using the numbers of his questions):

1. The net premium principle can be viewed as a limit of the family of exponential principles ($a \rightarrow 0$). Thus it has all the nice properties of the exponential principle except one: it does not provide for a loading.

2. The method of dispersal, which produces upper bounds, works for an arbitrary claim number distribution. However, the method of truncation, which produces lower bounds, depends on the Poisson assumption.

3. The original claim amount distribution can be any distribution, not necessarily discrete. Then, for a given span, we assign to it two arithmetic claim amount distributions, one with the method of dispersal and the other with the method of truncation.

4 and 5. The classical approximations (normal, Edgeworth, Esscher, and Bowers) produce "point estimates" for the stop-loss premium, and it is difficult or impossible to establish rigorous bounds for the errors involved. In contrast to these methods, our method produces upper and lower bounds (or an "interval estimate") for the stop-loss premium.

We are glad that Mr. Panjer mentions Jensen's inequality. This inequality has still not received the recognition it deserves by actuaries. For example, the celebrated inequality between a life annuity and the corresponding annuity-certain follows immediately from Jensen's inequality. Regarding Mr. Panjer's comment about Section II, a distinction should be made between numerical knowledge of $F(x)$ for certain values of x and implicit knowledge of F.