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**NOTE ON DERIVATION OF UNISEX ANNUITY VALUES,
AND EARLY RETIREMENT AND JOINT
AND SURVIVOR OPTION FACTORS**

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THE purpose of this note is to demonstrate the derivation of unisex single life and joint life annuity values, and early retirement and joint and survivor option factors based on certain assumptions. The assumptions are as follows: Let

- κ = Proportion of plan members retiring at age ρ who are male; then $1 - \kappa$ is the proportion of members retiring at age ρ who are female;
- l_ρ = Number of members (male and female) retiring at age ρ ;
- $l_\rho^m, l_{\rho+1}^m, \dots$ = Number living at ages $\rho, \rho + 1, \dots$, according to the male mortality table;
- $l_\rho^f, l_{\rho+1}^f, \dots$ = Number living at ages $\rho, \rho + 1, \dots$, according to the female mortality table.

If a_ρ^x represents the unisex straight life annuity value at retirement age ρ (payments at the end of the year), where κ is the proportion of retiring members who are male,

$$\begin{aligned}
 a_\rho^x &= \left[v\kappa l_\rho \frac{l_{\rho+1}^m}{l_\rho^m} + v(1 - \kappa) l_\rho \frac{l_{\rho+1}^f}{l_\rho^f} + v^2 \kappa l_\rho \frac{l_{\rho+2}^m}{l_\rho^m} + v^2(1 - \kappa) l_\rho \frac{l_{\rho+2}^f}{l_\rho^f} + \dots \right] \\
 &\quad \div [\kappa l_\rho + (1 - \kappa) l_\rho] \\
 &= [v\kappa p_\rho^m + v(1 - \kappa) p_\rho^f + v^2 \kappa {}_2p_\rho^m + v^2(1 - \kappa) {}_2p_\rho^f + \dots] \quad (1) \\
 &\quad \div [\kappa + (1 - \kappa)] \\
 &= \kappa(v p_\rho^m + v^2 {}_2p_\rho^m + \dots) + (1 - \kappa)(v p_\rho^f + v^2 {}_2p_\rho^f + \dots) \\
 &= \kappa a_\rho^m + (1 - \kappa) a_\rho^f.
 \end{aligned}$$

Therefore, at age ρ , the unisex straight life annuity value is simply the weighted average of straight life annuity values from the single life mortality tables.

Let $a_{\rho+t}^x$ represent the unisex straight life annuity value at age $\rho + t$,

where ρ is the retirement age and κ is the proportion of retiring members at age ρ who were male.

Then

$$\begin{aligned}
 a_{\rho+t}^{\kappa} &= \left[v\kappa l_{\rho} \frac{l_{\rho+t+1}^m}{l_{\rho}^m} + v(1-\kappa) l_{\rho} \frac{l_{\rho+t+1}^f}{l_{\rho}^f} + v^2 \kappa l_{\rho} \frac{l_{\rho+t+2}^m}{l_{\rho}^m} \right. \\
 &\quad \left. + v^2(1-\kappa) l_{\rho} \frac{l_{\rho+t+2}^f}{l_{\rho}^f} + \dots \right] \div \left[\kappa l_{\rho} \frac{l_{\rho+t}^m}{l_{\rho}^m} + (1-\kappa) l_{\rho} \frac{l_{\rho+t}^f}{l_{\rho}^f} \right] \quad (2) \\
 &= [v\kappa {}_{t+1}p_{\rho}^m + v(1-\kappa) {}_{t+1}p_{\rho}^f + v^2 \kappa {}_{t+2}p_{\rho}^m + v^2(1-\kappa) {}_{t+2}p_{\rho}^f \\
 &\quad + \dots] \div [\kappa {}_t p_{\rho}^m + (1-\kappa) {}_t p_{\rho}^f] \\
 &= [\kappa {}_t p_{\rho}^m a_{\rho+t}^m + (1-\kappa) {}_t p_{\rho}^f a_{\rho+t}^f] \div [\kappa {}_t p_{\rho}^m + (1-\kappa) {}_t p_{\rho}^f].
 \end{aligned}$$

Note that the unisex straight life annuity value at age $\rho + t$ is *not* the weighted average of straight life annuity values from the single life mortality tables. The reason for this is, of course, the difference in mortality rates for males and females since retirement.

If ${}_{65-\rho}|a_{\rho}^{\kappa}$ represents the unisex life annuity value deferred to age 65 for a pensioner at retirement age ρ , where κ is the proportion of retiring members who are male, then

$$\begin{aligned}
 {}_{65-\rho}|a_{\rho}^{\kappa} &= \left[v^{66-\rho} \kappa l_{\rho} \frac{l_{66}^m}{l_{\rho}^m} + v^{66-\rho} (1-\kappa) l_{\rho} \frac{l_{66}^f}{l_{\rho}^f} + v^{67-\rho} \kappa l_{\rho} \frac{l_{67}^m}{l_{\rho}^m} \right. \\
 &\quad \left. + v^{67-\rho} (1-\kappa) l_{\rho} \frac{l_{67}^f}{l_{\rho}^f} + \dots \right] \div [\kappa l_{\rho} + (1-\kappa) l_{\rho}] \quad (3) \\
 &= [v^{66-\rho} \kappa {}_{66-\rho}p_{\rho}^m + v^{66-\rho} (1-\kappa) {}_{66-\rho}p_{\rho}^f + v^{67-\rho} \kappa {}_{67-\rho}p_{\rho}^m \\
 &\quad + v^{67-\rho} (1-\kappa) {}_{67-\rho}p_{\rho}^f + \dots] \div [\kappa + (1-\kappa)] \\
 &= \kappa {}_{65-\rho}|a_{\rho}^m + (1-\kappa) {}_{65-\rho}|a_{\rho}^f.
 \end{aligned}$$

Similar to the unisex straight life annuity value at age ρ , the unisex deferred life annuity value at age ρ is the weighted average of the deferred life annuity values based on the single life mortality tables.

On the basis of equations (1) and (3) above, the unisex early retirement factor for a member retiring at age ρ with a normal retirement age 65 is as follows:

Early retirement factor at age ρ

$$= [\kappa {}_{65-\rho}|a_{\rho}^m + (1-\kappa) {}_{65-\rho}|a_{\rho}^f] \div [\kappa a_{\rho}^m + (1-\kappa) a_{\rho}^f]. \quad (4)$$

Let ${}_{65-\rho-t}|a_{\rho+t}^{\kappa}$ be the unisex life annuity value at age $\rho + t$ deferred to age 65, where ρ is the retirement age and κ is the proportion of retiring members at age ρ who were male. Then

$$\begin{aligned}
 {}_{65-\rho-t}|a_{\rho+t}^{\kappa} &= \left[v^{66-\rho-t} \kappa l_{\rho} \frac{l_{66}^m}{l_{\rho}^m} + v^{66-\rho-t} (1 - \kappa) l_{\rho} \frac{l_{66}^f}{l_{\rho}^f} + v^{67-\rho-t} \kappa l_{\rho} \frac{l_{67}^m}{l_{\rho}^m} \right. \\
 &\quad \left. + v^{67-\rho-t} (1 - \kappa) l_{\rho} \frac{l_{67}^f}{l_{\rho}^f} + \dots \right] \div \left[\kappa l_{\rho} \frac{l_{\rho+t}^m}{l_{\rho}^m} + (1 - \kappa) l_{\rho} \frac{l_{\rho+t}^f}{l_{\rho}^f} \right] \\
 &= [v^{66-\rho-t} \kappa {}_{66-\rho}p_{\rho}^m + v^{66-\rho-t} (1 - \kappa) {}_{66-\rho}p_{\rho}^f + v^{67-\rho-t} \kappa {}_{67-\rho}p_{\rho}^m \\
 &\quad + v^{67-\rho-t} (1 - \kappa) {}_{67-\rho}p_{\rho}^f + \dots] \div [\kappa {}_t p_{\rho}^m + (1 - \kappa) {}_t p_{\rho}^f] \\
 &= [\kappa {}_t p_{\rho}^m {}_{65-\rho-t}|a_{\rho+t}^m + (1 - \kappa) {}_t p_{\rho}^f {}_{65-\rho-t}|a_{\rho+t}^f] \\
 &\quad \div [\kappa {}_t p_{\rho}^m + (1 - \kappa) {}_t p_{\rho}^f]. \quad (5)
 \end{aligned}$$

The above annuity values were calculated on the basis of the assumption that a payment would be paid at the end of each year. The continuous annuity value \bar{a}_{ρ} can be approximated by adding one-half of an immediate payment of 1 to the a_{ρ} . On this basis,

$$\bar{a}_{\rho}^m = \frac{1}{2} + a_{\rho}^m; \quad (6)$$

$$\bar{a}_{\rho}^f = \frac{1}{2} + a_{\rho}^f \quad (\text{etc.}). \quad (7)$$

Table 1 shows the unisex straight life annuity values for ages 65-75, where $\rho = 65$ and $\kappa = 0.70$. The values are based on the 1974 George B. Buck Mortality Table—5 per cent. For comparative purposes, values

TABLE 1

Age 65 + t	$a_{65+t}^{0.70}$ (1)	a_{65+t}^m (2)	a_{65+t}^f (3)	Weighted Average [0.70(2)+0.30(3)] (4)
t = 0	10.279	9.668	11.705	10.279
1	9.973	9.355	11.396	9.967
2	9.666	9.044	11.082	9.655
3	9.361	8.735	10.762	9.343
4	9.058	8.430	10.438	9.032
5	8.757	8.130	10.111	8.724
6	8.460	7.835	9.780	8.419
7	8.166	7.545	9.447	8.116
8	7.875	7.261	9.112	7.816
9	7.588	6.982	8.777	7.521
10	7.306	6.710	8.442	7.230

are given based on the male mortality table, the female mortality table, and the weighted average (0.70-0.30) of the single life annuity values.

Table 2 indicates the unisex life annuity values, deferred to age 65, for ages 55-65, where $\rho = 55$ and $\kappa = 0.70$. Values are also based on the male mortality table, the female mortality table, and the weighted average (0.70-0.30) of the single life annuity values.

Table 3 demonstrates the derivation of the unisex early retirement factors for members retiring at alternative ages 55, 56, . . . , 64, where $\kappa = 0.70$ at their respective retirement age and where benefits would

TABLE 2

Age $55+t$	${}_{10-t} d_{55+t}^{0.70}$ (1)	${}_{10-t} d_{55+t}^m$ (2)	${}_{10-t} d_{55+t}^f$ (3)	Weighted Average [0.70(2)+0.30(3)] (4)
$t=0$	5.644	5.188	6.708	5.644
1	5.967	5.490	7.075	5.966
2	6.313	5.816	7.465	6.311
3	6.685	6.167	7.880	6.681
4	7.085	6.546	8.323	7.079
5	7.517	6.957	8.794	7.508
6	7.984	7.404	9.298	7.972
7	8.491	7.892	9.837	8.476
8	9.043	8.428	10.415	9.024
9	9.646	9.017	11.036	9.623
10	10.308	9.668	11.705	10.279

TABLE 3

RETIREMENT AGE ρ	$d_{\rho}^{0.70}$ (1)	${}_{65-\rho} d_{\rho}^{0.70}$ (2)	EARLY RETIREMENT FACTOR			
			(2) ÷ (1) (3)	Male Annuity Table (4)	Female Annuity Table (5)	Weighted Average [0.70(4) +0.30(5)] (6)
55	13.230	5.644	0.4266	0.4078	0.4653	0.4251
56	12.957	5.966	0.4604	0.4416	0.4990	0.4588
57	12.679	6.311	0.4978	0.4790	0.5358	0.4960
58	12.395	6.681	0.5390	0.5207	0.5761	0.5373
59	12.105	7.079	0.5848	0.5672	0.6202	0.5831
60	11.810	7.508	0.6357	0.6192	0.6687	0.6341
61	11.510	7.972	0.6926	0.6777	0.7221	0.6910
62	11.207	8.476	0.7563	0.7437	0.7812	0.7550
63	10.900	9.024	0.8279	0.8184	0.8465	0.8268
64	10.591	9.623	0.9086	0.9032	0.9191	0.9080
65	10.279	10.279	1.0000	1.0000	1.0000	1.0000

normally begin at age 65. Columns 1-3 show the unisex straight life annuities, the unisex deferred life annuities, and the early retirement factors respectively. Columns 4-6 show the early retirement factors based on the male mortality table and the female mortality table, and the weighted average (0.70-0.30) of the single life early retirement factors, respectively.

In order to calculate the joint and survivor option factor, the joint life annuity value must first be derived. For this purpose, we will assume that every male member who selects the option when he retires at age ρ has a female contingent annuitant and every female member who retires at age ρ has a male contingent annuitant. Let c be the age of the contingent annuitant at the time the member retires at age ρ . Assume that κ is the proportion of members retiring at age ρ , selecting a joint and survivor option, who are male, and $1 - \kappa$ then is the proportion of members retiring at age ρ , selecting a joint and survivor option, who are female.

If $a_{\rho c}^{\kappa}$ represents the unisex joint life annuity value,

$$\begin{aligned}
 a_{\rho c}^{\kappa} &= \left\{ v \left[\kappa l_{\rho} \frac{l_{\rho+1}^m}{l_{\rho}^m} + (1 - \kappa) l_{\rho} \frac{l_{\rho+1}^f}{l_{\rho}^f} \right] \left[\kappa l_{\rho} \frac{l_{c+1}^f}{l_c^f} + (1 - \kappa) l_{\rho} \frac{l_{c+1}^m}{l_c^m} \right] \right. \\
 &\quad + v^2 \left[\kappa l_{\rho} \frac{l_{\rho+2}^m}{l_{\rho}^m} + (1 - \kappa) l_{\rho} \frac{l_{\rho+2}^f}{l_{\rho}^f} \right] \left[\kappa l_{\rho} \frac{l_{c+2}^f}{l_c^f} + (1 - \kappa) l_{\rho} \frac{l_{c+2}^m}{l_c^m} \right] \\
 &\quad \left. + \dots \right\} \div [\kappa l_{\rho} + (1 - \kappa) l_{\rho}] [\kappa l_{\rho} + (1 - \kappa) l_{\rho}] \tag{8} \\
 &= \{ v [\kappa p_{\rho}^m + (1 - \kappa) p_{\rho}^f] [\kappa p_c^f + (1 - \kappa) p_c^m] \\
 &\quad + v^2 [\kappa {}_2p_{\rho}^m + (1 - \kappa) {}_2p_{\rho}^f] [\kappa {}_2p_c^f + (1 - \kappa) {}_2p_c^m] + \dots \} \div 1 \\
 &= [v \kappa^2 p_{\rho}^m p_c^f + v \kappa (1 - \kappa) p_{\rho}^m p_c^m + v \kappa (1 - \kappa) p_{\rho}^f p_c^f + v (1 - \kappa)^2 p_{\rho}^f p_c^m] \\
 &\quad + \dots \\
 &= \kappa^2 a_{\rho c}^{mf} + \kappa (1 - \kappa) a_{\rho c}^{mm} + \kappa (1 - \kappa) a_{\rho c}^{ff} + (1 - \kappa)^2 a_{\rho c}^{fm} .
 \end{aligned}$$

Note that, if $\kappa = 1$,

$$a_{\rho c}^{\kappa} = a_{\rho c}^{mf} ,$$

and, if $\kappa = 0$,

$$a_{\rho c}^{\kappa} = a_{\rho c}^{fm} ;$$

it follows that

$$\bar{a}_{\rho c}^{mf} = \frac{1}{2} + a_{\rho c}^{mf} \quad (\text{etc.}),$$

and

$$\bar{a}_{\rho c}^{\kappa} = \kappa^2 \bar{a}_{\rho c}^{mf} + \kappa (1 - \kappa) \bar{a}_{\rho c}^{mm} + \kappa (1 - \kappa) \bar{a}_{\rho c}^{ff} + (1 - \kappa)^2 \bar{a}_{\rho c}^{fm} .$$

The unisex 100 per cent joint and survivor option factor for a male retiring at age ρ and selecting a contingent annuitant aged c , where κ is the proportion of retiring members selecting the option who are male, is as follows:

100 per cent joint and survivor option factor

$$\begin{aligned} &= \bar{a}_\rho^\kappa \div (\bar{a}_\rho^\kappa + \bar{a}_c^{1-\kappa} - \bar{a}_{\rho c}^\kappa) \\ &= [\kappa \bar{a}_\rho^m + (1 - \kappa) \bar{a}_\rho^f] \div [\kappa \bar{a}_\rho^m + (1 - \kappa) \bar{a}_\rho^f + (1 - \kappa) \bar{a}_c^m + \kappa \bar{a}_c^f] \quad (9) \\ &\quad - \kappa^2 \bar{a}_{\rho c}^{mf} - \kappa(1 - \kappa) \bar{a}_{\rho c}^{mm} - \kappa(1 - \kappa) \bar{a}_{\rho c}^{ff} - (1 - \kappa)^2 \bar{a}_{\rho c}^{fm}]. \end{aligned}$$

Note that, if $\kappa = 1$, expression (9) is equal to

$$\bar{a}_\rho \div (\bar{a}_\rho^m + \bar{a}_c^f - \bar{a}_{\rho c}^{mf}),$$

the option factor for a male pensioner aged ρ with a female contingent annuitant aged c .

If $\kappa = 0$, expression (9) is equal to

$$\bar{a}_\rho^f \div (\bar{a}_\rho^f + \bar{a}_c^m - \bar{a}_{\rho c}^{fm}),$$

the option factor for a female pensioner aged ρ with a male contingent annuitant aged c .

The number living (l^x) at age $\rho + s$ would be

$$l_{\rho+s}^\kappa = \kappa l_\rho \frac{l_{\rho+s}^m}{l_\rho^m} + (1 - \kappa) l_\rho \frac{l_{\rho+s}^f}{l_\rho^f} \quad (10)$$

and at age $\rho + s + 1$ would be

$$l_{\rho+s+1}^\kappa = \kappa l_\rho \frac{l_{\rho+s+1}^m}{l_\rho^m} + (1 - \kappa) l_\rho \frac{l_{\rho+s+1}^f}{l_\rho^f}. \quad (11)$$

Therefore, for the single life unisex mortality table,

$$p_{\rho+s}^\kappa = \frac{l_{\rho+s+1}^\kappa}{l_{\rho+s}^\kappa} = \frac{\kappa {}_{s+1}p_\rho^m + (1 - \kappa) {}_{s+1}p_\rho^f}{\kappa {}_s p_\rho^m + (1 - \kappa) {}_s p_\rho^f}.$$

Since

$${}_{s+1}p_\rho^m = {}_s p_\rho^m {}_s p_{\rho+s}^m \quad \text{and} \quad {}_{s+1}p_\rho^f = {}_s p_\rho^f {}_s p_{\rho+s}^f,$$

then

$$\begin{aligned} p_{\rho+s}^\kappa &= [\kappa {}_s p_\rho^m {}_s p_{\rho+s}^m + (1 - \kappa) {}_s p_\rho^f {}_s p_{\rho+s}^f] \div [\kappa {}_s p_\rho^m + (1 - \kappa) {}_s p_\rho^f] \\ &= \frac{\kappa {}_s p_\rho^m}{{}_\kappa {}_s p_\rho^m + (1 - \kappa) {}_s p_\rho^f} p_{\rho+s}^m + \frac{(1 - \kappa) {}_s p_\rho^f}{{}_\kappa {}_s p_\rho^m + (1 - \kappa) {}_s p_\rho^f} p_{\rho+s}^f. \end{aligned} \quad (12)$$

The derivation of the p^* 's based on the joint life unisex mortality table is analogous to formula (12).

The number living ($l_{\rho+s:c+s}^*$) at ages $\rho + s, c + s$ would be

$$l_{\rho+s:c+s}^* = \left[\kappa l_{\rho} \frac{l_{\rho+s}^m}{l_{\rho}^m} + (1 - \kappa) l_{\rho} \frac{l_{\rho+s}^f}{l_{\rho}^f} \right] \left[\kappa l_{\rho} \frac{l_{c+s}^f}{l_{\rho}^f} + (1 - \kappa) l_{\rho} \frac{l_{c+s}^m}{l_{\rho}^m} \right],$$

$$l_{\rho+s+1:c+s+1}^* = \left[\kappa l_{\rho} \frac{l_{\rho+s+1}^m}{l_{\rho}^m} + (1 - \kappa) l_{\rho} \frac{l_{\rho+s+1}^f}{l_{\rho}^f} \right] \times \left[\kappa l_{\rho} \frac{l_{c+s+1}^f}{l_{\rho}^f} + (1 - \kappa) l_{\rho} \frac{l_{c+s+1}^m}{l_{\rho}^m} \right].$$

Therefore, for the joint life unisex mortality table,

$$p_{\rho+s:c+s}^* = \frac{l_{\rho+s+1:c+s+1}^*}{l_{\rho+s:c+s}^*},$$

and it can be shown that

$$p_{\rho+s:c+s}^* = \left[\frac{\kappa {}_s p_{\rho}^m}{{}_\kappa {}_s p_{\rho}^m + (1 - \kappa) {}_s p_{\rho}^f} {}_s p_{\rho+s}^m + \frac{(1 - \kappa) {}_s p_{\rho}^f}{{}_\kappa {}_s p_{\rho}^m + (1 - \kappa) {}_s p_{\rho}^f} {}_s p_{\rho+s}^f \right] \times \left[\frac{\kappa {}_s p_c^f}{{}_\kappa {}_s p_c^f + (1 - \kappa) {}_s p_c^m} {}_s p_{c+s}^f + \frac{(1 - \kappa) {}_s p_c^m}{{}_\kappa {}_s p_c^f + (1 - \kappa) {}_s p_c^m} {}_s p_{c+s}^m \right]. \tag{13}$$

TABLE 4

PENSIONER AGE ρ	CONTINGENT ANNUITANT AGE c	100% JOINT AND SURVIVOR OPTION FACTOR			
		Unisex (1)	Separate Tables ρ = Male c = Female (2)	Separate Tables c = Male ρ = Female (3)	Weighted Average [0.70(2) +0.30(3)] (4)
65	55	.6964	.6445	.8145	.6955
65	56	.7049	.6526	.8229	.7037
65	57	.7135	.6610	.8313	.7121
65	58	.7224	.6697	.8396	.7207
65	59	.7316	.6787	.8479	.7295
65	60	.7409	.6880	.8561	.7384
65	61	.7504	.6976	.8641	.7476
65	62	.7601	.7076	.8720	.7569
65	63	.7699	.7177	.8797	.7663
65	64	.7798	.7281	.8872	.7758
65	65	.7898	.7388	.8945	.7855

It can be seen that equation (12) is a weighted average of the respective p 's based on the single life male and female mortality tables. Equation (13) is a product of two such weighted averages.

Table 4 shows the unisex 100 per cent joint and survivor option factors for retirement age $\rho = 65$ and contingent annuitant ages $c = 55, \dots, 65$, where κ , the proportion of members retiring at age 65 and selecting joint and survivor options, is 0.70. Analogous factors are also given based on the separate life mortality tables and on the weighted average (0.70–0.30) of such option factors.

DISCUSSION OF PRECEDING PAPER

RICHARD S. HESTER, SR.:

Mr. Toussaint's paper neglected one area of increasing importance in pension plans in which unisex tables should be used. This is the widow's benefit or "dead horse option." There is an extremely simple formula for making the charge against normal benefits due to election of this option, as long as one is willing to make such a charge annually.

Let

- B_x^r = Accrued benefit at age x payable at r ($r > x$);
- E_x^r = Early retirement adjustment at age x for normal retirement age r such that the benefit commencing at age x is $B_x^r E_x^r$;
- $f_{x,y}^{50}$ = Factor to be applied to a single life annuity for a life aged x in order to produce an annuity payable to a life aged x if living, with 50 per cent of the annuity payable to a life aged y if surviving.

The cost of the widow's benefit as a one-year term benefit is

$$WB_x = B_x^r E_x^r \left(\frac{1}{2}\right) f_{x,v}^{50} a_v q_x. \quad (1)$$

The reduction to be made at age r in exchange for such a benefit is

$$R_x^r = WB_x \frac{D_x}{D_r} \frac{1}{a_r}. \quad (2)$$

If E_x^r is assumed to be theoretically correct (regardless of whether such "correct" factors are actually used), then

$$E_x^r = \frac{D_r a_r}{D_x a_x}. \quad (3)$$

Substituting expression (3) for E_x^r in equation (1), and substituting for WB_x in equation (2), we have

$$R_x^r = B_x^r \frac{D_r a_r}{D_x a_x} \left(\frac{1}{2}\right) f_{x,v}^{50} a_v q_x \frac{D_x}{D_r} \frac{1}{a_r} \quad (4)$$

$$= B_x^r \left(\frac{1}{2}\right) f_{x,v}^{50} q_x \frac{a_v}{a_x}. \quad (5)$$

If $y = x$ and a unisex table is used, equation (5) can be reduced to

$$R_x^r = B_x^r \left(\frac{1}{2}\right) f_{x,x}^{50} q_x. \quad (6)$$

Equation (6) is independent of r . (For those plans which allow such a benefit beyond r , a slightly different formula, also independent of r , can be produced.) It depends solely on x and y and lends itself very nicely to a two-column table. The first column will contain values of $\frac{1}{2}f_{x,x}^{50} q_x$ and will show the reduction in benefit for every dollar of accrued benefit at age x if the spouse is the same age as the employee. The second column will contain constant additions to (or subtractions from) the first column for each year of age difference. These factors can be derived by setting $y = x - n$, calculating the required reduction from equation (5), subtracting the value in the first column, and dividing by n . A few trial values of n , within the normal range of age differences of husband and wife, will show that this does not greatly affect equity.

As to the topics covered by Mr. Toussaint, first I believe that most of us will continue to use simple early retirement factors such as $\frac{1}{2}$ per cent per month or $\frac{5}{8}$ per cent for each of the first sixty months and $\frac{5}{18}$ per cent for the next sixty months.

Second, I believe that any actuary with an eye for smoothness and fit (with, perhaps, more emphasis on the former) can produce a two-column table of joint and survivor factors and a table of annuity purchase rates that will satisfy the client, the government, and the participant's need for simplicity without having to resort to commutation columns. My reason for going through the equations above was solely to ensure some internal consistency between R_x^r and $f_{x,y}^{50}$ in much the same way that consistency should be maintained between $f_{x,y}^{50}$ and $f_{x,y}^{100}$ if both options are available.

(AUTHOR'S REVIEW OF DISCUSSION)

ROBERT C. TOUSSAINT:

In his discussion of my paper Mr. Hester referred to my omission of the application of unisex tables to widow's benefits. The main purpose of my paper was to illustrate one reasonable approach toward constructing a unisex single life mortality table and a unisex joint life mortality table. Once the table has been constructed, it is a simple matter to compute any desired factor. In my paper I discussed early retirement and joint and survivor options for illustrative purposes only.

Mr. Hester was correct in stating that, in most cases, it is more practical to use empirical factors. However, it is important first to derive the actuarially equivalent factors, based on a given set of assumptions, and then to adjust the factors to a simpler array of empirical factors or to a simple rule. This procedure is important only, of course, if our aim is to reflect actuarial equivalency in our factors.

Mr. Hester referred to *widow's benefits* in his discussion. I am sure that, as a result of ERISA, he would agree that we are now more interested in *spouse's benefits*.

Once the unisex tables are derived, Mr. Hester's method for computing empirical factors is a reasonable one. However, in order to derive his "second column," one essentially has to compute the actuarially equivalent factors and then estimate the differences for $y = x - n$ for the different values of x .

