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THE ADJUSTED ASSET BASE METHOD OF ALLOCATING INVESTMENT INCOME

THOMAS C. SUTTON

ABSTRACT

This paper describes a streamlined procedure for allocating investment income which permits the application of a single current investment-year interest rate, together with an "averaged" rollover rate, to an adjusted asset base to give essentially the same results as the traditional investment-year method. This general equivalence of results and the conditions under which it obtains are supported by a theoretical demonstration. The relationship between the adjusted asset base and market value is also explored. This procedure makes it practical to credit interest on individual contracts in a manner which recognizes the incidence of premium payments, the new-money interest rate in effect at the time of premium payment, and the rollover of invested assets.

INTRODUCTION

PRIOR to the 1960's there was little refinement in the allocation of investment income by insurance companies. The method in general use by North American companies was the mean fund method, which was based on an average company yield rate over a calendar year. Such an average rate could be applied to the mean assets of any block of funds to determine its share of the company investment income for that year. This method is simple and can easily be applied to a block of funds of any size from an entire annual statement subline to an individual contract.

In the early 1960's considerable interest developed in more refined methods of allocation which recognized the fact that funds received at different times are invested and reinvested at different rates of interest. The investment generation method accomplishes this recognition and was described by Edward A. Green in a 1961 paper, "The Case for Refinement in Methods of Allocating Investment Income" (*TSA*, XIII, 308). This more refined method is now used by many companies as an alternate to the mean fund method. However, unlike the latter, the investment generation method is difficult to apply and requires extensive record-keeping. As a result its use has usually been restricted to large blocks of funds such as Annual Statement sublines and sizable group pension cases. Within one of these large blocks, investment income is, when necessary, further allocated to subblocks by using an average yield rate (or mean fund method) for that block.

A modification of the investment generation method was presented in 1974 by Christopher D. Chapman in a paper on "Interest Allocation Using a Computer Model" (*TSA*, XXV, 331). In order to reduce the detail and complexity of application, a simplified model of a company's invested assets was substituted for the actual asset portfolio, and the investment generation method was then applied to the model assets to allocate investment income. But even with this simplification and additional pooling of investment results, the method is still practical only to allocate investment income among fairly large blocks of assets.

Currently there is a significant factor which suggests a need for a new method of investment income allocation which would (1) reflect appropriate variations in investment income due to the varying times at which funds are received and yet (2) be sufficiently simple in application that it can be extended to subblocks of small size, even to individual contracts. The 1974 Employee Retirement Income Security Act has increased considerably the market for individual annuities, and this has occurred at a time when the yields available on new investments exceed significantly the portfolio yields on blocks of in-force annuities. In order to compete effectively with other savings media, many companies have developed new annuity products under which the interest credited is well in excess of portfolio levels. Furthermore, these new contracts provide for wide flexibility in the incidence of premium payments. One approach adopted by some companies to allocate interest to these contracts is to group them by calendar year of issue and apply the investment generation method to that calendar-year block. Further allocation to individual contracts within the block is then accomplished by applying the average yield rate for the block to the mean assets for each contract. This approach also requires increased record-keeping and results in administrative difficulty and yet does not reflect variations in investment income among contracts issued in the same year arising from their different patterns of premium payment.

The adjusted asset base method of allocating investment income is a new method that reflects the incidence of receipt of funds, the yield rates of investments available at the time of receipt, and the rollover of invested assets. Further, for each subblock or contract to which investment income is to be allocated, a record of only one quantity is required—

the adjusted asset base for that subblock. Finally, both the calculation of investment income based on the adjusted asset base and the updating of the latter from one time period to the next are quite simple operations. Under certain "ideal" conditions this method produces the same results that would be obtained by using the investment generation method. To the extent that conditions are not "ideal" there is a somewhat greater "pooling" of investment results under the adjusted asset base method. The adjusted asset base for each block of assets is defined in such a way that multiplying it by the current new-money interest rate results in the amount of investment income for that block. Investment income under this procedure is relative to the initial or book value of the underlying assets, not relative to the market value of those assets. If market values were known, and if the current new-money interest rate were multiplied by the market value, the result would also be "investment income." However, the latter would include not only statutory accounting or book-value investment income but also a portion arising from the changing value of the original investment.

DEMONSTRATION OF METHOD

Example without Rollover

Consider an 8 per cent "interest only" bond purchased December 31, 1975, with interest payable at yearly intervals until December 31, 1985, when the principal is repaid. Further suppose that the new-money interest rate, which had been 8 per cent in 1975, changed to 10 per cent in 1976 and remained level for ten years, and that bond interest payments were reinvested in new 10 per cent bonds of the same type. Thus there is no rollover or return of principal during the ten-year period. The data are shown in Table 1. The book value and book interest are quite straightforward: book interest includes the $8.00 annual payments on the bond plus interest at 10 per cent on previously accumulated payments, and this total interest increases the book value. The market value, assuming that interest rates jump to 10 per cent on January 1, 1976, takes a drop on that date to a figure of $87.71. The latter is the present value at 10 per cent of future interest and principal payments on the bond. After January 1, 1976, the $87.71 can merely be accumulated at 10 per cent to obtain subsequent annual values, including the $227.50 in 1985. The market interest column shows the differences of the market-value column. The first entry, −$12.29, could be viewed as a market-value adjustment due to the change in interest rates. After that adjustment, interest credits proceed smoothly and are greater than the book interest, since over the ten-year period the −$12.29 is returned in "amortized" portions.

The market-value reduction of about 12 per cent for a 2 per cent increase in new-money interest rate is a relationship that is approximately true or at least representative of the change in many companies' portfolios, and is the genesis of the "6-for-1" rule used by some companies to approximate market values. In fact, at least one company uses the 6-for-1 rule and the market-value approach in crediting interest to group pension policyholders. This is in effect an approximate unit credit method of allocating interest (for a description of the unit credit method see

TABLE 1

Date or Year End	Book Value	Market Value	Adjusted Asset Base	Book Interest	Market Interest
December 31, 1975......	$100.00	$100.00	$100.00	....	....
January 1, 1976......	100.00	87.71	80.00	....	−$ 12.29
1976........	108.00	96.48	88.00	$ 8.00	8.77
1977........	116.80	106.13	96.80	8.80	9.65
1978........	126.48	116.74	106.48	9.68	10.61
1979........	137.13	128.42	117.13	10.65	11.68
1980........	148.84	141.26	128.84	11.71	12.84
1981........	161.72	155.39	141.72	12.88	14.13
1982........	175.90	170.92	155.90	14.18	15.53
1983........	191.49	188.02	171.49	15.59	17.10
1984........	208.64	206.82	188.64	17.15	18.80
1985........	227.50	227.50	207.50	18.86	20.68
				$127.50	$127.50
January 1, 1986......	227.50	227.50	227.50		

Chapman's paper) that overcomes the difficulty of frequent market valuations of all assets. Unfortunately, this method does not overcome the other principal drawback to this method: its inconsistency with the book-value approach to measuring investment income. In practice even a fairly sophisticated contract holder might have difficulty understanding why his $100 suddenly became $87.71. The amount of such adjustments can be reduced by applying the 6-for-1 rule more frequently, say monthly, but the conceptual problem, particularly for individual buyers, will still be there.

In contrast to the market value, the adjusted asset base does not represent a withdrawal amount or true value of the underlying assets but is a computational tool which can be used to obtain the book interest and market value. Turning back to Table 1, the adjusted asset base, like the market value, undergoes a "jump" at the time the interest rate changes. In fact, it changes in a ratio inversely proportional to the change in interest rates. Then it increases at a rate of 10 per cent per year, or by the amount of the book interest. It undergoes a final jump just as the principal on the 8 per cent bond is repaid. In such a simple situation, with one change in interest rate and no rollover or repayment of principal during the period, it is easy to see how the adjusted asset base works.

The amount actually received by the withdrawing contract holder depends on the contract provisions, but in a real sense the company has only the market value available for distribution. If the contract so specifies, the market value is payable and balance is automatically maintained:

TABLE 2

Date or Year End	Book Value	Market Value	Adjusted Asset Base	Book Interest	Market Interest
1980	$148.84	$141.26	$128.84	....	....
January 1, 1981	0	− 7.58	− 20.00	....	....
1981	− 2.00	− 8.34	− 22.00	−$ 2.00	−$ 0.76
1982	− 4.20	− 9.17	− 24.20	− 2.20	− 0.83
1983	− 6.62	− 10.09	− 26.62	− 2.42	− 0.92
1984	− 9.28	− 11.10	− 29.28	− 2.66	− 1.01
1985	− 12.21	− 12.21	− 32.21	− 2.93	− 1.11
				−$12.21	−$ 4.63
January 1, 1986	− 12.21	− 12.21	− 12.21		

if the contract calls for payment of book value, an imbalance of amount is created equal to the difference between market and book. In the latter case this difference will be spread, explicitly or implicitly, over remaining contract holders. Whatever the method, it is desirable to know the amount of difference between market and book. In the Table 1 example, suppose that the full book value was withdrawn on January 1, 1981. The various amounts could be recomputed and carried forward as shown in Table 2. That is, when the book value is withdrawn, the amount of withdrawal reduces both the market value and the adjusted asset value, so that both are negative. If the adjusted asset base is used to compute (negative) interest as described above, it accumulates to −$12.21 at bond maturity. This amount, when discounted back to withdrawal, equals the −$7.58 difference between market and book. This procedure will be referred to as a "discounted extrapolation."

Example with Rollover

Now consider a portfolio consisting solely of bonds which on each December 31 pay a stated rate of interest on the outstanding balance as of the previous December 31, along with 10 per cent of that outstanding balance. Suppose, further, that, $100 of net income was received under a particular subblock of business on December 31, 1975, and was invested immediately in bonds with an 8 per cent stipulated interest rate. As of January 1, 1976, and thereafter, all such new bonds have a stipulated interest rate of 10 per cent. Table 3 displays the results.

TABLE 3

Date or Year End	Book Value	Market Value	Adjusted Asset Base	Book Interest	Market Interest
December 31, 1975......	$100.00	$100.00	$100.00	....	....
January 1, 1976......	100.00	90.00	80.00	....	−$ 10.00
1976........	108.00	99.00	90.00	$ 8.00	9.00
1977........	117.00	108.90	100.80	9.00	9.90
1978........	127.08	119.79	112.50	10.08	10.89
1979........	138.33	131.77	125.21	11.25	11.98
1980........	150.85	144.95	139.04	12.52	13.18
1981........	164.75	159.45	154.12	13.90	14.50
1982........	180.16	175.40	170.59	15.41	15.95
1983........	197.22	192.94	188.61	17.06	17.54
1984........	216.08	212.23	208.33	18.86	19.29
1985........	236.91	233.45	229.93	20.83	21.22
				$136.91	$133.45
Ultimate*....	233.45	233.45	233.45		

* Discounted to year end 1985 at 10 per cent interest per year.

The book interest can be calculated in the usual way, using a simplified investment generation method, that is, keeping the book value in two pieces: the amount invested at 8 per cent and the amount invested at 10 per cent. The sum of the two products of the appropriate interest rate and the corresponding part of the book value gives the interest for that year. The book value itself increases each year by the amount of that interest. If $B(N)$ is the book value at the end of the Nth year, $B(N) = 100(1.1)^N - 10[(1.1)^N - (0.9)^N]$, and the interest in the Nth year is $B(N+1) - B(N)$. The $90 market value on January 1, 1976, is the present value at 10 per cent of the future stream of interest and rollover payments (close to but not quite the value predicted by the 6-for-1 rule). The $90 is accumulated thereafter at 10 per cent per year interest. As with the previous example, there is an initial "market adjustment" of −$10, which is returned in the interest on an "amortized" basis thereafter, making subsequent market interest amounts greater than the book interest amounts.

The adjusted asset base undergoes the same initial jump that it did in the previous example: that is, it is changed in inverse proportion to the interest rate change. Thereafter, however, it does not merely increase at 10 per cent as previously. Instead, it is increased not only by the interest but also by the rollover rate of 10 per cent times the previous difference between the book value and the adjusted asset base. For example,

TABLE 4

Date or Year End	Book Value	Market Value	Adjusted Asset Base	Book Interest	Market Interest
1980	$150.85	$144.95	$139.04	....	....
January 1, 1981	0	− 5.90	− 11.81	....	....
1981	− 1.18	− 6.49	− 11.81	−$ 1.18	−$ 0.59
1982	− 2.36	− 7.14	− 11.93	− 1.18	− 0.65
1983	− 3.55	− 7.85	− 12.16	− 1.19	− 0.71
1984	− 4.77	− 8.64	− 12.52	− 1.22	− 0.79
1985	− 6.02	− 9.50	− 13.00	− 1.25	− 0.86
Ultimate*	− 5.90	− 5.90	− 5.90	....	....

* Discounted to year end 1980 at 10 per cent interest per year.

in 1983 the adjusted asset base of $188.61 was calculated as follows: $170.59 + $17.06 + 0.10($180.16 − $170.59). The initial difference between the book value and the adjusted asset base was caused by a jump in interest rates; in fact, the extent of the initial difference is a measure of the size of the jump. Because of rollover of invested assets, however, the rate earned eventually on the book value will be 10 per cent—the same rate that is applied to the adjusted asset base to obtain the book interest. Hence, eventually the difference between these two will be zero. Since this diminishing difference is caused by rollover, it may not be too surprising that the difference decreases each year by a fraction equal to the rollover rate.

Now, as in the previous example, suppose that a withdrawal occurs as of year end 1980 and the book value is subtracted from the then current market value and adjusted asset base (Table 4). The subsequent amounts of book interest, adjusted asset base, and book value can be determined from an extrapolation of the process. It is fairly easy to show that the limit of this discounted value equals the adjusted asset base on January 1, 1981, times a factor of $i/(i + r)$, where i and r are the interest and rollover rates, respectively. In this example the factor would be 0.5. In a more complicated situation the factor may not be so simple in form, but the concept remains—that the market value can be obtained by taking the discounted extrapolated adjusted asset base. The general case is developed in the next section.

CONTINUOUS MODEL

Cash-Flow Functions and Book Value

Cash-flow or investable funds are composed of the following: (*a*) net cash flow generated by insurance operations, (*b*) reinvestment of assets invested at some prior time, (*c*) net investment income earned, and (*d*) net realized capital gains and losses. Let the rate of the above items at time t measured in dollars per year be denoted by $N(t)$, $R(t)$, $I(t)$, and $C(t)$, respectively. Thus, in the one-year period commencing at time y, $\int_y^{y+1} N(t)dt$ dollars of net cash flow are generated by the insurance operation. Similar expressions hold for the other functions.

Let $T(t)$ be the total rate at which funds become available for investment (or are actually invested, since all must be invested). Thus

$$T(t) = N(t) + R(t) + I(t) + C(t) . \tag{1}$$

Define $B(t)$ and $\epsilon(t)$ by the following equations:

$$\frac{dB(t)}{dt} = N(t) + I(t) + C(t) \quad \text{and} \quad B(0) = 0 \cdot \tag{2a}$$

$$\epsilon(t)B(t) = I(t) . \tag{2b}$$

$B(t)$ can be interpreted as the book value of assets, which is increased by net cash flow from insurance operations, net investment income, and realized net capital gains; $\epsilon(t)$ can be interpreted as the instantaneous value of the "portfolio" yield rate.

Among these functions, N may be regarded as the fundamental one, since it, together with characteristics of the underlying investment media, will in time give rise to R, I, C and therefore to T, B, and ϵ. All the derived functions depend upon the amounts and incidence of the net cash flow from insurance operations.

Characteristics of Underlying Assets

Assets available for investment at a given time t are assumed to have two relevant characteristic functions: (*a*) a rate of "survival" function $\gamma(t, s)$ and (*b*) a rate of return or yield function $\delta(t)$. These functions are defined relative to the book value of the associated assets as follows: (*a*) for each dollar of initial book value invested at time t, there remain, at later time s, $\gamma(t, s)$ dollars of book value still invested in the same assets; (*b*) during a small time interval ds, there will be $\delta(t)\gamma(t, s)ds$ dollars of net investment income arising from those remaining assets.

Generally, the yield function is "future time"–dependent, as is the survival function (that is, δ is a function of t and s). Further, a characteristic function for capital gains (clearly this function would be future time–dependent) could also be defined. For purposes of this development, it is assumed that $\delta(t)$ is the best estimate of the average yield rate over the entire future lifetime of investments made at time t, and thus by definition is not future time–dependent. Variations of this δ from actual future results, as well as actual capital gains, are to be reflected in the function $C(t)$, previously defined to include only the latter. As will be seen later, $C(t)$ is to be allocated in a "pooled" sense.

The survival function is to have the following restrictions:

$$\gamma(t, t) = 1 ; \tag{3a}$$

$$\frac{\partial\gamma(t, s)}{\partial s} \leq 0 \tag{3b}$$

$$\lim_{s \to \infty} \gamma(t, s) = 0. \tag{3c}$$

That is, the fraction surviving begins at unity, never increases, and eventually is zero.

The survival function $\gamma(t, s)$ can be used to determine a rate of rollover. In a small time interval ∂s, the remaining book value of a dollar of assets invested at time t will decline from $\gamma(t, s)$ to $\gamma(t, s + \partial s)$. The amount of decline is the rollover amount becoming available for reinvestment. The rate of such rollover is obtained by dividing the amount by the initial amount remaining and by the length of the time interval, that is,

$$\frac{-[\gamma(t, s) - \gamma(t, s + \partial s)]}{\gamma(t, s)\partial s}.$$

Let the limit of this fraction as ∂s approaches zero be $\bar{\gamma}(t, s)$. Then

$$\bar{\gamma}(t, s) = -\frac{1}{\gamma(t, s)}\frac{\partial\gamma(t, s)}{\partial s}. \tag{4}$$

Thus, for each \$1 of assets invested at time t, there are $\gamma(t, s)\bar{\gamma}(t, s)ds$ dollars of rollover at later time s. (The $\bar{\gamma}$ function is thus analogous to a select force of mortality.)

Basic Relationships

Consider a small time interval, du, in the past, during which $T(u)du$ dollars were invested. Due to the nature of the survival and rollover functions defined above, there will be $\gamma(u, t)\bar{\gamma}(u, t)T(u)du$ dollars of the original $T(u)du$ rolling over at time t. If these amounts are summed over all prior time, it follows that

$$R(t) = \int_0^t T(u)\gamma(u, t)\bar{\gamma}(u, t)du . \tag{5}$$

This equation may be regarded as the definition of R. Further, using equations (1), (2), (4), and (5), together with the fact that $\gamma(t, t) = 1$, the following results:

$$\begin{aligned} \frac{d}{dt}\left[\int_0^t T(u)\gamma(u, t)du\right] &= T(t)\gamma(t, t) + \int_0^t T(u)\frac{\partial\gamma(u, t)}{\partial t}du \\ &= T(t) - \int_0^t T(u)\gamma(u, t)\bar{\gamma}(u, t)du \\ &= T(t) - R(t) \\ &= \frac{d}{dt}B(t) . \end{aligned}$$

Setting $B(0) = 0$ for convenience and integrating the above,

$$B(t) = \int_0^t T(u)\gamma(u, t)du . \tag{6}$$

Of course, of the $T(u)du$ invested at time u, $T(u)\gamma(u, t)du$ are still invested in the same assets at some later time t. If all such remaining amounts are summed, the total must be the total book value of assets at time t.

It will prove convenient to define a "weighted" rollover function $\Gamma(t)$ such that

$$\Gamma(t) = \frac{\int_0^t T(u)\gamma(u, t)\bar{\gamma}(u, t)du}{\int_0^t T(u)\gamma(u, t)du} . \tag{7a}$$

From equations (5) and (6),

$$\Gamma(t)B(t) = R(t) . \tag{7b}$$

Thus $\Gamma(t)$ may be regarded as a weighted average of $\bar{\gamma}(u, t)$, where the weights are the remaining amounts of funds originally invested at time u and still invested in the same assets at later time t.

From equations (1), (2a), and (7b) it follows that

$$\frac{dB(t)}{dt} = T(t) - \Gamma(t)B(t)\,. \tag{8}$$

Define $I(t)$ as follows:

$$I(t) = \int_0^t \delta(u)T(u)\gamma(u, t)du\,. \tag{9}$$

Expressed verbally, this means that, of the funds $T(u)du$ originally invested at time u, there remain $T(u)\gamma(u, t)du$ still invested in the same assets at later time t and earning at time t net investment income at the rate of $\delta(u)$. The sum of these elements over all past time gives the total rate of net investment income.

Define another weighted rollover function $\bar{\Gamma}(t)$ such that

$$\bar{\Gamma}(t)I(t) = \int_0^t \delta(u)T(u)\gamma(u, t)\bar{\gamma}(u, t)du\,. \tag{10}$$

The integral can be interpreted as the rate at which net investment earnings are "lost" or are rolling over as a result of the rollover of previously invested assets.

Now, by differentiating equation (9) and using equation (10), the following emerges:

$$\frac{dI(t)}{dt} = \delta(t)T(t) - I(t)\bar{\Gamma}(t)\,. \tag{11}$$

Adjusted Asset Base

The adjusted asset base, $A(t)$, is defined to be such that

$$A(t)\delta(t) = I(t)\,. \tag{12}$$

It is desired that $dA(t)/dt$, and therefore $A(t)$, be calculable from "prior" information modified by two adjusting functions $X(t)$ and $Y(t)$ such that

$$\frac{dA(t)}{dt} = \frac{dB(t)}{dt} + X(t)A(t) - Y(t)[A(t) - B(t)]\,. \tag{13}$$

A condition will be imposed below that will enable X to be regarded as a rate of adjustment arising from "current changes" and Y as a rate of amortization of accumulated past adjustments.

To proceed, differentiate equation (12) and use equations (12) and (13) to eliminate $dA(t)/dt$ and $A(t)$:

$$\delta(t)\left\{\frac{dB(t)}{dt} + X(t)\frac{I(t)}{\delta(t)} - Y(t)\left[\frac{I(t)}{\delta(t)} - B(t)\right]\right\} + \frac{d\delta(t)}{dt}\frac{I(t)}{\delta(t)} = \frac{dI(t)}{dt}\,.$$

Substitute from equation (11) to eliminate $dI(t)/dt$, from (8) to eliminate $dB(t)/dt$, and from (2b) to eliminate $B(t)$, resulting in:

$$Y(t)[\epsilon(t) - \delta(t)] = [\epsilon(t)\bar{\Gamma}(t) - \delta(t)\Gamma(t)] + \epsilon(t)\left[X(t) + \frac{\delta'(t)}{\delta(t)}\right],$$

where $\delta'(t) = d\delta(t)/dt$.

Clearly a family of solutions for X and Y exists, but it is desirable to impose the criterion that, if $\delta' = 0$, then $X = 0$. This corresponds to a previously stated interpretation of X as depending on current changes. The only solution which accomplishes this is

$$X(t) = -\frac{\delta'(t)}{\delta(t)}, \tag{14}$$

$$Y(t) = \frac{\epsilon(t)\bar{\Gamma}(t) - \delta(t)\Gamma(t)}{\epsilon(t) - \delta(t)}. \tag{15}$$

Having found these adjusting functions, we clearly can reverse the derivation above in order to prove the following: With $X(t)$ and $Y(t)$ determined by equations (14) and (15), equation (13) can be used to determine $A(t)$; the function $A(t)$ so determined will satisfy equation (12).

Use of the Adjusted Asset Base to Allocate Interest

Suppose that the block of assets $B(t)$ was generated under a number of distinct contracts, and that the net cash flow from insurance operations was known to be $N^i(t)$ for the ith contract ($i = 1, 2, 3, \ldots, k$). Then define $A^i(t)$, $B^i(t)$, and $I^i(t)$ to satisfy the following:

$$\frac{dA^i(t)}{dt} = \frac{dB^i(t)}{dt} + X(t)A^i(t) - Y(t)[A^i(t) - B^i(t)]; \tag{16a}$$

$$\frac{dB^i(t)}{dt} = N^i(t) + I^i(t)\left[1 + \frac{C(t)}{I(t)}\right]; \tag{16b}$$

$$I^i(t) = A^i(t)\delta(t). \tag{16c}$$

These three equations in the three unknowns may be solved for A^i, B^i, and I^i. The procedure may be viewed as a continuous iterative process:

a) Using the known prior value (infinitesimally prior) of A^i, equation (16c) can be used to compute I^i.

b) Then, using equation (16b), the change in B^i can be determined using N^i and I^i.

c) Then the current value of B^i can be determined on the basis of its prior value and its change in value.

d) The change in A^i can be determined from equation (16a).

e) The current value of A^i can be determined, and the process repeated.

The interest I^i so determined is the presumed or allocated share of the total interest which is attributed to the ith contract. The essential decision in developing this allocated amount is that the adjusting functions X and Y are to be applied to each and every subset of assets corresponding to each and every contract. A judgment at this point about the equity or appropriateness of the method hinges on the appropriateness of this decision.

For the function X it is easy to conclude that its application to all subsets of assets is appropriate because X is independent of the amount and incidence of actual income from insurance operations, either in aggregate or by subclass. In practice, of course, "available investments" may be different depending on the aggregate amount of available funds for investment. To that extent, applying X to all subsets of assets involves a judgmental decision.

The function Y is different in that it is in general dependent on the amounts and incidence of income from insurance operations. In that sense, it is possible to compute a Y^i for each subset of assets, using the methods developed above. To the extent that Y^i deviates from Y, some "distortion" is introduced.

If the rollover rate is constant (or, more generally, if it depends only on the current time and not on past time), then Y will equal Y^i regardless of the amount or incidence of income. Or, if each contract continually contributes a regular and level periodic amount, Y will equal Y^i regardless of the size of the rollover rate. To the extent that these two situations do not hold, there will be some variance between Y and Y^i. However, a judgmental decision may be made that variations in interest allocated arising from the variance between Y and Y^i are not sufficiently significant to warrant a more complex method.

The factor $[1 + C(t)/I(t)]$ in equation (16b) allocates realized net capital gains in proportion to the investment income. If the relative magnitude of such gains is small, a judgmental decision may be made that this method is adequate. As previously discussed, $C(t)$ also includes any variations arising from differences between the estimate δ and the actual rate of return. Thus these differences (for example, arising from mortgage prepayment) are pooled and allocated along with capital gains in proportion to the investment income as developed above.

A Special Case

Suppose that $\partial\bar{\gamma}(t, s)/\partial t = 0$, that is, the rollover rate depends only on the time at which rollover occurs, not on the time at which the investments now subject to rollover were made. From equation (7a) it follows that $\Gamma(t) = \bar{\gamma}(u, t)$; and from equations (9) and (10) it follows that $\bar{\Gamma}(t) = \bar{\gamma}(u, t)$; thus $\bar{\Gamma}(t) = \Gamma(t)$, and from equation (15)

$$Y(t) = \Gamma(t) = \bar{\Gamma}(t) = \bar{\gamma}(u, t) \quad \text{for any} \quad u \leq t .$$

Thus in this case the adjusted asset base is subject to a continuous modification due to (*a*) the current fractional rate of change of the "interest rate" and (*b*) amortization of all prior adjustments at a rate equal to the current rollover rate on underlying assets. Even without the restricting assumption of this special case, Y may be regarded as an aggregate weighted rollover rate, and a statement similar to the above can be made. Further in this special case, Γ and $\bar{\Gamma}$ and thus Y are not dependent on the amount or incidence of previously invested funds.

Market Value

Consider the portfolio of invested assets at time t. If no further cash flows from insurance operations occur, and if no further reinvestments of assets take place, the portfolio may be regarded as equivalent to a future income stream. At some future time s, the rate of the income stream will be the sum of interest, $I(s)$, and funds rolling over, $R(s)$ (again assume no future capital gains), where:

$$R(s) = \int_0^t T(u)\gamma(u, s)\bar{\gamma}(u, s)du$$

$$I(s) = \int_0^t T(u)\delta(u)\gamma(u, s)du .$$

The value at time t of this stream of future income can be determined by discounting the stream of amounts with interest at the current rate $\delta(t)$. This resulting value is the equivalent single sum or market value of the future income stream and hence of the portfolio at time t. Denote this market value by $M(t)$.

$$\begin{aligned} M(t) &= \int_t^\infty e^{-\delta(t)(s-t)}[R(s) + I(s)]ds \\ &= \int_t^\infty e^{-\delta(t)(s-t)} \int_0^t T(u)\gamma(u, s)[\bar{\gamma}(u, s) + \delta(u)]duds \qquad (17) \\ &= \int_0^t T(u)\gamma(u, t) \int_t^\infty \frac{\gamma(u, s)}{\gamma(u, t)} e^{-\delta(t)(s-t)}[\bar{\gamma}(u, s) + \delta(u)]dsdu . \end{aligned}$$

In order to make this expression more manageable, define a new function

$\lambda(u, t)$ such that:

$$\lambda(u, t) = \int_t^{\infty} e^{-\delta(t)(s-t)} \frac{\gamma(u, s)}{\gamma(u, t)} ds . \tag{18}$$

Verbally, $\lambda(u, t)$ is the present value at time t of the future rollover and interest income stream arising for each \$1 of funds remaining at time t after being invested at time u. Further, using equation (4) and integrating by parts, we have the following:

$$1 - \delta(t)\lambda(u, t) = \int_t^{\infty} e^{-\delta(t)(s-t)} \frac{\gamma(u, s)}{\gamma(u, t)} \bar{\gamma}(u, s) ds . \tag{19}$$

Adding equation (19) and the product of $\delta(u)$ and equation (18) gives

$$1 + [\delta(u) - \delta(t)]\lambda(u, t) = \int_t^{\infty} \frac{\gamma(u, s)}{\gamma(u, t)} e^{-\delta(t)(s-t)}[\bar{\gamma}(u, s) + \delta(u)]ds . \tag{20}$$

Substituting this on the right-hand side of equation (17) and using equation (6), we finally obtain

$$M(t) = B(t) + \int_0^t T(u)\gamma(u, t)\lambda(u, t)[\delta(u) - \delta(t)]du . \tag{21}$$

It will prove interesting to develop an expression for the derivative of $M(t)$. Again, the form of expression will be simplified by using two new quantities defined below:

$$\bar{\lambda}(u, t) = \int_t^{\infty} e^{-\delta(t)(s-t)} \frac{\gamma(u, s)}{\gamma(u, t)} (s - t)ds \tag{22a}$$

$$\phi(t) = \frac{\int_0^t T(u)\gamma(u, t)\{\bar{\lambda}(u, t)[\delta(u) - \delta(t)] + \lambda(u, t)\} du}{\int_0^t T(u)\gamma(u, t)\{\lambda(u, t)[\delta(u) - \delta(t)] + 1\} du} . \tag{22b}$$

The first quantity is similar in form to λ and the second, while appearing rather peculiar, will reduce to a very simple form in the special case. But when we differentiate equation (21) and substitute (22a), (22b), and (20), the following emerges:

$$\frac{dM(t)}{dt} = N(t) + \delta(t)M(t) - \delta'(t)\phi(t)M(t) . \tag{22c}$$

Verbally, the change in market value equals new cash plus interest on the market value less a fractional amount of market value if the rate of interest is changing. If the 6-for-1 rule held exactly, $\phi(t)$ would equal 6.

In discussing the example, the phrase "discounted extrapolated amount" was used. More precisely, the discounted extrapolated market value would be

$$\lim_{s\to\infty} e^{-\delta(t)(s-t)} M(s) \,. \tag{23}$$

But if no further cash flow from insurance occurs and interest rates do not change after time t, then equation (22c) simplifies to $dM(s)/ds = \delta(t)M(s)$. Integrating gives $M(s) = M(t)e^{\delta(t)(s-t)}$, so that $e^{-\delta(t)(s-t)}M(s) = M(t)$. Thus expression (23) is merely the limit of a constant, $M(t)$. Further, equation (21) can be written as

$$M(s) - B(s) = \int_0^t T(u)\, \lambda(u, s)\, \gamma(u, s)[\delta(u) - \delta(t)]du \,.$$

The limit of the right-hand side as s becomes large is zero, since the limit of $\gamma(u, s)$ as s becomes large is zero and all other integrands are bounded. Hence

$$\lim_{s\to\infty} e^{-\delta(t)(s-t)}B(s) = \lim_{s\to\infty} e^{-\delta(t)(s-t)}M(s) = M(t) \,.$$

Stated verbally, the discounted extrapolated book value is the market value.

Market Value with Constant Rollover

Suppose that rollover rates are constant and equal to $\bar{\gamma}$. Then from equation (4) it follows that:

$$\gamma(t, s) = e^{-\bar{\gamma}(s-t)} \,, \tag{24a}$$

and substituting this in equation (18) and (22a) and simplifying gives

$$\lambda(u, t) = \frac{1}{\delta(t) + \bar{\gamma}} \,; \tag{24b}$$

$$\lambda(u, t) = \frac{1}{[\delta(t) + \bar{\gamma}]^2} \,. \tag{24c}$$

Substituting these in equation (22b) gives

$$\phi(t) = \frac{1}{\delta(t) + \bar{\gamma}} \,. \tag{25}$$

This can be related to the example shown in Table 4, when extended to the discrete case. Formula (22c) indicates that the fractional change in market value due to a change in interest rate is $-\delta'(t)\phi(t)$, or $-\delta'(t)/[\delta(t) + \bar{\gamma}]$. Applying this "intuitively" to the discrete case gives $-(0.10 - 0.08)/(0.10 + 0.10) = -0.1$. That is, the market value decreases by 10 per cent. Relating again to the 6-for-1 rule, clearly, if it were valid, $\frac{1}{6} = \delta(t) + \bar{\gamma}$, or the sum of the rollover rate and interest rate is about 17 per cent. For some companies this has been true with, say, interest at 8 per cent and rollover at 9 per cent.

Adjusted Asset Base and Market Value

Again, if no further cash flow from insurance occurs and if no interest rate changes occur after time t, equation (13) with rearrangement becomes

$$\frac{d}{ds}[A(s) - B(s)] = -Y(s)[A(s) - B(s)] .$$

Integrating gives

$$A(s) - B(s) = [A(t) - B(t)] \exp\left[-\int_t^s Y(u)du\right] . \qquad (26)$$

From equation (15) and the definitions of the functions in (15) it is clear that Y is bounded and positive, so that, as s increases without limit, the right-hand side tends toward zero and so A tends toward B. Hence the discounted extrapolated adjusted asset base equals the discounted extrapolated book value, which equals the market value at time t.

Differentiate equation (26) with respect to s, substitute $\delta(t)A(s)$ for $B'(s)$ (this is permissible, since $N(s)$ is assumed zero and $I(s) = \delta(t)A(s)$), and integrate using standard techniques to obtain

$$A(s)e^{-\delta(t)(s-t)} \qquad (27)$$
$$= A(t) - [A(t) - B(t)]\int_t^s Y(v) \exp\left\{-\int_t^v [Y(u) + \delta(t)]du\right\}dv .$$

The limit of the left-hand side as s increases is $M(t)$, so that, if the limit of equation (27) is taken and $B(t)$ subtracted from both sides,

$$M(t) - B(t) \qquad (28)$$
$$= [A(t) - B(t)]\left[\!\left[1 - \int_t^\infty Y(v) \exp\left\{-\int_t^v [Y(u) + \delta(t)]du\right\}dv\right]\!\right] .$$

Adjusted Asset Base with Constant Rollover

With constant rollover, $\bar{\gamma} = Y$, equation (28) becomes

$$M(t) - B(t) = [A(t) - B(t)]\frac{\delta(t)}{\bar{\gamma} + \delta(t)} .$$

This can be used as the basis of an approximation to obtain the market value from the adjusted asset base: use the current value of Y as an estimate of a level future rollover rate, and let

$$M^*(t) = B(t) + [A(t) - B(t)]\frac{\delta(t)}{Y(t) + \delta(t)} . \qquad (29)$$

$M^*(t)$ is an estimate of $M(t)$.

DISCONTINUOUS MODEL

In practice, a discontinuous formulation is necessary for a period of specified duration, and can be obtained either as a special case of the continuous model, or developed separately along the same lines as above.

The analogue of equation (13) will be

$$\Delta A_{n-1} = \Delta B_{n-1} + X_{n-1}A_{n-1} - Y_{n-1}(A_{n-1} - B_{n-1}) . \qquad (30)$$

However, the exact form of X and Y will depend on assumptions made concerning the incidence of rollover (i.e., at the beginning or end of the period) and the treatment of interest during the period. One analogue of equation (12) can be

$$I_n = (A_{n-1} + \bar{N}_n)\delta_n , \qquad (31)$$

where $\bar{N}_n$ is the "time-weighted" value of the cash flow during the period. With this particular form, it is necessary to have A^i_{n-1} to distribute I_n, the interest in the Nth year. To compute the A^i_{n-1} from equation (30) it is necessary to have A^i_{n-2}, B^i_{n-2}, B^i_{n-1}, X_{n-2}, and Y_{n-2}. However if I_n, δ_n, and $\bar{N}_n$ are known, A_{n-1} can be computed from equation (31). Substituting this value in equation (30) gives an equation with two unknowns, X_{n-2} and Y_{n-2}. If X_{n-2} is easily computed from the new-money interest rates as it is in the continuous model, then the equation can be solved for Y_{n-2}. Then, with the value of Y_{n-2} so obtained, equation (30) can be used to determine the A^i_{n-1}'s and equation (31) can be used to determine the I^i_n's. Applying the method in this manner, one does not need to know or compute the analogues of Γ, $\bar{\Gamma}$, or ϵ to determine Y.

CONCLUSION

The adjusted asset base method of allocating investment income has the following characteristics:

1. It reflects variations in investment income due to the varying times at which funds are received.
2. It results in an allocation similar to that which would result under the investment generation method.
3. It is consistent with the book value measurement of assets.
4. It is very simple to administer.
5. It provides an approximation to the current market value.

Although the method was intended to allocate interest among individual annuities, it may have wider applications. If the greater degree of pooling of results under the adjusted asset base method as compared with the investment generation method is acceptable, the former could be used to allocate interest by subline (excluding the allocation of interest already earmarked by subline, such as policy loans). In this case a further refinement of the method is possible by permitting Y to vary for large blocks of assets. For example, in the continuous case $Y = (\epsilon\bar{\Gamma} - \delta\Gamma)/(\epsilon - \delta)$, where the symbols on the right apply to the entire block of assets. However, ϵ for a particular subblock may be approximated by extrapolating past values of ϵ. This value can then be used together with aggregate values for Γ, $\bar{\Gamma}$, and δ, thus providing a closer estimate of Y for the subblock. The total adjusted asset base computed with these Y's according to equation (13) may not satisfy equation (12) exactly, so that it may be necessary to apply an aggregate "off-balance" factor to adjust each Y.

DISCUSSION OF PRECEDING PAPER

CHARLES E. FARR:

Early in his timely paper Mr. Sutton comments that "at least one company uses the 6-for-1 rule and the market-value approach in crediting interest to group pension policyholders." The historical development of this practice and a few words as to its effect on group pension contract holders and participants may be of interest.

The first insured profit-sharing plans used the conventional group annuity contract as the funding vehicle. This generally involved the purchase each year of an amount of deferred annuity for each participant, the premium being the amount of the contribution allocated to the participant. Later the form of the contract was changed so that an account was maintained for each participant. The amount in the account was equal to the accumulation with interest of the net contributions made in his behalf, and was applied at an appropriate annuity rate at retirement.

In the early years of this latter arrangement, distributable surplus under participating contracts was determined annually and added to the plan contribution of the following year for allocation to then-existing participants. This distributable surplus generally consisted of portfolio average interest earnings in excess of guarantees and mortality gain or loss from retired lives.

The incidence of the allocation of distributable surplus to individual participants was altered with the practice, begun a little later, of determining a dividend interest rate to be credited to continuing participants' accounts after the close of each yearly accounting period. From this interest rate was derived the interest assumption for the annuity purchase rate at which the accounts of retiring participants would be applied. An interim interest accumulation rate was chosen for any required determination of account values during the accounting period.

The investment-year method of allocating investment income to a group annuity contract, as opposed to the portfolio average interest method, posed no significant problems in this process. Its use, however, once again changed the incidence of the allocation of distributable surplus to individual participants, but this time starting at the contract level. The dividend interest rate to be credited to continuing participants' accounts became a function of the interest credited to the group contract, based on its own money flow over the years. An interest rate for interim benefit payments was still necessary, related to the aggregate equivalent of the portion of the investment-year family applicable to the contract.

Introduction of the investment-year method brought another concept to the foreground, namely, that of market value in the general account as opposed to book value. The methods discussed to this point involved only participants' accounts held at book value. The aggregate interest rate credited to a participant's account under the investment-year method reflected the effects of the plan's plus and minus cash flows at different interest rates. When a terminating participant took his book-value account from such a plan, a gain or loss was introduced, to be shared by remaining participants over future years. The gain or loss was a function of the aggregate rate and of future reinvestment and new-money rates. In fact, the interest credited to this terminating participant's account in prior years had also been affected by similar cash withdrawals of others.

Two other developments deserve mention. One is the lump-sum disbursement of plan funds for the purpose of moving those funds to another insurer or to an uninsured arrangement. This option, as an alternative to payments spread over a period of years, permitted a more rapid movement of plan funds by the plan sponsor. It also introduced the concept of a market value as the fair value to be transferred at the direction of the terminating group contract holder.

The other development involved the concepts of allocation of "shares" in a common stock separate account, and group contracts permitting a decision at the participant level to transfer part of an existing general account accumulation to the separate account. Logic leads to the conclusion that a market value is also the fair amount for a transfer in this situation.

It can easily be concluded that maintenance of participants' accumulations at market value in the general account as well as in the separate account is a way to preserve equity among continuing, transferring, and terminating participants or among groups of participants. "Equity" as used here involves less pooling than in previous approaches to allocating investment results to plan participants. It involves both new-money interest and change in asset value, the combination of which becomes "investment income." The 6-for-1 method of adjusting asset values at the time of a change in interest rate is a convenient basis that has sufficient accuracy to be feasible.

DAVID G. ADAMS:

My discussion of Mr. Sutton's paper is directed toward the use of this method with individual deferred annuity contracts. It appears to me that using the adjusted asset base method to allocate net investment income to individual deferred annuity contracts will pose two problems:

1. Maintaining a record of the adjusted asset base for each contract could prove to be administratively burdensome, particularly if an investment rollover assumption is used.
2. The adjusted asset base allocates total net investment income. This is appropriate for annual statement purposes but is generally not appropriate for updating policy values on individual annuities. In crediting interest to individual annuities, most companies maintain a spread between the net rate earned on invested assets and the "current" rate credited to contracts. In this way, companies can provide for expenses not covered by explicit expense charges deducted from the premium and generate a profit for the company. Having determined the spread needed to achieve a particular profit objective, a company will need some method of monitoring the "portfolio" rate on annuity assets to indicate when the current rate must be adjusted to maintain the desired spread.

To minimize these problems, I suggest allocating net investment income for annual statement purposes by using an annuity portfolio rate applied to the book value of annuity assets. This portfolio can be compared periodically to the current interest rate credited to cash values to verify that the appropriate spread is being maintained.

To demonstrate that the portfolio rate approach will achieve the same results as the adjusted asset base method, I will refer to the author's example in his Table 3. In that table, net investment income (or book interest) is equal to the adjusted asset base multiplied by the new-money rate. As the author suggests in equation (2b), book interest is also equal to the portfolio yield rate multiplied by the book value of assets. Solving for the portfolio yield rate in Table 3 produces the values shown in the accompanying tabulation. Note that the portfolio rate for a particular year is, in effect, a weighted average yield rate, reflecting the amounts invested at each rate. This suggests the possibility of determining the portfolio rate directly, on the basis of annuity cash flow and the corresponding new-money rates.

Determination of the book value of annuity assets and the weighted average portfolio rate would involve the same cash-flow items suggested by the author: net cash flow from operations, reinvested assets, net investment income, and realized capital gains and losses. Certain of the cash-flow items could be identified directly from accounting records, while others would have to be estimated. Just as with the adjusted asset base method, the degree of precision used in assessing cash flow will determine the degree of accuracy in the end result. Significantly, the book value of annuity assets and the weighted average new-money rate are determined for the block of business as a whole, and no extra records are required for individual contracts.

To summarize, the investment income allocable to the annuity line can be determined by applying the annuity portfolio rate (determined as a weighted average new-money rate) to the book value of annuity assets. The portfolio rate approach offers reduced record-keeping and a built-in profitability monitor.

Year End	Assets Invested		Book Value	Book Interest	Portfolio Yield Rate
	At 8%	At 10%			
1975	$100.00	$ 0.00	$100.00	....	....
1976	90.00	18.00	108.00	$ 8.00	8.00%
1977	81.00	36.00	117.00	9.00	8.33
1978	72.90	54.18	127.08	10.08	8.62
1979	65.61	72.72	138.33	11.25	8.85
1980	59.05	91.80	150.85	12.52	9.05
1981	53.14	111.61	164.75	13.90	9.21
1982	47.83	132.33	180.16	15.41	9.35
1983	43.05	154.17	197.22	17.06	9.47
1984	38.74	177.34	216.08	18.86	9.56
1985	34.87	202.04	236.91	20.83	9.64

(AUTHOR'S REVIEW OF DISCUSSION)

THOMAS C. SUTTON:

Mr. Farr's presentation of the history and effect of the 6-for-1 rule is a valuable addition to the paper, and I thank him. His addition is particularly appropriate because the idea for the adjusted asset base method was in large measure generated by the 6-for-1 rule, or at least by its two principal features as I saw them.

The first feature was practicality. Indeed, the motivation for developing the adjusted asset base method was entirely pragmatic, a fact which has undoubtedly been obscured by the formulas in the paper. However, at my company we felt a pressing need for an individual annuity with interest credits based on an investment-year approach. We wanted to maintain the natural equity of such an approach in some manner that was relatively simple to administer and control. The 6-for-1 rule was an alternative considered, but it was rejected because it implies immediate recognition of realized and unrealized capital gains or losses, a procedure as yet unusual for individual annuities. The second feature of the 6-for-1 rule really accounted for its simplicity—the calculation of investment income as the product of the current new-money rate and an "asset base" that is easy to obtain. This feature led to a "no-rollover" illustration similar to that shown in Table 1 of the paper, and subsequently to the modification incorporating rollover. The equations for the general case followed after the basic ideas were set.

In actual application to individual annuities we use an interest rate based on current new-money rates but reduced by provision for federal income tax, overhead expense, liquidity charge, and margin.

Mr. Adams' main point is that a portfolio rate allocation basis is simpler than the adjusted asset base method. Undoubtedly this is true. However, Mr. Adams further states that the two methods achieve the same results, basing this conclusion on my equation (2b), which defines a portfolio rate for the entire block of business as the net investment income for that block divided by the corresponding asset book values. Mr. Adams' conclusion is right in the sense that the total investment income for a block of business adds up to the same amount regardless of how it is allocated among smaller units within that block. This fact is not the result of a method but rather a general criterion imposed on any method. The results of a method are measured by the relative allocations to those smaller units under one allocation method as compared with another. To make this clear, assume that the initial $100 deposit in Mr. Adams' example was made on behalf of Policy A and, further, that a deposit of $100 on behalf of Policy B was made on January 1, 1976. Then the 1976 results for each policy and for the two policies combined are as follows:

	Assets Invested*		Book Value	Book Interest	Portfolio Yield Rate
	At 8%	At 10%			
Policy A	$100	$ 0	$100	$ 8	....
Policy B	0	100	100	10	....
Total	$100	$100	$200	$18	9.00%

* At beginning of year.

At the end of the year the known quantities include the deposits made on behalf of each contract and the total book interest earned during the year. The portfolio approach involves computing the portfolio yield rate, which is 9 per cent, and then applying that rate to the mean assets of each policy to obtain its interest allocation. This clearly results in a $9 allocation to each policy. The objective of an investment-year approach (and of the adjusted asset base approach) is to allocate the $18 in such a way as to obtain approximately $8 for Policy A and $10 for Policy B, bearing in mind that the results of this allocation are not known a priori.