TRANSACTIONS OF SOCIETY OF ACTUARIES
1976 VOL. 28

A PROBABILISTIC MODEL FOR (LIFE) CONTINGENCIES AND A DELTA-FREE APPROACH TO CONTINGENCY RESERVES

HANS U. GERBER

ABSTRACT

The first part of the paper is an attempt to give to some ideas of classical life contingencies a common probabilistic denominator. In the second part, *generalized reserves*, and in particular *exponential reserves*, are developed. Like Horn's system, exponential reserves produce a release of risk. Unlike Horn's system, they are compatible with risk theory.

I. INTRODUCTION

THE purpose of this paper is twofold. In the first part some of the main principles of classical life contingencies are generalized and formulated in the language of probability theory. This is an extension of the line of thought that has been initiated by others [9, 12]. In order to avoid technicalities, the discrete time approach has been chosen.

In the second part of the paper the concept of generalized reserves is developed. In particular, exponential reserves seem to be a meaningful alternative to the system developed by Horn [10]. Both systems show a release of risk. While Horn's system requires the determination of the deltas, exponential reserves require the determination of the parameter a. Since a can be interpreted as an adjustment coefficient (a term familiar to every Part 5 student; see [14]), its determination is an easy task. Exponential reserves are related to premium calculation principles that are based on an exponential utility function (see [5, 6, 7]).

II. WHAT IS A POLICY?

For our purposes a policy will consist of a sequence of real random variables X_0, X_1, X_2, . . . and a sequence of random vectors S_0, S_1, S_2, . . . (say k-dimensional vectors) such that X_t is measurable with respect to the σ-field that is generated by $S_0, S_1, \ldots, S_t$. Intuitively, X_t is the balance (benefits plus expenses, etc., minus premiums) payable at time t. Furthermore, the values of S_0, S_1, . . . , S_t provide information about the development of the risk (or, more generally, of the "world") up to time t. Then the interpretation of the measurability condition above is simply that the cash flow at any time is a function of past and present states of the risk but is otherwise independent of the future. Supposedly the policy is initiated at time 0, so we assume that S_0 and X_0 are constant. The following examples illustrate this concept.

Example 1: Consider a whole life policy (issue age x, face amount z, annual premium P). Disregarding expenses and so on, set

$$\begin{aligned} S_t &= 1 \quad \text{if the person survives to age } x+t \\ &= 0 \quad \text{if the person does not survive to age } x+t \end{aligned} \tag{1}$$

and

$$\begin{aligned} X_t &= -P \quad && \text{if } S_t = 1 \\ &= z && \text{if } S_{t-1} = 1 \text{ and } S_t = 0 \\ &= 0 && \text{if } S_{t-1} = 0 \,. \end{aligned} \tag{2}$$

Observe that $S_0 = 1$ and $X_0 = -P$ are constant.

Example 2: A policy provides the following survivorship benefit for k lives: as long as person 1 is alive, and as long as at least one of the other $k-1$ persons is alive, an annual premium of P is payable. If person 1 is dead, person i receives an annuity of a_i ($i = 2, 3, \ldots, k$). Here it is convenient to define S_t as a vector,

$$S_t = (S_t^1, S_t^2, \ldots, S_t^k) \,, \tag{3}$$

where

$$\begin{aligned} S_t^i &= 1 \quad \text{if person } i \text{ is alive at the end of policy year } t \\ &= 0 \quad \text{if person } i \text{ is dead at the end of policy year } t \end{aligned} \tag{4}$$

($i = 1, 2, \ldots, k$). Then

$$\begin{aligned} X_t &= -P \quad && \text{if } S_t^1 = 1 \text{ and } \sum_{i=2}^{k} S_t^i \geq 1 \\ &= 0 && \text{if } \sum_{i=2}^{k} S_t^i = 0 \\ &= \sum_{i=2}^{k} a_i S_t^i && \text{if } S_t^1 = 0 \,. \end{aligned} \tag{5}$$

Example 3: The reader is invited to choose any discrete time policy from [11] (including the ones described in chap. 16, "A Generalized Model") and to define it in terms of two sequences $\{S_t\}$ and $\{X_t\}$.

Example 4: This example relates to casualty insurance. Let S_t denote the claims to be covered in policy year t ($S_0 = 0$), and let P_0 be the initial premium. Then a credibility premium payable at time t would be of the form

$$P = \frac{S_1 + \ldots + S_t}{t + c} + \frac{c}{t + c} P_0 , \tag{6}$$

where c is some constant (see [2]). Thus we may set $X_t = S_t - P_t$.

We conclude this section with a technical remark. In order to avoid any discussions about convergence, we shall assume that the random variables X_t are of bounded range and that only finitely many among them are not identically zero. From a practical point of view, this is not a real restriction.

III. RESERVES AS EXPECTED VALUES

For simplicity assume a constant rate of interest i, with a discount factor $v = 1/(1 + i)$. (More generally, one could assume that the rate of interest for policy year t is a random variable that is measurable with respect to the σ-field generated by $S_0, S_1, \ldots, S_t$.)

The *initial reserve* at time t, say V_t, is defined as a conditional expected value,

$$V_t = E\Big[\sum_{n=t+1}^{\infty} v^{n-t} X_n \,|\, S_0, \ldots, S_t\Big] . \tag{7}$$

Remarks

1. This definition corresponds to the *prospective formulas* for reserves. There is no obvious way to adapt the *retrospective formulas* to our model.
2. Since we do not distinguish formally between benefits and premiums, we can define only the *initial reserve* (rather than the *terminal reserve*).
3. V_t, like all the other random variables that are indexed by t, is measurable with respect to $S_0, \ldots, S_t$.

We shall now derive a general *recurrence relation between successive reserves* (special cases of this formula can be found in [11]; see sec. 5 of chap. 5). First apply the iterative rule for expectations and rewrite definition (7) as follows:

$$V_t = E\Big[E\Big[\sum_{n=t+1}^{\infty} v^{n-t} X_n \,|\, S_0, \ldots, S_{t+1}\Big] \Big| S_0, \ldots, S_t\Big] . \tag{8}$$

Now split off the first term of this sum and make use of the fact that X_{t+1} is a function of $S_0, \ldots, S_{t+1}$:

$$V_t = E\Big[vX_{t+1} + vE\Big[\sum_{n=t+2}^{\infty} v^{n-t-1}X_n \,|\, S_0, \ldots, S_{t+1}\Big] \Big| S_0, \ldots, S_t\Big]. \quad (9)$$

But the conditional expected value inside is V_{t+1}. Thus

$$V_t = E[vX_{t+1} + vV_{t+1} \mid S_0, \ldots, S_t], \quad (10)$$

which is the desired relationship. (Strictly speaking, eq. [10] as well as most of the other equations in this paper holds only "almost surely"; for practical purposes this is an irrelevant qualification and can be ignored.)

For numerical purposes the recursive formula (10) is very handy: it is easily programmable, and avoids the use of commutation functions and other special devices.

Let us denote the present values of X_t and V_t at time $t = 0$ by Y_t and W_t:

$$Y_t = v^t X_t, \quad (11)$$

$$W_t = v^t V_t. \quad (12)$$

Thus equation (10) can be rewritten as follows:

$$W_t = E[Y_{t+1} + W_{t+1} \mid S_0, \ldots, S_t]. \quad (13)$$

With the understanding that $L_0 = X_0 + V_0$, define

$$\begin{aligned} L_t &= Y_t + W_t - W_{t-1} \\ &= v^t[X_t + V_t - (1+i)V_{t-1}], \end{aligned} \quad (14)$$

which is the present value of the loss incurred at time t ($t = 0, 1, \ldots$).

IV. HATTENDORF'S THEOREM REVISITED

As an application we derive Hattendorf's theorem (see [9], for example). A simple proof is based on the following properties of the sequence $\{L_t\}$.

LEMMA 1. (*a*) *For* $t = 0, 1, 2, \ldots$ *and* $h = 1, 2, \ldots$, $E[L_{t+h} \mid S_0, \ldots, S_t] = 0$. (*b*) Cov $(L_t, L_{t'}) = 0$ *for* $t \neq t'$.

Proof:

a) A glance at equations (13) and (14) shows that

$$\begin{aligned} &E[L_{t+h} \mid S_0, \ldots, S_{t+h-1}] \\ &\qquad = E[Y_{t+h} + W_{t+h} \mid S_0, \ldots, S_{t+h-1}] - W_{t+h-1} \quad (15) \\ &\qquad = 0. \end{aligned}$$

Using this and the iterative rule for expectation, we see that

$$E[L_{t+h} \mid S_0, \ldots, S_t]$$
$$= E[E[L_{t+h} \mid S_0, \ldots, S_{t+h-1}] \mid S_0, \ldots, S_t] \qquad (16)$$
$$= 0 .$$

b) Suppose $t < t'$. Since $E[L_{t'}] = 0$,

$$\operatorname{Cov}(L_t, L_{t'}) = E[L_t L_{t'}] . \qquad (17)$$

Because of the iterative rule, this is

$$E[E[L_t L_{t'} \mid S_0, \ldots, S_t]] . \qquad (18)$$

Since L_t is a function of $S_0, \ldots, S_t$, this is the same as

$$E[L_t E[L_{t'} \mid S_0, \ldots, S_t]] . \qquad (19)$$

Using part *a*, we see that the conditional expected value vanishes. Q.E.D.

HATTENDORF'S THEOREM. *Let $n < m$. Then*

$$\operatorname{Var}\left[\sum_{t=n}^{m} L_t\right] = \sum_{t=n}^{m} \operatorname{Var}[L_t] .$$

Since the L_t's are by no means independent, this result has come as a surprise to many people. Its proof follows immediately from the formula

$$\operatorname{Var}\left[\sum_{t=n}^{m} L_t\right] = \sum_{t=n}^{m} \operatorname{Var}[L_t] + 2 \sum_{n \le t < t' \le m}\sum \operatorname{Cov}(L_t, L_{t'}) \qquad (20)$$

and part *b* of Lemma 1.

V. GENERALIZED RESERVES

Generalized reserves are based on the concept of *reserving principles*. A reserving principle R is a functional that assigns a real number to any random variable Y of bounded range (given by its distribution). Symbolically,

$$Y \rightarrow R(Y) . \qquad (21)$$

Thus, mathematically, a reserving principle is the same as a principle of premium calculation (see [2]). Some examples are

a) *The expected value principle:*

$$R(Y) = E[Y] ;$$

b) *The variance principle:*

$$R(Y) = E[Y] + \alpha \operatorname{Var}[Y] ;$$

c) The exponential principle:

$$R(Y) = \frac{1}{a} \ln E[e^{aY}] , \quad a \neq 0 .$$

Then, given a reserving principle R, define the present value of the generalized reserves in the following way:

$$W_t^R = R\left(\sum_{n=t+1}^{\infty} Y_n \,|\, S_0, \ldots , S_t \right) . \tag{22}$$

This means that the functional R is applied to the conditional distribution of $\Sigma_{n=t+1}^{\infty} Y_n$ (given $S_0, \ldots , S_t$).

Now the generalized reserve at time t, V_t^R, is defined from the relation $W_t^R = v^t V_t^R$:

$$V_t^R = (1 + i)^t R\left(\sum_{n=t+1}^{\infty} v^n X_n \,|\, S_0, \ldots , S_t \right) . \tag{23}$$

Observe that in example *a* above these formulas bring us back to formulas (7) and (12) of Section III. If the variance principle is applied, we obtain formulas that are related to recent suggestions in *ARCH* (see [3, 8]). The author has a strong preference for the exponential principle, which leads to the formulas

$$W_t^R = \frac{1}{a} \ln E\left[\exp\left(\sum_{n=t+1}^{\infty} a Y_n \right) \Big| \, S_0, \ldots , S_t \right] \tag{24}$$

and

$$V_t^R = \frac{1}{av^t} \ln E\left[\exp\left(\sum_{n=t+1}^{\infty} av^n X_n \right) \Big| \, S_0, \ldots , S_t \right] . \tag{25}$$

In this case we shall speak of *exponential reserves.*

VI. WHY EXPONENTIAL RESERVES?

For computational purposes as well as for theoretical reasons it is desirable that the generalized reserves satisfy a recurrence relation in the spirit of formulas (10) and (13). Therefore our *postulate* is that

$$W_t^R = R(Y_{t+1} + W_{t+1}^R \,|\, S_0, \ldots , S_t) \tag{26}$$

or, equivalently, that

$$V_t^R = (1 + i)^t R(v^{t+1} X_{t+1} + v^{t+1} V_{t+1}^R \,|\, S_0, \ldots , S_t) . \tag{27}$$

Thus the expected value principle satisfies this postulate. But so does the exponential principle: from formula (24) and the iterative rule for expectations we obtain

$$W_t^R = \frac{1}{a}\ln E\Big[E\Big[\exp\Big(\sum_{n=t+1}^{\infty} aY_n\Big)\Big| S_0, \ldots, S_{t+1}\Big]\Big| S_0, \ldots, S_t\Big]. \quad (28)$$

Now use the definition of W_{t+1}^R and the fact that Y_{t+1} and W_{t+1}^R are functions of $S_0, \ldots, S_{t+1}$, and obtain

$$W_t^R = \frac{1}{a}\ln E[\exp(aY_{t+1} + aW_{t+1}^R) \mid S_0, \ldots, S_t], \quad (29)$$

which shows the validity of formula (26) for exponential reserves. The equivalent formula (27) then reads as follows:

$$V_t^R = \frac{1}{av^t}\ln E[\exp(av^{t+1}X_{t+1} + av^{t+1}V_{t+1}^R) \mid S_0, \ldots, S_t]. \quad (30)$$

Thus both the expected value and the exponential principle satisfy the recursive equations (26) and (27). But, to some extent, the converse is also true: under a mild additional condition, these are the only reserving principles that satisfy that postulate (see [13]). So the postulate leads in a natural way to exponential reserves!

Exponential reserves also have another, very desirable property: they are *additive*, that is, the reserve for a portfolio of independent policies is just the sum of the individual reserves (to show this, one makes use of the fact that the expected value of a product of independent random variables is the product of the individual expected values). Of course the generalized reserves that are based on the variance principle are also additive. In fact, they may be interpreted as a first approximation (with $\alpha = a/2$) to the exponential reserves (see [7]).

VII. FURTHER DISCUSSION OF EXPONENTIAL RESERVES

As in Section IV, let us examine the present value of the loss incurred at time t:

$$\begin{aligned} L_t^R &= Y_t + W_t^R - W_{t-1}^R \\ &= v^t[X_t + V_t^R - (1+i)V_{t-1}^R]. \end{aligned} \quad (31)$$

The following two properties may be viewed as the counterpart of Lemma 1.

LEMMA 2. *For* $t = 0, 1, 2, \ldots$ *and* $h = 1, 2, \ldots$, (*a*) $E[\exp(aL_{t+h}^R) \mid S_0, \ldots, S_t] = 1$; (*b*) $E[L_{t+h}^R \mid S_0, \ldots, S_t] \leq 0$ *if* $a > 0$, ≥ 0 *if* $a < 0$.

Proof:

a) By the iterative formula the left-hand side of part *a* becomes

$$E[E[\exp(aL_{t+h}^R) \mid S_0, \ldots, S_{t+h-1}] \mid S_0, \ldots, S_t]. \quad (32)$$

Now use the definition of L^R_{t+h}. Since W^R_{t+h-1} is a function of $S_0, \ldots, S_{t+h-1}$, the last expression equals

$$E[\exp(-aW^R_{t+h-1})E[\exp(aY_{t+h} + aW^R_{t+h}) \mid S_0, \ldots, S_{t+h-1}] \mid S_0, \ldots, S_t]. \tag{33}$$

But because of the recurrence equation (29), this is the expected value of 1—and equals 1.

b) By taking logarithms in part *a* and using Jensen's inequality, we see that

$$\begin{aligned} 0 &= \ln E[\exp(aL^R_{t+h}) \mid S_0, \ldots, S_t] \\ &\geq aE[L^R_{t+h} \mid S_0, \ldots, S_t], \end{aligned} \tag{34}$$

which is equivalent to the assertion *b*. Q.E.D.

Remark

Part *b* of Lemma 2 strongly suggests that the parameter a should be positive. Then we observe a *release from risk* at any time (with the possible exception of $t = 0$) and under all circumstances, as far as expected values are concerned. For the sequel we assume $a > 0$.

The following result is reminiscent of Hattendorf's theorem.

ANALOGUE OF HATTENDORF'S THEOREM. *Let* $n < m$. *Then*

$$E\left[\exp\left(a\sum_{t=n}^{m} L^R_t\right)\right] = \prod_{t=n}^{m} E[\exp(aL^R_t)].$$

While the random variables $\exp(aL^R_t)$ are by no means independent, the expected value of their product is nevertheless the product of their expected values! The proof is based on part *a* of Lemma 2, but its details are left to the reader.

VIII. HOW TO CHOOSE a

The quantification of the parameter a is rather easy. In this section we shall see that the parameter a plays the role of an *adjustment coefficient* (see [14], for example). For this purpose let u denote the initial capital that the insurance company sets aside for the line of business in question. We are interested in the event that "ruin" occurs, that is, that

$$\sum_{n=0}^{t} L^R_n \geq u \tag{35}$$

for some t. Let T be the first time when this happens (T is called the time of "ruin"), with the understanding that $T = \infty$ if ruin does not occur. Then $\phi(u) = P[T < \infty]$ is the probability of ruin. The following result suggests how a can be determined from the initial capital and a prescribed upper bound for the probability of ruin.

THEOREM. $\phi(u) \leq \exp[-a(u - L_0^R)]$.

This is a version of the famous inequality in risk theory, which is due to Lundberg and Cramér on the one hand and DeFinetti on the other hand (see [4]). For its proof one uses part a of Lemma 2 to show that

$$E\left[\exp\left(a\sum_{n=1}^{T} L_n^R\right)\right] = 1\ . \tag{36}$$

Obtain a lower bound by considering only the case where $T < \infty$. But there

$$\sum_{n=1}^{T} L_n^R \geq u - L_0^R\ . \tag{37}$$

Using this estimate in equation (36), we see that

$$P[T < \infty] \exp[a(u - L_0^R)] \leq 1\ , \tag{38}$$

which is the desired inequality. Q.E.D.

IX. A NUMERICAL ILLUSTRATION

The purpose of this section is to show the practical implementation of exponential reserves for a twenty-year term policy (face amount z, issue age 30, financed by a single premium).

Recall that the additivity property (see Sec. VI) requires independence of the risks. It is quite reasonable to assume that the risks are *independent with respect to mortality;* however, they are *dependent with respect to investment performance, expenses, and so on.* Therefore it is suggested that loaded interest and expense rates and so on be used (see [10]), but that unloaded mortality rates be used and the mortality risk covered by means of exponential reserves. (It is still necessary to determine the deltas for interest and expenses, but the delta for mortality is now zero.)

The numerical calculations are based on the 1965–70 Basic Table ([1]; see Table 7, Males and Females Combined). The rate of interest i is conservatively assumed as 0.04, and expenses and so on are neglected for this example. The parameter a was chosen as 0.00005. This is consistent with an initial surplus of \$92,000 (if the probability of ruin should be less than 0.01) or with \$140,000 (if the probability of ruin should be less than 0.001). The sequences $\{S_t\}$ and $\{X_t\}$ can be defined as follows:

$$\begin{aligned} S_t &= 1 \quad \text{if the person is alive at age } 30 + t \\ &= 0 \quad \text{if the person is dead at age } 30 + t\ , \end{aligned} \tag{39}$$

and, for $t \geq 1$,

$$\begin{aligned} X_t &= z \quad \text{if } S_{t-1} = 1, S_t = 0, t \leq 20 \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{40}$$

For simplicity assume that the single premium equals the initial reserve, $X_0 = -V_0^R$, so that $L_0^R = 0$ (no underwriting loss).

The net reserves V_t and the exponential reserves V_t^R can be obtained recursively from formulas (10) and (30). Obviously $V_{20} = V_{20}^R = 0$, and $V_t = V_t^R = 0$ if $S_t = 0$. Let ${}_1V_t$ and ${}_1V_t^R$ denote the values of the reserves if $S_t = 1$. Equations (10) and (30) reduce to the recursive formulas

$${}_1V_t = v(zq_{30+t} + {}_1V_{t+1}\ p_{30+t}) \tag{41}$$

and

$${}_1V_t^R = \frac{1}{av^t} \ln \left[\exp\left(av^{t+1}z\right)q_{30+t} + \exp\left(av^{t+1}\ {}_1V_{t+1}^R\right)p_{30+t}\right]. \tag{42}$$

Tables 1–6 in Appendix II exhibit age in column 1, the net reserves in column 2, and the exponential reserves in column 3. Column 4 shows the exponential reserves as a percentage of the net reserves. We observe that, unlike net reserves, *exponential reserves are not proportional:* if the face amount is doubled, for example, exponential reserves have to be more than doubled. This makes sense: it shows why policies with a high face amount should be reinsured.

Also computed was

$$G_t^R = (1 + i)\ {}_1V_{t-1}^R - zq_{30+t-1} - {}_1V_t^R p_{30+t-1}\,, \tag{43}$$

which is the conditional expected value (given $S_{t-1} = 1$) of the risk released at time t. Column 5 shows G_t^R/p_{t-1}, which is the gain per survivor at time t. Finally, column 6 exhibits the expected value of the present value of the release of risk at time t,

$$-E[L_t^R] = v^t G_t^R\ {}_{t-1}p_{30}\,. \tag{44}$$

Observe that the total in column 6 equals ${}_1V_0^R - {}_1V_0$, which has an obvious interpretation and may be used for checking purposes.

X. ACKNOWLEDGMENTS

The author had the benefit of stimulating discussions with Hans Bühlmann and Donald A. Jones.

REFERENCES

1. "1965–70 Basic Tables." *TSA, 1973 Reports*, pp. 199–223.
2. BÜHLMANN, H. *Mathematical Methods in Risk Theory.* New York: Springer-Verlag, 1970.

3. CARLSON, D. L. "Direct Calculation of Contingency Margins for Gross Premiums and Contingency Reserves Resulting from Those Premiums." *ARCH* 1975.2.
4. DUBOURDIEU, J. *Théorie mathématique du risque dans les assurances de répartition*. Paris: Gauthier-Villars, 1952.
5. FERRA, C. "Considerazioni sulle funzioni di utilità in connessione con la teoria del rischio," *J. Italian Inst. Actuaries*, XXVII (1964), 51–70.
6. FREIFELDER, L. R. "Statistical Decision Theory and Credibility Theory Procedures." In *Proceedings of the Berkeley Actuarial Research Conference*. New York: Academic Press, 1975.
7. GERBER, H. "On Additive Premium Calculation Principles," *ASTIN*, VII (1974), 215–22.
8. HICKMAN, J. C. "Notes on Individual Risk Theory and Released from Risk Reserves." *ARCH* 1975.1.
9. ———. "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory," *TSA*, XVI (1964), 1–16.
10. HORN, R. G. "Life Insurance Earnings and the Release from Risk Policy Reserve System," *TSA*, XXIII (1971), 391–99.
11. JORDAN, C. W. *Life Contingencies*. Chicago: Society of Actuaries, 1967.
12. NESBITT, C. J., AND BICKNELL, W. S. "Premiums and Reserves in Multiple Decrement Theory," *TSA*, VIII (1956), 344–75.
13. Problem No. 10 in *ARCH*. *ARCH* 1975.3.
14. WOODDY, J. C. "Study Notes on Risk Theory." Chicago: Society of Actuaries, 1971.

APPENDIX I

RETROSPECTIVE FORMULAS FOR RESERVES

Under the assumption that the premiums are net, that is, $X_0 + V_0 = 0$, or

$$E\left[\sum_{n=0}^{\infty} v^n X_n\right] = 0 , \tag{1}$$

one can derive formulas that are analogous to Jordan's formulas (5.3) and (5.18). Let us introduce

$$K_n = E[X_n + V_n \mid S_0, \ldots, S_{n-1}] - X_n - V_n \tag{2}$$

for $n = 1, 2, \ldots$. The random variable K_n (which is measurable with respect to $S_0, \ldots, S_n$) should be interpreted as the *cost of insurance on the "path" $S_0, S_1, \ldots, S_n$ based upon the net amount at risk in the nth policy year*. Provided that condition (1) holds, the following formula extends Jordan's formula (5.18):

$$V_t = -\sum_{n=0}^{t} (1+i)^{t-n} X_n - \sum_{n=1}^{t} (1+i)^{t-n} K_n . \tag{3}$$

Proof: By the definition of K_n, the right-hand side of equation (3) may be rewritten as

$$-(1+i)^t X_0 + \sum_{n=1}^{t} (1+i)^{t-n}\{V_n - E[X_n + V_n \mid S_0, \ldots, S_{n-1}]\} \,. \quad (4)$$

Because of formula (10) of the paper, this is

$$-(1+i)^t X_0 + \sum_{n=1}^{t} (1+i)^{t-n}[V_n - (1+i)V_{n-1}] \,, \quad (5)$$

which can be simplified to

$$-(1+i)^t X_0 - (1+i)^t V_0 + V_t \,. \quad (6)$$

Thus the validity of equation (3) is equivalent to the validity of condition (1). Q.E.D.

To obtain a formula in the spirit of Jordan's (5.3), one considers $V_t = V_t(s_0, \ldots, s_t)$ as a function of the path $(s_0, \ldots, s_t)$. Then, for a given path, we identify the event $[S_i = s_i$ for $i = 0, 1, \ldots, n]$ with being alive at the end of policy year n, and the event $[S_i = s_i$ for $i = 0, 1, \ldots, n-1$ and $S_n \neq s_n]$ with death occurring in the nth policy year. In this way elementary principles can be applied to obtain the desired formula, and details are left to the reader.

APPENDIX II

TABLE 1

FACE AMOUNT $10,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	288.84	337.56	116.9	0.0	0.0
31	289.03	336.68	116.5	3.08	2.96
32	289.14	335.79	116.1	2.96	2.73
33	289.24	335.00	115.8	2.82	2.50
34	289.16	334.07	115.5	2.74	2.33
35	288.59	332.62	115.3	2.73	2.23
36	287.31	330.42	115.0	2.74	2.15
37	285.01	327.06	114.8	2.85	2.14
38	281.44	322.30	114.5	2.95	2.13
39	276.36	315.84	114.3	3.09	2.14
40	269.42	307.30	114.1	3.25	2.16
41	260.03	296.02	113.8	3.50	2.24
42	247.91	281.65	113.6	3.76	2.31
43	232.72	263.91	113.4	3.99	2.35
44	214.04	242.28	113.2	4.30	2.42
45	191.32	216.17	113.0	4.60	2.49
46	164.15	185.16	112.8	4.92	2.55
47	132.13	148.77	112.6	5.30	2.63
48	94.73	106.47	112.4	5.64	2.68
49	51.06	57.30	112.2	6.02	2.74
50	0.0	0.0	....	6.53	2.84
Total	....	....	....	....	48.71

TABLE 2

FACE AMOUNT $20,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	577.69	796.40	137.9	0.0	0.0
31	578.07	791.36	136.9	14.44	13.87
32	578.27	786.56	136.0	13.80	12.73
33	578.49	782.27	135.2	13.10	11.60
34	578.32	777.82	134.5	12.69	10.79
35	577.18	772.36	133.8	12.55	10.26
36	574.62	765.28	133.2	12.60	9.89
37	570.02	755.68	132.6	12.90	9.72
38	562.89	742.93	132.0	13.34	9.65
39	552.73	726.43	131.4	13.86	9.63
40	538.84	705.22	130.9	14.60	9.73
41	520.07	677.84	130.3	15.62	9.99
42	495.81	643.63	129.8	16.65	10.21
43	465.44	601.79	129.3	17.77	10.46
44	428.09	551.34	128.8	18.96	10.69
45	382.63	490.91	128.3	20.31	10.98
46	328.30	419.62	127.8	21.69	11.23
47	264.26	336.56	127.4	23.04	11.43
48	189.45	240.47	126.9	24.49	11.63
49	102.12	129.20	126.5	26.23	11.92
50	0.0	0.0	....	28.32	12.31
Total	....	....	....	....	218.71

TABLE 3

FACE AMOUNT $40,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	1,155.37	2,260.44	195.6	0.0	0.0
31	1,156.14	2,226.00	192.5	80.75	77.55
32	1,156.55	2,194.39	189.7	76.13	70.22
33	1,156.98	2,166.37	187.2	71.23	63.10
34	1,156.65	2,139.80	185.0	67.88	57.75
35	1,154.36	2,111.76	182.9	66.35	54.21
36	1,149.25	2,080.44	181.0	65.83	51.64
37	1,140.04	2,043.26	179.2	66.60	50.17
38	1,125.77	1,998.52	177.5	68.05	49.22
39	1,105.46	1,944.57	175.9	70.07	48.65
40	1,077.67	1,878.98	174.4	72.98	48.63
41	1,040.13	1,797.91	172.9	77.31	49.43
42	991.63	1,699.65	171.4	81.89	50.22
43	930.89	1,582.48	170.0	86.65	50.97
44	856.17	1,443.90	168.6	91.87	51.82
45	765.26	1,280.62	167.3	97.83	52.88
46	656.59	1,090.56	166.1	103.91	53.82
47	528.53	871.56	164.9	110.06	54.60
48	378.90	620.56	163.8	116.64	55.40
49	204.23	332.34	162.7	124.42	56.55
50	0.0	0.0	....	133.95	58.23
Total	....	....	....	....	1,105.05

TABLE 4

FACE AMOUNT $60,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	1,733.06	4,895.65	282.5	0.0	0.0
31	1,734.21	4,772.09	275.2	255.06	244.96
32	1,734.83	4,661.06	268.7	236.90	218.51
33	1,735.47	4,563.89	263.0	218.46	193.53
34	1,734.97	4,474.85	257.9	205.21	174.58
35	1,731.54	4,386.76	253.3	197.82	161.62
36	1,723.87	4,295.25	249.2	193.70	151.97
37	1,710.06	4,194.43	245.3	193.66	145.89
38	1,688.66	4,080.72	241.7	195.67	141.51
39	1,658.19	3,950.76	238.3	199.36	138.41
40	1,616.51	3,799.65	235.1	205.54	136.96
41	1,560.20	3,619.44	232.0	215.94	138.06
42	1,487.44	3,407.10	229.1	226.91	139.17
43	1,396.33	3,159.54	226.3	238.38	140.22
44	1,284.26	2,872.01	223.6	251.25	141.70
45	1,147.89	2,538.18	221.1	266.25	143.93
46	984.89	2,154.32	218.7	281.61	145.86
47	792.79	1,716.41	216.5	297.36	147.51
48	568.35	1,218.68	214.4	314.40	149.32
49	306.35	650.96	212.5	334.97	152.24
50	0.0	0.0	....	360.32	156.63
Total	....	....	....	....	3,162.56

TABLE 5

FACE AMOUNT $80,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	2,310.75	9,404.92	407.0	0.0	0.0
31	2,312.29	9,081.23	392.7	617.64	593.19
32	2,313.11	8,793.60	380.2	567.52	523.47
33	2,313.96	8,544.11	369.2	517.52	458.45
34	2,313.30	8,319.93	359.7	480.50	408.80
35	2,308.72	8,105.58	351.1	457.85	374.07
36	2,298.50	7,891.88	343.3	443.33	347.82
37	2,280.08	7,666.94	336.3	438.52	330.34
38	2,251.55	7,423.82	329.7	438.70	317.28
39	2,210.92	7,156.29	323.7	442.85	307.44
40	2,155.35	6,855.28	318.1	452.78	301.69
41	2,080.27	6,506.16	312.8	472.17	301.88
42	1,983.26	6,103.81	307.8	493.03	302.40
43	1,861.78	5,642.93	303.1	515.25	303.09
44	1,712.34	5,115.21	298.7	540.82	305.02
45	1,530.52	4,509.53	294.6	571.29	308.82
46	1,313.19	3,819.34	290.8	603.02	312.33
47	1,057.05	3,037.44	287.4	636.24	315.62
48	757.80	2,153.46	284.2	672.86	319.57
49	408.46	1,149.03	281.3	717.87	326.27
50	0.0	0.0	....	774.30	336.58
Total	....	....	....	....	7,094.11

TABLE 6

FACE AMOUNT $100,000

Age (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (%) (4)	Release of Risk (5)	(6)
30	2,888.44	16,481.49	570.6	0.0	0.0
31	2,890.36	15,815.25	547.2	1,228.44	1,179.81
32	2,891.39	15,223.55	526.5	1,125.61	1,038.24
33	2,892.45	14,711.44	508.6	1,021.61	905.01
34	2,891.62	14,255.47	493.0	942.67	801.99
35	2,885.90	13,827.16	479.1	891.92	728.72
36	2,873.12	13,409.73	466.7	857.35	672.64
37	2,850.10	12,981.76	455.5	841.99	634.29
38	2,814.43	12,531.06	445.2	836.56	605.02
39	2,763.65	12,046.70	435.9	839.25	582.64
40	2,694.19	11,513.07	427.3	853.38	568.61
41	2,600.34	10,905.12	419.4	885.88	566.39
42	2,479.08	10,214.18	412.0	921.86	565.42
43	2,327.22	9,431.21	405.3	961.25	565.44
44	2,140.43	8,541.96	399.1	1,007.81	568.40
45	1,913.15	7,527.18	393.4	1,064.87	575.64
46	1,641.49	6,375.03	388.4	1,125.79	583.09
47	1,321.31	5,072.10	383.9	1,191.42	591.03
48	947.25	3,599.30	380.0	1,265.65	601.10
49	510.58	1,923.40	376.7	1,358.52	617.44
50	0.0	0.0	....	1,477.18	642.12
Total	....	....	....	....	13,593.01

DISCUSSION OF PRECEDING PAPER

HARRY H. PANJER:

Dr. Gerber provides a probabilistic cash-flow model that is most intriguing. The notation corresponds to that of Bühlmann.[1] Since no reference is made to the distributions of the random variables, the model can be applied in situations not involving life contingencies.

The model when applied to life contingencies can be also formulated using the approach of Hickman,[2] which, of course, will be equivalent to Gerber's formulation. This is demonstrated as follows (in the continuous case, for convenience):

Consider, for simplicity, (*a*) an ordinary life policy, (*b*) an *n*-year term policy, and (*c*) a single premium immediate life annuity with an *n*-year certain period, each issued to a life aged *x*. The probability density function of the random variable *T*, the time of death of the individual aged *x*, is

$$f(t) = {}_tp_x\,\mu_{x+t}\,, \quad t > 0\,.$$

The net premium for the above three plans is *P*, which satisfies the equation

$$E[g(T)] = 0\,,$$

where $g(T)$ is the random variable of the following form for each of the above cases:

$$a)\ g(t) = v^t - P\bar{a}_{\overline{t|}}\,, \quad t > 0\,,$$

$$\begin{aligned} b)\ g(t) &= v^t - P\bar{a}_{\overline{t|}}\,, & 0 < t < n \\ &= 0\,, & t \geq n\,, \end{aligned}$$

$$\begin{aligned} c)\ g(t) &= \bar{a}_{\overline{n|}} - P\,, & 0 < t < n \\ &= \bar{a}_{\overline{t|}} - P\,, & t \geq n\,. \end{aligned}$$

Note that, as in Gerber's paper, the function whose expected value is taken is just the present value of future cash flow. The expectation is taken with respect to the distribution of $g(T)$ or, equivalently, with

[1] H. Bühlmann, *Mathematical Methods in Risk Theory* (New York: Springer-Verlag, 1970).

[2] J. C. Hickman, "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory," *TSA*, XVI (1964), 1–16.

respect to the distribution of T. For example, in case b above, the net premium P satisfies

$$\int_0^n \{v^t - P\bar{a}_{\overline{t|}}\} \; {}_tp_x \; \mu_{x+t} \, dt = 0 \, .$$

With respect to net reserves, the same model can be used, except, of course, that the distribution is now conditional on the individual aged x being alive. For the three plans, the net reserves at time s are $E[g(T)]$, where $g(t)$ can be expressed as:

$$\begin{aligned} a)\ g(t) &= v^t - P\bar{a}_{\overline{t|}}\,, && t > 0\,, \\ b)\ g(t) &= v^t - P\bar{a}_{\overline{t|}}\,, && 0 < t < n - s \\ &= 0\,, && t \geq n - s\,, \\ c)\ g(t) &= \bar{a}_{\overline{n-s|}}\,, && 0 < t < n - s \\ &= \bar{a}_{\overline{t-s|}}\,, && t \geq n - s\,, \end{aligned}$$

and where the associated probability density function is

$$f(t) = {}_tp_{x+s} \; \mu_{x+s+t} \, .$$

Gerber's exponential reserves in the above three cases are then of the form $\{\ln E[e^{ag(T)}]\}/a$. It is particularly interesting to interpret this function from a statistical point of view. The function $\ln E[e^{ag(T)}]$ considered as a function of the variable a is the cumulant generating function of the random variable $g(T)$. When it is divided by a, it can be written as an expansion of the form

$$\sum_{h=1}^{\infty} \frac{K_h}{h!} a^{h-1} \, ,$$

where the K_h's are the cumulants. The cumulants are simply functions of the moments, with

$$K_1 = E[g(T)] \, , \quad K_2 = \mathrm{Var}\,[g(T)] \, , \quad \text{etc.} \, ,$$

and with each higher cumulant taking into account the corresponding higher moments. Since a distribution is determined uniquely if all its moments are known, it can be argued that the exponential reserving principle takes into account all characteristics of the underlying distribution. The expected value principle takes into account only the first moment, while the variance principle takes into account the first two. If a distribution is particularly skewed, adding higher moments may add valuable information. When a small value of a is used, it is clear from the above series that a decreasing emphasis is placed on successively higher moments, which is natural.

Realizing this, one can then generate a whole range of reserving principles by successively adding terms of the expansion given above. Each reserving principle will have the additivity property postulated by Gerber, but each will suffer the defect that it does not satisfy the recurrence relation of Gerber. The resulting sequence of reserves will converge to the exponential reserves proposed by Gerber; each will result in a different release from risk. It can be argued that, since the exponential reserves take into account all characteristics of the mortality risk (via the distribution), the exponential reserving principle is the most reasonable one to use, at least from a theoretical standpoint.

I thank the author for a most stimulating paper.

GOTTFRIED BERGER:*

In many elegantly written papers, Hans Gerber has advocated the exponential principle. Like the expected value principle, the exponential principle has the very desirable property of additivity. Unlike the former, the latter is not proportional; that is, exponential premiums and exponential reserves are not proportional to the corresponding face amounts. In my opinion, this restricts the application of the exponential premium principle to nonproportional covers like stop-loss reinsurance.

The exponential principle is closely tied to risk theory, in that the parameter a is related to the available funds (capital plus surplus), depending upon the ruin probability that the insurance company wishes to tolerate. This makes the exponential principle attractive to both parties of a *reinsurance* transaction, at least as far as nonproportional covers are concerned.

However, the *insurance* market is different. A glance at Tables 3 and 5 of the paper raises the question: Why should a prospect purchase from one company an $80,000 policy at the single premium of $9,405 if he can obtain from different companies two $40,000 policies at $2,260 each? Besides, nonproportional pricing would create a severe administrative burden to the direct insurer—think of the "exponential" size of the ratebook that would be needed!

The foregoing arguments refer to some practical aspects of exponential premiums, but not necessarily to exponential reserves. After all, the choice between admissible reserving methods is an internal affair of the

* Dr. Berger, not a member of the Society, is a member of the American Academy of Actuaries, and President, Cologne Life Reinsurance Company.

insurance company. I feel that this paper opens exciting new aspects. The following are some thoughts that occurred to me while reading it:

1. Gerber defines prospective initial reserves under certain reserve principles. These reserves depend upon a vector of random variables X_t that represent the balance of benefits less premiums. Thus, premiums are defined a priori, and the same (gross) premiums apply to any reserve principle. In that case, the variance principle yields higher reserves than the expected value principle, provided that $a > 0$. (For a similar statement with respect to exponential reserves, see the last sentence of Sec. VI of the paper.)
2. The numerical illustrations presented in Section IX of the paper refer to the special case of a single premium payment mode. Here Gerber departs from the concept of a priori premiums. Instead, he assumes that $P^R = V_0^R$; that is, premiums P^R depend upon the reserve principle R in such a way that they equal the initial reserve V_0^R.
3. To generalize, let R be any reserve principle and P^R be a level premium (single or periodic) that satisfies the condition $P^R = V_0^R$. We may call such premium P^R the "reserve premium" related to a given reserve principle R. Intuitively, if the reserve calculation is based upon a reserve premium, prospective reserves are equal to retrospective reserves.
4. In the case of a single premium payment, the reserve premium P^R under the exponential *reserve* principle equals the gross premium derived under the exponential *premium* principle.
5. To my knowledge, the exponential premium principle has been applied so far to the single premium payment mode only. It seems to be a logical extension to define generally the exponential reserve premium P^R as the exponential gross premium.
6. If this idea is accepted, it would be necessary to study the properties of these generalized exponential premiums. It is intuitively clear that the condition $P^R = V_0^R$ has one unique solution P^R for a broad class of policies. However, the computation is somewhat awkward, since trial-and-error methods seem inevitable.

As an example, consider a \$100,000 three-year endowment policy with level annual premiums. Assumed mortality: $q_0 = 0.005$, $q_1 = 0.010$, q_2 arbitrary. As in the paper, $i = 4$ per cent and $a = 0.00005$. The accompanying tabulation corresponds to Tables 1–6 of the paper. Added is a

Time t (1)	Net Reserves (2)	Exponential Reserves (3)	(3)/(2) (4)	Release of Risk (5)	(6)	Alternative Release (6′)
0	31,027.86	31,971.32	103.0%	0.00	0.00	943.46
1	62,956.47	62,577.12	99.4	2,308.95	2,209.04	902.64
2	96,153.85	96,153.85	100.0	544.95	496.31	859.25
Total					2,705.35	2,705.35

new column (6′), which shows, alternatively, the effect of a uniform release of the premium loading. The latter is \$31,971.32 − \$31,027.86 = \$943.46. Note that columns 6 and 6′ should have the same total, namely, the present value of the loading contained in the assumed gross premium.

Finally, let us consider the effect of different assumed gross premiums on the exponential reserves and on the corresponding reserve release:

Time t	Exponential Reserves, if Gross Premium Equals			Release of Risk		
	31,027.86	31,971.32	33,000.00			
(1)	(3a)	(3b)	(3c)	(6a)	(6b)	(6c)
0......	33,500.89	31,971.32	30,324.07	−2,473.03	0.00	2,675.93
1......	63,439.65	62,577.12	61,638.84	2,010.76	2,209.04	2,443.64
2......	96,153.85	96,153.85	96,153.85	462.27	496.31	535.48
Total	....	....	....	0.00	2,705.35	5,655.05

As expected, the totals of columns 6 equal the present values of the respective premium loadings. Test:

$$2{,}705.35/(31{,}971.32 - 31{,}027.86)$$
$$= 5{,}655.05/(33{,}000.00 - 31{,}027.86) = 2.8675\,.$$

Furthermore, it appears that under the exponential reserve principle a substantial portion of the present value of the difference between the assumed gross premium and the reserve premium (i.e., \$31,971.32) is recognized immediately as profit or loss.

(AUTHOR'S REVIEW OF DISCUSSION)

HANS U. GERBER:

It was a pleasure to read the discussions by Dr. Panjer and Dr. Berger.

The first part of Harry Panjer's discussion is an excellent illustration of how probabilistic methods can be used in the theory of life contingencies. If this point of view is adopted, formulas like the one in problem 7, chapter 2, of Jordan's *Life Contingencies* become trivial. In the second part Dr. Panjer shows how exponential reserves can be approximated in terms of the first n cumulants of a given distribution. The quality of such an approximation for a fixed n is not uniformly good for all distributions. This means that the number of terms used should depend on the distribution.

Gottfried Berger points out the practical difficulties in connection with nonproportional premiums. I agree that it would be difficult to explain to the public (or the agents) why the premium rate for an \$80,000 policy should be different from that for a \$40,000 policy. However, the situation is different if an \$80,000 policy is compared with an \$8,000,000 policy. An insurer may refuse to sell the latter policy under the same conditions, at least in the absence of reinsurance, since a claim could jeopardize the financial stability of the company. Of course, if reinsurance is available at that rate (for example, from one large reinsurer who uses a smaller value of a or from ninety-nine friendly insurance companies that use the same rates), it may still be possible to offer the \$8,000,000 at the said rate.

The numerical example at the end of Dr. Berger's discussion is very instructive. In the case of annual premiums, the ratio of exponential reserves to net reserves may be less than 1 (see col. 4 of his first table), which is somewhat surprising. Obviously, the explanation is that the future premiums will be higher in the case of exponential reserves (\$31,971.32) than in the case of net reserves (\$31,027.86).

I would like to add a word of caution to item 3 of Dr. Berger's discussion, and also to Appendix I of the paper. While the formulas of Appendix I are valid, generally they are not sufficient to define the reserves retrospectively. It seems that "prospective reserves" are more generally applicable than "retrospective reserves."

In conclusion, I would like to thank the two discussants for their very competent remarks.