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A NEW COLLECTIVE RISK MODEL

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ABSTRACT

A mathematical model is constructed to study the deviations of claims, investment performance, operating expenses, and lapse expenses as random (stochastic) processes. The model does allow for interdependence among the processes. It is primarily concerned with random deviations, but one section is concerned with provisions for a deterministic trend. An expression for the global variation is given in terms of the component variations, and the covariances between the processes. Under certain assumptions, these quantities are expressed in very simple terms. This model is one attempt to quantify the loading for adverse deviations.

I. INTRODUCTION

The Committee on Financial Reporting Principles of the American Academy of Actuaries asked the Joint Committee on the Theory of Risk to help develop more knowledge and techniques for measuring the risk of adverse deviation—that is, the quantification of the "deltas" in Richard Horn's paper "Life Insurance Earnings and the Release from Risk Policy Reserve System" [16]. After benefiting from some excellent memoranda and conversations with members of the joint committee, and with other actuaries (see Acknowledgments), the author devised a mathematical model which attempts to describe and analyze the random components which contribute to adverse deviations.

II. A NEW COLLECTIVE RISK MODEL

After a premium is set for a new policy, its performance over time $t \geq 0$ begins. It has long been recognized that the evolving patterns of claims, investment income, operating expenses, and lapse expenses are somewhat random in nature. Not only is the aggregate performance over a fixed number of years (say five) random, but the aggregate performance at any intermediate time point is random. Thus the perfor-

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mance viewed over time is a random (stochastic) process. It is becoming more common now to view the claims process as a stochastic process. One exposition of this idea is given by John A. Beekman, in "Collective Risk Results" [2]. A stochastic process is an infinite collection of random variables $\{X(t): t \in A\}$. Frequently the set $A = [0, T]$ or $[0, \infty)$ and the process are thought of as describing some natural phenomenon evolving in the time interval A. To discuss the adverse deviations of claims, investments, expenses, and lapses, one must quickly admit that these deviations vary from one time to another. Although one may be forced to settle for simple measures of each deviation, and simple relationships between the deviations, at least initially a model should be considered that treats the four random processes equally—that is, one that recognizes that the four components are subject to random deviations at each point of time.

One of the functions of collective risk theory is to determine the loading for the net premium applicable to a set of policies. We will now repeat some of reference [2]. Assume that $\{X_i\}$ is a sequence of independent, identically distributed random variables with a common distribution function $P(z)$. The X_i's represent the claims. Assume that $E\{X_i\} = p_1$. Assume that $\{N(t), t \geq 0\}$ is a Poisson stochastic process, independent of $\{X_i\}$, with $E\{N(t)\} = t$. This describes the random *number* of claims over time. The collective risk stochastic process

$$\left\{\sum_{i=1}^{N(t)} X_i, \quad 0 \leq t < \infty\right\}$$

is a model for the *aggregate* claim randomness. The *aggregate* gross premium equals $p_1t + \lambda t$, where p_1t is the aggregate net premium and λt is the aggregate security loading. The loading λ for one unit of operational time can be determined so that the probability that the greatest of the differences at each point in time between aggregate claims and aggregate premiums is greater than the initial reserve u is ϵ, where $\epsilon = 0.001$ or some other appropriately small number. In symbols, this is expressed as

$$P\left\{\underset{0 \leq t < \infty}{\text{supremum}} \left[\sum_{i=1}^{N(t)} X_i - t(p_1 + \lambda)\right] > u\right\} = \epsilon\,.$$

The "initial reserve u" may be considered to be an amount of money which management has assigned (perhaps informally) to the development of this class of policies.

Actuaries have long recognized that the λt term provided a measure of safety against adverse deviations from their assumptions, and have expressed this in various ways. To attempt to split λ into several parts, we must back up and create three new processes to accompany the claim process. Let

$$C(t) = \sum_{i=1}^{N(t)} X_i , \quad 0 \leq t < \infty .$$

This stochastic process describes the evolution of claim patterns, recognizing both the random number of claims, $N(t)$, in time t, as well as the random nature of the claims, the X_i's. Let $C^*(t) = C(t) - tp_1$, $0 \leq t < \infty$. It is apparent that this process describes the claim deviations over time. Note that its mean function $E\{C^*(t)\}$ is equal to zero for all $t \geq 0$. Let $\{I(t), 0 \leq t < \infty\}$ be a stochastic process describing the investment performance deviations over time. Just as a claim pattern may vary in either direction, we will allow the investment deviations to be both adverse and favorable, but we will assume that $E\{I(t)\} = 0$ for all t. Let $\{O(t), 0 \leq t < \infty\}$ be a third process describing the deviations from the operating expense assumptions. Again, we will assume that $E\{O(t)\} = 0$ for all t. Finally, let $\{L(t), 0 \leq t < \infty\}$ be a fourth random process describing the deviations from lapse expense assumptions. Although we will assume that $E\{L(t)\} = 0$ for all t, actuaries are well aware that this process may deviate greatly from their assumptions.

Let us now define a new collective risk process as

$$R(t) = C(t) - I(t) + O(t) + L(t), \quad 0 \leq t < \infty . \tag{1}$$

This time-ordered set of random variables recognizes the randomness of claims, investment deviations, operating expense deviations, and lapse expense deviations. We use $-I(t)$ rather than $+I(t)$, since adverse investment results would be valued negatively. Since it is natural to determine gross premiums from expected values, and deviations from those expected values, let us now determine $E\{R(t)\}$ and Var $\{R(t)\}$.

$$\begin{aligned} E\{R(t)\} &= E\{C(t)\} - E\{I(t)\} + E\{O(t)\} + E\{L(t)\} \\ &= p_1 t, \quad 0 \leq t < \infty , \end{aligned} \tag{2}$$

by the linearity of the E operator. For each fixed t,

$$\begin{aligned} \text{Var}\,\{R(t)\} = \text{Var}\,\{C(t)\} &+ \text{Var}\,\{I(t)\} + \text{Var}\,\{O(t)\} + \text{Var}\,\{L(t)\} \\ &- 2\,\text{Cov}\,\{C(t), I(t)\} + 2\,\text{Cov}\,\{C(t), O(t)\} \\ &+ 2\,\text{Cov}\,\{C(t), L(t)\} - 2\,\text{Cov}\,\{I(t), O(t)\} \\ &+ 2\,\text{Cov}\,\{O(t), L(t)\} - 2\,\text{Cov}\,\{I(t), L(t)\} , \end{aligned} \tag{3}$$

where Cov $\{X, Y\} = E\{(X - \mu_X)(Y - \mu_Y)\} = E\{XY\} - \mu_X\mu_Y$. This formula for the variance of a sum of random variables may be found in many probability and/or statistics books. For example, it appears on page 230 of Feller's textbook [11]. We have made use of the fact that Cov $\{-X, Y\} = -\text{Cov}\{X, Y\}$. In the above formula we know that Var $\{C(t)\} = p_2 t$, where $E\{X_i^2\} = p_2$.

We may also define a deviations stochastic process as

$$D(t) = C^*(t) - I(t) + O(t) + L(t), \quad 0 \leq t < \infty . \tag{4}$$

Since $E\{C^*(t)\} = 0$, we quickly see that

$$E\{D(t)\} = 0, \quad 0 \leq t < \infty . \tag{5}$$

Now Var $\{C^*(t)\}$ = Var $\{C(t)\}$, and

$$\begin{aligned} \text{Cov}\{C^*(t), I(t)\} &= E\{[C^*(t) - E\{C^*(t)\}][I(t) - E\{I(t)\}]\} \\ &= E\{[C(t) - tp_1 - 0][I(t) - E\{I(t)\}]\} \\ &= \text{Cov}\{C(t), I(t)\} . \end{aligned}$$

Furthermore, Cov $\{C^*(t), O(t)\}$ = Cov $\{C(t), O(t)\}$, and Cov $\{C^*(t), L(t)\}$ = Cov $\{C(t), L(t)\}$. Combining these results, we see that

$$\text{Var}\{D(t)\} = \text{Var}\{R(t)\}, \quad 0 \leq t < \infty . \tag{6}$$

The actuary is interested in a gross premium G such that the probability that the greatest difference between $R(t)$ and tG at each point in time is greater than the initial reserve u (as interpreted earlier) is appropriately small, say 0.001. Thus one chooses G so that

$$P\{\underset{0 \leq t < \infty}{\text{supremum}} [R(t) - tG] > u\} = 0.001 . \tag{7}$$

Obviously $G = p_1 + \lambda$ for some $\lambda > 0$, so equation (7) is equivalent to

$$P\{\underset{0 \leq t < \infty}{\text{supremum}} [D(t) - t\lambda] > u\} = 0.001 . \tag{8}$$

In words, we seek the loading λ such that the probability that the greatest timed difference between the random deviations and the provision for deviations is greater than the initial reserve is appropriately small.

It has frequently been assumed that $\{N(t), t \geq 0\}$ is a Poisson process, for two reasons: (1) in many cases this fits the facts fairly well, and (2) it leads to fairly simple mathematics. If one is to capitalize on the considerable amount of research recently completed on stochastic processes, similar considerations must be acknowledged in choosing models for the other three processes. Available data and general reasoning would suggest that the other three processes are Gaussian (normal) and hence symmetrically distributed around a mean function. With respect to investment deviations, we shall assume for an illustration of the theory that the investment income deviations are normally distributed. The total investment deviations will also swing either way because of the changing values of the portfolio. Although individual items in the portfolio may change drastically in value, for simplicity we shall assume that the aggregate of the items enjoys normal deviations. Thus, at each time point, the total investment deviation will be the sum of two normally distributed random variables representing investment income deviations and portfolio value deviations. Although these random variables are (in general) not independent, it is still reasonable to assume that the total investment deviation at each time point has a normal distribution. (See the Appendix for a more technical exposition.) In general, future values for the processes are dependent on the past. It is quite difficult to recognize that dependence in complete detail. However, if we assume that, as a first approximation, future values can be predicted on the basis of present positions, our models are Markovian processes. It must be emphasized that the Markovian property is more of an approximation than is the normality. (See the Appendix for a more technical exposition.)

The most widely studied and used stochastic process with these properties is the Wiener process $\{w(t), 0 \leq t < \infty\}$ with mean function $E\{w(t)\} = 0$ and covariance function $E\{w(s)w(t)\} = \min(s, t)$. A very readable account of the Wiener stochastic process will be found on pages 27–29 of reference [21]. It is appropriate to work with a process with a zero mean function. However, $\text{Var}\{w(t)\} = t$, which implies unbounded variation with evolving time. This seems unrealistic and is the main reason for using another Gaussian Markov stochastic process called the Ornstein-Uhlenbeck process. This was developed in references [25] and [26], when physicists decided to build a mathematical model for the velocities of Brownian motion because the Wiener process does not describe such velocities. These two papers, along with four others, appear in reference [27], which was compiled to serve those electrical engineers and physicists interested in learning how stochastic processes might be applied in their disciplines. Gaussian Markov processes have proved useful in mathematical statistics; reference [10] is an early paper on this subject. Reference [3] is concerned with the Ornstein-Uhlenbeck process and with other Gaussian Markov processes. The recent paper [1] lists sixteen papers dealing with this process. A further source is reference [14], which builds a Monte Carlo approximation to the process for the purpose of modeling meteorological phenomena. Reference [5] is devoted to the collective risk and Gaussian Markov processes and their applications in insurance, physics, statistics, and electrical engineering. The Ornstein-Uhlenbeck process has a constant variance function σ^2, as opposed to Var $\{w(t)\} = t$. Moreover, the Ornstein-Uhlenbeck process has the advantage that it models phenomena which react to offset excessive movements in any one direction, which is true of many economic phenomena is a free society. It is shown in the Appendix that the conditional mean function $E\{X(t) \mid X(s) = x\} = xe^{-\beta(t-s)}$ for $\beta > 0$. This implies a drift downward if the present position is positive and a drift upward if the present position is negative. Let us point out that the word "position" here will refer to a deviation from the expected value. By contrast, the conditional mean function for the Wiener process is $E\{X(t) \mid X(s) = x\} = x$, which does not reflect the stabilizing influences one would expect. Thus, if we use three Ornstein-Uhlenbeck processes, we have Var $\{I(t)\} = \sigma_I^2$, Var $\{O(t)\} = \sigma_O^2$, Var $\{L(t)\} = \sigma_L^2$. Empirical evidence would have to give us statistical estimates (confidence intervals) for the three constants. Luckily, we would not have to estimate variance functions evolving over time.

Now we approach the covariances. Assume that the evidence allows us to set $I(t) = a(t)O(t) + b(t)$, $0 \leq t < \infty$. This says that for each fixed time t, $I(t)$ is a linear function of $O(t)$. In some cases the a and b functions would not depend on time. If $a(t)$ should be positive (as in an inflationary period), then the correlation coefficient $\rho\{I(t), O(t)\} = +1$; since, for any random variables X and Y, $\rho\{X, Y\} = \text{Cov}\ \{X, Y\}/\sigma_X\sigma_Y$, this tells us that Cov $\{I(t), O(t)\} = \sigma_{I(t)}\sigma_{O(t)}$. If two random variables X and Y are linearly related ($Y = aX + b$), then $\rho\{X, Y\} = \pm 1$, where $+1$ is obtained if $a > 0$ and -1 is obtained if $a < 0$. This can easily be shown, starting with the definition of Cov $\{X, Y\}$. For each fixed time t, $I(t)$ and $O(t)$ are random variables, which could be designated X and Y. Furthermore, $\rho\{X, Y\} = \pm 1$ *only if* X and Y are linearly related (see Feller [11], pp. 236–37). It is to be emphasized that we do *not* require $a(t)$ and $b(t)$ to be linear functions. If we assume that $O(t) = c(t)L(t) + d(t)$ for a positive function $c(t)$, this would reflect the fact that, as lapse expenses grow, so do operating expenses. Thus Cov $\{O(t), L(t)\} = \sigma_O\sigma_L$. By contrast, if lapse expenses diminish with rising interest rates, it may be possible to express $L(t) = g(t)I(t) + h(t)$ for some function $g(t)$ taking only negative values. Thus $\rho\{L(t), I(t)\} = -1$, and Cov $\{L(t), I(t)\} = -\sigma_L\sigma_I$. So far we have said nothing about the relation of the $C(t)$ process to the other three processes. At least in theory, one would hope that this process is independent of the other three. For simplicity we will make this assumption, and hence the other three covariances are all zero. Combining these several results, we obtain

$$\text{Var}\{R(t)\}$$
$$= p_2t + \text{Var}\, I + \text{Var}\, O + \text{Var}\, L - 2\sigma_I\sigma_O + 2\sigma_O\sigma_L - 2\sigma_L\sigma_I\,. \quad (9)$$

Furthermore, we do not have to determine the functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, $g(t)$, $h(t)$ explicitly. However, evidence must support (1) the three linear relations and (2) the positive nature of $a(t)$ and $c(t)$ and the negative nature of $g(t)$ if we are to enjoy the simplicity of the correlation coefficients of $+1$, $+1$, and -1, respectively. Obviously, one can still measure adverse deviations by expression (3), the previous linear combination of variances and covariances, if evidence does not support the simplifying assumptions.

We have now succeeded in obtaining Var $\{R(t)\}$ for each t. Professor Hans Bühlmann has explained several principles for obtaining premiums. These premiums involve a loading for contingencies but no loading for expenses. Thus they exceed net premiums but are less than gross premiums. If a risk involves an accumulated claim process S_t, then the premium P_t for the assumption of the claim experience depends on the distribution of S_t. One way to express this dependence is

$$P_t = E\{S_t\} + \beta\, \text{Var}\, \{S_t\} \quad (10)$$

(see pp. 85 and 86 of ref. [8]). Bühlmann calls this the variance principle for calculating premiums. He also states a standard deviation principle

$$P_t = E\{S_t\} + \alpha(\text{Var}\, \{S_t\})^{1/2}\,. \quad (11)$$

Let us return to the simple mortality model $C(t)$. Since Var $\{C(t)\} = p_2t$, if one is concerned with premiums for risks over a time interval $[0, T]$, premium $= p_1T + \beta p_2T$. This capitalizes on the monotonically increasing nature of the mean and variance functions, that is,

$$\max_{0\le t\le T} E\{C(t)\} = E\{C(T)\}\,, \qquad \max_{0\le t\le T} \text{Var}\, \{C(t)\} = \text{Var}\, \{C(T)\}\,.$$

Mathematically, this need not be the case for the $R(t)$ process variance. Thus a more conservative form of the variance principle would be

$$\text{Premium over } [0, T] = \text{Mean value at } T + \beta \max_{0\le t\le T} \text{Var}\, \{R(t)\}\,. \quad (12)$$

From a practical point of view, $\max_{0\le t\le T}$ Var $\{R(t)\}$ probably can be approximated in most cases by Var $\{R(T)\}$. Since the variance measures dispersion in squared units, obviously β is of dimension unit^{-1}, so that the premium is not a mixture of units and squared units. The standard deviation principle seems more natural, and we will use it in an example in Section V.

III. PROVISION FOR A TREND

Edward A. Lew has observed that the deltas added for adverse deviations should allow for a possible trend as well as for random oscillations. This suggests that one may wish to consider a model which is partially random and partially deterministic. This could be accomplished by modifying the $R(t)$ process as follows:

$$R^*(t) = f(t) + R(t), \quad 0 \leq t < \infty , \tag{13}$$

where the $f(t)$ term gives one method of allowing for a trend, for example, inflation if $f(t) \geq 0$, for $t \geq 0$. We quickly see that

$$E\{R^*(t)\} = f(t) + p_1 t , \tag{14}$$

$$\text{Var}\{R^*(t)\} = \text{Var}\{R(t)\} . \tag{15}$$

Let us also modify the deviations process:

$$D^*(t) = f(t) + D(t), \quad 0 \leq t < \infty . \tag{16}$$

One obtains easily

$$E\{D^*(t)\} = f(t) , \tag{17}$$

$$\text{Var}\{D^*(t)\} = \text{Var}\{D(t)\} = \text{Var}\{R(t)\} . \tag{18}$$

If we assume that the $f(t)$ term is monotonically increasing, the addition of the trend term increases the premiums for T units of time to

$$\text{Net premium} = Tp_1 + f(T) , \tag{19}$$

$$\text{Gross premium } (G^*) = Tp_1 + f(T) + \lambda T . \tag{20}$$

It must be stressed that expressions (13) and (16) are models which have random and deterministic components. Presumably one could obtain a reliable trend function from observed data, using various numerical analysis techniques. It should be observed that the nonrandomness of the trend function provides a pleasant reduction in various probability statements. Thus, assume that one wants the gross premium G^* such that the probability that the greatest timed difference between $R^*(t)$ and tG^* is greater than the initial reserve u is appropriately small, say 0.001. This means that one seeks G^* such that

$$P\{\underset{0 \leq t < \infty}{\text{supremum}} [R^*(t) - tG^*] > u\} = 0.001 . \tag{21}$$

But

$$R^*(t) - tG^* = f(t) + R(t) - tp_1 - f(t) - \lambda t = D(t) - \lambda t , \tag{22}$$

and hence equation (21) reduces to

$$P\{\underset{0\le t<\infty}{\text{supremum}}\,[D(t) - t\lambda] > u\} = 0.001\ . \tag{23}$$

The reader will see that this agrees with equation (8).

Let us consider two examples of $f(t)$. Let us assume that an insurance line expects 60 claims per year, that is, $t = 60$ corresponds to one year. Let us assume that past statistics lead the actuary to expect an inflationary trend of 2 per cent of the average claim. This amounts to expecting $0.02p_1 \times 60$ as the *aggregate* cost of inflation for one year. Mathematically, this can be achieved by letting $f(t) = 0.02p_1t$, $0 \le t \le 60$. The same example can be refined to include a cycle added to the linear trend. Thus, let $f(t) = 0.02p_1t + 0.075p_1 \sin(\pi t/30)$, $0 \le t \le 60$. The graph of this $f(t)$ appears in Figure 1. As t increases from 120 through 180,

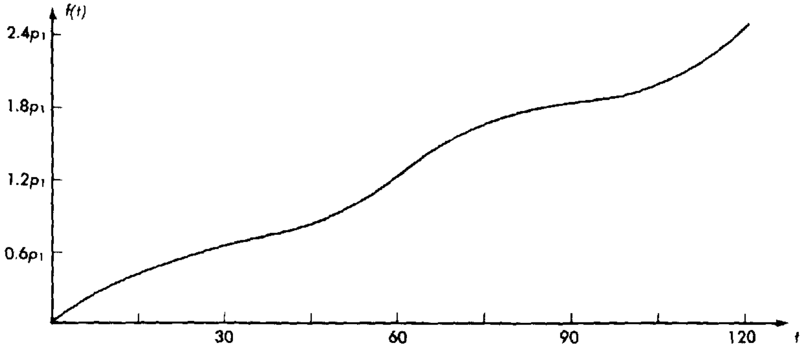

FIG. 1

$\sin(\pi t/30)$ will again grow from zero through $+1$, then down through zero to -1 and up again to zero. Indeed, one can refine $f(t)$ even more by adding a t^2 or t^3 term, or a cosine term, and so on. However, as far as the premium calculation is involved, only the function value at the last time point, namely, $f(T)$, is involved.

The process of obtaining $f(t)$ from observations $D^*(t)$ which involve deviations is similar to what the actuary does in the graduation of collected data. For example, on page 4 of Morton Miller's monograph [20], the process of graduation involves striving for the true values V_x from the observed values u_x'' which contain error terms e_x.

$$u_x'' = V_x + e_x\ . \tag{24}$$

This is analogous to our equation

$$D^*(t) = f(t) + D(t)\ . \tag{25}$$

For the regular $R(t)$ process, the variation or standard deviation gives a precise expression for a "unit of risk deviation." The addition of α or β of these units transforms a net premium into a gross premium. When one also considers a trend, equation (12) becomes

$$\text{Premium over } [0, T] = \text{Mean value at } T + \beta \max_{0 \le t \le T} \text{Var } \{R^*(t)\}$$

$$\doteq Tp_1 + f(T) + \beta \text{ Var } \{R(T)\}$$

$$= Tp_1 + \beta[\, f(T)/\beta + \text{Var } \{R(T)\}] \,.$$

Thus $f(T)/\beta + \text{Var } \{R(T)\}$ or $f(T)/\beta + \text{std. dev. } \{R(T)\}$ would be the unit of adverse deviation for T units of time.

It should be observed that one can adopt the view that $f(T)$ is part of the net premium rather than part of the provision for adverse deviations. Obviously, the mathematics remains unchanged. There are cases where the provision for an inflationary trend should be larger than the provision for adverse random fluctuations. This is possible in our mathematical model for adverse deviations. Even though β would usually be an integer greater than unity, the provision for an adverse trend, namely, $f(T)$, could exceed the provision for adverse random fluctuations, namely, β Var $\{R(T)\}$. The reader is referred to Gordon D. Shellard's Discussion on Underwriting the Catastrophe Accident Hazard [23] for an interesting discussion involving a trend in claims and one usage of collective risk theory. The constant α or β will always have to be determined by a combination of actuarial judgment and competition.

This paper has attempted to quantify one basic unit "risk dispersion," a multiple of which equals the sum of the deltas in Horn's paper. The theoretical multiple is that number k such that the probability that the greatest difference between $R(t)$ and the gross premium is greater than the initial reserve u is appropriately small, say 0.001. John Wooddy's monograph [30] contains techniques used to calculate the probability distributions for the $C(t)$ process by itself. The bibliography in reference [30] contains references to papers on that subject by Dwight Bartlett, Newton Bowers, Harald Cramér, Paul Kahn, Hilary Seal, and the present author, as well as twenty other references to risk theory. It will be some time before researchers can develop comparable techniques for the $R(t)$ process. As Cramér has pointed out [9], the $C(t)$ process enjoys the advantages of having stationary and independent increments. This proved most beneficial in deriving probability distributions of functionals of the $C(t)$ process. Ornstein-Uhlenbeck processes do not have independent increments, as demonstrated by Beekman ([4], p. 791). Mercifully, we do not need to know the probability distributions to calculate the basic unit "risk dispersion."

IV. LENGTH OF PREMIUM-PAYING PERIOD AND EFFECTS OF INTEREST

Despite the fact that the probability statements are phrased in terms of an infinite observation period or infinite planning horizon, frequently one may think of a relatively short period of time, and no sizable error will be committed. There are now at least six references on this point: Olof Thorin, "Analytical Steps towards a Numerical Calculation of the Ruin Probability for a Finite Period when the Risk Process Is of the Poisson Type or the More General Type Studied by Sparre Andersen" [24]; Nils Wikstad, "Exemplification of Ruin Probabilities" [29]; John A. Beekman and Newton L. Bowers, "An Approximation to the Finite Time Ruin Function" [6, 7]; David G. Halmstad, Discussion on Underwriting the Catastrophe Accident Hazard [15]; and Hilary L. Seal, "Numerical Calculation of the Probability of Ruin in the Poisson/Exponential Case" [22]. In essence, these references show that, if a line of business has a reasonable number of claims per year, then probability statements with respect to a relatively few calendar years are little different from probabilities based on an "infinite" planning horizon.

Dr. Hans Gerber added a multiple of the Wiener process to the collective risk process [13]. He stated that this allowed another dimension of variability. Gerber has also studied the effects on the potential management decisions when the collective risk process is modified to discount all future claims for interest [12]. Possibly the effects of interest could be recognized here also, although it would make this presentation more complicated. The author has some preliminary results in this area.

V. PRACTICAL IMPLEMENTATION

First, what does all this mean for the practicing actuary charged with computing premiums which provide for the adverse deviations of mortality, investment performance, operating expenses, and lapses? Second, does the actuary have to think in terms of stochastic processes, operational time, and the like? The reader undoubtedly suspects what the answers to these questions are. However, let us pause a moment to reflect on some of the values of mathematical models. An excellent statement in this regard is given by the noted probabilist Mark Kac [18]: "Models are, for the most part, caricatures of reality, but if they are good, then, like good caricatures, they portray, though perhaps in distorted manner, some of the features of the real world. The main role of models is not so much to explain and to predict—though ultimately these are the main functions of science—as to polarize thinking and to pose sharp questions." The collective risk model which Filip Lundberg created in 1903 [19] has forced many actuaries to see some of their problems in a different light. Furthermore, it is now generally accepted that the model provides more accurate answers to some of the actuary's problems. This is partially due to the great amount of research which has been done in various countries to refine the collective risk model and obtain distributions for the functionals of the $C(t)$ process. The reader may consult John Wooddy's work [30] for the results of some of this research.

The answer to the second question above is no. With respect to the first question, the actuary should go through some thinking and research on the ingredients in equation (3) as well as a possible trend function. This does *not* mean that he has to think about variance (aggregate claims) as $p_2 t$, which involves the operational time t. It does mean that he must think about and study, and form an estimate for, the variance of aggregate claims over a reasonable amount of time. With respect to the investment income deviation, he does *not* have to think about its variance as the variance of an Ornstein-Uhlenbeck process. However, if one obtains a single number for the variance over time, that single number may also be interpreted as the constant variance function of an Ornstein-Uhlenbeck process. Similar remarks apply to the operating expense and lapse expense deviation. If one obtains a single number close to $+1$ or -1, where the "closeness" is judged by the usual statistical tests for sample correlation coefficients, for the correlation coefficient of the investment income deviation and the operating expense deviation, that single number may also be interpreted as the correlation coefficient function of two Ornstein-Uhlenbeck processes which are linearly related for each member of the time interval. As stressed earlier, the basic relation (3) can be used whether or not the various correlation coefficients are "close" to $+1$ or -1.

These preparations of estimating variances, covariances, and a possible trend will provide the actuary with an estimate of the basic risk dispersion unit. A multiple of this unit is the actuary's quantification of the provisions for adverse deviations.

Let us assume that the actuary has some information on lapse, investment, and operating expense deviations. For example, we shall denote the lapse expense deviations by L_i, $i = 1, \ldots, n_L$ with frequencies

$f(i)$, $i = 1, \ldots, n_L$. These can be used to compute

$$m(1, L) = \sum_{i=1}^{n_L} L_i f(i) , \quad m(2, L) = \sum_{i=1}^{n_L} L_i^2 f(i) .$$

Although we have assumed that the theoretical mean is zero, the sample mean $m(1, L)$ probably will not equal zero. We may now estimate the quantity σ_L^2 by $\hat{\sigma}_L^2 = m(2, L) - [m(1, L)]^2$. One may use observed deviations and frequencies to obtain also the estimates $\hat{\sigma}_I^2 = m(2, I) - [m(1, I)]^2$ and $\hat{\sigma}_O^2 = m(2, O) - [m(1, O)]^2$. It will prove convenient to express these estimates as multiples of p_2: $\hat{\sigma}_L^2 = k_1 p_2$; $\hat{\sigma}_I^2 = k_2 p_2$; $\hat{\sigma}_O^2 = k_3 p_2$. For simplicity, assume that the distribution of claims consists of equal 0.25 weights at the values 5, 10, 25, and 50. We are measuring in \$1,000 units. Then the average claim $p_1 = 22.5$ units, and $p_2 = 812.5$. Let us assume that we expect 100 claims per year. Then the aggregate net premium for one year without provision for deviations is $p_1 t = 2{,}250$. Let us assume that we have computed $\hat{\sigma}_L^2$, $\hat{\sigma}_I^2$, $\hat{\sigma}_O^2$ and that $k_1 = 75$, $k_2 = 25$, and $k_3 = 10$. We must now examine our evidence about the correlation coefficients. The simplest case would be that for which they are all zero. Then, for a one-year period, Var $\{R(100)\} = 812.5(100 + 75 + 25 + 10)$. If existing evidence allows us to use the simple correlation coefficients involved in expression (9), we obtain Var $\{R(t)\} = p_2[t + k_2 + k_3 + k_1 - 2\sqrt{(k_2 k_3)} + 2\sqrt{(k_3 k_1)} - 2\sqrt{(k_1 k_2)}]$. For our values of k_1, k_2, k_3, Var $\{R(100)\} = 812.5(146.6)$. As a provision for trend, let us use $0.05 p_1 t = 112.5$. Combining these quantities, a net premium would be $p_1 t + f(t) + \alpha\sqrt{[\text{Var}\ \{R(t)\}]} = 2{,}250 + 112.5 + \alpha(344.85)$.

What α should be used? Theoretically, this could be determined from equation (7), but this would be very difficult at the present time. In view of the fact that Var $\{C(t)\} <$ Var $\{R(t)\}$, it should be conservative to approximate $\alpha\sqrt{[\text{Var}\ \{R(t)\}]}$ by $\bar{\alpha}\sqrt{[\text{Var}\ \{R(t)\}]}$, where $\bar{\alpha}$ is determined by the condition that $P\{C(t) - p_1 t > \bar{\alpha}\sqrt{(p_2 t)}\} = 0.001$. This amounts to determining $\bar{\alpha}$ such that $F(p_1 t + \bar{\alpha}\sqrt{(p_2 t)},\ t) = 0.999$. There are a variety of good techniques for doing so. Let us use the asymptotic expansion involving the normal distribution and its first six derivatives. This can be found in references [1] and [9] and elsewhere. We find that $\bar{\alpha} = 3.30$. Under the various assumptions we have made, the *aggregate* provision for deviations is now 1,138.01. Thus the net premium for one year with provision for deviations and trend is 3,500.51.

The next problem is to distribute the provision for deviations and trend equitably among the various policyholders. One possible method (among several) is the following. Since the larger-risk policies contribute more to the mortality deviation, they should bear a greater proportion of the provision for total deviations and trend. The accompanying tabulation shows a mathematical way of doing this. A second step would

Class of Policy	Proportion of Aggregate Provision	Provision
$ 5,000 (5)	$\frac{1}{4}$(5/22.5)	69.473
10,000 (10)	$\frac{1}{4}$(10/22.5)	138.946
25,000 (25)	$\frac{1}{4}$(25/22.5)	347.364
50,000 (50)	$\frac{1}{4}$(50/22.5)	694.728
Total	....	1,250.51

be to divide these class provisions into individual premium provisions according to age at issue. It would seem logical that these should increase with age. One method of allowing for this would be to multiply the class provision by ratios r_x, where r_x is the net single premium for issue age x divided by the aggregate net single premiums for the class. The reader should be cautioned that the various numbers in this example are for illustrative purposes only.

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APPENDIX

In a Gaussian process $\{X(t), t \in A\}$ any finite collection of random variables has a multivariate normal distribution. This means that for any integer n and any subset $\{t_1, t_2, \ldots, t_n\}$ of A, the n random variables $X(t_1), \ldots, X(t_n)$ possess a joint probability density given, for any real numbers $x_1, x_2, \ldots, x_n$, by

$$\frac{1}{(2\pi)^{n/2}}\frac{1}{|C|^{1/2}}\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}(x_j-m_j)C^{jk}(x_k-m_k)\right],$$

where, for $j, k = 1, 2, \ldots, n$, $m_j = E\{X(t_j)\}$, and $C_{jk} = \text{Cov}\,\{X(t_j), X(t_k)\}$,

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix},$$

$$C^{-1} = \begin{bmatrix} C^{11} & C^{12} & \dots & C^{1n} \\ C^{21} & C^{22} & \dots & C^{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C^{n1} & C^{n2} & \dots & C^{nn} \end{bmatrix},$$

and $|C|$ is the determinant of the matrix C. Since the multivariate normal density is completely determined by the m_j's and the C_{jk}'s, it is clear that the mean-value function $E\{X(t)\}$, $t \in A$, and the covariance function Cov $\{X(s), X(t)\}$, $s \in A$, $t \in A$, determine the complete probability law of the process. A stochastic process is Markovian if, for any integer $n \geq 1$, if $t_1 < t_2 < \dots < t_n$ are points in A, $P\{X(t_n) \leq \lambda | X(t_1), X(t_2), \dots, X(t_{n-1})\} = P\{X(t_n) \leq \lambda | X(t_{n-1})\}$. For a Gaussian process to be Markovian, the covariance function $r(s, t) = E\{[X(s) - m(s)][X(t) - m(t)]\}$ must factor as follows:

$$r(s, t) = \begin{cases} u(s)v(t) , & s \leq t \\ u(t)v(s) , & t \leq s , \end{cases}$$

where $u(t) \geq 0$, $v(t) > 0$, and $u(t)/v(t)$ is strictly increasing on A. The best-known examples of Gaussian Markov processes are the following:

1. The Wiener process [28]:

$$u(t) = t, \quad v(t) = 1, \quad 0 \leq t < \infty .$$

2. The Doob-Kac process (see refs. [10], [17]):

$$u(t) = t, \quad v(t) = 1 - t, \quad 0 \leq t \leq 1 .$$

3. The Ornstein-Uhlenbeck family of processes:

$$u(t) = \sigma^2 e^{\beta t}, \quad v(t) = e^{-\beta t}, \quad \sigma^2 > 0, \quad \alpha > 0, \quad 0 \leq t < \infty .$$

The Doob-Kac process is also referred to as the tied-down Wiener process or the Brownian bridge, because not only does $X(0) = 0$ as in the Wiener process but also $X(1) = 0$. J. L. Doob used the process in computing the distributions of nonparametric statistics, specifically Kolmogorov-Smirnov statistics.

For the Ornstein-Uhlenbeck process, the transition density function ($s < t$) is

$$p(x, s; y, t) = \frac{\partial}{\partial y} P\{X(t) \leq y \mid X(s) = x\}$$

$$= [2\pi A(s, t)]^{-1/2} \exp\left(-\frac{\{y - x \exp[-\beta(t-s)]\}^2}{2A(s, t)}\right),$$

where $A(s, t) = \sigma^2\{1 - \exp[-2\beta(t - s)]\}$, and $\sigma^2 > 0$, $\beta > 0$. This yields a conditional mean function

$$E\{X(t) \mid X(s) = x\} = \int_{-\infty}^{\infty} y p(x, s; y, t) dy = x e^{-\beta(t-s)}.$$

The variance function is $u(t)v(t) = \sigma^2$, for *all* $t \geq 0$.

For the Wiener process,

$$E\{X(t) \mid X(s) = x\} = \int_{-\infty}^{\infty} y \frac{\exp\{-(y - x)^2/[2(t - s)]\}}{\sqrt{[2\pi(t - s)]}} dy = x.$$

The transition density function is a dynamic method for describing the evolution of the process in time.

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DISCUSSION OF PRECEDING PAPER

RICHARD W. ZIOCK:

This addition of investment, withdrawal, and expense processes to the traditional collective risk model is a welcome contribution to the actuarial literature. It seems to me that modeling a portfolio of individual life policies with the new collective risk model may have a shortcoming. In the collective risk model, the four processes are given mean values. Yet on the individual policy all four processes vary by duration. In particular, mortality increases with duration, lapse decreases, expense decreases at first then may increase with inflation, and investment income increases substantially as the assets build up on permanent policies.

Nevertheless, it is quite common in practical actuarial work to replace a known variable with its mean or median and proceed with one's calculations, in order to avoid undue complexity and detail. This is the great practical value of this model.

Traditionally—in risk theory—alpha, the loading, has been determined so that the ruin probability, given an initial capital, is very small, say 0.001. The loadings—or deltas—in the natural reserve premium under GAAP accounting are to serve another purpose. Their purpose is the assurance that the reserves will be sufficient, with some large probability, say 75 per cent, to mature a given block of policies at a profit. Knowledgeable people tell me this.

This requirement—to mature a block of policies—drives the collective risk model to the wall, since collectivization has destroyed the identity of individual blocks of business. However, if we can assume that the company possesses a fair number of such blocks of business which remain in a stationary relationship to one another through time, then it seems to me that an equivalent requirement is that 75 per cent of the time the collective of policies shows a profit (that is, alpha flows through as a gain and is not consumed by adverse experience).

This leads me to propose that alpha be determined so that there is both a 75 per cent chance of a profit each year and not more than a 0.001 chance that the capital and surplus are ever negative.

ETHAN STROH:

The author has taken on the problem of dealing with risks other than mortality within the context of collective risk theory. In setting gross premiums, an insurer must make provision for unfavorable risks in the areas of investment return, expense rates, and lapse experience. On the other hand, excessive zeal in making such provision can lead to an unfavorable competitive position. For this reason the quantification of the risks associated with premium loading items is a useful enterprise. My comments are addressed to the consequences of introducing these elements into a "pure" collective risk model.

Collective risk theory involves the making of two separate guesses about mortality experience. The first guess is about the rate of claim; the second concerns the amount of money which is paid on incidence of one claim. These two assumptions, or sets of assumptions, are then married by means of a compound Poisson process.

It seems natural to suspect that, since investment risk, expenses, and lapse rate are related to the aggregate payout, they might be incorporated directly into the classical formulas of collective risk theory.

Consider formula (1) of the paper:

$$R(t) = C(t) - I(t) + O(t) + L(t) ,$$

where $C(t)$ is the claim process, with probability distribution as conventionally derived, and $I(t)$, $O(t)$, $L(t)$ are the net premium "loading" processes covering investment, expense, and lapse; if these are Gaussian, the probability distributions are normal distributions. Let $F(x)$ be the probability distribution of the claim process for a unit of operational time, and let $N_1(x)$, $N_2(x)$, $N_3(x)$ be the probability distributions of the loading processes; $R(x)$ is the probability distribution of total insurer cost, including claims and expenses and including provision for investment deviation and lapsation. Then, if the four processes are independent,

$$R(x) = F(x) * N_1(x) * N_2(x) * N_3(x) ,$$

where the operator $*$ denotes convolution, that is, for two probability distributions $G(x)$, $H(x)$ with independent random variables $g(t)$, $h(t)$, the probability distribution of $g(t) + h(t)$ is

$$G(x) * H(x) = \int_0^x G(x - s)dH(s) .$$

It is well known that the sum of two independent normally distributed variables is again normally distributed. Accordingly, we may define $N(x) = N_1(x)*N_2(x)*N_3(x)$ as the aggregate loading distribution, so that

$$R(x) = F(x) * N(x)$$

Over a unit of operational time the distribution $F(x)$ has a moment generating function $\Phi(s) = e^{-1+v(s)}$, where $v(s)$ is the moment generating function of the probability distribution $v(x)$ giving the likelihood that one death claim at random will be for the amount \$$X$ or less. The normal distribution $N(x)$ has the moment generating function $n(s) = \exp(ms + \sigma^2 s^2/2)$. The convolution $R(x)$ of $F(x)$ and $N(x)$ has the moment generating function

$$\begin{aligned}\rho(s) &= \exp[-1 + v(s)] \exp(ms + \sigma^2 s^2/2) \\ &= \exp[-1 + v(s) + ms + \sigma^2 s^2/2] .\end{aligned}$$

If we define $v^*(s) = v(s) + ms + \sigma^2 s^2/2$, we see that $\rho(s) = \exp[-1 + v^*(s)]$, which is the moment generating function of a compound Poisson process similar to that associated with net claims.

It appears, then, that loadings of various sorts (positive and negative) may be introduced into the probability distribution of claims simply as adjustments to the individual claim amount distribution, provided that the loadings are governed by Gaussian processes.

Turning to the question of determining the net premium loading needed in order to have an appropriately small probability of ruin, we see that the modified distribution described above has a mean value equal to the gross premium.

This "net" gross premium, so to speak, should be loaded to cover contingencies. Using the modified distribution discussed above, we may (*a*) compute the standard deviation in the usual manner for collective risk theory distribution and apply one of the principles for obtaining premiums cited by the author or (*b*) go directly to ruin function approximations in order to obtain a rational loading formula.

I am not sure that I agree fully with the author's Section III. The whole question of introducing a claim trend into a collective risk theory model seems to me to be more complicated than is suggested. The classical model presupposes the independence of a mean number of claims over an interval of time and the exact instant at which the interval commences; for example, expected numbers of claims over the periods 1850–60 and 1970–80 would be the same. This means that a single portfolio distribution of insurance amounts for one claim may be constructed and extended, by convolution, to cover any number of claims in any time interval. The Poisson process is stationary in its nature in much the same way as a stationary population. Introducing the notion of a trend requires that the commencement of time intervals be identified, a concept inimical to the Poisson process.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. BEEKMAN:

The author appreciates Mr. Ziock's and Mr. Stroh's discussions of his paper.

With respect to Mr. Ziock's first point, it should be stressed that the new collective risk model was designed for the *deviations* of the four processes, not for the processes themselves. Even though mortality, lapse risk, expenses, and investment income will vary in time for any one person, the collective deviations may not vary. The suggestion that alpha (or lambda) "be determined so that there is both a 75 per cent chance of a profit each year and not more than a 0.001 chance that the capital and surplus are ever negative" is most interesting. Let us express this in probability phrases similar to equations (7) and (8) of the paper. Assume that the expected number of claims in one year is T. Then the first requirement on λ is that $P\{p_1 + \lambda) - R(T) > 0\} = 0.75$. The second requirement is that

$$P\{\underset{0 \leq t < \infty}{\text{supremum}} [R(T) - t(p_1 + \lambda)] > u\} = 0.001 .$$

The first requirement can be converted into the statement $P\{D(T) < \lambda T\}$. The second (long-range) requirement is not so simple. However, a suggestion for making some of the calculations will be given briefly.

Mr. Stroh has suggested several interesting ideas. In the formula $R(t) = C(t) - I(t) + O(t) + L(t)$, he refers to $I(t)$, $O(t)$, and $L(t)$ as the loading processes. Under the assumption that the four processes are independent, he has derived the distribution of $R(t)$ for one unit of operational time. I think that his derivation is very worthwhile. Since submitting my paper for publication, I have obtained various results about the distribution of $R(t)$, which will be explained briefly. Mr. Stroh observes how the mean and standard deviation of the distribution of $R(t)$ can be used in setting a premium which allows for adverse deviations. Mr. Stroh has very accurately stated the implication of the stationarity of the compound Poisson process. However, I wanted the model to contain a deterministic component allowing for trend. This was so introduced that it does not affect the stationarity of the $C(t)$ process. The trend function was added to allow for inflationary (deflationary) movements in any or all of the four processes. The over-all probability statements are not changed, as revealed in equations (21), (22), and (23).

Motivated by Mr. Stroh's and Mr. Ziock's discussions, let us consider a further result about the $D(t)$ process and its application to the example in Section V of the paper. We will calculate the asymptotic distribution of $D(t)$ as $t \to \infty$. The deviations process $D(t)$ is defined in equation (4). Using the independence of the four processes, the characteristic function reduces as follows:

$$E\{\exp[i\theta D(t)]\} = \exp(-i\theta p_1 t) E\{\exp[i\theta C(t)]\}$$
$$\times E\{\exp[-i\theta I(t)]\} E\{\exp[i\theta O(t)]\} E\{\exp[i\theta L(t)]\} .$$

It is shown in references [9], [1], and elsewhere that

$$E\{\exp[i\theta C(t)]\} = \exp\left\{t\left[\int_{-\infty}^{\infty} \exp(i\theta y) dP(y) - 1\right]\right\},$$

where $P(y)$ is the common distribution of individual claims. Since $I(t)$, $O(t)$, and $L(t)$ have been assumed to be normal variates with zero means and constant variances σ_I^2, σ_O^2, σ_L^2, the last three expectations are given by $\exp(-\theta^2\sigma_I^2/2)$, $\exp(\theta^2\sigma_O^2/2)$, and $\exp(\theta^2\sigma_L^2/2)$.

Assuming that we can integrate term by term,

$$t\left[\int_{-\infty}^{\infty} \exp(i\theta y) dP(y) - 1\right] = i\theta p_1 t - \theta^2 p_2 t/2! - i\theta^3 p_3 t/3!$$
$$+ \theta^4 p_4 t/4! + i\theta^5 p_5 t/5! + \dots .$$

Combining these facts would give an expression for the characteristic function of $D(t)$.

Now consider the standardized random variable

$$\frac{D(t) - E[D(t)]}{\{\mathrm{Var}[D(t)]\}^{1/2}} = \frac{D(t)}{(p_2 t + \sigma_I^2 + \sigma_O^2 + \sigma_L^2)^{1/2}} .$$

This makes use of equations (5), (6), and (9) of the paper. This random variable has as its characteristic function $E\{\exp[i\theta^* D(t)]\}$, where $\theta^* = \theta[p_2 t + \sigma_I^2 + \sigma_O^2 + \sigma_L^2)^{-1/2}$. Using the previous results,

$$E\{\exp[i\theta^* D(t)]\}$$
$$= \exp\left[-\frac{(\theta^*)^2}{2!}(p_2 t + \sigma_I^2 + \sigma_O^2 + \sigma_L^2)\right.$$
$$\left. - i\frac{(\theta^*)^3}{3!} p_3 t + \frac{(\theta^*)^4}{4!} p_4 t + \frac{i(\theta^*)^5}{5!} p_5 t - \frac{(\theta^*)^6}{6!} p_6 t + \dots\right]$$
$$= \exp\left(-\frac{\theta^2}{2}\right)\left[1 - \frac{i(\theta^*)^3}{3!} p_3 t + \frac{(\theta^*)^4}{4!} p_4 t + \frac{i(\theta^*)^5}{5!} p_5 t\right.$$
$$\left. - \frac{(\theta^*)^6}{6!} p_6 t + O(t^{-5/2})\right],$$

where the last term indicates that |Remainder| $< At^{-5/2}$ for large t and for some positive constant A. It will prove convenient also to think of the series with $(\theta^*)^j$ replaced by θ^j/k^j, where $k = (p_2 t + \sigma_I^2 + \sigma_O^2 + \sigma_L^2)^{1/2}$.

If

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \quad \text{and} \quad \Phi^{(j)}(x) = \frac{d^j\Phi(x)}{dx^j},$$

it is shown in many references that

$$\int_{-\infty}^{\infty} \exp(i\theta x) d\Phi(x) = \exp(-\theta^2/2)$$

and

$$\int_{-\infty}^{\infty} \exp(i\theta x) d\Phi^{(j)}(x) = (-i\theta)^j \exp(-\theta^2/2) .$$

Thus our previous series expression for $E\{\exp[i\theta^* D(t)]\}$ also equals

$$\int_{-\infty}^{\infty} \exp(i\theta x) d\left[\Phi(x) - \frac{p_3 t}{3!k^3}\Phi^{(3)}(x) + \frac{p_4 t}{4!k^4}\Phi^{(4)}(x) - \frac{p_5 t}{5!k^5}\Phi^{(5)}(x) + \ldots\right].$$

Hence, by the complete equality of functions with the same Fourier transform,

$$P\left\{\frac{D(t)}{(p_2 t + \sigma_I^2 + \sigma_O^2 + \sigma_L^2)^{1/2}} \le x\right\} = \Phi(x) - \frac{p_3 t}{3!k^3}\Phi^{(3)}(x) + \frac{p_4 t}{4!k^4}\Phi^{(4)}(x) + O(t^{-3/2}).$$

This gives an easy method of computing the distribution of $D(t)$, if the processes are assumed independent and t is fairly large. A readily available source of $\Phi^{(j)}(x)$ values is *Handbook of Tables for Probability and Statistics*, edited by William H. Beyer (2d ed.; Cleveland, Ohio: Chemical Rubber Co., 1968).

Let us reconsider the example contained in Section V of the paper and now assume that the four processes are independent. Then $k = [812.5(210)]^{1/2} \doteq 413.07$, and |Remainder| $< A/1{,}000$ for the series approximation to the probability. Let us determine x so that $P\{D(t) > x(413.07)\} = 0.001$. It is easy to calculate the needed constants: $p_3 t/6k^3 = 0.00838$ and $p_4 t/24k^4 = 0.00238$. With a few trial calculations, one obtains $x = 2.97$. This says that, under the previous assumptions, there is a probability of only 0.001 that the aggregate deviations will exceed $x\sigma_{D(t)} = 1{,}226.818$. This number may be contrasted with $\bar{\alpha}\sigma_{D(t)} = 3.30(413.07) = 1{,}363.131$, produced by the Section V method for determining $\bar{\alpha}$ if the four processes are independent. This is further evidence for the statement that the Section V method for calculating $\bar{\alpha}$ is conservative.

The author wishes to thank Mr. Stroh and Mr. Ziock for their thought-provoking discussions.