**TRANSACTIONS OF SOCIETY OF ACTUARIES
1976 VOL. 28**

INTRODUCTION TO THE DYNAMICS OF PENSION FUNDING

NEWTON L. BOWERS, JR., JAMES C. HICKMAN, AND CECIL J. NESBITT

ABSTRACT

This paper is presented in seven sections. Section I, "Purpose and Scope," indicates sources of the ideas and how we undertook to develop them. Section II, "The Model Pension Plan," gives a general mathematical setting for studying pension funding under dynamic conditions. Section III defines, and expresses the derivatives of, five basic functions of time that are utilized in pension funding theory. Section IV makes application to individual cost methods and Section V to aggregate cost methods, the latter being linked in each case to a corresponding individual cost method by means of an accrual function. Section VI, "The Exponential Growth Case," examines the special relations that exist when all the growth functions are exponential. The concluding section notes that only a foundation has been laid and there is much room for further mathematical exploration.

I. PURPOSE AND SCOPE

In 1952 Trowbridge's paper "Fundamentals of Pension Funding" [13], utilizing a simplified mathematical model of a pension plan operating in a stationary population, greatly clarified the basic principles that underlie the funding methods employed by pension actuaries. Since that time demographers have refined the theory of stable populations subject to fixed rates of fertility and mortality (see, for example, Keyfitz [7]). This theory may be used to provide a setting more general than stationary populations within which to study the operation of funding principles. During the same period inflation of prices and wages has had profound effects on the actuarial management of pension funds and social security systems. A number of actuaries have touched on these matters in various publications [3–6, 8–11, 14, 16], but an overall mathematical exploration has not appeared. The purpose of this paper is to provide an introduction to mathematical principles applicable to pension funding under dynamic conditions of population growth, inflation, and automatic adjustment of benefits.

The introductory aspect of this paper will be sought by restricting the presentation to basic ideas. The model that will be developed will be deterministic, with no attention paid to contingency margins, to experience deviations from the assumptions, or to estimation of the parameters of the model. There appear to be many branches from the ideas of this paper that could be explored in discussions or future papers. In particular, with some complications, the theory could be extended to the case of pensions expressed in units which are tied to some time-dependent index such as the consumer price index. In the simplified model presented here, the adjustment of pensions may depend on duration since retirement but not intrinsically on time of payment.

Generality will be obtained by using the concept of the cumulative pension purchase function, developed by Cooper and Hickman [2]. At the same time, by this approach simplicity is gained for expressing the accrued liability of the model plan. For ease of mathematical development, a continuous rather than a discrete model will be presented.

The presentation will be mathematical throughout, including the applications. This is for the convenience of readers who want to follow in detail the mathematical developments. However, it is hoped that readers who seek only a general understanding of the paper can achieve this by selective skipping of mathematical details. The mathematics is elementary, and many of the mathematical statements lend themselves to immediate verbal interpretations, which will be supplied. Other authors have worked on or are developing numerical illustrations of these and related concepts [1, 6, 9, 16]. In contrast, this paper sets up a basis for the mathematical discussion of pension funding under conditions of growth.

II. THE MODEL PENSION PLAN

In this section the model plan that will be used in our development will be described. For this purpose, certain functions expressing the dynamic economic and demographic forces influencing the plan will be defined. Some typical actuarial functions for the valuation of individual unit benefits will be introduced to serve in the development of the basic actuarial functions for the plan as a whole.

For the model plan, entry is fixed at age a and retirement at age r. As in Trowbridge's paper [13], only retirement benefits will be considered, that is, no account will be taken of death, disability, or withdrawal benefits. (A slightly different approach was taken by Humphrey et al. [6], C.3, p. 212.) Participants are assumed to survive as members of the plan according to a time-independent survivorship function l_x, which applies to both the active period from age a to age r and the retired period from age r. Further, the radix of the function l_x is chosen so that l_r represents at time 0 the density of participants at age r, that is, at time 0 the number of participants between ages r and $r + dr$ is approximately $l_r dr$.

Growth in the number of participants will be measured in terms of growth in the density of participants at age r by means of a function $g_1(t)$, with $g_1(0) = 1$. Then the density of participants aged r at time t is given by the expression $g_1(t)l_r$, and the density of participants aged x at time t by the formula

$$g_1(t + r - x)l_x\,, \qquad a \leq x\,, \tag{1}$$

since such participants will reach (or have reached) age r at time $t + r - x$.

The density of participants aged x at time 0 is, from formula (1), equal to $g_1(r - x)l_x$. For $x = r$ this density is l_x, but for other ages x it may differ from l_x. As an example, if the density of participants at age r is growing at the annual rate α, so that $g_1(t) = e^{\alpha t}$, then the density of participants aged x at time t is $e^{\alpha(t+r-x)}l_x = e^{\alpha t}e^{\alpha(r-x)}l_x$, where $e^{\alpha(r-x)}l_x$ is the density of participants aged x at time 0. It should be noted that, if the density of participants at age r is growing at the annual rate α, then the density at all other ages $x \geq a$, and hence the total covered group, is also growing at rate α. This is a consequence of the survivorship function l_x being invariant over time.

It is assumed further that the annual rate of salary for a participant aged x at time 0 is $s(x)$, $a \leq x \leq r$, and that $s(x)$ remains as a base factor for salary at age x at times $t \geq 0$. The function $s(x)$ describes the pattern of salary over a working lifetime if we ignore the impact of dynamic economic forces (compare Humphrey et al. [6], p. 212).

Growth in salaries over time will be represented by means of a function denoted as $g_2(t)$, with $g_2(0) = 1$. Then the annual rate of salary for a participant aged x at time t is

$$g_2(t)s(x)\,, \qquad a \leq x \leq r\,. \tag{2}$$

Formula (2) indicates that salaries at any given point of time vary by age in proportion to the function $s(x)$.

It is important to note the difference between formulas (1) and (2). In formula (1) the age interval $r - x$ enters the calculation, but in formula (2) it does not. This is due to the fact that $g_1(r - x)l_x$, not l_x, represents the density of participants at time 0, while $s(x)$ does represent the actual salary rates at time 0. In the case of participants, growth is determined on a generation basis, in fact, by the generation reaching age r at time t. In the case of salary rates, growth is determined on a year-of-experience basis, the fixed pattern at time 0 defined by $s(x)$ being amplified over time by the factor $g_2(t)$. A generation pattern for participants and a year-of-experience pattern for salaries was our choice of model, but under some circumstances one might choose other models using, say, a generation pattern for both participants and salaries.

The density of salaries at age x and time t is given by combining formulas (1) and (2) to obtain

$$g_1(t+r-x)g_2(t)l_x s(x)\,, \quad a \le x \le r\,, \tag{3}$$

and the total annual payroll $\mathbf{W}(t)$ at time t is then expressible as[1]

$$\mathbf{W}(t) = \int_a^r g_1(t+r-x)g_2(t)l_x s(x)dx\,. \tag{4}$$

Further description of the model plan and its actuarial aspects requires three key functions, which will be defined and explained in turn.

The Pension Incurrence Density Function $\mathbf{h}(t)$

The symbol $\mathbf{h}(t)$ represents the density at time t of the amount of newly incurred age r pensions. Thus $\mathbf{h}(10) = 1{,}000$ implies that in the moment $(10,\ 10+dt)$ the amount of new age r pensions which come into effect under the model plan is approximately $1{,}000dt$.

From formula (1) the density of new retirees at time $t+r-x$ from participants aged x at time t is

$$g_1(t+r-x)l_r\,, \tag{5}$$

and from formula (2) each of these will, at time $t+r-x$, have annual salary rate

$$g_2(t+r-x)s(r)\,. \tag{6}$$

It will be assumed for the model plan that pensions are a flat percentage, represented by b, of final salary. Then the definition of $\mathbf{h}(t)$ and formulas (5) and (6) show that the density of new pensions incurred, that is, entering benefit status, at time $t+r-x$ for those who at time t are aged x $(x<r)$, or who at time $t+r-x$ were aged r $(x \ge r)$, can be expressed as

$$\mathbf{h}(t+r-x) = g_1(t+r-x)g_2(t+r-x)l_r s(r)b\,. \tag{7}$$

For $x = r$ equation (7) becomes

$$\mathbf{h}(t) = g_1(t)g_2(t)l_r s(r)b \tag{8}$$

and represents the density of new pensions incurred at time t.

[1] Boldface functions such as $\mathbf{W}(t)$ relate to the whole group rather than to a unit of initial pension. This convention is followed throughout the paper.

Equations (7) and (8) embody a key idea and deserve some discussion. In a stationary population model with no inflation or other growth of salaries over time, $g_1(t) = g_2(t) = 1$, and $h(t)$ is the constant $l_r s(r) b$. The stationary population model can be replaced by one relative to a stable population. The latter is the steady state realized in a group with constant but unequal rates of entry and of termination applicable to the existing population. In this steady state the percentage distribution by age is fixed, but the size of the group can be growing at a fixed rate α. Then $g_1(t)$ can be taken in the form $e^{\alpha t}$, $g_2(t)$ is again assumed to be 1, and the density (7) becomes

$$e^{\alpha(t+r-x)} l_r s(r) b \; . \tag{9}$$

If the population is stationary but salaries have been increasing over time at the annual rate γ, then $g_1(t) = 1$, $g_2(t) = e^{\gamma t}$, and

$$h(t + r - x) = e^{\gamma(t+r-x)} l_r s(r) b \; . \tag{10}$$

Note that the right-hand side of equation (10) is of the same form as (9) but has developed from different growth assumptions.

Equations (7) and (8) may also apply to some simple cases of immature groups. Such a case is that in which $g_1(t) = 0$, $t < 0$, $g_1(t) = 1$, $t \geq 0$, and $g_2(t) = 1$ for all t. Then $h(t)$ is the constant $l_r s(r) b$ for $t \geq 0$.

In general, it will be assumed that the density function $h(t)$ is differentiable. However, as seen in the immature case, $h(t)$ may have points of discontinuity, but in such cases it will be assumed that sufficient one-sided differentiability remains to carry out the purpose on hand.

The simple observation that

$$\begin{aligned} \frac{\partial}{\partial t} h(t + r - x) &= \lim_{\Delta \to 0} \frac{h(t + r - x + \Delta) - h(t + r - x)}{\Delta} \\ &= -\frac{\partial}{\partial x} h(t + r - x) \end{aligned} \tag{11}$$

will be much used in the theory to follow.

The Pension Adjustment Function $\beta(x)$

This function expresses the annual rate of pension payable at age x per unit of annual pension rate at the initial retirement age r. Thus $\beta(x) = 1$, $x = r$. If $\beta(x) = 1$, $x > r$, then pensions remain fixed at their age r levels. If $\beta(x) = e^{\beta(x-r)}$, then pensions are adjusted continuously by a constant rate of increase β. This latter form for the rate of benefit payment function is plausible for several types of postretirement adjustments.

If the adjustment of pensions were tied to a time-dependent index, say $I(t)$, then a function $\beta(x, t)$ of two variables would be required, with

$$\begin{aligned} \beta(x, t) &= 1\,, & x &= r\,, \\ &= I(t)/I(t + r - x)\,, & x &> r\,. \end{aligned}$$

The theory developed on the basis of the two-variable function $\beta(x, t)$ is more complex than when only the single-variable function $\beta(x)$ is used. Its applications would require more detailed assumptions about the future than one might choose to make. If $I(t)$ is taken as the exponential function $ke^{\beta t}$, then $\beta(x, t)$ becomes the single-variable function $e^{\beta(x-r)}$. In view of these observations, the theory will be developed in terms of the simpler function $\beta(x)$.

The Accrual (of Liability) Function $M(x)$

To express the accrual of liability for pensions from age r under the actuarial cost method to be employed, the accrual function $M(x)$ will be used. $M(x)$ is a nondecreasing, right-continuous function of only the age variable and is such that $0 \leq M(x) \leq 1$ for all $x \geq a$. It will be assumed that $M(a) = 0$ except for initial funding, and that $M(x) = 1$, $x \geq r$, except when pay-as-you-go funding is considered. This accrual function is identical with the cumulative pension purchase function described by Cooper and Hickman [2]. Their function was defined in terms of a pension purchase density function $m(x)$ such that

$$M(x) = \int_a^x m(y)dy\,. \tag{12}$$

In some cases, statements may be more understandable in terms of $m(x)$ rather than $M(x)$, and for such cases $m(x)$ will be utilized. In general, it will be assumed that $m(x)$ is continuous for $a < x < r$ and is right-continuous at a and left-continuous at r, and that $m(x) = 0$, $x > r$. For initial and terminal funding, all funding is concentrated at a single age, and for such cases some easy adaptations of formulas are required. For the continuous case it is clear from formula (12) that

$$m(x) = M'(x)\,, \quad a < x < r\,, \tag{13}$$

and is the right derivative at $x = a$ and the left derivative at $x = r$.

The advantage of introducing the accrual function $M(x)$ is that thereby one can develop pension funding theory simultaneously for a whole family of actuarial cost methods instead of working out the theory separately for each method. From the general theory, one can obtain the theory for various cost methods by specializing $M(x)$. Thus pay-as-you-go funding is given by taking $M(x) = 0$, $a \leq x$; terminal funding by $M(x) = 0$, $a \leq x < r$, and $M(x) = 1$, $x \geq r$; unit credit funding by

$$M(x) = \int_a^x s(y)dy \Big/ \int_a^r s(y)dy \,;$$

and entry age normal cost funding by

$$M(x) = \int_a^x s(y)D_y dy \Big/ \int_a^r s(y)D_y dy \,.$$

The last two formulas for $M(x)$ are based on the fixed salary pattern described by the function $s(x)$. When salaries are growing according to the function $g_2(t) = e^{\gamma t}$ (compare formula [2]), one can redefine $M(x)$ to have $s(y)e^{\gamma y}$ in place of $s(y)$ in the two formulas. Thereby, $M(x)$ remains independent of the time variable t, and level costs in relation to salary emerge for individual entry age normal cost funding. If $g_2(t)$ is other than an exponential function, one might revert to the $M(x)$ formulas based on the fixed salary pattern given by the function $s(x)$, but then one would lose some of the level character of entry age normal cost funding.

Additional Functions Relating to a Unit of Initial Pension

In this subsection statements in pension funding terminology will be made regarding a unit pension. The theory is essentially that of individual life annuities (generally deferred) and notations A, V, and P will be used to express concepts analogous to present value, reserve, and annual premium rate for a unit benefit. All functions are on a continuous basis, and, since this is a general assumption throughout, it will not normally be specified in the notation.

The present value of future payments for each unit of initial pension from age r payable while a participant aged x survives is denoted by $A(x)$, where

$$A(x) = \int_{r-x}^{\infty} e^{-\delta u} \; {}_u p_x \beta(x+u)du = \frac{D_r}{D_x}\bar{a}_r^{\beta} \,, \qquad x \leq r \,, \tag{14}$$

and

$$A(x) = \int_{0}^{\infty} e^{-\delta u} \; {}_u p_x \beta(x+u)du = \bar{a}_x^{\beta} \,, \qquad x \geq r \,. \tag{15}$$

Here δ denotes the assumed force of interest, and the superscript β on the annuity value indicates that annuity payments are adjusted by the function $\beta(x)$.

The accrued liability for a unit of initial pension from age r in regard to a participant aged x is denoted by $V(x)$, where

$$V(x) = A(x)M(x) \,. \tag{16}$$

For $x \geq r$, $M(x) = 1$ (except for pay-as-you-go funding) and $V(x) = A(x)$.

The normal cost rate $P(x)$ in regard to a participant aged x for a unit of initial pension from age r is given in the continuous case by the relations

$$P(x) = A(x)m(x)\,, \qquad a \leq x \leq r\,, \tag{17}$$

$$P(x) = 0\,, \qquad x > r\,. \tag{18}$$

For initial funding a single premium $A(a)$ is payable at age a, and for terminal funding a single premium $A(r)$ is payable at age r. In the case of pay-as-you-go funding, $P(x) = 0$, $x < r$, and $P(x) = \beta(x)$, $x \geq r$.

The present value $(Pa)(x)$ of future normal costs in regard to a participant aged x for a unit of initial pension from age r is given by the expression

$$\begin{aligned}(Pa)(x) &= \int_0^{r-x} \frac{D_{x+u}}{D_x} P(x+u)du \\ &= \int_0^{r-x} \frac{D_{x+u}}{D_x} \frac{D_r}{D_{x+u}} \bar{a}_r^\beta m(x+u)du \\ &= A(x)\int_0^{r-x} dM(x+u) \\ &= A(x)[1 - M(x)] = A(x) - V(x)\,. \end{aligned} \tag{19}$$

In other words,

$$V(x) = A(x) - (Pa)(x)\,, \tag{20}$$

which expresses the accrued liability for a unit of initial pension from age r for a participant aged x as the present value of that unit of pension less the present value of future normal costs for the unit pension. This is the usual prospective principle for a benefit being purchased by annual premiums, while formula (16) expresses the liability in terms of single premiums. Formula (16) will usually be the simpler one to employ, but there will be some occasions when it will be convenient to use formula (19) or formula (20).

The derivatives of $A(x)$, $V(x)$, and $P(x)$ will be needed. From equation (14), for $a \leq x < r$, one finds

$$\frac{dA(x)}{dx} = A(x)(\mu_x + \delta)\,, \tag{21}$$

and from equation (15), for $x \geq r$,

$$\frac{dA(x)}{dx} = \frac{d}{dx}\frac{\bar{N}_x^\beta}{D_x}\,,$$

where $D_x^\beta = D_x\beta(x)$ and $\bar{N}_x^\beta = \int_x^\infty D_y^\beta dy$. Then, for $x \geq r$,

$$\begin{aligned}\frac{dA(x)}{dx} &= \bar{N}_x^\beta \frac{d}{dx}\left(\frac{1}{D_x}\right) + \frac{1}{D_x}\frac{d\bar{N}_x^\beta}{dx}\\ &= \frac{\bar{N}_x^\beta}{D_x}(\mu_x + \delta) - \frac{D_x^\beta}{D_x}\\ &= A(x)(\mu_x + \delta) - \beta(x)\,. \qquad (22)\end{aligned}$$

Formulas (21) and (22) indicate that $A(x)$ increases through the forces of termination and interest and, for $x \geq r$, decreases by outgoing payments at rate $\beta(x)$.

For $V(x)$, with $a < x < r$,

$$\begin{aligned}\frac{dV(x)}{ax} &= \frac{d}{dx}[A(x)M(x)]\\ &= A(x)(\mu_x + \delta)M(x) + A(x)m(x)\\ &= V(x)(\mu_x + \delta) + P(x)\,. \qquad (23)\end{aligned}$$

Formula (23) holds also for the right derivative at a and the left derivative at r. For $x > r$ and $M(x) = 1$, $dV(x)/dx = dA(x)/dx$ is given by formula (22). Formula (23) indicates that, for $a < x < r$, $V(x)$ increases under the forces of termination and interest and also increases by reason of incoming normal cost.

For $a < x < r$, and for the right derivative at a and the left derivative at r,

$$\begin{aligned}\frac{dP(x)}{dx} &= \frac{d}{dx}[A(x)m(x)]\\ &= A(x)(\mu_x + \delta)m(x) + A(x)m'(x)\\ &= P(x)(\mu_x + \delta) + A(x)m'(x)\,. \qquad (24)\end{aligned}$$

When $x > r$ and $M(x) = 1$, then $P(x) = 0 = m(x)$ and no derivative is required.

The model plan can now be summarized as follows. The active group extends over the ages a to r, with all new entrants coming in at age a and all retirements occurring at age r. Only retirement benefits are considered. For both the active and the retired participants survivorship is in accordance with the function l_x, which does not depend on the time variable t. At time 0 the density of participants aged r is l_r, and thereafter this density increases by a factor $g_1(t)$. This establishes a generation pattern of growth for the participants. Salaries at time 0 are represented by the function $s(x)$, and thereafter they increase by a factor $g_2(t)$, which establishes a year-of-experience pattern of growth for salaries. Initial pensions are a fixed percentage of final salaries and increase during retirement by a factor $\beta(x)$ dependent on age x. For $a \leq x < r$, the density of new pensions to be incurred at time $t + r - x$ in respect to the survivors of participants aged x at time t is given by the function $h(t + r - x)$. For $x \geq r$, $h(t + r - x)$ is the density of new pensions incurred at time t for those who were then aged r and who may or may not be surviving at age x at time t. The function $h(t + r - x)$ is proportional to the product $g_1(t + r - x)g_2(t + r - x)$. The stage is now set to develop the theory of pension funding under conditions of growth expressible by the functions g_1 and g_2.

III. BASIC FUNCTIONS OF TIME AND THEIR DERIVATIVES

In this section there will be discussed five basic functions of time related to the valuation of the model pension plan subject to growth in accordance with the functions g_1 and g_2. A sixth function, $W(t)$, the total annual payroll at time t, has appeared already, in equation (4).

$B(t)$, the Annual Rate of Pension Outgo at Time t

The density of pensions for participants aged x at time t, $x \geq r$, is determined by the density of new pensions at time $t - (x - r) = t + r - x$, by survivorship from age r to age x, and by the adjustment factor $\beta(x)$. Hence this density is given by the formula

$$h(t + r - x)\frac{l_x}{l_r}\beta(x)\,, \tag{25}$$

and it follows that the annual rate of pension outgo at time t is

$$B(t) = \int_r^\infty h(t + r - x)\frac{l_x}{l_r}\beta(x)dx\,. \tag{26}$$

The derivative of $B(t)$ is obtained as follows:

$$\frac{dB(t)}{dt} = \int_r^\infty \frac{\partial}{\partial t}h(t + r - x)\frac{l_x}{l_r}\beta(x)dx\,.$$

By use of formula (11), this can be rearranged as

$$\frac{dB(t)}{dt} = \int_r^\infty \left[-\frac{\partial}{\partial x}h(t + r - x)\right]\frac{l_x}{l_r}\beta(x)dx$$

$$= -\int_r^\infty \frac{l_x}{l_r}\beta(x)dh(t + r - x)\,,$$

where $dh(t+r-x)$ denotes the differential of $h(t+r-x)$ with respect to x. Then

$$\frac{dB(t)}{dt} = -\frac{l_x}{l_r}\beta(x)h(t+r-x)\Big|_r^\infty + \int_r^\infty \frac{h(t+r-x)}{l_r}\,d[l_x\beta(x)]$$

$$= h(t) + \int_r^\infty \frac{h(t+r-x)}{l_r}[l_x d\beta(x) - \beta(x)l_x\mu_x dx] ,$$

or

$$\frac{dB(t)}{dt} = h(t) - \int_r^\infty h(t+r-x)\beta(x)\frac{l_x}{l_r}\mu_x dx + \int_r^\infty h(t+r-x)\frac{l_x}{l_r}\,d\beta(x) . \tag{27}$$

Each term of the right-hand member of formula (27) can be interpreted with regard to the rate of change of the pension outgo function $B(t)$. The first term (see formula [8]) is the density of new pensions incurred at time t. The middle term represents the pensions terminated by deaths among the retirees. The first two terms can be considered to comprise a *replacement effect*, such that pensions terminated by death are replaced by new pensions represented by $h(t)$. The third term in the right-hand member of equation (27) expresses a *pension adjustment effect.*

$A(t)$, the Present Value of Future Pension Payments to Participants Covered by the Plan at Time t

While only a single expression was developed for the pension outgo function $B(t)$, there are several ways of expressing $A(t)$, depending on the starting data and on the discount factors employed. A direct formula for $A(t)$ is given by

$$A(t) = \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^\beta dx + \int_r^\infty h(t+r-x)\frac{l_x}{l_r}\bar{a}_x^\beta dx . \tag{28}$$

In the first term $h(t+r-x)\bar{a}_r^\beta dx$ represents the value at time $t+r-x$ of new pensions incurred momently at that time for survivors from those aged x at time t, and v^{r-x} brings this value down to time t. The second term represents the value of future pension payments for those participants already retired at time t.

Alternatively, one can consider the density,

$$h(t+r-x)\frac{l_x}{l_r} = g_1(t+r-x)l_x g_2(t+r-x)s(r)b , \quad a \le x < r , \tag{29}$$

of potential new pensions to be incurred at time $t + r - x$ in regard to participants aged x at time t, where such potential new pensions are to be discounted for survivorship from age x to age r. Then, by use of formulas (14) and (15),

$$A(t) = \int_a^r h(t + r - x) \frac{l_x}{l_r} A(x)dx + \int_r^\infty h(t + r - x) \frac{l_x}{l_r} A(x)dx$$

$$= \int_a^\infty h(t + r - x) \frac{l_x}{l_r} A(x)dx . \tag{30}$$

One could also bring in salary growth factors from age x to age r to get additional expressions for $A(t)$, but formulas (28) and (30) are possibly the most useful ones.

Each of the formulas for $A(t)$ leads to an expression for the derivative $dA(t)/dt$. The expression coming from formula (28) can be obtained by procedures similar to those for $dB(t)/dt$. Thus

$$\begin{aligned}
\frac{dA(t)}{dt} &= - \int_a^r v^{r-x}\bar{a}_r^\beta dh(t + r - x) - \int_r^\infty \frac{l_x}{l_r} \bar{a}_x^\beta dh(t + r - x) \\
&= -v^{r-x}\bar{a}_r^\beta h(t + r - x)\Big|_a^r + \int_a^r h(t + r - x)\delta v^{r-x}\bar{a}_r^\beta dx \\
&\quad - \frac{l_x}{l_r} \bar{a}_x^\beta h(t + r - x)\Big|_r^\infty \\
&\quad + \int_r^\infty \frac{h(t + r - x)}{l_r} d\Big[(1 + i)^x \int_x^\infty D_y\beta(y)dy\Big] \\
&= h(t + r - a)v^{r-a}\bar{a}_r^\beta + \delta \int_a^r h(t + r - x)v^{r-x}\bar{a}_r^\beta dx \\
&\quad + \int_r^\infty \frac{h(t + r - x)}{l_r} \Big[\delta(1 + i)^x \int_x^\infty D_y\beta(y)dy \\
&\qquad\qquad - (1 + i)^x D_x\beta(x)\Big]dx \\
&= h(t + r - a)v^{r-a}\bar{a}_r^\beta + \delta A(t) - B(t) ,
\end{aligned}$$

or

$$\frac{dA(t)}{dt} = h(t + r - a) \frac{l_a}{l_r} A(a) + \delta A(t) - B(t) . \tag{31}$$

Here the first term in the right-hand member of equation (31) represents the present value of future pensions for new entrants, the second term takes account of assumed interest, and the third term expresses the rate of pension outgo.

Starting from formula (30), one obtains

$$\begin{aligned}\frac{dA(t)}{dt} &= -h(t+r-x)\frac{l_x}{l_r}A(x)\Big|_a^\infty + \int_a^\infty \frac{h(t+r-x)}{l_r}\,d[l_xA(x)] \\ &= h(t+r-a)\frac{l_a}{l_r}A(a) \\ &\quad + \int_a^\infty \frac{h(t+r-x)}{l_r}\,[l_x dA(x) - A(x)l_x\mu_x dx] \\ &= h(t+r-a)\frac{l_a}{l_r}A(a) + \int_a^\infty h(t+r-x)\frac{l_x}{l_r}\,dA(x) \\ &\quad - \int_a^\infty h(t+r-x)A(x)\frac{l_x}{l_r}\,\mu_x dx\,. \qquad (32)\end{aligned}$$

Here the first term in the right-hand member is the same as for formula (31), the second term represents the change in the present value through aging, and the third member represents present values released by terminations. The equivalence of formulas (32) and (31) can be shown by substituting from formulas (21) and (22) for $dA(x)$.

$P(t)$, the Annual Rate of Normal Cost for the Plan at Time t

As for $A(t)$, alternative forms are available for $P(t)$, the annual rate of plan normal cost at time t; for example,

$$P(t) = \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^\beta m(x)dx\,, \qquad (33)$$

or, by using formulas (14) and (17),

$$P(t) = \int_a^r h(t+r-x)\frac{l_x}{l_r}P(x)dx\,. \qquad (34)$$

Differentiation of formula (33) by the procedures previously utilized for $B(t)$ and $A(t)$ leads to

$$\begin{aligned}\frac{dP(t)}{dt} &= h(t+r-a)v^{r-a}\bar{a}_r^\beta m(a) - h(t)\bar{a}_r^\beta m(r) + \delta P(t) \\ &\quad + \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^\beta dm(x)\,, \end{aligned} \qquad (35)$$

while differentiation of formula (34) produces

$$\frac{dP(t)}{dt} = h(t+r-a)\frac{l_a}{l_r}P(a) - h(t)P(r) - \int_a^r h(t+r-x)\frac{l_x}{l_r}\mu_x P(x)dx + \int_a^r h(t+r-x)\frac{l_x}{l_r}dP(x) \,. \tag{36}$$

The first two terms of the right-hand members of equations (35) and (36) are equal and represent the normal cost for new entrants offset by the normal cost for retirees, while the remaining terms differ. The equivalence of the right-hand members can be demonstrated by substituting for $dP(x)$ from formula (24). The right-hand member of formula (36) is possibly the more meaningful, with the first three terms representing a replacement effect of normal cost for new entrants offset by normal cost for those retiring and for those terminating, and with the fourth term representing the result of change with age of the normal cost rate $P(x)$.

$V(t)$, the Accrued Liability of the Plan as of Time t

The accrued liability of the plan at time t can be expressed as

$$V(t) = \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^\beta M(x)dx + \int_r^\infty h(t+r-x)\frac{l_x}{l_r}\bar{a}_x^\beta dx \,, \tag{37}$$

or, since $M(x) = 1$, $x \geq r$, $V(t)$ can, by use of formulas (14)–(16), be condensed to

$$V(t) = \int_a^\infty h(t+r-x)\frac{l_x}{l_r}V(x)dx \,. \tag{38}$$

Differentiation of formula (37) may be performed directly or by observing that the first integral is similar to the right-hand member of (33), but with $m(x)$ replaced by $M(x)$, and the second integral is the second term of formula (28) for $A(t)$. Formulas (35) and (13) and the calculations for $dA(t)/dt$ then indicate that

$$\frac{dV(t)}{dt} = v^{r-a}h(t+r-a)\bar{a}_r^\beta M(a) + \delta V(t) + P(t) - B(t) \,. \tag{39}$$

Unless initial funding is involved, $M(a) = 0$, and the formula simplifies to

$$\frac{dV(t)}{dt} = P(t) + \delta V(t) - B(t) \,. \tag{40}$$

Differentiation of the alternative formula (38) yields

$$\frac{dV(t)}{dt} = -h(t+r-x)\frac{l_x}{l_r}V(x)\Big|_a^\infty + \int_a^\infty \frac{h(t+r-x)}{l_r}\,d[l_x V(x)]\,,$$

and, if $M(a) = V(a) = 0$,

$$\frac{dV(t)}{dt} = 0 + \int_a^\infty \frac{h(t+r-x)}{l_r}[l_x dV(x) - V(x)l_x\mu_x dx]$$

$$= \int_a^\infty h(t+r-x)\frac{l_x}{l_r}dV(x) - \int_a^\infty h(t+r-x)V(x)\frac{l_x}{l_r}\mu_x dx\,. \quad (41)$$

The first term of the right-hand member represents the change in the accrued liability through aging, and the second term indicates the accrued liability released by terminations. Substitution for $dV(x)$ from formulas (23) and (22) transforms formula (41) into formula (40).

There are many comments that can be made about formula (40). First, its interpretation could be foreseen—namely, that the rate of change of the plan accrued liability is expressed by the rates of inflow of plan normal cost and assumed interest less the rate of outflow of pension payments. It should be emphasized that the concepts of the accrued liability and its derivative are mathematical and that no mention has been made yet of the actual fund which the plan has developed. One can consider the accrued liability to represent the fund that should be on hand under the chosen funding method and actuarial assumptions, but, in practice, equality of the accrued liability and the actual fund on hand may not be maintained. At this stage, only the mathematical concepts are under discussion.

A second comment is that formula (40) generalizes to a model plan subject to growth factors the equation of maturity stated by Trowbridge for a model plan in a stationary condition [13]. This equation (40) plays a key role in both theory and applications. In Trowbridge's mature case, $\boldsymbol{V}(t)$, $\boldsymbol{P}(t)$, and $\boldsymbol{B}(t)$ are constant, and the equation reduces to

$$\boldsymbol{P} + \delta\boldsymbol{V} = \boldsymbol{B}\,. \quad (42)$$

In this paper, $\boldsymbol{V}$, $\boldsymbol{P}$, and $\boldsymbol{B}$ can vary with t, and

$$\boldsymbol{P}(t) + \delta\boldsymbol{V}(t) = \boldsymbol{B}(t) + \frac{d\boldsymbol{V}}{dt}\,, \quad (43)$$

which demonstrates that the inflow of plan normal cost and assumed interest suffices to meet the outflow of pension payments and the growth in the accrued liability. Also, equation (40) applies in some immature as well as in mature or stable cases. For this reason, equation (40) will be called the "liability growth equation," rather than the "equation of equilibrium" or "equation of stability," which might be appropriate in special cases.

$(\boldsymbol{Pa})(t)$, the Present Value at Time t of Future Normal Costs of the Plan

Various expressions can be obtained for $(\boldsymbol{Pa})(t)$. The simplest procedure for this purpose may be to start with

$$(\boldsymbol{Pa})(t) = \int_a^r h(t + r - x) \frac{l_x}{l_r} (Pa)(x)dx \tag{44}$$

and, by use of formula (19), convert this to

$$(\boldsymbol{Pa})(t) = \int_a^r h(t + r - x) \frac{l_x}{l_r} [A(x) - V(x)]dx . \tag{45}$$

From formulas (30) and (38), it then follows that

$$(\boldsymbol{Pa})(t) = \boldsymbol{A}(t) - \boldsymbol{V}(t) . \tag{46}$$

Also, substitution from formulas (28) and (37) in formula (46) gives

$$(\boldsymbol{Pa})(t) = \int_a^r h(t + r - x)v^{r-x}\bar{a}_r^\beta[1 - M(x)]dx . \tag{47}$$

Since the derivatives of $\boldsymbol{A}(t)$ and $\boldsymbol{V}(t)$ are known already, formula (46) can be used to obtain the derivative of $(\boldsymbol{Pa})(t)$. Thus, from formulas (31) and (40),

$$\frac{d(\boldsymbol{Pa})(t)}{dt} = h(t + r - a) \frac{l_a}{l_r} A(a) + \delta(\boldsymbol{Pa})(t) - \boldsymbol{P}(t) . \tag{48}$$

Formula (48) indicates that the present value of future normal costs increases by the present value of the future pensions for new entrants at age a and by assumed interest, and decreases by the normal cost currently being received.

This completes the discussion of the five basic functions $\boldsymbol{B}(t)$, $\boldsymbol{A}(t)$, $\boldsymbol{P}(t)$, $\boldsymbol{V}(t)$, and $(\boldsymbol{Pa})(t)$. In the following two sections individual and aggregate cost funding methods will be analyzed in terms of these functions and three additional functions, namely, $\boldsymbol{F}(t)$, the fund on hand for the plan at time t; $\boldsymbol{C}(t)$, the annual rate of contribution at time t to the fund of the plan; and $\boldsymbol{U}(t)$, the unfunded accrued liability of the plan at time t.

From these definitions, it follows that

$$\boldsymbol{U}(t) = \boldsymbol{V}(t) - \boldsymbol{F}(t) . \tag{49}$$

The five basic functions $B(t)$, $A(t)$, $P(t)$, $V(t)$, and $(Pa)(t)$ have indicated a theoretical structure for the funding of the model plan. The classical problem is that initially, and perhaps for a long time thereafter, there are not sufficient funds on hand to match the accrued liability. The new functions $F(t)$, $C(t)$, and $U(t)$ are needed to describe the building up of funds to meet the requirements of the theoretical structure. The discussion will remain at the level of basic theory and, for instance, it will be supposed that the actuarial assumptions are exactly realized. In practice, gains and losses occur and a series of adjustments is required to bring the funding into congruence with the actual experience.

IV. INDIVIDUAL COST METHODS

By an individual cost method is meant one under which a normal cost, accrued liability, and possibly contribution can be determined for each individual participant. In contrast, an aggregate cost method determines the contribution on a group rather than an individual basis, and there may not be explicit determination of a normal cost or accrued liability. By means of the accrual function $M(x)$, a whole family of individual cost methods can be defined, and the preceding theory has been essentially in terms of individual cost methods. In this section, some further comment will be made concerning individual cost funding, and in the next section aggregate cost funding, defined by means of the accrual function, will be considered. In practice, funding may follow a mixture of individual and aggregate concepts because of the amortization of unfunded accrued liability and the adjustments for gains and losses.

For an individual participant of the model plan, aged x at time t and with salary at the annual rate $g_2(t)s(x)$, the normal cost rate defined by the accrual function $M(x)$ is

$$bg_2(t + r - x)s(r)A(x)m(x) \, , \tag{50}$$

and the accrued liability is

$$bg_2(t + r - x)s(r)A(x)M(x) \, . \tag{51}$$

The contribution with respect to this individual may then be fixed by the normal cost rate (50) and by some amortization process for the unfunded accrued liability.

The fund on hand for the plan at time t will be defined by the equation

$$\frac{dF(t)}{dt} = C(t) + \delta F(t) - B(t) \tag{52}$$

and an assumed initial value $F(0)$.[2] Here, as always in this paper, it is supposed that the actuarial assumptions are realized exactly; in particular, interest is earned at the force δ and benefit outgo is given by equation (26). If it is assumed further that the contribution rate $C(t)$ is maintained at the level

$$C(t) = P(t) + \delta U(t) , \quad t \geq t_0 , \tag{53}$$

that is, $C(t)$ covers the normal cost rate and interest on the unfunded accrued liability, then

$$\frac{dF(t)}{dt} = P(t) + \delta V(t) - B(t) . \tag{54}$$

Subtraction of equation (54) from equation (40) gives $dU(t)/dt = 0$, which can be solved as

$$U(t) = U(t_0) . \tag{55}$$

Formula (55) is the actuarially obvious statement that the unfunded accrued liability remains stationary if the contribution rate equals the normal cost rate plus interest on the unfunded accrued liability (and the actuarial assumptions are realized exactly). In particular, if $U(t_0) = 0$, then $C(t) = P(t)$, $t \geq t_0$, will maintain $F(t) = V(t)$.

V. AGGREGATE COST METHODS

The accrual function $M(x)$ may be used in defining a family of aggregate cost methods which by proper choice of $M(x)$ are equivalent (at least asymptotically) to some of the familiar funding methods. For this purpose one defines a mean temporary annuity value $\bar{a}(t)$ by the formula

$$\bar{a}(t) = \frac{(Pa)(t)}{P(t)} . \tag{56}$$

The $M(x)$ for initial funding is excluded as it is not applicable. As an example of formula (56), if $g_1(t)$, $g_2(t)$, $s(x)$ are all constant and equal to 1, and if $M(x) = (\bar{N}_a - \bar{N}_x)/(\bar{N}_a - \bar{N}_r)$, then from formulas (34), (17), (44), and (19), one finds

$$\bar{a}(t) = \int_a^r l_x \bar{a}_{x:\overline{r-x}|} dx \Big/ \int_a^r l_x dx . \tag{57}$$

In this stationary case, $\bar{a}(t)$ does not depend on t. The expression (57) also illustrates $\bar{a}(t)$ as a mean temporary annuity.

[2] For convenience it is assumed that inception of the plan occurs at time 0.

The aggregate contribution rate $C(t)$ is determined by the relation

$$C(t)\bar{a}(t) = A(t) - F(t) , \tag{58}$$

which implies that a mean annuity of $C(t)$ per year is to provide the difference between the present value of future pension payments (for participants covered at time t) and the fund on hand. By use of formula (56), formula (58) can be rearranged as

$$C(t) = \frac{A(t) - F(t)}{(Pa)(t)} P(t)$$

or, on substitution from formula (46),

$$C(t) = \frac{A(t) - F(t)}{A(t) - V(t)} P(t) . \tag{59}$$

It follows immediately from formula (59) that, if $F(t) \rightarrow V(t)$ from below, then $C(t) \rightarrow P(t)$ from above.

Formula (52) now takes the form

$$\frac{dF(t)}{dt} = \frac{A(t) - F(t)}{A(t) - V(t)} P(t) + \delta F(t) - B(t) . \tag{60}$$

Subtraction of formula (60) from formula (40) yields

$$\frac{dU(t)}{dt} = \frac{F(t) - V(t)}{A(t) - V(t)} P(t) + \delta U(t) , \tag{61}$$

which can be rearranged as

$$\frac{dU(t)}{dt} = - \frac{U(t)}{(Pa)(t)} P(t) + \delta U(t) ,$$

$$= - U(t)\left[\frac{1}{\bar{a}(t)} - \delta\right] , \tag{62}$$

or, replacing t by u,

$$\frac{d \ln U(u)}{du} = -\left[\frac{1}{\bar{a}(u)} - \delta\right] .$$

Integration and exponentiation give

$$U(t) = U(0) \exp \left\{-\int_0^t \left[\frac{1}{\bar{a}(u)} - \delta\right] du\right\} , \tag{63}$$

or, on use of relation (49),

$$F(t) = V(t) - [V(0) - F(0)] \exp \left\{-\int_0^t \left[\frac{1}{\bar{a}(u)} - \delta\right] du\right\} . \tag{64}$$

Equation (64) shows that if $F(0) = V(0)$, then $F(t) = V(t)$, $t > 0$, and from formula (59), $C(t) = P(t)$ (under the assumptions for the model plan).

Typically, $F(0) < V(0)$, and convergence of $F(t)$ to $V(t)$ depends upon the integral $\int_0^t [1/\bar{a}(u) - \delta]du$ becoming large as t increases. If $\bar{a}(u) \leq 1/\delta - m = \bar{a}_{\overline{\infty}|} - m$, m some positive number, then $1/\bar{a}(u) - \delta$ will have a positive lower bound, and $\exp\{-\int_0^t [1/\bar{a}(u) - \delta]du\}$ becomes small as t grows large and $F(t) \to V(t)$. Since the funding will usually be chosen so that $\bar{a}(u) < \bar{a}_{\overline{r-a}|} < \bar{a}_{\overline{\infty}|}$, the condition for convergence of $F(t)$ to $V(t)$ is usually satisfied.

Note that $V(t)$ can be interpreted as the accrued liability under an individual cost funding method determined by the accrual function $M(x)$, and that the aggregate cost funding method defined by formulas (58) and (59) is asymptotically equivalent to such individual cost method in the sense that $F(t) \to V(t)$ and $C(t) \to P(t)$ (it being assumed that convergence occurs).

Some common funding methods involve a modification of the aggregate cost method. General treatments of such methods are to be found in papers by Trowbridge [15] and Taylor [12] and in the accompanying discussions.

At the inception of a plan using some of these modified aggregate cost methods, the present value of future benefits, $A(0)$, is divided into two parts. Some arbitrariness is permitted in this division, to be denoted by $L(0)$ and $A(0) - L(0)$. The quantity $L(0)$ will usually be capable of interpretation as a measure of initial accrued liability.

The unfunded portion of this initial accrued liability will be denoted by $L(t)$, $0 \leq t$. The cost of funding $L(0)$ may be kept, in the absence of experience gains and losses, separate from future normal costs. The differential equation for $L(t)$ is

$$\frac{dL(t)}{dt} = \delta L(t) - E(t)\,, \quad 0 \leq t\,, \tag{65}$$

where $E(t)$ is the payment rate used to amortize $L(0)$. The amount $E(t)$ is often chosen as the constant $L(0)/\bar{a}_{\overline{m}|\delta}$, where $L(m) = 0$ and m is typically equal to 20 (with time measured in years).

The remaining present value, $A(t) - L(t)$, is funded by the aggregate cost method. If the contribution rate for this purpose is denoted by $C^*(t)$, then

$$C^*(t)\bar{a}(t) = [A(t) - L(t)] - F(t)\,, \tag{66}$$

so that

$$C^*(t) = \frac{A(t) - L(t) - F(t)}{A(t) - V(t)} P(t)\,. \tag{67}$$

The differential equation for the fund $F(t)$ must be modified with $C(t) = C^*(t) + E(t)$ and becomes

$$\frac{dF(t)}{dt} = \delta F(t) + C^*(t) + E(t) - B(t)\,. \tag{68}$$

With $U^*(t)$ defined as $V(t) - L(t) - F(t)$, that is, with the remaining unfunded initial liability taken as part of the assets of the plan, the use of equations (40), (65), and (68) produces

$$\frac{dU^*(t)}{dt} = \delta U^*(t) + P(t) - C^*(t)\,. \tag{69}$$

Simplifying by use of formula (67) gives

$$\frac{dU^*(t)}{dt} = -U^*(t)\left[\frac{1}{\bar{a}(t)} - \delta\right].$$

This is of the same form as equation (62), and the solution is formula (63) with $U^*(t)$ in place of $U(t)$. Substitution for $U^*(t)$ leads to

$$F(t) = V(t) - L(t) - [V(0) - L(0) - F(0)] \times \exp\left\{-\int_0^t \left[\frac{1}{\bar{a}(u)} - \delta\right]du\right\}. \tag{70}$$

Convergence arguments can be adapted from those following formula (64). In essence, the theory is the same as for the aggregate cost method with the inclusion in the fund assets of an amortization arrangement for $L(0)$.

Several observations are in order. If $L(0) = V(0)$, and $F(0) = 0$, then, in the absence of gains and losses, the fund would equal the accrued liability minus the unfunded initial liability. If $V(t)$ were valued by the entry age normal funding method, and $L(0) = V(0)$, the resulting method is called entry age normal funding with frozen initial liability, and involves a spread adjustment of experience gains and losses.

If $L(0) < V(0)$ and $F(0) = 0$, the fund will have more characteristics of aggregate funding. Subject to the usual conditions on the exponent, $F(t) \rightarrow V(t)$ as $L(t) \rightarrow 0$ and $t \rightarrow \infty$. The value of $F(t)$ would depend on the accrued liability function employed in evaluating $V(t)$.

This would be the case if $V(0)$ were valued by the entry age normal funding method and $L(0)$ were set equal to $V(0)$ as determined by unit credit funding. Such a combination is called attained age normal funding.

VI. THE EXPONENTIAL GROWTH CASE

A simple way to represent long-term growth is by exponential functions, and there are various circumstances that make such representation reasonable. In this section we will discuss the case where

$$g_1(t) = e^{\alpha t}, \qquad (71)$$

$$g_2(t) = e^{\gamma t}, \qquad (72)$$

and

$$\beta(x) = e^{\beta(x-r)}, \qquad (73)$$

where in the right-hand member β represents a constant rate of increase. The $e^{-\beta r}$ factor in formula (73) is introduced to simplify the writing of formulas. A constant factor could be applied to the various functions, but again for simplicity of formulas such factors will be omitted.

Introduction of expressions (71)–(73) in the formulas of the preceding sections leads to a great many special formulas and interpretations for this exponential growth case, and the reader is invited to explore some of these. In the first subsection some general observations will be made, and in the second subsection the special case where the force of interest equals the total force of growth will be discussed.

General Observations

a) Effect of growth rates.—Substituting from equations (71) and (72) into equation (8) yields

$$h(t) = e^{(\alpha+\gamma)t} l_r s(r) b . \qquad (74)$$

It is convenient to set

$$\tau = \alpha + \gamma , \qquad (75)$$

and then equation (74) becomes[3]

$$h(t) = e^{\tau t} l_r s(r) b . \qquad (76)$$

Here τ is a total force of growth composed of a force α applicable to the covered population and a force γ applicable to salaries. It is to be noted that α and γ enter $h(t)$ in the same way and that a change in α is equivalent to the corresponding change in γ as far as $h(t)$ is concerned. Since the functions $\boldsymbol{B}(t)$, $\boldsymbol{A}(t)$, $\boldsymbol{P}(t)$, $\boldsymbol{V}(t)$, and $(\boldsymbol{Pa})(t)$ are defined by means of $h(t)$, the same remark applies to them. However, from formulas (4), (71), (72), and (75), the total payroll function $W(t)$ takes the form

$$W(t) = \int_a^r e^{(\alpha+\gamma)t} e^{\alpha(r-x)} l_x s(x) dx = \int_a^r e^{\tau t} e^{\alpha(r-x)} l_x s(x) dx , \qquad (77)$$

and for this function a change in α, the population growth rate, is not equivalent to a corresponding change in γ, the salary growth rate.

[3] Unfortunately τ and r look alike, and the reader will have to distinguish them by the context.

Nevertheless, in this case

$$\frac{dW(t)}{dt} = \tau W(t) , \tag{78}$$

so that

$$W(t) = W(0)e^{\tau t} . \tag{79}$$

Here

$$W(0) = \int_a^r e^{\alpha(r-x)} l_x s(x) dx , \tag{80}$$

and the remark still holds that a change in α is not equivalent to a corresponding change in γ.

b) Fixed ratios.—Formulas corresponding to (78) and (79) for $W(t)$ are easily found for the basic functions $B(t)$, $A(t)$, $P(t)$, $V(t)$, and $(Pa)(t)$. This implies that, if $K_1(t)$, $K_2(t)$ are any pair from the six functions (including $W(t)$), then the ratio

$$\frac{K_1(t)}{K_2(t)} = \frac{K_1(0)}{K_2(0)} \tag{81}$$

is independent of t. In particular, it can be calculated that

$$\frac{B(t)}{W(t)} = \frac{b\bar{a}'_r}{{}^s\bar{s}_{a:\overline{r-a}|\alpha}} , \tag{82}$$

where $\bar{a}'_r$ is valued at the force of interest $\delta' = \tau - \beta$ and

$${}^s\bar{s}_{a:\overline{r-a}|\alpha} = \int_a^r e^{\alpha(r-x)} \frac{l_x s(x)}{l_r s(r)} dx$$

is calculated at force of interest α and takes the salary function into account. Also, another calculation shows that, for

$$M(x) = \int_a^x s(y) D_y dy \Big/ \int_a^r s(y) D_y dy ,$$

we have

$$\frac{P(t)}{W(t)} = {}^s\pi(a) ({}^s\bar{s}_{a:\overline{r-a}|\tau} / {}^s\bar{s}_{a:\overline{r-a}|\alpha}) , \tag{83}$$

where ${}^s\pi(a)$ is the normal cost rate as a percentage of salary (based on $s(x)$, l_x, and δ) for an initial pension of $bs(r)$ from age r. Equations (82) and (83) illustrate that, in the special case of exponential growth, benefit payments in relation to wages and normal costs in relation to wages do not vary over time.

c) Aggregate cost funding.—The key function here is $\bar{a}(t) = (Pa)(t)/$

$P(t)$, which now will be independent of t. To find a more specific expression for $\bar{a}(t)$ in the exponential case, one can start with formulas (47) and (33) and simplify to

$$\bar{a}(t) = \int_a^r e^{-(\delta-\tau)(r-x)}[1 - M(x)]dx \Big/ \int_a^r e^{-(\delta-\tau)(r-x)} m(x)dx \,. \quad (84)$$

The numerator in the right-hand member of equation (84) can be integrated by parts to give

$$\frac{1}{\delta - \tau}\left[-e^{-(\delta-\tau)(r-a)} + \int_a^r e^{-(\delta-\tau)(r-x)} m(x)dx\right].$$

Then

$$\bar{a}(t) = \frac{1}{\delta - \tau}\left[1 - 1 \Big/ \int_a^r e^{(\delta-\tau)(x-a)} m(x)dx\right].$$

But, by the theorem of the mean for integrals,

$$\int_a^r e^{(\delta-\tau)(x-a)} m(x)dx = e^{(\delta-\tau)(\xi-a)} \int_a^r m(x)dx$$
$$= e^{(\delta-\tau)(\xi-a)} \,,$$

since $\int_a^r m(x)dx = 1$, and hence

$$\bar{a}(t) = \bar{a}_{\overline{\xi-a}|\delta-\tau} \,, \quad (85)$$

where, in general, $a < \xi < r$, and the annuity value is calculated at force of interest $\delta - \tau$. For $M(x)$ leading to terminal funding, $\xi = r$, and the $M(x)$ for initial funding is excluded. It now follows from equation (64) that

$$F(t) = V(t) - [V(0) - F(0)] \exp\left[-\left(\frac{1}{\bar{a}_{\overline{\xi-a}|\delta-\tau}} - \delta\right)t\right]. \quad (86)$$

Convergence of $F(t)$ to $V(t)$ will require $\bar{a}_{\overline{\xi-a}|\delta-\tau}$ to be less than $\bar{a}_{\overline{\infty}|} = 1/\delta$, and for τ close to δ a rapid funding method that will produce a small value of $\xi - a$ may be required. This result may be achieved by selecting a funding method characterized by a function $M(x)$ that approaches 1 rapidly as x goes from a to r.

d) The liability growth equation.—Since now $dV(t)/dt = \tau V(t)$, the liability growth equation (40) becomes

$$P(t) + (\delta - \tau)V(t) = B(t) \,. \quad (87)$$

This can be interpreted to mean that only $(\delta - \tau)V(t)$ of the assumed interest is available to meet current pension outgo, the remaining assumed interest income, namely, $\tau V(t)$, being required for the growth of liability. If τ exceeds δ, then the growth of liability absorbs more than the assumed interest income and $P(t)$ exceeds the pay-as-you-go cost $B(t)$. This is not unexpected under conditions of growth, in particular, in the case of immature plans. It also suggests the need for study of more refined growth models in which the exponential factors are eventually dampened to some appropriate levels.

The situation in which a pension plan has a long-term investment income rate δ, less than γ, the long-term growth rate over time in salaries, seems both artificial and unrealistic. Nevertheless, this is the short-term situation faced by some pension plans in recent years. The implication seems to be that if this condition persists, that is, if the long-term wage inflation rate exceeds the long-term investment income rate, pay-as-you-go funding will result in lowest cost. Related reasoning may be used by those convinced of the necessity for savings through pension funds, to argue that investments with indexed yield rates to keep $\delta - \gamma$ a positive constant are necessary to elicit savings and investments through pension plans in the face of wage inflation.

The Special Case $\delta = \tau$

A number of special and usually simple relations can be obtained for the crossover case where $\delta = \tau$.

a) ***P***(*t*) *the same for all accrual functions.*—If the force of interest is equal to the sum of the population and salary growth rates, $\delta = \tau = \alpha + \gamma$, the liability growth equation (87) reduces to

$$P(t) = B(t) , \tag{88}$$

which implies that the plan normal cost is the same no matter what accrual function $M(x)$ is used. This is an intriguing situation and seems to imply that all funding methods have come down to pay-as-you-go funding. To the contrary, all funding methods except pay-as-you-go define an accrued liability not equal to zero for the present participants, and, as indicated previously, the classical problem of pension funding is to establish and maintain a fund equal to the accrued liability. If this has been accomplished in the $\delta = \tau$ case, then the investment income $\delta V(t)$ suffices to match the growth $\tau V(t)$ of the accrued liability, and the plan normal cost $P(t)$ can be used for the pension outgo $B(t)$.

b) *Formulas for* ***A***(*t*), ***V***(*t*), (***Pa***)(*t*).—From formulas (28) and (76) with $\delta = \tau$, it can be shown that

$$A(t) = bl_r s(r) e^{\delta t} [\bar{a}_r'(r - a) + (I\bar{a})_r'] , \tag{89}$$

where $(\bar{I}\bar{a})'_r$ is at force of interest $\delta - \beta = \tau - \beta$. A similar calculation for $V(t)$, starting from formulas (37) and (76), yields

$$V(t) = bl_r s(r) e^{\delta t}[\bar{a}'_r(r - \bar{x}) + (\bar{I}\bar{a})'_r] , \tag{90}$$

where

$$\bar{x} = \int_a^r x m(x) dx . \tag{91}$$

Subtraction of formula (90) from (89) then gives

$$(\boldsymbol{Pa})(t) = bl_r s(r) e^{\delta t} \bar{a}'_r(\bar{x} - a) . \tag{92}$$

Thus, although $P(t) = B(t)$ for all accrual functions when $\delta = \tau$, the value of future normal costs for the *current actives* depends on the accrual function $M(x)$ and can differ from $A(t)$ so that $V(t) \neq 0$.

c) Aggregate funding.—When $\delta = \tau$, formula (84) becomes

$$\begin{aligned} \ddot{a}(t) &= \int_a^r [1 - M(x)] dx \Big/ \int_a^r m(x) dx \\ &= [1 - M(x)]x \Big|_a^r + \int_a^r x m(x) dx \end{aligned}$$

since $\int_a^r m(x) dx = 1$. Then, by formula (91),

$$\ddot{a}(t) = \bar{x} - a . \tag{93}$$

Now according to the discussion following formula (64), the convergence of $F(t)$ to $V(t)$ depends on $\exp\{-[1/(\bar{x} - a) - \delta]t\}$. If $1/(\bar{x} - a) > \delta$, $F(t) \to V(t)$; if $1/(\bar{x} - a) = \delta$, then the unfunded accrued liability $U(t)$ remains stationary; and if $1/(\bar{x} - a) < \delta$, then $U(t)$ increases by the factor $\exp\{[\delta - 1/(\bar{x} - a)]t\}$.

VII. CONCLUSION

In this paper a foundation has been laid for the mathematical exploration of pension funding under conditions of growth which may arise from inflation or other sources. The theory has been applied to the relatively simple case of exponential growth. The theory itself could be developed further, and more complex applications are possible, for example, to cases of covered groups which have not reached maturity, to exponential growth with a dampening factor, or to situations where the actuarial assumptions are not realized in experience. It is hoped that this paper will stimulate such further developments.

REFERENCES

1. ALLISON, GLENN D., AND WINKLEVOSS, HOWARD E. "The Interrelationships among Inflation Rates, Salary Rates, Interest Rates, and Pension Costs," *TSA*, forthcoming.
2. COOPER, STEVEN L., AND HICKMAN, JAMES C. "A Family of Accrued Benefit Actuarial Cost Methods," *TSA*, XIX (1967), 53.
3. FRANCIS, P. R., AND SCHOLEY, J. K. "Pension Fund Finance, Equalization of Burdens and Accumulation of Assets," *Transactions of the International Congress of Actuaries*, I (1968), 651.
4. HICKMAN, JAMES C. "Funding Theories for Social Insurance," *PCAS*, LV (1968), 303.
5. HICKMAN, JAMES C., AND MONTGOMERY, DAVID B. "Pension Funding under Wage and Price Inflation," *ARCH*, forthcoming.
6. HUMPHREY, G. T., LANGHAM, F. R., SNELSON, R. E., AND SPARKS, J. E. "Pensions and Company Finance," *JIA*, Vol. XCVI (1970), Appendix C.
7. KEYFITZ, NATHAN. *Introduction to the Mathematics of Population*. Reading, Mass.: Addison-Wesley Publishing Co., 1968.
8. ———. "Mathematics and Population," *Society for Industrial and Applied Mathematics Review*, XV (1971), 370.
9. MYERS, ROBERT J. "Actuarial Analysis of Pension Plans under Inflationary Conditions," *Transactions of the International Congress of Actuaries*, I (1960), 301.
10. NESBITT, CECIL J. "Are There Ideas in Population Mathematics Which Are Adaptable to Pension Funding Theory?" paper presented at Harvard Actuarial Conference, 1973, distributed as a supplement to *ARCH*, 1974.
11. NOWLIN, PAUL W. "Insufficient Premiums," *TSA*, XI (1959), 100.
12. TAYLOR, JOHN R. "The Generalized Family of Aggregate Cost Methods for Pension Funding," *TSA*, XIX (1967), 1.
13. TROWBRIDGE, CHARLES L. "Fundamentals of Pension Funding," *TSA*, IV (1952), 17.
14. ———. "Social Security Amendments, 1969–72," *TSA*, XXV (1974), 625.
15. ———. "The Unfunded Present Value Family of Pension Funding Methods," *TSA*, XV (1963), 151.
16. TROWBRIDGE, CHARLES L., AND FARR, CHARLES E. "Theory and Practice of Pension Funding." Unpublished manuscript.

DISCUSSION OF PRECEDING PAPER

HARRY M. SARASON:

This paper has started me thinking. Since thinking is painful, I view the authors' effort with mixed emotions. What I have been thinking is this: What about a paper or two, or three, or more, to follow the "Introduction"? The titles could be "The Dynamics of Pension Funding," "Advanced Dynamics of Pension Funding," "An Analysis of Literature on the Dynamics of Pension Funding," and, finally (unless I am reincarnated), "The Dynamics of Pension Funding in the Twenty-second Century."

To get out of the "introductory" area, what about discontinuance of plans? What about discontinuance of the underlying economic entity that supports the plans? What about the economic entity moving away from the individual member, who does not like to move and who is not just a statistic, at least so far as he and his family are concerned?

And what about the interrelationships, in what I like to call the "pay-as-you-go *nonfunding* method," of nonfunding with inflation and with plan discontinuance or partial discontinuance, which may involve either a reduction in benefits or a reduction in the number of plan participants?

RICHARD K. KISCHUK:

The authors are to be congratulated on a paper that seems destined to be one of the landmark papers on the subject of pension mathematics. It seems appropriate, instead of offering specific criticisms of the paper, to enter into the spirit in which the paper was written and to offer some extension of the theory.

Basic Functions of Time

One technique that would appear to be useful in the context of some applications is the partitioning of some of the basic functions of time into two or more parts. As an example of what is meant, consider the formula for $A(t)$ given by formula (28) of the paper:

$$A(t) = \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^{\beta}dx + \int_r^{\infty} h(t+r-x)\frac{l_x}{l_r}\bar{a}_x^{\beta}dx\,.$$

An obvious subdivision of formula (28) is the following:

$$(Aa)(t) = \int_a^r h(t+r-x)v^{r-x}\bar{a}_r^{\beta}dx$$

and

$$(Ar)(t) = \int_r^\infty h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx .$$

In these formulas, $(Aa)(t)$ represents the present value at time t of future pension payments to currently active participants, and $(Ar)(t)$ represents the present value at time t of future pension payments to those participants already retired.

Equations corresponding to equation (31) are

$$\frac{d(Aa)(t)}{dt} = h(t + r - a) \frac{l_a}{l_r} A(a) - h(t)A(r) + \delta(Aa)(t)$$

and

$$\frac{d(Ar)(t)}{dt} = h(t)A(r) + \delta(Ar)(t) - B(t) .$$

The terms $-h(t)A(r)$ in the first equation and $h(t)A(r)$ in the second represent the transfer of present value of future pension payments from $(Aa)(t)$ to $(Ar)(t)$ due to new pensions incurred at time t.

Similar equations may be written relative to $V(t)$, the accrued liability of the plan as of time t. $V(t)$ is defined by formula (37) of the paper as

$$V(t) = \int_a^r h(t + r - x)v^{r-x}\bar{a}_r M(x)dx$$
$$+ \int_r^\infty h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx .$$

This may, in turn, be subdivided as follows:

$$(Va)(t) = \int_a^r h(t + r - x)v^{r-x}\bar{a}_r^\beta M(x)dx$$

and

$$(Vr)(t) = \int_r^\infty h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx ,$$

where $(Va)(t)$ represents the accrued liability attributable to future pension payments for participants currently active at time t and $(Vr)(t)$ represents the accrued liability attributable to future pension payments for participants already retired at time t.

Corresponding to equation (40), we have

$$\frac{d(Va)(t)}{dt} = P(t) + \delta(Va)(t) - h(t)A(r)$$

and

$$\frac{d(Vr)(t)}{dt} = h(t)A(r) + \delta(Vr)(t) - B(t) .$$

Corresponding to equation (46), we have

$$(Pa)(t) = (Aa)(t) - (Va)(t) .$$

In addition, it can be noted that

$$(Vr)(t) = (Ar)(t) .$$

This type of partitioning of $A(t)$ and $V(t)$ makes sense for those accrual functions, $M(x)$, where $M(x) = 1$ and $x \geq r$. This, of course, excludes pay-as-you-go funding.

Thus, the actuarial cost methods under consideration include as an objective the full funding of the present value of future pension payments for all participants retired at time t. Assuming that $F(t) \geq (Ar)(t) = (Vr)(t)$, then $F(t)$ can be subdivided as follows:

$$\frac{d(Fa)(t)}{dt} = C(t) + \delta(Fa)(t) - h(t)A(r)$$

and

$$\frac{d(Fr)(t)}{dt} = h(t)A(r) + \delta(Fr)(t) - B(t) ,$$

where $(Fa)(t)$ is the fund on hand for the plan at time t to offset $(Va)(t)$, and $(Fr)(t)$ is the fund on hand for the plan at time t to offset $(Vr)(t)$. At time t, an amount $h(t)A(r)$ is transferred from $(Fa)(t)$ to $(Fr)(t)$ because of new pensions incurred at time t. This transfer is sufficient to maintain $(Fr)(t) = (Vr)(t)$. Consequently,

$$U(t) = (Va)(t) - (Fa)(t) .$$

We may summarize the development thus far as follows:

$$(Aa)(t) = (Pa)(t) + (Fa)(t) + U(t) , \qquad \text{(a)}$$

$$(Ar)(t) = (Vr)(t) = (Fr)(t) . \qquad \text{(b)}$$

In this situation, the funding problem concerns currently active participants only. Payments, $C(t)$, to the active life fund, $(Fa)(t)$, will consist of normal cost payments, $P(t)$, plus payments toward the unfunded accrued liability, $U(t)$.

This refinement of the basic model is especially useful when one is considering the case where there are no retired participants at $t = 0$. This situation is more typical of the conditions present at plan inception than is the basic model of the paper. For example, the following conditions may be defined to apply at $t = 0$:

$$F(0) = (Fa)(0) = (Fr)(0) = 0 ,$$

$$(Ar)(0) = (Vr)(0) = 0 ,$$

$$U(0) = (Va)(0) = (Aa)(0) - (Pa)(0) .$$

For $t > 0$, a modified formula for $(Ar)(t)$ is

$$(Ar)(t) = \int_r^{r+t} h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx .$$

Of course, formulas (a) and (b) still apply. The funding problem still relates only to the currently active participants; in other words, it is unaffected by the modified assumptions with respect to retired lives. As noted earlier, this is a useful refinement of the basic model because it is more typical of the usual situation at plan inception.

Inflation, Interest, and the Exponential Growth Case

It is useful to consider the impact of certain economic interrelationships in the context of the exponential growth case of the basic model presented in the paper. First, it is necessary to restate the formula for $(Aa)(t)$ in terms of the exponential growth case:

$$(Aa)(t) = e^{\tau t}(Aa)(0) ,$$

where

$$(Aa)(0) = \int_a^r e^{-\delta'(r-x)} l_r s(r) b \int_0^\infty e^{-\delta'' u} {}_u p_r du dx ,$$

$$\delta' = \delta - \tau ,$$

and

$$\delta'' = \delta - \beta .$$

A popular concept in economics is that interest rates consist of a basic component, a risk component, and an inflation component. The inflation component represents the investor's expectation of inflation during the period of investment. The risk component represents the risk premium for the particular type of investment. The basic component represents the risk-free, inflation-free interest rate, and is generally assumed to be in the 2–3 per cent range.

Assuming pension assets to be invested in risk-free securities, and the inflation component to be a perfect predictor of future inflation, one may assume that

$$\delta = \delta_0 + \rho ,$$

where δ_0 is the basic component of force of interest δ and ρ is the inflation component of force of interest δ.

Assuming further that salaries grow over time at the same rate as prices (i.e., there are no increases in wages due to productivity) and that there is no growth in the number of participants, then we have $\tau = \gamma = \rho$. Similarly, it might be assumed that pensions increase after retirement at the same rate as prices increase, so that $\beta = \rho$. On the basis of these assumptions, it may be concluded that

$$\delta' = \delta - \tau = \delta_0 = \delta - \beta = \delta''$$

and that

$$(Aa)(0) = \int_a^r e^{-\delta_0(r-x)} l_r s(r) b \int_0^\infty e^{-\delta_0 u} \, {}_u p_r du dx \, . \qquad \text{(c)}$$

This equation is independent of the absolute level of interest, δ, and inflation, ρ, assumed; it depends only upon the differential between these two assumptions, δ_0.

Units of Measurement

The equations shown in the paper are expressed in terms of a basic unit of measurement, "dollars." The dollar is becoming meaningless as inflation becomes a permanent fixture in the economy. To perform any meaningful analysis of economic and cost variables, it has become necessary to transform them into some form of "real" or "constant" dollars.

For example, one might consider expressing the basic pension cost functions in terms of "constant dollars of purchasing power." Specifically, the present value of future pension payments to active participants covered by the plan at time t may be expressed as

$$(\overline{Aa})_\rho(t) = \frac{(Aa)(t)}{e^{\rho t}} \, ,$$

where the notation $(\overline{})_y(t)$ indicates that a given function of t is being expressed in "constant dollars" with respect to variable y. Retaining the assumptions $\delta = \delta_0 + \rho$ and $\tau = \gamma = \beta = \rho$, we get

$$\begin{aligned}(\overline{Aa})_\rho(t) &= \frac{(Aa)(t)}{e^{\rho t}} \\ &= \frac{e^{\rho t}(Aa)(0)}{e^{\rho t}} \\ &= (Aa)(0) \, .\end{aligned}$$

Thus, although $(Aa)(t)$ (in terms of "dollars") increases over time, $(\overline{Aa})_\rho(t)$ (in terms of "constant dollars") is constant over time and equal to the initial value $(Aa)(0)$, which, it may be recalled, is independent of the absolute levels of the inflation and interest assumptions used under the special set of assumptions selected here.

Similarly, we may choose to select a unit of measurement which is adjusted for growth in salaries and in number of participants. This may be accomplished by dividing a given function by $e^{\tau t}$. For example,

$$(\overline{Aa})_\tau(t) = \frac{(Aa)(t)}{e^{\tau t}} .$$

Under the assumptions of the exponential growth case, and on the basis of the formula corresponding to (79) of the paper,

$$\begin{aligned}(\overline{Aa})_\tau(t) &= \frac{(Aa)(t)}{e^{\tau t}} \\ &= \frac{e^{\tau t}(Aa)(0)}{e^{\tau t}} \\ &= (Aa)(0) .\end{aligned}$$

Thus, expressed in these terms, the present value of future pension payments to active participants covered by the plan remains constant over time. This observation corresponds to the stability of the basic functions $B(t)$, $A(t)$, $P(t)$, $V(t)$, and $(Pa)(t)$, in relation to wages, which was noted by the authors.

This may seem to be a theoretical consideration. However, it is very relevant. At first blush, it often seems that pension costs have increased substantially from one year to the next. However, after costs are expressed in "real" terms, it is often possible to demonstrate to a pension client that his costs have not really increased to any great extent but have remained basically stable.

Stated another way, "current dollars" is an appropriate unit of measurement when various cost alternatives are compared, all of which have been determined as of the same time t, or when one is determining how large a contribution should be made at time t, expressed in "current dollars." However, in comparing pension costs determined at two different points in time, it is almost always necessary to adjust costs to some form of "constant dollars," in order to gain a true perspective of the "real" change in cost. The units of measurement selected will, of course, depend on the type of perspective desired.

General Comments

This discussion has been limited mainly to the basic function $A(t)$ and related special functions derived from $A(t)$. Similar analyses can, of course, be performed with respect to other basic functions of time.

The results of these brief examples will, it is hoped, give some idea of the interesting and useful perspectives that can be gained by exploring the model which the authors have developed. There is reason to believe that the authors' expectations will be fulfilled and that this paper will prove to be a springboard for many future developments in the theory of pension funding.

C. L. TROWBRIDGE:

Professors Bowers, Hickman, and Nesbitt have produced a most welcome, and sorely needed, addition to pension actuarial literature.

It is my surmise that the efforts which culminated in this important paper had their beginnings at the 1973 Actuarial Research Conference, which all three authors (and this discussant) attended. Adjustments in pension funding for demographic changes and for conditions of inflation were matters of great concern to social security actuaries in 1973; but OASDI is essentially pay-as-you-go, and my social security presentation at Harvard did not involve the complications of pension funding. The three academic authors have now developed the necessary extension of theory for the private pension system.

Although there is much of interest throughout the mathematical development, the eye of this discussant focuses first on equation (40) (or its equivalent, eq. [43]). This is a statement of the "equation of equilibrium" known to actuaries (for a stationary population) for at least twenty-five years. In the generalized model introduced by this paper, the population is growing, as are salary levels and the pension fund. Contributions and investment earnings must provide not only the benefits but also the required fund growth.

The authors have not attempted numerical illustrations of the important principles to which their mathematics leads. One reason they have given for not doing so is that such illustrations are being developed by others. One such attempt has been made by Charles E. Farr and this discussant. A rather complex set of computer-calculated illustrations will soon be published as a part of *Theory and Practice of Pension Funding* (Homewood, Ill.: Richard D. Irwin, 1976). The mathematical model underlying the Trowbridge-Farr (TF) illustrations is similar, but not identical, to the Bowers-Hickman-Nesbitt (BHN) model. Readers interested in seeing the BHN results in numerical form can study the TF illustrations, but as they do they should be aware of several differences in the models chosen.

The BHN model is more general in that the function $g_1(t)$ makes possible the study of a population of participants changing in size. Trowbridge and Farr set $g_1(t)$ always equal to unity, and hence illustrate an active population of constant size. On the other hand, TF illustrations show an initially immature population gradually maturing, while the BHN paper discusses a population mature from its beginning.

Other differences arise because of two TF attempts to put more "realism" into the model. TF models presuppose a pension plan vested after ten years, and a pension benefit based on a final ten year average. The BHN paper considers a nonvested plan based on final pay. It has been found that these differences are responsible for what otherwise might appear to be certain conflicts between the BHN theory and the TF computer simulation.

Drs. Bowers, Hickman, and Nesbitt are to be congratulated on their fine development of important theory. Although the demographic aspects of their theory extensions may have little practical use in the field of private plans (they will always be important in social insurance), the practicality of their treatment of wage and price inflation cannot be denied.

It is now up to practicing pension actuaries to put to good use the new tools that our academic colleagues have provided us.

(AUTHORS' REVIEW OF DISCUSSION)

NEWTON L. BOWERS, JR., JAMES C. HICKMAN, AND CECIL J. NESBITT:

One idea that emerges from review of the discussion is that we are only beginning to appreciate the flexibility and generality of the mathematical model presented in the paper. The function $g_1(t)$ that controls the density of new retirees at time t can take on many forms to represent a variety of situations. For instance, some choices of $g_1(t)$ could be made to denote discontinuance or partial discontinuance of a plan and, thereby, at least partially answer Mr. Sarason's questions concerning our theory. Further, as noted in the paragraph following formula (10) of the paper, choosing $g_1(t) = 0$ for $t < 0$ will allow the model to handle immature cases despite the impression gathered by Messrs. Kischuk and Trowbridge that the model relates to mature cases only. It is true that the exponential case is defined for a mature situation, but, again, by use of the definition $g_1(t) = 0, t < 0$, one could adapt the theory to an immature case. Thereby, one obtains Mr. Kischuk's modified formula

$$(Ar)(t) = \int_r^{r+t} h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx ,$$

which, however, equals

$$\int_r^\infty h(t + r - x) \frac{l_x}{l_r} \bar{a}_x^\beta dx ,$$

since $h(t + r - x) = 0$ for $x > r + t$ under the assumption regarding $g_1(t)$. As far as we have verified, the formulas for the basic functions and their derivatives hold for immature as well as mature cases.

Mr. Kischuk's splitting of the function $A(t)$ into components $(Aa)(t)$ and $(Ar)(t)$ relating to the active participants and the retired participants, respectively, is a natural idea which gives additional insights into the workings of the model. We note that his formula (c), which applies in a particular instance of the exponential growth case, can be condensed to

$$(Aa)(0) = l_r s(r) b \bar{a}_{\overline{r-a}|} \bar{a}_r ,$$

where both annuity values are based on force of interest δ_0.

We welcome Mr. Trowbridge's comments contrasting our model with that underlying the Trowbridge-Farr illustrations. In our model only retirement benefits are considered, and a variety of other models could be developed for death, disability, and vested benefits. The question then arises as to how one defines a composite model to cover all four types of benefits, including the retirement benefit. There are a number of mathematical ways of doing this which would lead to the same total functions but which would result in various decompositions into component functions for the individual benefits (see, for instance, W. S. Bicknell and C. J. Nesbitt, "Premiums and Reserves in Multiple Decrement Theory," *TSA*, VIII, 344, or J. C. Hickman, "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory," *TSA*, XVI, 1). A special subcase is that in which there is immediate and exact vesting of the total reserve.

It seems to us that, in addition to extension of the model to include multiple benefits and more general retirement amounts, there is a second major project demanding attention. The developments in this paper have implicitly assumed that the actuarial assumptions are realized precisely. The decomposition of total gains or losses into components that reasonably may be interpreted as being associated with the various actuarial assumptions of the general model is the second project. To relate such gain and loss analysis to ordinary actuarial practice, a discrete form of the general model may be needed.

For even the simple retirement benefit model there is much left to explore—for example, the effect on the basic functions and their relationships if there is variation in the assumed rates of growth and decrement. The Trowbridge-Farr and other forthcoming illustrations will give insights regarding the effects of these variations.

There is a plethora of ideas and alternatives, but, fortunately, there are also unifying principles such as those embodied in formulas (27) and (40) of the paper.

We thank Messrs. Kischuk, Sarason, and Trowbridge for contributing their discussions. Special thanks are due Mr. Trowbridge, whose writings, ideas, and leadership helped to shape our thoughts in preparing the paper.