TRANSACTIONS OF SOCIETY OF ACTUARIES
1977 VOL. 29

ANOTHER LOOK AT GROUP PENSION PLAN GAIN AND LOSS

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ABSTRACT

This paper shows how gain and loss formulas for pension plans may be developed precisely from definitions of the unfunded accrued liability and the normal cost rate. The derivation is general enough to cover the many variations in plan benefits, cost methods, and assumptions that may be met in practice. The first task is to establish a nomenclature that will permit a generalized approach without loss of precision. Standard recursion formulas for the liabilities are then developed before proceeding with the analysis.

The resulting formulas have been tested by a computer program that develops the liabilities by standard methods and sums them in appropriate groups based on status codes at the beginning and end of the plan year. They are of general applicability to both insured and trusteed plans and so should be useful to the actuary who is responsible for pension plan valuations and to the student being exposed to the subject for the first time.

I. INTRODUCTION

RECENTLY a number of papers on pension plan gain and loss have been published in the *Transactions*. Occasional reference will be made here to two such papers, Arthur W. Anderson's "A New Look at Gain and Loss Analysis," *TSA*, XXIII (1971), 7, and Josiah M. Lynch, Jr.'s "A Practical Approach to Gains Analysis," *TSA*, XXVII (1975), 423. Mr. Anderson's paper is distinguished by its mathematical rigor and Mr. Lynch's by its applicability to a variety of plans. This paper will attempt a synthesis of these two approaches, incorporating elements of each yet charting an essentially independent course.

Definitions of the various gains and losses are clearest if they are in terms of the commonly accepted actual-versus-expected values. The notation used here permits a rigorous derivation of the formulas but at the same time is general enough to cover either trusteed or insured plans and all benefits, assumptions, and closed-group cost methods. Gain and loss formulas may then be developed in detail, but concisely and clearly, from the initial definition of the gain and from the recursion relations connecting beginning- and end-of-year assets, liabilities, and status. Any necessary modifications may be made easily.

I must acknowledge my debt to Mr. Anderson and Mr. Lynch for their rigorous analyses of pension plan gain and loss, particularly for aggregate methods. The derivation of aggregate gains and losses in Section VI of this paper makes clear that two quite different definitions of gains and individual accrued liabilities are possible and equally valid for aggregate methods. Mr. Anderson uses one definition and Mr. Lynch another. The reader will find references to these two definitions only in the discussions of Mr. Anderson's paper. The distinction is important and merits the attention it gets later in this paper.

Section II explains the notation and should be studied carefully before proceeding. Section III defines the sets or status groups over which the summations will be made. Section IV develops recursion formulas connecting liabilities at the beginning and end of the plan year. Section V develops gain and loss formulas for those cost methods that define the accrued liability for each life (individual methods). Section VI develops gain and loss formulas for aggregate methods and shows that they are analogous to those developed for individual methods, provided that we make the appropriate definition of the accrued liability for each life. Section VII reviews the gain and loss formulas in more detail and discusses how they should be applied in practice, and Section VIII contains some brief summary remarks. The implications of charging a term cost for disability are discussed in Appendix I. Appendix II presents the functions defined by Mr. Lynch in terms of the notation adopted here.

II. NOTATION

The subscripts 0 and 1 denote, for the valuation year being analyzed, the beginning of the year (BOY) and the end of the year (EOY), respectively. Subscripted variables are denoted by the subscript t. Subscripts enumerating the population over which summations are to be made have been omitted for the sake of brevity. The assets may be employer contributions held by an insurance company in a contractual and unapplied fund, or total contributions held in a trusteed plan for active and retired lives. Employee money reduces employer cost for insured plans, unless we are dealing with a commingled fund. Benefits guaranteed by an insurance company must be purchased and the fund reduced. Occasionally, because of cash-flow problems, the purchase may be deferred and the benefits paid directly from the fund. For simplicity, assume that benefits are purchased at retirement, if at all.

General Definitions

i = Valuation interest rate.

NRA = Normal retirement age, which might be used as the assumed retirement age for all incidental benefits.

ARA = Assumed retirement age for the pension benefit, or the last age at which retirement is used as a decrement.

Assets and Liabilities

F_t = Funds used as assets, excluding employee contributions unless these are commingled with employer money.

AL_t = Accrued liability for pension and all incidental benefits, as defined by the cost method. If future benefits are projected to retirement, the accrued liability is the present value of benefits to be paid in the future less the present value of future normal cost; otherwise it is the present value of accrued benefits.

PVB_t = Present value of benefits, either projected to retirement (for active lives) or accrued; it includes the present value of paid-up benefits for lives no longer active.

$UAL_t = \Sigma\, AL_t - F_t$

= Unfunded accrued liability.

UL_t = Unfunded frozen initial liability (FIL cost methods only).

Financial Transactions

I = Investment results.

K = Employer contributions.

D = Miscellaneous asset changes, such as dividends or other credits not used as contributions and experience adjustments.

E = Expenses charged to the fund.

B = Benefits purchased or paid directly from the fund, exclusive of annuity payments. These may be subdivided further by type of benefit, for example, B^d, B^w, B^r, and B^h for payments at death, payments at withdrawal (severance), purchases at retirement, and purchases at disability, respectively.

P = Life or disability annuity payments to lives included in the valuation.

$F_1 = F_0 + I + K + D - E - B - P.$

${}^iK$, ${}^iD$, ${}^iE$, ${}^iB$, and ${}^iP$ = Financial transactions described above, increased with interest at valuation rate i to EOY.

Items Related to Normal Cost

PVA_t = Present value of annuity during future active service, where the first payment is \$1 and future payments escalate according to a salary scale s_t. Thus $PVA_t = ({}^sN_{x+t} - {}^sN_y)/{}^sD_{x+t}$, where the prefixed superscript s denotes the salary scale s_t. The commutation functions are based on a service table.

S_t = Current annual salary.

PVS_t = Present value of future salary
$= S_t PVA_t$.

$1 + r_0 = s_1/s_0$.

NCR_t = Normal cost rate per life or per unit of salary.

$NC_t = NCR_t S_t$ for salary-based plans, which will be assumed unless otherwise noted.

NC = Normal cost. The subscript is omitted whenever the constraint that $NC_t = NCR_t S_t$ is removed. See the comments following the development of recursion formula (2) in Section IV.

AC = Administration charge added to the normal cost at BOY, so that ${}^iAC = AC(1 + i)$.

Expected Values

EP_1 = Expected annuity payments payable during the plan year, increased with interest to time 1. Let $P_0^{(m)}$ be the annual annuity, payable m times a year, in force at time 0 on all lives who are valued as immediate annuitants, even if they are still active. Then EP_1 is equal to the following:

$P_0^{(1)}(1 + i)$ for an annual life annuity due;

$(1 + i)P_0^{(12)}(N_x^{(12)} - N_{x+1}^{(12)})/D_x$ for a monthly life annuity (this is equivalent to $P_0^{(12)}(1 + \frac{13}{24}i - \frac{11}{24}q_0^d)$);

$P_0^{(12)}\ddot{s}_{\overline{1}|}^{(12)}$ for a monthly annuity-certain;

etc.

p_t^s and q_t^s = Probability of remaining active and probability of decrement, respectively, according to service table s. Four sources of decrement are commonly considered in the service table, namely, death, withdrawal, retirement, and disability, as indicated by the superscript. Thus $q_t^s = q_t^d + q_t^w + q_t^r + q_t^h$. Note that expected values can arise only from those decrements used to discount the benefits. For instance, disability is used as a decrement in evaluating the temporary disability annuity benefit, whereas pension benefits for active lives may be discounted only for death, withdrawal, and retirement, and vested paid-up benefits may be discounted only for death. Accordingly, expected values involving q^h arise only in connection with the temporary disability annuity benefit; they do not arise in connection with the pension benefits.

$B_1^s + L_1^s$ = Value at time 1, based on data known at time 0, of all benefits that would be incurred if decrement were to occur between time 0 and time 1. This is divided into benefits payable before time 1 (B_1^s) and the liability that would be set up for benefits payable after time 1 (L_1^s). Exact values can be given to B_1^s and L_1^s only when the form of PVB_0 is known.

$q_0^s B_1^s$ = Benefits expected to be paid between time 0 and time 1 because of decrement between time 0 and time 1, increased with interest to time 1. This is really shorthand for $q_0^d B_1^d + q_0^w B_1^w + q_0^r B_1^r + q_0^h B_1^h$.

$q_0^s L_1^s$ = Liability expected to be set up at time 1 for payments due after time 1 because of decrement between time 0 and time 1. Such payments could arise because of survivor benefits at death, vested benefits at withdrawal, retirement benefits at retirement, or temporary disability annuity benefits at disability. Accordingly, this is shorthand for $q_0^d L_1^d + q_0^w L_1^w + q_0^r L_1^r + q_0^h L_1^h$.

Projected Values

Projected values differ from expected values in that *no* probabilities of decrement (or survivorship) are applied between time 0 and time 1.

PVB_1^0 = Present value of benefits at time 1 as projected from time 0, assuming no projected benefit, plan, or status changes but considering projected changes in salary. Projected accruals cease at the ARA.

PVS_1^0 = Present value of future salary at time 1 as projected from time 0, assuming no status changes but considering projected changes in salary (projected salary and normal cost cease at the ARA).
$= S_0(1 + r_0)PVA_1$.

AL_1^0 = Accrued liability at time 1 as projected from time 0, assuming no projected benefit, plan, or status changes, or changes in the normal cost rate, but considering projected changes in salary. For individual projected benefit cost methods, $AL_1^0 = PVB_1^0 - (NC_0/S_0)PVS_1^0$. For aggregate cost methods, $AL_1^0 = PVB_1^0 - (NCR)PVS_1^0$, where NCR is the aggregate normal cost rate at either time 0 or time 1 depending upon the definition of total gains, as will be explained in Section VI. Note that the recursion formulas developed in Section IV link AL_0 and AL_1^0, PVB_0 and PVB_1^0, and PVS_0 and PVS_1^0. Any differences between AL_1^0 and AL_1, etc., arise from unscheduled changes in benefits, salary, etc. AL_1^0 is calculated at time 1 based on time 0 data.

III. STATUS GROUPS

It is convenient to define the groups for which we are summing liabilities in standard set notation, where $\cap$ means "intersection of," $\cup$ "union of," and $\subset$ "is a subset of." Thus $X \cap Y$ means those lives common to both sets; $X \cup Y$ means all lives in either set, and is equivalent to $X + Y$ if the sets are mutually exclusive; and $X \subset Y$ means all lives in X are also in Y, so that set X may be subtracted from set Y. Brackets around a set mean that the set is to be treated as an ordinary number.

Liability summations are made over groups of lives, or "sets," that are defined by status at the beginning and end of the year. First enumerate the sets defined by their status at valuation time t; these are as follows:

A_t: Active lives below the ARA.
B_t: Active lives at or over the ARA.
H_t: Disabled lives.
V_t: Vested paid-up lives.
R_t: Retired lives and survivors entitled to benefits.
Z_t: All lives included in the valuation at time t; $Z_t = A_t + B_t + H_t + V_t + R_t$.

All active lives at or past the ARA will be in set B_t. They will be treated as retired lives with unpurchased benefits, since a retired life liability will be held for them. If retired lives are not included in the valuation, set R_t above will be empty.

The other sets to be defined are transitional sets, which reflect events occurring during the plan year and are defined by differing status codes at the beginning and end of the year. The designations of these sets have no subscripts and are as follows:

N: New entrants; $N \subset A_1$.
W: Withdrawals; $W \subset A_0$.
V: Withdrawals with vested benefits; $V \subset W$.
H: Disabilities; $H \subset A_0$.
D: Deaths; $D \subset (A_0 + B_0 + H_0 + V_0 + R_0)$.
R: Retirements; $R \subset (A_0 + B_0 + H_0 + V_0)$.
T: Terminations; $T = W + H + D + R$.

VDB: "Vested" death benefits if periodic payments to survivors are included in the valuation after death of the original annuitant; $VDB \subset D$.

VPU: "Vested" paid-up lives; $VPU = (V + H + R + VDB) \subset T$. This category includes terminations either with vested withdrawal benefits or with disability, survivor, or retirement benefits.

Recoveries from disability and reinstatements of vested paid-up lives are not given separate set designations above, but they are counted numerically as negative $[H]$ and $[V]$, respectively, in the following recursion relations.

Recursion Relations

$$\begin{aligned}[A_1] &= [A_0] + [N] - [T \cap A_0] - [A_0 \cap B_1] \\ &= [A_0] + [N] - [W] - [H] - [D \cap A_0] - [R \cap A_0] \\ &\quad - [A_0 \cap B_1].\end{aligned}$$

$$[B_1] = [B_0] - [D \cap B_0] - [R \cap B_0] + [A_0 \cap B_1].$$

$$[H_1] = [H_0] + [H] - [D \cap H_0] - [R \cap H_0].$$

$$[V_1] = [V_0] + [V] - [D \cap V_0] - [R \cap V_0].$$

$$[R_1] = [R_0] + [R] - [D \cap R_0] + [VDB].$$

$$\begin{aligned}[Z_1] &= [Z_0] + [N] - [W] - [D] + [V] + [VDB] \\ &= [Z_0] + [N] - ([W] + [H] + [D] + [R]) \qquad (1) \\ &\quad + ([V] + [H] + [R] + [VDB]) \\ &= [Z_0] + [N] - [T] + [VPU].\end{aligned}$$

This last relation, formula (1), will be used in Sections V and VI.

IV. RECURSION FORMULAS CONNECTING LIABILITIES

Note the following textbook formulas connecting liabilities at the beginning and end of the year for a life aged x at BOY:

1. Level premium insurance reserve for a benefit of \$1 payable at the end of the year of death:

$$({}_0V + P)(1 + i) = p_x({}_1V) + q_x B_1.$$

2. Single premium annuity reserve for a benefit of \$1 payable annually:

$$\ddot{a}_x(1 + i) = p_x \ddot{a}_{x+1} + (1 + i).$$

3. Single premium annuity reserve for a benefit of \$1 payable in monthly installments:

$$\ddot{a}_x^{(12)}(1+i) = p_x\ddot{a}_{x+1}^{(12)} + (1 + \tfrac{13}{24}i - \tfrac{11}{24}q_x^d).$$

These formulas may all be considered as special cases of the generalized recursion formula:

$$(1+i)\left(\sum_{Z_0} AL_0 + NC\right) = \sum_{Z_0} [p_0^s AL_1^0 + q_0^s(B_1^s + L_1^s) + EP_1]\,. \quad (2)$$

The general principle is that the reserve at the beginning of the year plus the level premium due at the beginning of the year, increased with interest for one year, provides the projected accrued liability for the expected survivors plus the expected benefits (see *Life Contingencies* by C. W. Jordan, [5.13]). NC in formula (2) can be either the single premium for one year's benefit accrual, for a unit credit case, or a level premium if benefits are projected to retirement.

Note that the summation is over the lives in the BOY valuation. The EOY liability AL_1^0 does not recognize unscheduled benefit changes or changes in the normal cost between time 0 and time 1. That is, if $AL_t = PVB_t - (NCR)PVS_t$, the normal cost rate defining AL_0 and AL_1^0 in recursion formula (2) may be quite arbitrary as long as it does not change between time 0 and time 1. In the following simple proof, the level premium P is the analogue of NC and may be defined arbitrarily.

Let

$$V_0 = {}^iB(A_x) - P\ddot{a}_x, \qquad V_1 = {}^iB(A_{x+1}) - P\ddot{a}_{x+1}\,.$$

Since

$$(1+i)A_x = p_xA_{x+1} + q_x$$

and

$$(1+i)\ddot{a}_x = p_x\ddot{a}_{x+1} + (1+i),$$

then

$$\begin{aligned}(V_0 + P)(1+i) &= [{}^iB(A_x) - P\ddot{a}_x + P](1+i)\\ &= {}^iB(p_xA_{x+1} + q_x) - Pp_x\ddot{a}_{x+1}\\ &= p_xV_1 + q_x({}^iB). \qquad \text{Q.E.D.}\end{aligned}$$

The corresponding recursion formula for the PVB's is

$$(1+i)\sum_{Z_0} PVB_0 = \sum_{Z_0} [p_0^s PVB_1^0 + q_0^s(B_1^s + {}^rL_1^s) + EP_1] \quad (3)$$

and for the PVS's

$$(1+i)\sum_{A_0} PVS_0 = \sum_{A_0} [p_0^s PVS_1^0 + S_0(1+i)]$$

Note that the last term is $S_0(1+i)$ and not $S_0(1+\frac{13}{24}i-\frac{11}{24}q_0^d)$, because the normal cost is generally assumed to be payable at the beginning of each year, and of course the salary will be multiplied by the normal cost rate. Rearranging terms, note that

$$(1+i)\sum_{A_0}(PVS_0-S_0)=\sum_{A_0}p_0^s PVS_1^0=\sum_{A_0}PVS_1^e\,, \tag{4}$$

or the expected present value of future salary at the end of the year. Summations of S and PVS over A_0 are equivalent to summations over Z_0, since these terms are zero for lives not in A_0.

V. GAIN AND LOSS FORMULAS FOR INDIVIDUAL METHODS

Pension plan gains may be described as the excess of the expected over the actual unfunded accrued liability at the end of the period to be analyzed. Then

$$\begin{aligned} G &= UAL_1^e - UAL_1 \\ &= (UAL_0+NC_0)(1+i)+{}^iAC-{}^iK-UAL_1 \\ &= F_1-F_0(1+i)-{}^iK+(1+i)\Big(\sum_{Z_0}AL_0+NC_0\Big) \\ &\qquad +{}^iAC-\sum_{Z_1}AL_1\,. \end{aligned} \tag{5}$$

Recursion formula (2) connecting AL_0 and AL_1^0 is

$$(1+i)\Big(\sum_{Z_0}AL_0+NC\Big)=\sum_{Z_0}[p_0^s AL_1^0+q_0^s(B_1^s+L_1^s)+EP_1]\,.$$

Substituting this relation in equation (5), we have

$$\begin{aligned} G = F_1-F_0(1+i)-{}^iK+\sum_{Z_0}AL_1^0-\sum_{Z_0}q_0^s(AL_1^0-B_1^s-L_1^s) \\ +\sum_{Z_0}EP_1+{}^iAC-\sum_{Z_1}AL_1\,. \end{aligned} \tag{6}$$

Since $[Z_0]=[Z_1]+[T]-[VPU]-[N]$ from formula (1), we will now change $\Sigma\, AL_1^0$ from summation over Z_0 to summation over the sets on the right-hand side, so that

$$\begin{aligned} G &= F_1-F_0(1+i)-{}^iK \\ &\quad +\sum_{Z_1}AL_1^0+\sum_{T}AL_1^0-\sum_{VPU}PVB_1^0-\sum_{N}AL_1^0 \\ &\quad -\sum_{Z_0}q_0^s(AL_1^0-B_1^s-L_1^s)+\sum_{Z_0}EP_1+{}^iAC-\sum_{Z_1}AL_1\,. \end{aligned} \tag{7}$$

Add and subtract $({}^iB + {}^iD + {}^iE + {}^iP)$ and rearrange terms. Then

$$
\begin{aligned}
G = {} & \sum_{Z_1} AL_1^0 - \sum_{Z_1} AL_1 - \sum_{N} AL_1^0 + {}^iD \\
& + \sum_{Z_0} EP_1 - {}^iP + {}^iAC - {}^iE \\
& + F_1 - [F_0(1+i) + {}^iK + {}^iD - {}^iE - {}^iB - {}^iP] \\
& + \sum_{T} AL_1^0 - \sum_{VPU} PVB_1^0 - \sum_{Z_0} q_0^s(AL_1^0 - L_1^s) \\
& \qquad - {}^iB + \sum_{Z_0} q_0^s B_1^s .
\end{aligned}
\tag{8}
$$

By suitable groupings of terms the gains may now be written as follows:

1. *Gain from miscellaneous liability changes:* $\Sigma_{Z_1} AL_1^0 - \Sigma_{Z_1} AL_1$.
   This will usually be a loss for salary-based plans, resulting from salary increases exceeding those expected according to the salary scale.
2. *Gain from new entrants:* $-\Sigma_N AL_1^0$
   This will usually be a loss under unit credit and individual entry age normal (EAN) methods. Where birth and entry dates are evenly distributed, age nearest birthday will increase between entry and valuation dates roughly half the time, so that the EAN accrued liability will be about half the normal cost for new entrants.
3. *Gain from miscellaneous asset changes:* ${}^iD$
   See the description of D in Section II. The gain includes interest at the valuation rate from date of crediting.
4. *Gain from expenses:* ${}^iAC - {}^iE$
5. *Gain from annuity payments:* $\Sigma_{Z_0} EP_1 - {}^iP$
   EP_1 for annual payments is $P_0^{(1)}(1+i)$, for monthly payments is $P_0^{(12)}(1 + \frac{13}{24}i - \frac{11}{24}q_0^d)$, and for monthly payment annuities-certain is $P_0^{(12)} \ddot{s}_{\overline{1}|}^{(12)}$.
6. *Gain from interest:* $F_1 - [F_0(1+i) + {}^iK + {}^iD - {}^iE - {}^iB - {}^iP]$
7. *Gain from terminations:* $(\Sigma_T\ AL_1^0 - \Sigma_{Z_0}\ q_0^s AL_1^0) - (\Sigma_{VPU}\ PVB_1^0 - \Sigma_{Z_0}\ q_0^s L_1^s) - ({}^iB - \Sigma_{Z_0}\ q_0^s B_1^s)$

VI. GAIN AND LOSS FORMULAS FOR AGGREGATE METHODS

For aggregate methods, the accrued liability is known only in total for all lives. It is expressed (omitting subscripts) as

$$AL = PVB - (NCR)PVS .$$

Since $NCR = [PVB - (F + UL)]/PVS$, then $AL = F + UL$, and $AL = FIL$ (frozen initial liability) at issue when $F = 0$. For gain and loss analysis we must define the accrued liability for each life in order to make the proper summations. We will find this accrued liability as a direct result of our definition of total aggregate gain and loss, which is usually measured by the change in the normal cost rate. To give this aggregate gain a dollar value, we will develop a spread factor (SF) such that the gain $G = (NCR_0 - NCR_1)SF$. By definition,

$$NCR_0 = \frac{\sum_{Z_0} PVB_0 - (F_0 + UL_0)}{\sum_{A_0} PVS} = \frac{NC_0}{\sum_{A_0} S_0},$$

so that

$$NCR_0(1+i) \sum_{A_0} (PVS_0 - S_0) = (1+i)\Big(\sum_{Z_0} PVB_0 - F_0 - UL_0 - NC_0\Big). \tag{9}$$

Also by definition,

$$NCR_1 \sum_{A_1} PVS_1 = \sum_{Z_1} PVB_1 - (F_1 + UL_1). \tag{10}$$

Subtracting equation (10) from equation (9), we obtain

$$\begin{aligned} - NCR_1 \sum_{A_1} PVS_1 + NCR_0(1+i) \sum_{A_0} (PVS_0 - S_0) &= F_1 + UL_1 - \sum_{Z_1} PVB_1 \\ &\quad + (1+i)\Big(\sum_{Z_0} PVB_0 - F_0 - UL_0 - NC_0\Big). \end{aligned} \tag{11}$$

Adding $NCR_0\ \Sigma_{A_1} PVS_1$ to both sides of equation (11) and rearranging terms, we have

$$\begin{aligned} (NCR_0 - NCR_1) \sum_{A_1} PVS_1 = {} & F_1 + UL_1 \\ & - (1+i)(F_0 + UL_0 + NC_0) \\ & + (1+i)\Big(\sum_{Z_0} PVB_0\Big) - \sum_{Z_1} PVB_1 \\ & - NCR_0\Big[(1+i) \sum_{A_0} (PVS_0 - S_0) \\ & \qquad - \sum_{A_1} PVS_1\Big]. \end{aligned} \tag{12a}$$

Subtracting $NCR_1(1+i)\,\Sigma_{A_0}\,(PVS_0 - S_0)$ from both sides of equation (11) and rearranging terms results in

$$
\begin{aligned}
(NCR_0 - NCR_1)(1+i) \sum_{A_0} (PVS_0 - S_0) \\
&= F_1 + UL_1 - (1+i)(F_0 + UL_0 + NC_0) \\
&\quad + (1+i)\Big(\sum_{Z_0} PVB_0\Big) - \sum_{Z_1} PVB_1 \\
&\quad - NCR_1\Big[(1+i)\sum_{A_0}(PVS_0 - S_0) - \sum_{A_1} PVS_1\Big].
\end{aligned}
\tag{12b}
$$

The left-hand sides of equations (12a) and (12b) are in the form of $(NCR_0 - NCR_1)SF$, where $SF = \Sigma_{A_1}\, PVS_1$ in (12a) and $SF = (1+i)\,\Sigma_{A_0}\,(PVS_0 - S_0)$ in (12b). We will define these as two alternative expressions for the gain.

The right-hand sides of equations (12a) and (12b) are identical except for the normal cost rate. Henceforth, NCR without a subscript will mean NCR_0 if (12a) is chosen as the expression for the gain, and NCR_1 if (12b) is preferred. Since $NC = (NCR)S_0$ and $AL = PVB - (NCR)PVS$, the gain can be expressed as

$$
\begin{aligned}
G = F_1 + UL_1 - (1+i)(F_0 + UL_0 + NC_0) \\
+ (1+i)\Big(\sum_{Z_0} AL_0 + NC\Big) - \sum_{Z_1} AL_1 .
\end{aligned}
\tag{13}
$$

Note the distinction between NC_0 and NC, namely, $NC_0 = NCR_0S_0$, but $NC = (NCR)S_0$, where NCR can be either NCR_0 or NCR_1 depending on the choice of formula (12a) or (12b) as the expression for the total gain. This choice also determines whether NCR_0 or NCR_1 is to be used in the above formula in defining the accrued liabilities AL_0 and AL_1.

The use of $\Sigma_{A_1}\, PVS_1$ for the spread factor rather than the more complicated expression $(1+i)\,\Sigma_{A_0}\,(PVS_0 - S_0)$ has two strong arguments in its favor. The more important argument is that it is easier to explain to clients who may want to have dollar gains and losses translated into changes in their normal cost rate. The other argument is that NCR_0 (not NCR_1) defines the accrued liability at the end of the year (AL_1), so that the use of $\Sigma_{A_1}\, PVS_1$ preserves the independence of the gains. Otherwise, for example, a high termination rate may produce a gain, which reduces the normal cost rate, and this would affect gains from all other sources of decrement.

Formula (4) in Section IV shows that the spread factor $(1+i)\,\Sigma_{A_0}$

$(PVS_0 - S_0)$ is equal to $p_0^s PVS_1^0$, the expected present value of future salary at time 1, or PVS_1^e. There can be a substantial difference between PVS_1^e and PVS_1 whenever actual terminations or salary changes differ markedly from expected. Thus total gains are a matter of definition, as Mr. Anderson points out, and there is no definition generally agreed upon for aggregate methods. Whichever definition is accepted will not affect the analysis that follows. In order to obtain explicit expressions for each gain, make use of recursion formula (2) connecting AL_0 and AL_1^0. The formula is

$$(1+i)\left(\sum_{Z_0} AL_0 + NC\right) = \sum_{Z_0} [p_0^s AL_1^0 + q_0^s(B_1^s + L_1^s) + EP_1] .$$

Transform this formula to the following:

$$\begin{aligned}(1+i)\left(\sum_{Z_0} AL_0 + NC\right) - \sum_{Z_1} AL_1 = {} & - \sum_{Z_0} q_0^s AL_1^0 \\ & + \sum_{Z_0} q_0^s(B_1^s + L_1^s) + \sum_{Z_0} EP_1 + \sum_{Z_0} AL_1^0 - \sum_{Z_1} AL_1 .\end{aligned}$$

Since $[Z_0] = [Z_1] + [T] - [VPU] - [N]$ from formula (1), change $\Sigma\, AL_1^0$ from summation over Z_0 to summation over the sets on the right-hand side, so that

$$\begin{aligned}(1+i)\left(\sum_{Z_0} AL_0 + NC\right) - \sum_{Z_1} AL_1 = {} & \sum_{T} AL_1^0 \\ & - \sum_{VPU} PVB_1^0 - \sum_{N} AL_1^0 - \sum_{Z_0} q_0^s AL_1^0 \\ & + \sum_{Z_0} q_0^s(B_1^s + L_1^s) + \sum_{Z_0} EP_1 + \sum_{Z_1} AL_1^0 - \sum_{Z_1} AL_1 .\end{aligned} \tag{14}$$

Substituting the right-hand side of equation (14) for the terms $(1 + i)$ $(\Sigma_{Z_0} AL_0 + NC) - \Sigma_{Z_1} AL_1$ in (13) results in

$$\begin{aligned}G = {} & F_1 + UL_1 - (1+i)(F_0 + UL_0 + NC_0) \\ & + \sum_{T} AL_1^0 - \sum_{VPU} PVB_1^0 - \sum_{N} AL_1^0 \\ & - \sum_{Z_0} q_0^s AL_1^0 + \sum_{Z_0} q_0^s(B_1^s + L_1^s) + \sum_{Z_0} EP_1 \\ & + \sum_{Z_1} AL_1^0 - \sum_{Z_1} AL_1 .\end{aligned} \tag{15}$$

By adding and subtracting $({}^iK + {}^iD + {}^iE + {}^iB + {}^iP + {}^iAC)$, we obtain from formula (15)

$$
\begin{aligned}
G = {} & F_1 - [F_0(1+i) + {}^iK + {}^iD - {}^iE - {}^iB - {}^iP] \\
& + {}^iD + ({}^iAC - {}^iE) \\
& + UL_1 - [(UL_0 + NC_0)(1+i) + {}^iAC - {}^iK] \\
& + \sum_T AL_1^0 - \sum_{VPU} PVB_1^0 - \sum_{Z_0} q_0^s AL_1^0 + \sum_{Z_0} q_0^s(B_1^s + L_1^s) - {}^iB \\
& \qquad - \sum_N AL_1^0 + \sum_{Z_0} EP_1 - {}^iP + \sum_{Z_1} AL_1^0 - \sum_{Z_1} AL_1 .
\end{aligned}
\tag{16}
$$

To facilitate comparison with the formulas developed for individual methods, we will itemize the terms in the expression for the gain G in formula (16).

1. $\sum_{Z_1} AL_1^0 - \sum_{Z_1} AL_1$
2. $-\sum_N AL_1^0$
3. ${}^iD$
4. ${}^iAC - {}^iE$
5. $\sum_{Z_0} EP_1 - {}^iP$
6. $F_1 - [F_0(1+i) + {}^iK + {}^iD - {}^iE - {}^iB - {}^iP]$
7. $\left(\sum_T AL_1^0 - \sum_{Z_0} q_0^s AL_1^0\right) - \left(\sum_{VPU} PVB_1 - \sum_{Z_0} q_0^s L_1^s\right)$
   $- \left({}^iB - \sum_{Z_0} q_0^s B_1^s\right)$
8. $UL_1 - [(UL_0 + NC_0)(1+i) + {}^iAC - {}^iK]$

Note that the total gain G is not dependent on the values assigned to ${}^iK$, ${}^iD$, ${}^iE$, ${}^iB$, ${}^iP$, or ${}^iAC$. Only the individual gains by source depend upon these values. Therefore, interest adjustments to cash transactions may be approximated without loss of accuracy in the total gain.

Gains 1–7 are the same as were developed before for individual methods. Gain 8, which is unique for aggregate methods, may be defined as the gain from contributions in excess of the normal cost. This gain will be zero for the FIL method, since

$$UL_1 = (UL_0 + NC_0)(1 + i) + {}^iAC - {}^iK ,$$

but for the pure aggregate method it will be

$${}^iK - [NC_0(1 + i) + {}^iAC] ,$$

since $UL_1 = UL_0 = 0$.

For the FIL method, in the year when amortization of the unfunded FIL is completed and $UL_1 = 0$, the gain will be

$${}^iK - [(UL_0 + NC_0)(1 + i) + {}^iAC] .$$

The fact that the formulas for the gain under individual methods are similar to the corresponding formulas under aggregate methods justifies the definition of aggregate gains as

$$(NCR_0 - NCR_1)SF ,$$

where the choice of the spread factor SF determines the definition of the accrued liability AL. For individual methods,

$$\sum_{Z_1} AL_1 = \sum_{Z_1} PVB_1 - \sum_{A_1} NC_1PVA_1$$

and

$$\sum_{Z_1} AL_1^0 = \sum_{Z_1} PVB_1^0 - \sum_{A_1} NC_0PVA_1$$

For aggregate methods,

$$\sum_{Z_1} AL_1 = \sum_{Z_1} PVB_1 - NCR \sum_{A_1} PVS_1$$

and

$$\sum_{Z_1} AL_1^0 = \sum_{Z_1} PVB_1^0 - NCR \sum_{A_1} PVS_1^0 ,$$

where NCR is either NCR_0 or NCR_1 depending upon the choice of spread factors. Note the difference between individual and aggregate methods in either case.

Mr. Anderson defines the spread factor as $(1 + i)\ \Sigma_{A_0}\ (PVS_0 - S_0)$, the same as in formula (12b), so he uses NCR_1 in his definition of the accrued liability. Mr. Lynch defines the spread factor as $\Sigma_{A_1}\ PVS_1$, so the formulas for aggregate gains in his paper may be derived directly from formula (12a), allowing for differences in notation. This is the commonly accepted definition. In fact I have not seen Mr. Anderson's definition used anywhere else in the literature. For further comment see

Donald A. Lockwood's discussion of Mr. Anderson's paper (*TSA*, XXIII, 160).

If the normal cost rate is based on lives rather than on salary, $r_t = 0$ and $S_t = s_t = 1$ (i.e., $1 + r_t = s_{t+1}/s_t = 1$). The spread factor $\Sigma_{A_1}\ PVS_1$ becomes $\Sigma_{A_1}\ PVA_1$, where $PVA_1 = (N_{x+1} - N_y)/D_{x+1}$ and the commutation functions are based on a service table but not on a salary scale. The spread factor $(1 + i)\ \Sigma_{A_0}\ (PVS_0 - S_0) = \Sigma_{A_0}\ p_0^s PVS_1^0$ becomes $(1 + i)\ (\Sigma_{A_0}\ PVA_0 - [A_0]) = \Sigma_{A_0}\ p_0^s PVA_1$, where $PVA_0 = (N_x - N_y)/D_x$. Since this is merely the special case where both the salary and the salary scale are identically equal to 1, it is convenient to derive formulas for the general case where the normal cost rate and the spread factor are salary-based.

So far no distinction has been made between benefits provided by employees and those provided by employers. Insured plans frequently have separate employee funds. If only the employer fund is included in the plan assets, the plan liabilities should be offset by the value of the accrued or projected employee contributions, accumulated with interest at the guaranteed rate. The terms AL and PVB then should be understood to be reduced by the amount of any such offset. Alternatively, the use of the valuation rate to accumulate employee money anticipates the investment gains expected on these funds. Gains arising from employee money are due primarily to salary increases higher than expected. Such gains appear as "miscellaneous liability changes."

VII. DETAILS OF GAIN AND LOSS FORMULAS

This section reviews the gain and loss formulas in more detail and discusses how they should be applied in practice.

1. *Gain from miscellaneous liability changes:* $\Sigma_{Z_1}\ AL_1^0 - \Sigma_{Z_1}\ AL_1$

   Reflected in this term will be any differences between actual and projected salaries, benefits, social security offsets, and employee contributions used as an offset to employer costs. The portion due to unexpected salary increases can be determined by a double valuation if desired. For example, say that AL_1' is the liability at time 1 projected from time 0, but using the actual salary at time 1. Then the portion of the gain above resulting from salary changes can be expressed as $\Sigma_{Z_1}\ AL_1^0 - \Sigma_{Z_1}\ AL_1'$.

   It is convenient to assume that the projected liability for all lives leaving a valuation group is released, and that the projected liability for all lives entering a valuation group, such as new entrants and vested paid-up lives, equals the actual liability. These assumptions will introduce no error as long as they are applied consistently in determining the gain from new entrants and terminations. This gain therefore will be defined as zero for new entrants and terminations, and those who change status, such as those becoming vested paid-up or disabled, will be considered in the gain from terminations. Thus, the summation is over all lives in the same status group at the beginning and end of the year. To narrow it down even further, summation over the group $(A_0 + B_0) \cap (A_1 + B_1)$ may be sufficient if there are no changes in retired or paid-up benefits.

2. *Gain from new entrants:* $-\Sigma_N\ AL_1^0$

   To be consistent with the assumptions made for the gain in item 1 above, redefine this gain as $-\Sigma_N\ AL_1$. For aggregate methods this becomes $-\Sigma_N\ PVB_1 + NCR\ \Sigma_N\ PVS_1$, which can be a gain or a loss depending upon whether the normal cost rate for new entrants as a group is less than or greater than the aggregate normal cost rate. The new-entrant rate is $\Sigma_N\ PVB_1/\Sigma_N\ PVS_1$. The expected normal cost rate for new entrants is either NCR_0 or NCR_1 as explained in Section VI.

3. *Gain from miscellaneous asset changes:* ${}^iD$

   As explained in Section II, this term accounts for any change in assumed assets that cannot be described as an employer contribution, benefit purchase or payment, expense charge, or investment result. The gain includes interest at the valuation rate from the date of crediting.

4. *Gain from expenses:* ${}^iAC - {}^iE$

   AC is the estimated administration expense added to the normal cost. The fund is charged with the actual expense E. In practice, three situations are commonly met. Trusteed plans generally are charged with actual expenses, in which case $AC = 0$. Insured plans are of two types. Under deposit administration contracts, expenses are not charged directly to the fund, but frequently there is a contract administration charge. Unless determined separately by analysis of the dividend, $E = AC$ under these contracts, and the gain is zero. Under contracts with immediate participation guarantees, there is no explicit expense charge, so $AC = E = 0$. Actual expenses are considered in the experience adjustment and can be determined separately if desired; otherwise there is no gain.

5. *Gain from annuity payments:* $\Sigma_{Z_0}\ EP_1 - {}^iP$

   The summation is really over only those lives eligible for immediate annuity payments, or $(B_0 + R_0 + H_0)$, since EP_1 is zero for other lives. The gain may be written as

$$\sum_{B_0+R_0+H_0} EP_1 - {}^iP$$

if retired and disabled lives are included in the valuation. Alternatively, $\Sigma_{B_0}\ EP_1$ can be considered as a retirement gain.

6. *Gain from interest:* $F_1 - [F_0(1+i) + {}^iK + {}^iD - {}^iE - {}^iB - {}^iP]$
This is the excess of investment income over the interest that would have been credited to the fund at the valuation rate. It includes any recognized market value changes. Since $F_1 = F_0 + \mathrm{I} + K + D - E - B - P$, the gain from interest also can be written as

$$\mathrm{I} - iF_0 + ({}^iP - P) + ({}^iB - B) + ({}^iE - E) - ({}^iK - K) - ({}^iD - D)\,.$$

7. *Gain from terminations:* $(\Sigma_T\ AL_1^0 - \Sigma_{Z_0}\ q_0^s AL_1^0) - (\Sigma_{VPU}\ PVB_1^0 - \Sigma_{Z_0}\ q_0^s L_1^s) - ({}^iB - \Sigma_{Z_0}\ q_0^s B_1^s)$
This is the excess of the actual over expected accrued liability released by the various decrements, less the actual over expected liability for accrued benefits set up at death, withdrawal, disability, or retirement, less the actual over expected claims paid and annuity purchases made. To be consistent with the assumptions made for gain 1, replace $\Sigma_{VPU}\ PVB_1^0$ with $\Sigma_{VPU}\ PVB_1$. This gain may be subdivided further into the gains from deaths, withdrawals, retirements, and disabilities, as shown below.

In applying these formulas, it is important to realize that the expected values must reflect the service tables used to discount these benefits. Assume that the service table recognizes four sources of decrement: deaths, withdrawals, retirements, and disabilities. However, suppose, for example, that retirement is used as a decrement only in valuing the retirement benefit and future normal cost, whereas incidental active life benefits terminate at an assumed retirement age, except for any postretirement death benefit not valued as part of the retirement benefit. Then all terms involving q_0^r in the gain from retirement must be summed, over set A_0, only for the retirement benefit and future normal cost, since, for incidental benefits, $q_0^r = 0$ except at the ARA. For set B, $q_0^r = 1$ and $AL_1^0 = L_1^r$, so that $q_0^r AL_1^0 = q_0^r L_1^r$.

In subdividing gain from terminations by individual source of decrement, it is important to check the summations to make sure that no groups are left out. Thus Σ_T in gain 7 can be replaced by $\Sigma_W + \Sigma_H + \Sigma_D + \Sigma_R$ in gains 7(*a*)–7(*d*) below, since $T = W + H + D + R$ (Sec. III). Σ_{VPU} in gain 7 can be replaced by $\Sigma_V + \Sigma_R + \Sigma_H + \Sigma_{VDB}$, since $VPU = V + R + H + VDB$ (also Sec. III). Expected values, as noted above, need be summed over only those groups for whom the probability of decrement exists.

a) Gain from deaths: $(\Sigma_D \ AL_1^0 - \Sigma_{Z_0} \ q_0^d AL_1^0) - (\Sigma_{VDB} \ PVB_1 - \Sigma_{Z_0} q_0^d L_1^d) - ({}^iB^d - \Sigma_{Z_0} \ q_0^d B_1^d)$

In the above expression,

${}^iB^d$ = Death benefits paid or purchased, with interest to EOY.

B_1^d = Death benefits payable if death occurs during the current year, with interest to EOY.

L_1^d = Liability that will be set up, if death occurs in the current year, for survivor or contingent benefits and remaining term certain for unpurchased retired life benefits.

Expected mortality may not be the same for all benefit and status groups. If different mortality tables are used to value preretirement and postretirement benefits and disabled life temporary annuity benefits, the mortality rates in the above expression are understood to be the appropriate rates for those benefits and employee groups.

b) Gain from withdrawals: $(\Sigma_W \ AL_1^0 - \Sigma_{A_0} \ q_0^w AL_1^0) - (\Sigma_V \ PVB_1 - \Sigma_{A_0} \ q_0^w L_1^w) - ({}^iB^w - \Sigma_{A_0} \ q_0^w B_1^w)$

In the above expression,

${}^iB^w$ = Severance benefits paid, with interest to EOY.

B_1^w = Severance benefits payable if withdrawal occurs in the current year, with interest to EOY.

L_1^w = Liability that will be set up for vested retirement benefits if withdrawal occurs in the current year.

c) Gain from retirements: $(\Sigma_R \ AL_1^0 - \Sigma_{Z_0} \ q_0^r AL_1^0) - (\Sigma_R \ PVB_1 - \Sigma_{Z_0} \ q_0^r L_1^r) - ({}^iB^r - \Sigma_{Z_0} \ q_0^r B_1^r)$

In the above expression,

${}^iB^r$ = New retirement benefits paid or purchased, with interest to EOY. Alternatively, unpurchased payments may be included in ${}^iP$.

B_1^r = Annuity benefits payable before time 1 if retirement occurs between time 0 and time 1, with interest to EOY.

L_1^r = Liability that will be set up for retirement benefits if retirement occurs in the current year.

If retired lives are included in the valuation at age nearest birthday, $B_1^r = 0$ for group A_0, since retirement normally is expected to occur at integral ages, hence at the end of a plan year.

Of course $q_0^r = 0$ for group R_0. Setting $q_{\text{ARA}}^r = 1$ implies that, when the ARA is reached, only one source of decrement is expected thereafter, namely death, and that $AL_1^0 = L_1^r$. Even if the life remains active, the liability held at the ARA and later is for an immediate annuity valued on the postretirement mortality table and at the valuation interest rate.

d) Gain from disabilities: $(\Sigma_H\ AL_1^0 - \Sigma_{A_0}\ q_0^h AL_1^0) - (\Sigma_H\ PVB_1 - \Sigma_{A_0}\ q_0^h L_1^h) - ({}^iB^h - \Sigma_{A_0}\ q_0^h B_1^h)$

In the above expression,

${}^iB^h$ = New disability temporary annuity benefits paid or purchased, with interest to EOY. Alternatively, unpurchased payments may be included in ${}^iP$.

B_1^h = Disability annuity benefits payable before time 1 if disability occurs between time 0 and time 1, with interest to EOY.

L_1^h = Liability that will be set up for disability temporary annuity benefits if disability occurs in the current year.

Note that, if disability is used as a decrement only for valuing the temporary annuity, AL_1^0 and L_1^h in the terms $\Sigma_{A_0}\ q_0^h AL_1^0$ and $\Sigma_{A_0}\ q_0^h L_1^h$ involve only the liability for the temporary annuity, and there is no PVS term for expected disabilities in $\Sigma_{A_0}\ q_0^h AL_1^0$.

8. *Gain from contributions in excess of the normal cost:* $UL_1 - [(UL_0 + NC_0)(1 + i) + {}^iAC - {}^iK]$

This gain exists only for aggregate methods, as explained in Section VI, and is other than zero only for the pure aggregate method or in the final year of FIL funding. It of course can be negative (that is, a loss can arise from this source) in the case of the pure aggregate method if the contributions are less than the normal cost.

9. *Gain from changes in assumptions or plan benefits*

This gain should be determined by a double valuation using both the old and the new assumptions and plan benefits. For analysis purposes, the assumptions and plan benefits should not change between time 0 and time 1.

VIII. SUMMARY

Gains and losses developed under individual and aggregate methods have a common ground. This is best shown by analogy with the plan sponsor's profit statement. Corporate earnings may be described as the increase in the balance-sheet surplus (profit and loss) account. In the special notation used here for pension plans, corporate earnings are $G = (F_1 - AL_1) - (F_0 - AL_0)$.

If the pension plan is thought of as a financial entity, can gains be equated to earnings? To continue the analogy, the distribution of gains by source is similar to the income account, which is closed out at the end of the accounting period when a trial balance is struck and credited to the surplus account. The analogy is close but not exact. Except for the pure aggregate method, pension plan gains do not include employer contributions in excess of the normal cost, nor do they include expected investment earnings on such contributions or on the BOY surplus. The analogy is closer if (1) excess employer contributions are deemed to be new paid-in capital and (2) interest dividends on surplus are charged to the income account rather than to the surplus account. In practice, plan surplus is usually negative, so the increase in the unfunded accrued liability due to interest at the valuation rate is not considered a loss to the plan. The gain then may be written as follows:

$$
\begin{aligned}
G &= (F_1 - AL_1) - (1+i)(F_0 - AL_0) - [{}^iK - (1+i)(NC + AC)] \\
&= F_1 - F_0(1+i) - {}^iK + (1+i)(AL_0 + NC) + (1+i)AC - AL_1 .
\end{aligned}
$$

This is formula (5), the formula for the total gain for individual methods.

If we define UL_1 as $(1+i)(UL_0 + NC + AC) - {}^iK$, and substitute $UL_1 - (1+i)UL_0$ for $(1+i)(NC + AC) - {}^iK$ in the pension plan gain formula above, we have

$$
\begin{aligned}
G &= (F_1 - AL_1) - (1+i)(F_0 - AL_0) + [UL_1 - (1+i)UL_0] \\
&= F_1 + UL_1 - (1+i)(F_0 + UL_0 + NC) + (1+i)(AL_0 + NC) - AL_1 .
\end{aligned}
$$

This is formula (13), the formula for the total gain for aggregate methods, when $NC = (NCR_0)S_0$. Note that UL is treated like a plan asset in defining FIL aggregate gains. To define pure aggregate gains, let $UL_1 = UL_0 = 0$. The result is

$$
G = (F_1 - AL_1) - (1+i)(F_0 - AL_0),
$$

which exceeds corporate earnings only by $iAL_0 - iF_0$, interest at the required rate on liabilities less the expected rate on assets.

The reason for the identity of the formulas under individual and aggregate methods now should be apparent.

For aggregate methods the definition of the accrued liability is closely tied to the definition of the spread factor. Two possible definitions have been explored, and a choice recommended. The accrued liability for an individual life is artificial and even can be negative, since it ignores the true normal cost for that life, but the method gives the correct result for all lives combined.

In practice many more status groups will be encountered than were considered here; these would result from recoveries, reinstatements, deferred purchases, and so on. A complete breakdown by status group and details of the formulas for calculating liabilities are of more concern to the computer programmer than to the developer of the theory. The breakdowns and details will differ depending upon the case, but the information can be supplied readily by the user.

In conclusion, I would like to thank Arthur Baldwin, William Horbatt, and Larry Lang, who have helped me test the theory presented here by developing computer programs for gain and loss analysis of actual cases. Without their help, I would not be half so sure that the theory actually works in practice.

APPENDIX I

TERM COST OF DISABILITY

It is common practice to earmark a portion of assets as a disability fund in lieu of including disability as one of the benefits in the liability for incidental benefits. The disability fund is increased each year by the term cost of disability and decreased by the cost of purchasing temporary annuities for newly disabled lives. The normal cost is also increased by the term cost of disability. How does this affect the gain and loss analysis?

Refer to formulas (5) and (13) for the total gain. Let H_t equal the disability fund at time t, and let TCD equal the term cost of disability. Increasing $\Sigma_{Z_0} AL_0$ or decreasing F_0 by H_0, and increasing $\Sigma_{Z_1} AL_1$ or decreasing F_1 by H_1, adds $(1 + i)H_0 - H_1$ to the total gain. TCD added to NC increases the expected UL_1 or UAL_1 and therefore the total gain. For the pure aggregate method, where $UL_1 = 0$, TCD increases the disability gain and is offset by a reduction in the gain from excess contributions. The cost of new disability claims ${}^iB^h$ decreases F_1 and therefore decreases the gain. Thus, referring to gain 7(d) of Section VII, the total gain from disability becomes

$$\sum_{H} AL_1^0 - \sum_{H} PVB_1 + (1 + i)H_0 - H_1 + TCD(1 + i) - {}^iB^h .$$

The expected terms are zero, since disability is not being valued as an incidental benefit and therefore disability is not treated as a decrement.

If we define the disability fund H_1 as $H_1 = (H_0 + TCD)(1 + i) - {}^iB^h$, the gain from disability is

$$\sum_{H} AL_1^0 - \sum_{H} PVB_1 .$$

This method substitutes a known and stable annual charge, the term cost of disability, for widely fluctuating gains and losses. It thus is particularly appropriate for small cases. Frequently a limit is placed on the size of the fund.

Once the limit is reached and there are no further claims, the term cost of disability should no longer be added to the normal cost. On the other hand, once the fund has been wiped out by excessive claims, any further claims result in losses.

APPENDIX II

COMPARISON OF NOTATION

The following is appended for those interested in comparing the formulas developed here with those developed in Mr. Lynch's paper. It will be convenient to assume that retired lives and disabled lives receiving temporary annuity benefits are excluded from the valuation. Then his notation translates into the notation used here as follows:

$$V = AL_1 ;$$

$$V' = AL_1^0 ;$$

$$NL = B_1^s + L_1^s ;$$

$$EV = (1 + i)(AL_0 + NC_0)$$

$$= p_0^s AL_1^0 + q_0^s(B_1^s + L_1^s) ;$$

$$GV = p_0^s AL_1^0 + q_0^s(B_1^s + L_1^s) - AL_1 ;$$

$$AR = AL_1^0 - AL_1 ;$$

$$ER = q_0^s(AL_1^0 - B_1^s - L_1^s) ;$$

$$EE = AC ;$$

$$G_I = F_1 - [F_0(1 + i) + {}^iK + {}^iD - {}^iB - {}^iE] ;$$

$$G_E = AC(1 + i) - {}^iE ;$$

$$\sum_K G_K = \Big(\sum_{Z_0} AL_0 + NC_0\Big)(1 + i) - \sum_{Z_1} AL_1 - {}^iB + {}^iD ;$$

$$G = G_I + G_E + \sum_K G_K$$

$$= G \text{ as defined by formulas (5) and (13) .}^1$$

[1] Make the substitution $UL_1 = (UL_0 + NC_0 + AC)(1 + i) - {}^iK$ in formula (13).

DISCUSSION OF PRECEDING PAPER

LARRY LANG:

Mr. Street's paper on pension plan gain and loss analysis is certainly a major contribution to the Society's pension literature, and one that links effectively recent papers by Mr. Anderson and Mr. Lynch. My comments are intended to amplify and supplement the paper's concepts and will focus on the following areas: status groups, decrement assumptions, and notation.

Status Groups

The recursion relations of Section III indicate the interaction of various transitional sets of lives with the BOY and EOY status sets. Formula (1) summarizes the net effect. We see that the total BOY population, Z_0, can be increased only by new entrants, N, from, shall we say, the universal set. Z_0 can be decreased only by the excess of terminating lives, T, over vested paid-up lives, VPU. Algebraically, we can see that this latter component, representing "leakage" from Z_0 to the universal set, is equal to the nonvested withdrawals plus the deaths of all Z_0 lives whose beneficiaries are not eligible for survivor benefits. Figure 1 consists of a Venn diagram that will help the reader to visualize the interaction of the various sets.

Mr. Street indicates that recoveries from disability and reinstatements of vested paid-up lives are not given separate set designations but are counted numerically as negatives. I would add that it might be possible for a disabled life to recover and join the vested rather than the active status. Figure 1 indicates both of these effects by dotted lines.

The author notes that if retired lives are not included in the valuation, set R_t will be empty. We might restate this by saying that purchased retirement benefits represent additional leakage from the system (although, for immediate participation guaranteed contracts, such purchased benefits may be "unpurchased" for valuation purposes). If there are purchased disability or vested benefits, there similarly will be leakage from H_t or V_t, respectively.

Decrement Assumptions

Referring to the author's *gain from terminations* of Section VII, we note a very significant concept requiring some amplification. Mr. Street states that "it is important to realize that the expected values must

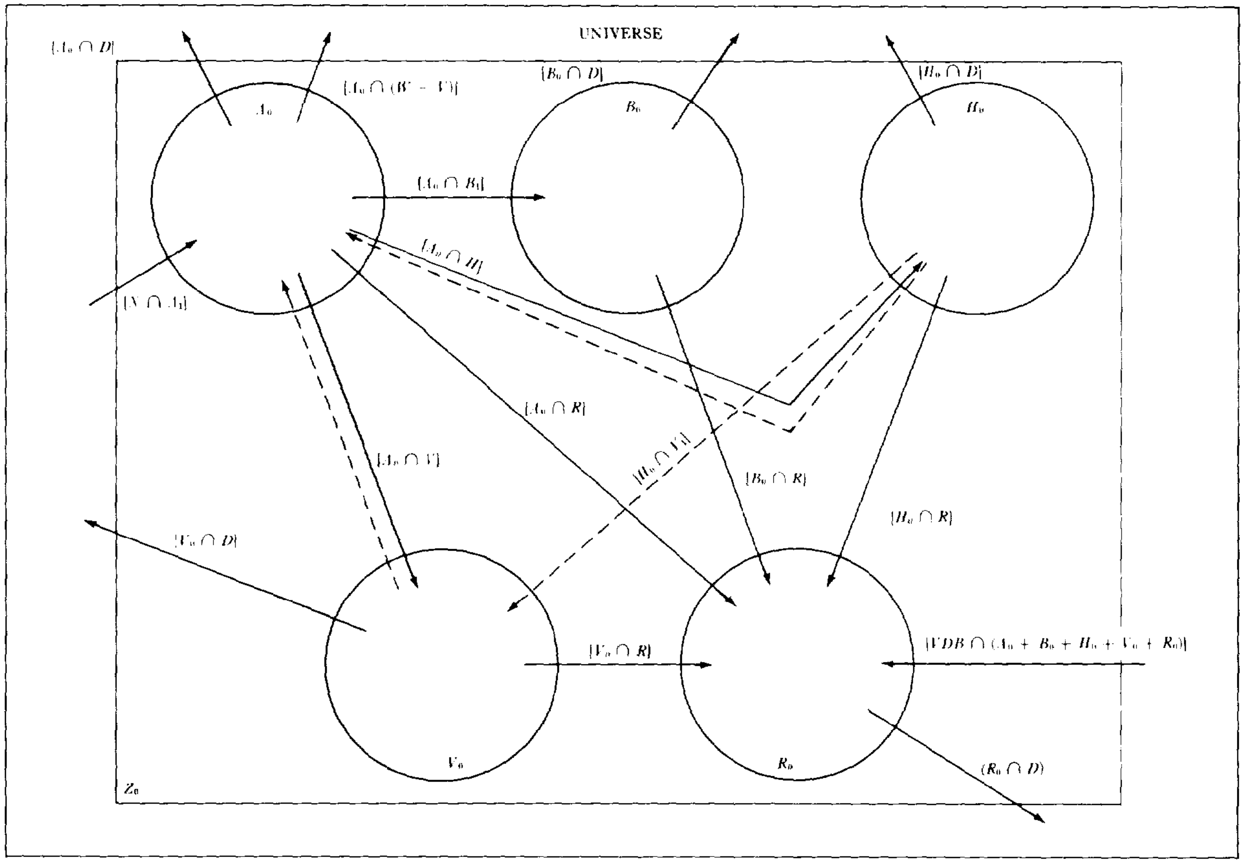

FIG. 1.—Venn diagram of status groups

reflect the service tables used to discount these benefits." This is of little consequence if the actuarial assumptions used to determine the present values of the retirement, disability, vesting, and death benefits and of the future salaries are completely consistent. Invariably, however, such assumptions are not consistent. The author cites as an example the common situation where retirement is a decrement for only the retirement benefit and the present value of future salary. Another example might be the use of the disability decrement for only the disability benefit. Because of the existence of this approach for valuing these terms that is theoretically inconsistent but necessary from a practical standpoint, a given individual at a given age is assumed to have more than one associated probability of death, retirement, disability, or withdrawal. Referring to the gains by decrement (7[*a*]-[*d*] of Sec. VII), the author's concept is applied easily by considering $q_0^j\ AL_1^0$ or $q_0^j\ B_1^j$ to be the sum of various products. Each of these products consists of a term of AL_1^0 or B_1^j multiplied by the appropriate q_0^j, where j represents one of the four contingencies.

Notation

This is often as important as the actuarial concepts that it conveys. Certainly the inability of pension actuaries to agree on terms and notation has complicated communications unduly. Although the Society's Committee on Pensions recently published its "Report on Actuarial Terminology for Pension Plans" in *TSA*, Volume XXVIII, continued efforts in this area certainly are needed.

Mr. Street's notation is meaningful and readily comprehendible. In contrast, Mr. Anderson's notation makes excessive use of commutation functions and Mr. Lynch's notation is somewhat simplistic. In particular, I would recommend the adoption of the author's notation with respect to the following: (1) financial transactions, increased with interest at valuation rate i to EOY (${}^iK$, ${}^iAC$, etc.); (2) projected-value notation (PVS_1^0, etc.); and (3) set notation.

In Appendix II Mr. Street wisely includes a comparison of Mr. Lynch's notation with his own in order to relate the two papers. Would it not be desirable if, through the continued efforts of our profession, the need for such comparisons were minimized?

(AUTHOR'S REVIEW OF DISCUSSION)

CHRISTOPHER C. STREET:

In performing a gain and loss analysis, it is important to define carefully the various status groups and the possible changes in status during the period of the study. Mr. Lang has shown how to make a diagrammatic check on the accuracy and completeness of the formulas in Section III, which connect the status at the beginning and end of the period. I thank him for this very useful aid, which he describes as a Venn diagram.

I must plead guilty to not adopting the new pension terminology, which Mr. Lang prefers for the sake of uniformity. This is a subject that is still under study by an interprofessional group, and the old terminology is embedded in the law. Most actuaries probably still feel more comfortable with the terms *normal cost* and *accrued liability* than with *annual actuarial value* and *supplemental present* (or actuarial) *value.*

Another comment I received concerned my definition of expected annuity payments, EP_1. The liability recursion formula for retired lives receiving monthly life annuity payments is

$$\ddot{a}_x^{(12)}(1+i) = p_x\ddot{a}_{x+1}^{(12)} + \left(1+\frac{13}{24}i - \frac{11}{24}q_x^d\right).$$

This can be rewritten as

$$\ddot{a}_x^{(12)}(1+i) = \ddot{a}_{x+1}^{(12)} - q_x^d\left(\ddot{a}_{x+1}^{(12)} + \frac{11}{24}\right) + \left(1+\frac{13}{24}i\right).$$

Other authors such as Mr. Anderson define the expected reserve released and expected payments as, respectively,

$$q_x^d\left(\ddot{a}_{x+1}^{(12)} + \frac{11}{24}\right) \quad \text{and} \quad \left(1+\frac{13}{24}i\right),$$

as the second formulation implies. I prefer to define the expected reserve released as the terminal rather than the mean reserve, for the sake of both consistency and simplicity. In the notation I have adopted, this becomes $\Sigma_{Z_0}\, q_0^s\, AL_1^0$, where AL_1^0 is the year-end value contemplated at time 0 should the participant survive in the same status to time 1. The expected benefit payments and/or the liability set up for paid-up benefits become

$$\sum_{Z_0}[q_0^s(B_1^s + L_1^s) + EP_1] \equiv \sum_{Z_0}(EB_1 + EP_1),$$

where the subscript 1 denotes that expected benefit payments have been improved with interest to the end of the year. In this connection note that, for monthly life annuity payments,

$$EP_1 = P_0^{(12)}(1+i)\frac{N_x^{(12)} - N_{x+1}^{(12)}}{D_x} = P_0^{(12)}\left(1+\frac{13}{24}i - \frac{11}{24}q_x^d\right),$$

which is the value expected at age x for annuity payments between age x and age $x+1$, improved with interest to the end of the year.

I now would like to make a few comments that may be helpful to actuaries seeking to apply these formulas to insured pension plans, which generally provide for purchase of benefits at retirement. Retired lives then will no longer be kept in the data file for actuarial valuation purposes. If it is desirable to include retired lives in the valuation and also in the gain and loss analysis, pension plan assets and liabilities must be augmented by the amounts held for retired lives. The gain resulting from the inclusion of the retired life liability may be expressed in total as $(1 + i)\ \Sigma_{R_0}\ PVB_0 - \Sigma_{R_1}\ PVB_1$. Recursion formula (14) shows that this may be assigned to source as follows:

1. *Gain from miscellaneous liability changes:* $\Sigma_{R_1}\ PVB_1^0 - \Sigma_{R_1}\ PVB_1$
5. *Gain from annuity payments:* $\Sigma_{R_0}\ EP_1$
7(*a*). *Gain from mortality:* $\Sigma_{D\cap R_0}\ PVB_1^0 - \Sigma_{VDB}\ PVB_1 - \Sigma_{R_0}\ q_0^d PVB_1^0 + \Sigma_{R_0}\ EB_1^d$
7(*c*). *Gain from retirement:* $-\ \Sigma_R\ PVB_1$

Approximations will have to be made in assigning values to these "liability gains." There also will be an asset loss due to annuity payments ${}^iP$, which should be deducted from gain 5, and retired life death benefits, a part of ${}^iB^d$, which should be deducted from gain 7(*a*). In practice, retired life death benefits may be charged to annuity payments for accounting purposes and then may be difficult to isolate.

Also common to insured pension plans are retired life variable benefits that depend on an annuity unit value (AUV). These values generally assume an investment result of i', which is unlikely to be the same as the valuation rate i. Investment results greater or less than i' will cause, respectively, an increase or decrease in the AUV. An increase in the AUV will increase investment gains and liability losses. Ideally, assets equal exactly to the liability are held in a separate account, so that changes in the unit value will have no effect on the total gain. Since investment gains and liability losses both will be inflated by an increase in the AUV (or, conversely, lowered by a decrease), this portion of these gains and losses should be removed or at least noted. It may be shown that the effects of the change in unit value and the difference in interest rates can be approximated as

$$\left(\frac{r + i' - i}{2}\right)\left(PVVB_0 + \frac{PVVB_1}{1 + r}\right),$$

where $1 + r = AUV_1/AUV_0$ and $PVVB$ is the present value of variable benefits.

In the gain and loss analysis, all accrued active life benefits should be assumed to be fixed and valued at the valuation rate i. The fixed value of variable benefits at times 0 and 1 will be determined as if fixed-dollar accruals never had been converted to variable units. The adjustment to recognize that some of these benefits have been converted and are valued at rate i' should be done in bulk. If the variable value exceeds the fixed-dollar value by D_t at time t, the total effect of variable accruals is to increase the gains by $(1 + i)D_0 - D_1$, were D measures the additional liability resulting from a lower valuation interest rate (i') and a higher current market value (AUV) for variable benefits. The portion of the total adjustment $(1 + i)D_0 - D_1$ that results from the changes in the AUV and in the difference between i' and i may be measured by

$$\left(\frac{r + i' - i}{2}\right)\left(D_0 + \frac{D_1}{1 + r}\right).$$