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DIVIDEND FORMULAS IN GROUP INSURANCE

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ABSTRACT

The paper discusses the optimal design of dividend (or premium refund) formulas that occur in connection with group insurance contracts. Since the customer has full coverage for his liabilities, the claims, it is assumed that he has a risk-neutral attitude toward any dividends that he may receive on top of the claim payments. In contrast to this, the insurance company is supposedly a risk averter. As a consequence, the formulas described by Theorem 1 (for one-period contracts) and by Theorem 2 (for multiperiod contracts) are Pareto optimal.

It is shown how a result of the theory of random walks can be used to reduce the evaluation of a multiterm formula to the evaluation of one-term formulas. The paper concludes with illustrative examples for a portfolio of two hundred lives.

I. INTRODUCTION AND SUMMARY

The general idea of a policy featuring "dividends" or "premium refunds" can be summarized by noting two points. Compared with a nonparticipating policy, a participating policy requires higher premiums, on the one hand, but leads to higher security and thus to lower expected net cost for the insured, on the other.

What does the insured know about future dividends? In individual insurance he is given at most a vague forecast, since his dividends will depend partly on the claim experience of other customers. In group insurance many customers do not settle for such forecasts. They want to know exactly how their dividends are going to be a function of their claim experience. (We shall limit our discussion to one-year term insurance, since in the presence of actuarial reserves the dividends would of course depend also on the investment performance of the company.) Such a function is called a *dividend formula*. A very common dividend formula is of the form $D = (kP - S)_+$, guaranteeing dividends equal to the excess

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of a certain percentage, k, of the premiums, P, over the claims, S, whenever this excess is positive ([1], p. 514). In Section III we shall see that this kind of dividend formula is indeed optimal in a certain sense. In Section IV the result is generalized for n-year formulas. Finally, Section V is devoted to an illustrative example.

II. PRELIMINARIES

We shall assume that the insurer determines his preferences according to a *risk-averse utility function*, $u(x)$, $-\infty < x < \infty$, that is, a continuous, nondecreasing function whose derivative is nonincreasing ([4], p. 36). Intuitively, $u(x)$ is the utility that the insurer assigns to an income x. When the income is a random variable, the insurer will be interested in the expected utility.

A risk-averse utility function has the fundamental property that

$$u(x) - u(y) \leq u'(y)(x - y) \qquad \text{for all } x \text{ and } y\,. \tag{1}$$

A proof is readily formulated by distinguishing the cases $x > y$ and $x \leq y$. If $u(x)$ is a risk-averse utility function, then also $u^*(x) = u(ax + b)$, for $a > 0$, is a risk-averse utility function. As a consequence of this and the fact that we will consider the premiums as given constants, we will omit premiums in our considerations of Section IV.

If $f(x)$ is a real-valued function, then we shall denote its positive part by

$$f(x)_+ = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0\,. \end{cases}$$

III. ONE-YEAR CONTRACTS

3.1. *Formulation of the Problem*

In this section we consider a one-year term contract, for which the premium P is given and the distribution of the random variable S (aggregate claims) is known. The dividend D, payable after S has been paid, is another random variable that may assume only nonnegative values; that is, the insurer may not assess premium in addition to P even if claims are high. If D is a function of S, we may write, symbolically, $D = D(S)$; however, this is not required. We are interested in various designs for D.

We assume that the insurer is interested in the expected utility of the income that results from such a contract, namely, $E[u(P - S - D)]$. On the other hand, we assume that the insured has a risk-neutral attitude toward dividends, that is, he is only interested in $E[D]$. (There is some practical evidence that the customer is primarily concerned about $E[D]$ without regard to Var (D). For example, prominent consumer advocates compare only expected net costs.) We say that a dividend D is *Pareto optimal* if it cannot be improved for both sides at the same time.

Definition: A dividend D^* is Pareto optimal if, for any other dividend D with $E[D] \geq E[D^*]$ and $E[u(P - S - D)] \geq E[u(P - S - D^*)]$, we must have $E[D] = E[D^*]$ and $E[u(P - S - D)] = E[u(P - S - D^*)]$.

It is clear that insurer and insured should agree upon a Pareto-optimal dividend, at least as far as their preferences are accurately reflected by the above assumptions.

3.2. *Pareto-optimal Dividends*

THEOREM 1. *For any real number c, the dividend $D^*(S) = (c - S)_+$ is Pareto optimal.*

Proof:

If D is an arbitrary dividend, we have

$$\begin{aligned} u(P - D - S) - u(P - D^* - S) &\leq u'(P - D^* - S)(D^* - D) \\ &\leq u'(P - c)(D^* - D)\,. \end{aligned} \tag{2}$$

The first inequality follows from relation (1) and the second from the special form of D^*. The expected values of the members of relation (2) may be rearranged to obtain

$$\begin{aligned} &E[u(P - D^* - S)] + u'(P - c)E[D^*] \\ &\geq E[u(P - D - S)] + u'(P - c)E[D]\,, \end{aligned} \tag{3}$$

which proves the theorem.

REMARKS

1. The theorem says that, for each value of c, $D^*(S)$ is Pareto optimal for any P. It does not say how c should be chosen. The choice of c, which might be the subject of bargaining between insurer and insured, is influenced by factors such as the security loading contained in P and competitive aspects. In any case, c should be some fraction of P such that $E[S] + E[D^*] < P$. Note that $g(c) = E[(c - S)_+]$ is a continuous, nondecreasing (strictly increasing for c's such that $P(S \leq c) > 0$) function such that $g(0) = 0$ and $g(\infty) = \infty$; hence, for given $d > 0$, there is a unique c such that $E[(c - S)_+] = d$.
2. If the utility function is strictly concave and the claim distribution is not degenerate, inequality (3) is strict whenever $D \neq D^*$, in which case all Pareto-optimal dividends are of the form described in Theorem 1.
3. Theorem 1 says that $D^*(S)$ is Pareto optimal for any underlying risk-averse utility function. In the special case of a quadratic utility function, this means that $D^*(S)$ minimizes Var $(S + D)$ for a fixed $E[D]$.

3.3. *Geometrical Interpretation*

The "value" of any dividend D may be conveniently represented by a point in the plane which has coordinates $y = E[u(P - S - D)]$ and $x = E[D]$. The northeast boundary of the set of all such points which correspond to arbitrary dividends is the set of Pareto-optimal dividends. Inequality (3) shows that, at the point corresponding to $D^* = (c - S)_+$, the slope of the boundary is $-u'(P - c)$.

3.4. *Interpretation as Reinsurance*

A dividend formula of the form $D^* = (c - S)_+$ is equivalent to a stop-loss coverage with deductible c. The insured's gain under the dividend contract is

$$\begin{aligned} -P + D^* &= -P + (c - S)_+ \\ &= -(P - c) - S + (S - c)_+ . \end{aligned} \tag{4}$$

The right-hand side allows the following interpretation: the insured pays a stop-loss premium, $P - c$; then he pays total claims, S; and then he is reimbursed under the stop-loss coverage for claims in excess of c, $(S - c)_+$. Similarly, the insurer's gain under the contract with dividend can be written to allow the interpretation of a stop-loss coverage:

$$P - S - D^* = P - S - (c - S)_+ = (P - c) - (S - c)_+ . \tag{5}$$

This interpretation of the dividend contract is somewhat surprising. It is well known that a stop-loss coverage is a Pareto-optimal reinsurance contract for an *insured* with a *risk-averse* utility function and an *insurer* with a *linear* utility function ([2], p. 969, or, for the special case of a quadratic utility function, [7], p. 267, or [3], p. 104). We have found that an equivalent dividend contract is Pareto optimal for an *insured* with a *linear* utility function and an *insurer* with a *risk-averse* utility function.

This observation on the dividend contract does not imply that the equivalent stop-loss cover is a Pareto-optimal reinsurance contract for the risk-averse insurer and the insured with a linear utility function. In fact, if the reinsurance payment is required to be nonnegative and not in excess of total claims, then there is an interchangeability between the insured and the insurer which implies that in a Pareto-optimal contract the insured would issue a stop-loss cover on the insurer. Such a reinsurance agreement is not realistic, so some additional restrictions on the reinsurance payment may be made to obtain a more realistic contract for the Pareto-optimal solution. For example, if the ratio of the reinsurance payment, T, to some continuous, nondecreasing, positive function g of the total claims S is required to be nondecreasing, then a contract with reinsurance payments proportional to $g(S)$ would be Pareto optimal for the risk-averse insurer and the insured with a linear utility function. Miller [8] has shown this for $g(S) = S$.

IV. n-YEAR CONTRACTS

4.1. *Formulation of the Problem*

We assume that the aggregate claims $S_1, S_2, \ldots, S_n$ for an n-year period are independent random variables with known distributions, and the premiums $P_1, P_2, P_3, \ldots, P_n$ to cover these claims are given. An n-year dividend D consists of n nonnegative random variables $D_1, D_2, \ldots, D_n$ such that D_k is independent of $S_{k+1}, S_{k+2}, \ldots, S_n$ $(k = 1, 2, \ldots, n-1)$. Intuitively, D_k is the premium refund at the end of year k, and this is of course independent of future claims.

For the moment, we assume that the insurer is interested only in the expected utility of the over-all income resulting from the n-year operation. Since the premiums are given constants, it is sufficient and convenient to consider $E[u(-\Sigma_1^n (S_k + D_k))]$ rather than $E[u(\Sigma_1^n (P_k - S_k - D_k))]$. We also assume that the insured is interested only in $E[D_1], E[D_2], \ldots, E[D_n]$. Consequently, we state the following definition:

Definition: A dividend $D^* = (D_1^*, D_2^*, \ldots, D_n^*)$ is Pareto optimal if, for any other dividend $D = (D_1, D_2, \ldots, D_n)$, the inequalities

$$E[D_k^*] \leq E[D_k] \qquad \text{for} \qquad k = 1, 2, \ldots, n$$

imply

$$E\left[u\left(-\sum_1^n (S_k + D_k^*)\right)\right] \geq E\left[u\left(-\sum_1^n (S_k + D_k)\right)\right].$$

Thus, for given expected values of the premium refunds, a Pareto-optimal dividend maximizes the expected utility of the insurer's income.

4.2. *Pareto-optimal Dividends*

THEOREM 2. *For any n-tuple of real numbers $c_1, c_2, \ldots, c_n$, the dividend defined recursively by*

$$\begin{aligned} D_1^* &= (c_1 - S_1)_+ , \\ D_2^* &= (c_2 - S_1 - D_1^* - S_2)_+ , \\ &\vdots \\ D_n^* &= \left(c_n - \sum_1^n S_k - \sum_1^{n-1} D_k^*\right)_+ \end{aligned} \tag{6}$$

is Pareto optimal.

REMARKS

1. $E[D_k^*]$ is a continuous, nondecreasing, unbounded function of c_k which is zero for $c_k = 0$ and is independent of $c_{k+1}, \ldots, c_n$. It follows that, for any n-tuple of nonnegative numbers $(d_1, d_2, \ldots, d_n)$, there exists a dividend D^* of the form (6) such that $E[D_k^*] = d_k$, $k = 1, 2, \ldots, n$. Thus, for any dividend D, we can find one of form (6) which is at least as good as D.
2. It is remarkable that the dividends defined by equations (6) are independent of n. Thus, for given $E[D_1^*], \ldots, E[D_n^*]$, this dividend *maximizes simultaneously* $E[u(-\Sigma_1^j (S_k + D_k))]$ for $j = 1, 2, \ldots, n$. Also, the specific form of u is unimportant as long as it is risk-averse.
3. The proof of the theorem will be by induction and will show that an arbitrary n-year dividend is no better than one of form (6). First, the n-year situation will be reduced to $n - 1$ years by combining the claims of the first two years with the first year's arbitrary dividend, D_1, to form a new "first year's" claim. This is permissible, since the claims are not required to be identically distributed. The inductive hypothesis then implies that the last $n - 1$ premium refunds should be of form (6). Then an analysis shows that the first year's refund should also be in that form.

Proof by induction:

Theorem 1 proves the assertion for $n = 1$. For $n > 1$, let $(D_1, D_2, \ldots, D_n)$ be an arbitrary dividend. Define $D_1^* = (c_1 - S_1)_+$, where c_1 is chosen so that $E[D_1^*] = E[D_1]$. Define

$$D_1^\epsilon = (1 - \epsilon) D_1 + \epsilon D_1^* , \quad 0 \le \epsilon \le 1 . \tag{7}$$

Observe that, for $0 \le \epsilon < 1$, the following relationships among events hold:

$$\begin{aligned} \{D_1^* > D_1\} &\subseteq \{S_1 + D_1^\epsilon < c_1\} , \\ \{D_1^* < D_1\} &\subseteq \{S_1 + D_1^\epsilon > c_1\} . \end{aligned} \tag{8}$$

Define recursively for $k = 2, 3, \ldots, n$, and $0 \le \epsilon \le 1$,

$$D_k^\epsilon(x) = \left[c_k^\epsilon - x - \sum_2^k S_j - \sum_2^{k-1} D_j^\epsilon(x) \right]_+ , \tag{9}$$

where the c_k^ϵ's are such that

$$E[D_k^\epsilon(S_1 + D_1^\epsilon)] = E[D_k] .$$

Let

$$f(\epsilon, \eta) = E\left[u\left(- \sum_1^n S_k - D_1^\epsilon - \sum_2^n D_k^\eta(S_1 + D_1^\eta) \right) \right] . \tag{10}$$

Now $f(\epsilon, \epsilon)$ may be interpreted as the expected utility of the negative of the total claims and premium refunds for an $(n-1)$-"year" situation in which first-"year" claims are $S_1 + D_1^{\epsilon} + S_2$ and kth-"year" claims are S_{k+1}, $k = 2, 3, \ldots, n-1$, and for which the $n-1$ premium refunds have been set by equations (6).

By the inductive hypothesis,

$$f(\epsilon, \eta) \leq f(\epsilon, \epsilon) \quad \text{for} \quad 0 < \eta < 1 \,, \tag{11}$$

and

$$E\Big[u\Big(-\sum_1^n S_k - \sum_1^n D_k\Big)\Big] \leq f(0, 0) \,.$$

Since $f(1, 1)$ is also the expected utility of the negative of the total claims and premium refunds for the n-year situation whose n refunds have been set by equations (6), the proof will be complete if $f(0, 0) \leq f(1, 1)$. This inequality is an implication of the inequality $df(\epsilon, \epsilon)/d\epsilon \geq 0$ for $0 < \epsilon < 1$.

$$\frac{df(\epsilon, \epsilon)}{d\epsilon} = \frac{\partial f(\epsilon, \eta)}{\partial \eta}\Big|_{\eta=\epsilon} + \frac{\partial f(\epsilon, \eta)}{\partial \epsilon}\Big|_{\eta=\epsilon} \,. \tag{12}$$

Inequality (11) implies that the first term on the right-hand side of equation (12) is zero. The second term is

$$\frac{\partial f(\epsilon, \eta)}{\partial \epsilon}\Big|_{\eta=\epsilon} = E\Big[u'\Big(-\sum_1^n S_k + D_1^{\epsilon} - \sum_2^n D_k^{\epsilon}(S_1 + D_1^{\epsilon})\Big)(D_1 - D_1^*)\Big] \,.$$

For fixed outcomes of $S_2, S_3, \ldots, S_n$,

$$G_n(x) = x + \sum_2^n S_k + \sum_2^n D_k^{\epsilon}(x) \tag{13}$$

is a nondecreasing function of x. (This is intuitively clear when x is interpreted as the sum $(S_1 + D_1)$, and it may be verified by induction.)

In the following list of inequalities, each is now an immediate consequence of its predecessor:

$$D_1 < D_1^* \,,$$

$$S_1 + D_1^{\epsilon} < c_1 \quad \text{(see relations [8])} \,,$$

$$G_n(S_1 + D_1^{\epsilon}) \leq G_n(c_1) \,,$$

$$u'(-G_n(S_1 + D_1^{\epsilon})) \leq u'(-G_n(c_1)) \,,$$

$$u'(-G_n(S_1 + D_1^{\epsilon}))(D_1 - D_1^*) \geq u'(-G_n(c_1))(D_1 - D_1^*) \,. \tag{14}$$

A similar list started with $D_1 > D_1^*$ will again conclude with inequality (14), which thus holds for all outcomes. Since the expectation of the left-hand side of relation (14) is

$$\left.\frac{\partial f(\epsilon, \eta)}{\partial \epsilon}\right|_{\eta=\epsilon},$$

and the expectation of the right-hand side is zero, the proof is complete.

4.3 *Connection with Random Walks*

Let D^* be a dividend of form (6) for a set of values $c_1, c_2, \ldots$, and let $c_0 = 0$. The random variables defined by

$$X_k = c_k - c_{k-1} - S_k\,, \quad k = 1, 2, 3, \ldots, \tag{15}$$

are independent, since $S_1, S_2, \ldots$ are independent.

Let $Y_0 = 0$ and, for $k \geq 1$,

$$Y_k = \sum_1^k X_j \quad \text{and} \quad M_k = \max\,(Y_0, Y_1, \ldots, Y_k)\,. \tag{16}$$

It is easy to show by induction that the premium refund at the end of year k is

$$D_k^* = M_k - M_{k-1}\,, \quad k = 1, 2, \ldots. \tag{17}$$

4.4. *Application*

In the special case where the S_k's have identical distributions and where $c_k = k * c$ for $k = 1, 2, \ldots$, then the X_k's are also identically distributed. The following classical combinatorial result of the theory of random walks is applicable to our sequences (see [5], p. 287, or [6], p. 573).

$$E[M_k] = \sum_1^k \frac{1}{j} E[(Y_j)_+]\,. \tag{18}$$

From equations (15)–(18) we have

$$E[D_k^*] = \frac{1}{k} E\left[\left(kc - \sum_1^k S_j\right)_+\right]. \tag{19}$$

Thus, in this special case, the computation of $E[D_k^*]$, $k = 1, 2, \ldots, n$, reduces to the calculation of n one-year dividend formulas of the type discussed in Section III.

V. A PRACTICAL EXAMPLE

In this section several dividend formulas for the following portfolio of two hundred lives are discussed.

Amount at Risk	Mortality Rate	
	0.01	0.02
$1	20	40
$4	80	60

Thus twenty lives, each covered for $1, have a mortality rate of 0.01, and so on. The deaths are assumed to occur independently, and those lives who die are replaced at the end of the year by identical lives. Furthermore, there is to be no "aging," so that S_1, S_2, . . . are independent and identically distributed random variables, each being the sum of two hundred independent Bernoulli variables.

The following four types of dividends will be examined:

a) $D(S_1) = (c - S_1)_+$ (one-year formula).
b) D_1, D_2, D_3 defined as in equations (6) with $c_k = kc$ (three-year formula).
c) A five-year formula similar to type *b*.
d) $D_5(S) = (5c - S_1 - S_2 - S_3 - S_4 - S_5)_+$ (a one-period formula applied to a five-year period).

With $c = 10$, formula (19) produced the expected premium refunds shown in Table 1 without approximations. Table 1 suggests that for $c =$ 10 the lowest gross premiums can be offered with a formula of type *d*. Of course, this type has the disadvantage that no dividends are paid in the first four years. If this is not acceptable to the customer, he could be offered formulas of type *b* or type *c*. All three of these formulas have the common feature of losses being carried forward. If this feature is unacceptable, then, at a higher premium, the type *a* formula is available.

TABLE 1

	Formula Type			
	a	*b*	*c*	*d*
$E[D_1]$	2.852	2.852	2.852	0
$E[D_2]$	....	2.186	2.186	0
$E[D_3]$	....	1.895	1.895	0
$E[D_4]$	....	....	1.726	0
$E[D_5]$	....	....	1.611	8.056
Average per year	2.852	2.311	2.054	1.611

In practice, the problem is often reversed. Given the premiums (by regulation or competition), design a dividend formula that is attractive to the customer. To avoid the question of interest and acquisition costs, we reword the problem: Given an average expected dividend per year, say \$1.611, what value of c can be used in each of the four types of formulas? The answers are shown in Table 2.

TABLE 2

	FORMULA TYPE			
	a	b	c	d
c	7.664	8.783	9.265	10
$E[D_1]$	1.611	2.139	2.408	0
$E[D_2]$	....	1.493	1.753	0
$E[D_3]$	....	1.201	1.456	0
$E[D_4]$	....	....	1.279	0
$E[D_5]$	....	....	1.160	8.056
Average per year	1.611	1.611	1.611	1.611

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DISCUSSION OF PRECEDING PAPER

JAMES C. HICKMAN:

Each of us experiences moments of self-doubt. During such inevitable gloomy interludes, those of us who plan, manage, or study insurance systems sometimes ask ourselves whether these systems really contribute to the sum total of satisfaction in society. Do we simply push dollars around with no one really much better off for the effort?

By building their paper on a foundation of utility theory, Professors Gerber and Jones have helped all of us feel a little more useful. Utility theory provides an intellectually satisfying foundation for insurance and a plausible economic justification for risk-sharing. It would be redundant to attempt to augment in a short discussion the utility foundations that Professors Gerber and Jones provide in their exposition and reference list. However, it might be helpful to recall that in 1969 the Society of Actuaries held a stimulating session on utility theory which is recorded in *TSA*, XXI, D331–D363.

My discussion will center on one question and an alternative model to that employed by Professors Gerber and Jones. These are advanced solely to elicit the comments of the authors, for it is abundantly clear that they have thought deeply on the issues involved.

The question is very simple: Why does the model assume that the insured has a risk-neutral attitude? It would seem that the very fact that the insured has purchased insurance to cover his claims, S, is prima facie evidence that, in the interval of monetary values within which the insurance transaction may fall, the insured is risk-averse. Since an insurance premium must exceed the expected value of losses because of expense, contingency, and profit loadings, there would seem to be reason to assume that the insured's utility of wealth function has predominantly a negative second derivative on the relevant range of wealth. The contrary assumption is that the purchase of insurance would be an irrational action that would reduce the customer's expected utility.

The authors' practical observation that consumer advocates have locked onto expected net costs in comparing insurance contracts is true. Yet, since the very essence of insurance is the spreading of risk, it would seem that this decision is more an indication of shallow thinking on the part of some consumer advocates than an observation on the decision rule used by most purchasers of insurance.

Perhaps the answer is that an insurance purchase has taken place because the insured and the insurer have different probability distributions for the amount of claims. However, this possibility is not mentioned in the paper. In addition, this sort of inequality in probability distributions is closely related to the concept of "moral hazard" (the insured may know something that the insurer does not), and there are powerful forces at work in an open market to bring the two distributions together.

Employing the authors' symbols P for premium, D for dividend, and S for claim amount, I would like to suggest an alternative model. We will assume that the customer for insurance has a utility of wealth function, to be denoted by $u_c(x)$, that is ever increasing ($u_c'(x) > 0$) and concave down ($u_c''(x) < 0$), at least on the interval of wealth that may be attained during the period of the insurance contract. These assumptions guarantee that the customer will be willing to pay more than his expected losses for insurance. We will also assume that the insurance company, over the range of wealth levels that may be reached as a result of this contract, has a utility of wealth function that may be approximated by a straight line. Concentrating on the dividend function, this means that the customer will seek to maximize $u_c(S + D - P)$, while the company will seek to maximize $(P - D - S)$.

Definition: A dividend D^* is Pareto optimal if, for any other dividend D with

$$E[P - S - D] \geq E[P - S - D^*] \quad \text{and} \quad E[u_c(S + D - P)] \geq E[u_c(S + D^* - P)] ,$$

we must have

$$E[P - S - D] = E[P - S - D^*] \quad \text{and} \quad E[u_c(S + D - P)] = E[u_c(S + D^* - P)] .$$

THEOREM. *For any real number, c, the dividend $D^*(S) = (c - S)_+$ is Pareto optimal.*

Proof:

If D is an arbitrary dividend, we have

$$u_c(S + D - P) - u_c(S + D^* - P) \leq u_c'(S + D^* - P)(D - D^*) \leq u_c'(c - P)(D - D^*) .$$

The first inequality is a result of the assumption that the second derivative of the utility function is negative (eq. [1] in the paper). The second inequality may be proved by considering three cases:

Case 1. If $D^* = D$, then equality holds.

Case 2. If $D^* > D$, then $D^* > 0$, and $D^* + S = c$. In this case equality also holds.

Case 3. If $D^* < D$, then $D - D^* > 0$, and, since $S + D^* - P \geq c - P$,

$$u_c'(S + D^* - P) \leq u_c'(c - P)$$

because the second derivative of the utility function is negative. From these two facts, the inequality follows.

Much as in the paper, the expected values may be rearranged to yield

$$E[u_c(S + D - P)] - u_c'(c - P)E[D] \leq E[u_c(S + D^* - P)] - u_c'(c - P)E[D^*].$$

Adding $u_c'(c - P)E[P - S]$ to each side of the inequality yields

$$E[u_c(S + D - P)] + u_c'(c - P)E[P - S - D] \leq E[u_c(S + D^* - P)] + u_c'(c - P)E[P - S - D^*]\,,$$

from which the theorem follows.

JOHN A. MEREU:

Drs. Jones and Gerber are to be congratulated for disseminating the interesting term "Pareto optimal" in actuarial literature. This "everybody is happy" expression will be useful for livening up conversations.

Their theorem that a pure stop-loss dividend strategy is Pareto optimal was not intuitively obvious to me. As their elegant proof seemed to settle the matter too easily, I decided to work out a concrete example to gain a better understanding. To bring the subject into a more familiar perspective, I describe the dividend as the net premium less a claim charge and consider the problem as one of selecting a claim charge strategy. If we let CC be the claim charge, the income derived by the insurer from the contract $(P - D - S)$ reduces to $(CC - S)$.

By way of example, let us take a group life contract with the aggregate claim distribution described in Table 1 of this discussion. Let us assume a premium P of \$18,000 and a value for C of \$12,000. Under the Gerber-Jones (GJ) dividend strategy, the claim charge is equal to $P - (C - S)_+$. As can be seen from the table, this has an expected value of \$12,238.

Let us postulate a second strategy, II, with a claim charge defined as $P - K(P - S)_+$, with K so determined that the expected dividend and, as a consequence, the expected claim charge are the same as for the GJ strategy. The value of K turns out to be 0.589282, and the resulting claim charges are shown in Table 1.

Since both strategies have the same expected claim charge, the expected income to the insurer is the same, as well. Assuming a utility measure for

TABLE 1

COMPARISON OF GJ STRATEGY WITH STRATEGY II

Aggregate Claims	Probability of Occurrence	GJ Strategy			Strategy II		
		Claim Charge	Insurer Income	Utility	Claim Charge	Insurer Income	Utility
$0	.405	$ 6,000	$ 6,000	.451	$ 7,393	$ 7,393	.523
$4,000	.056	10,000	6,000	.451	9,750	5,750	.437
$6,000	.029	12,000	6,000	.451	10,929	4,929	.389
$8,000	.049	14,000	6,000	.451	12,107	4,107	.337
$10,000	.042	16,000	6,000	.451	13,286	3,286	.280
$12,000	.041	18,000	6,000	.451	14,464	2,464	.219
$14,000	.048	18,000	4,000	.330	15,643	1,643	.152
$16,000	.046	18,000	2,000	.181	16,821	821	.079
$18,000	.013	18,000	0	0	18,000	0	0
Over $18,000	.271	18,000	$18,000-S$	N.A.	18,000	$18,000-S$	N.A.
Expected values		$12,238		.305	$12,238		.296

income I of $[1 - \exp(-I/10{,}000)]$, the expected utility of the income under each strategy is computed and, as predicted by the paper, the GJ strategy has the greater utility.

Under the GJ strategy the income to the insurer is constant if the aggregate claims are \$12,000 or less. In moving from the GJ strategy to strategy II, the income, if the aggregate claims are zero, is increased and for other amounts it is decreased, in such a way that the expected income does not change. However, the gain in utility for the one contingency does not compensate for the loss in utility in the other contingencies. Additional dollars have diminishing utility under a risk-averse utility function.

Having now been convinced that the GJ strategy is Pareto optimal, what priority should an insurer give to adopting the GJ strategy? A couple of concerns on which the authors might comment come to mind.

The assumption that the insured has a risk-neutral attitude toward dividends is questionable. The insured would expect his dividend to bear some inverse relationship to claims. The GJ strategy cannot be faulted on this score, but the insurer may wish to compare potential strategies from the point of view of acceptability to the insured.

The insurer is interested more in the income derived from its whole portfolio. Does a strategy which is optimal for a single group continue to be optimal for a collection of groups, and how important is the difference in utility from one strategy to another?

In conclusion, I would like to say how much I enjoyed this education in utility theory.

(AUTHORS' REVIEW OF DISCUSSION)

DONALD A. JONES AND HANS U. GERBER:

Our thanks are due the discussants for their valuable contributions. We agree with Professor Hickman that an insured does not have a risk-neutral attitude per se. However, our considerations are based on the insured's attitude toward dividends: Given full coverage, at a known premium, our assumption is that the insured is interested only in the expected value of the premium refunds.

The theorem that Professor Hickman proved is closely related to the results of Kahn [7][1] and of Arrow [2]. Further, it shows that there are at least two ways to define the insured's income, as illustrated in the tabulation at the top of page 92.

In the first approach the claims are considered as a liability (that may be offset by the purchase of a policy). In the second approach there are

[1] Reference numbers refer to the list given in the authors' paper.

	Insured's Income	
	Without Coverage	With Coverage
Gerber-Jones	$-S$	$-P+D$
Hickman	0	$-P+S+D$

no "claims" a priori. A policy provides for certain payments contingent on certain events. These payments are treated as gains.

Mr. Mereu's example is an excellent illustration for our Theorem 1. His strategy II and the GJ strategy produce claim charges of the form

$$P - (kP - aS)_+ = \text{Max}\,[(1-k)P + aS,\, P]\,,$$

discussed by Ammeter [1]. The right-hand side of the above formula shows that for any member of this family ($a > 0$) the claim charge is a nondecreasing function of the claims. We agree with Mr. Mereu that this is a desirable property. Paul Kahn developed the conditions on the coefficients to maximize expected (quadratic) utility for the insured and again for the insurer (see "The Application of Utility Theory to Group Experience Rating," *Transactions of the Seventeenth International Congress of Actuaries*, pp. 578–91).

We received some requests for details regarding our numerical calculations for the n-year formulas. We hope that the tabulation below, which is a worksheet summarizing a computer printout, will be helpful.

x	$F(x)$	$\Sigma_0^x F(y)$	$F^{*2}(x)$	$\Sigma_0^x F^{*2}(y)$	$F^{*3}(x)$	$\Sigma_0^x F^{*3}(y)$
0	0.04854	0.04854	0.00236	0.00236	0.00011	0.00011
1	0.09798	0.14652	0.00716	0.00951	0.00046	0.00058
2	0.12269	0.26921	0.01200	0.02151	0.00099	0.00157
. . .						
9	0.60445	2.85199	0.14758	0.50884	0.02417	0.07113
. . .						
19	....	....	0.60083	4.37255	0.23469	1.24006
. . .						
29	....	....	....	....	0.62011	5.68455

The following formula for the expected value of $D_1 = (c - S_1)_+$ is useful:

$$E[D_1] = \sum_{0}^{c} (c - y)f(y) = \sum_{0}^{c-1} F(y)$$

(proof by summation by parts). From formula (19) we obtain, with $c = 10$, the figures shown in Table 1 of the paper:

$$E[D_1] = \sum_{0}^{9} F(y) = 2.85199 = 2.852 ;$$

$$E[D_2] = \frac{1}{2} \sum_{0}^{19} F^{*2}(y) = \tfrac{1}{2}(4.37255) = 2.186 ;$$

$$E[D_3] = \frac{1}{3} \sum_{0}^{29} F^{*3}(y) = \tfrac{1}{3}(5.68455) = 1.895 .$$