TRANSACTIONS OF SOCIETY OF ACTUARIES
1974 VOL. 26 PT. 1 NO. 75 AB

ON CALCULATING DELTA-IZED RESERVES— ACTUARIAL NOTE

WILLIAM A. BAILEY

ABSTRACT

The actuarial profession is faced with a problem of determining delta-ized reserves on a basis which will include appropriate provision for adverse deviations in mortality, lapses, interest rates, and expense rates. This paper describes briefly a method which could be used to solve this problem in a flexible manner. Initial research, using the suggested method, could take the form of a few computer runs to approximate the magnitude of ratios of (*a*) delta-ized reserve (that is, including appropriate provision for the adverse deviations) to (*b*) the corresponding reserve without provision for the adverse deviations. As additional information is gathered concerning variability of mortality rates, lapse rates, and interest rates, and as more computer runs become available, tables of the above-mentioned ratios, based on various bench-mark assumptions, could be published for use as a guide by the members of the Society of Actuaries and other interested persons.

INTRODUCTION

LET us assume that we are given a confidence level (k per cent) which the benefit reserves (${}_tV^B_{[x]}$) are to meet; that is, the benefit reserve under a policy is to be sufficient, together with future benefit premiums under the policy, to provide for the policy's share of the cost of benefits over the lifetime of the policy k per cent of the time.

The benefit premium (P^B), payable under a policy from issue to termination, would be defined to be the level premium which would, k per cent of the time, turn out to have been sufficient to provide for the policy's share of the cost of benefits over the lifetime of the policy. The benefit reserve at duration 0 would thus be zero.

In a sense, once we decide to deal with stochastic assumptions rather than mean-value assumptions, we really are no longer interested in so-called "most likely" assumptions. The stochastic assumptions become the "most likely" assumptions, even though they give rise to a frequency distribution of results. We could, of course, speak of "most likely" premium (or reserve), in the sense of the mean of a frequency distribution of premiums (or reserves). It might be of some interest to express a benefit premium, calculated to achieve a confidence level of k_1 per cent, in terms of a ratio of such benefit premium to the corresponding benefit premium calculated to achieve a confidence level of k_2 per cent (for example, 50 per cent). To the extent that these ratios exhibited some consistent patterns (with changes in parameters such as age at issue, duration, plan of insurance, and underwriting class, and changes in stochastic assumptions such as mortality, lapse, and investment income rates), they might prove to be quite helpful to the actuary in establishing GAAP benefit reserves.

In what follows, the main complicating factor is the handling of stochastic interest rates (new-money rates, old-money rates, rollover rates, and so on). The numerical processing would be considerably easier if a simpler model for interest rates could be used. However, a simpler model might be too unrealistic.

BENEFIT PREMIUMS

The Monte Carlo method (involving the generating of random numbers) represents one approach to the numerical solution to questions as to the likelihood that a particular level of benefit premiums (P^B) will be adequate to provide for future benefits. Another method would be to use a convolution approach to calculate discrete 5-dimensional frequency distributions; the results are those which would be produced by an infinite number of Monte Carlo trials. However, for the purpose of understanding the method in the first place, the Monte Carlo method may be more useful.

If we run enough Monte Carlo trials, where each trial represents a possible outcome in the life of one policy, we can become reasonably certain of having constructed a frequency distribution of all the possible outcomes which have a reasonable probability of occurring. Consider now one such trial.

Assume that a frequency distribution of mortality rates $q^d_{[x]+t-1}$ is available for each attained age and a frequency distribution of lapse rates $q^w_{[x]+t}$ for each policy year. The new-money rate in any policy year is assumed to be a random variable depending upon the new-money rate in the previous year. Investments are assumed to roll over at the rate of r per cent of the outstanding investments each year. More complicated assumptions could be made, but the implications for the number and type of variables to be followed should be investigated carefully. (Considerable analysis and judgment will be involved in constructing the frequency distributions which in this paragraph are assumed to be available.)

A random number is generated. Knowing the new-money rate in the year preceding the first policy year, we can enter the frequency distribution of new-money rates depending thereon and determine the new-money rate earned in the first policy year.

Another random number is generated. The frequency distribution of mortality rates for the first policy year is entered, and the mortality cost determined for the first policy year.

Another random number is generated. The frequency distribution of lapse rates for the first policy year is entered, and the surrender cost determined for the first policy year. (Note that, in the case of both mortality cost and surrender cost, we assess such costs to each policy which is exposed in the policy year under consideration, regardless of whether the particular policy we are valuing terminates at the end of the policy year by death or surrender, or persists into the next policy year.)

Another random number is generated. Using this random number and the same mortality rate that was determined above for the first policy year, we determine whether this particular policy terminates by death at the end of the first policy year. If not, then another random number is generated. Using this random number and the same lapse rate which was determined above for the first policy year, we determine whether this particular policy terminates by surrender at the end of the first policy year.

Continued application of the procedure described above (as covered in detail below for the second policy year) enables one to develop for each policy year a value for each of the following variables:

(1) New-money rate for the current policy year;[1]
(2) Old-money rate for (4) below, that is, an average interest rate being earned on (4) during the current policy year;
(3) Old-money rate for (5) below, that is, an average interest rate being earned on (5) during the current policy year;
(4) Value of $1 received at the beginning of each policy year and accumulated at generated interest rates to the end of the current policy year;
(5) Benefit costs accumulated at generated interest rates to the end of the current policy year.

We know whether this policy terminated by death or lapse in the first policy year or persisted into the second policy year. If the first trial ended by death or lapse in the first policy year, it produces the first entry in the 5-dimensional frequency distribution needed to calculate the benefit

[1] The current policy year is the particular policy year being processed at a given stage in the calculation process.

premium (P^B). If this policy persisted into the second policy year, we continue as indicated in the next paragraph.

A random number is generated. Knowing the new-money rate in the first policy year, we can enter the frequency distribution of new-money rates depending thereon and determine the new-money rate for the second policy year. Weight the old-money rate, (2) or (3), by $(1 - r/100)$, and the new-money rate just determined for the second policy year by $r/100$; the resulting interest rate is the new value for (2) or (3), respectively, for the second policy year. The new-money rate is the new value for (1). The previous value of (4) is brought forward for one policy year at the new value of (2), and the $1 received at the beginning of the second policy year is increased at the new-money rate, to produce a new value for (4). The previous value of (5) is brought forward for one policy year at the new value of (3); a mortality cost and a surrender cost will be added in to obtain the new value of (5).

Another random number is generated. The frequency distribution of mortality rates for the second policy year is entered, and the mortality cost determined for the second policy year.

Another random number is generated. The frequency distribution of lapse rates for the second policy year is entered, and the surrender cost determined for the second policy year. (Note again that we assess mortality costs and surrender costs against each policy which is exposed in the second policy year, regardless of whether the particular policy we are valuing terminates at the end of the second policy year by death or surrender or persists into the third policy year.)

Another random number is generated. Using this random number and the same mortality rate that was determined above for the second policy year, we determine whether this particular policy terminates by death at the end of the second policy year. If not, then another random number is generated. Using this random number and the same lapse rate that was determined above for the second policy year, we determine whether this particular policy terminates by surrender at the end of the second policy year. If the policy terminated by death or surrender in the second policy year, we have obtained the next entry in the 5-dimensional frequency distribution required in the calculation of the benefit premium (P^B). If this policy persisted into the third policy year, we continue the procedure essentially as indicated above for the second year.

We move through the "current" policy years of a particular policy until the end of the policy year in which the policy terminates by death or surrender. At that time, the entire procedure is repeated, commencing with the first policy year, in order to make another trial. This procedure continues until a sufficient number of trials have been completed. Of course, if the number of trials were to be, say, a million, it would probably be impractical to keep all the individual results. A practical solution would be to set up some arbitrary intervals and to capture each of the results in its appropriate interval, retaining the number and average of the values which end up in each interval.

When the desired 5-dimensional frequency distribution has been obtained, we proceed to transform such frequency distribution into a 1-dimensional frequency distribution, where the amount is (5)/(4) and the frequency is the same as in the 5-dimensional frequency distribution. Thus we would have a 1-dimensional frequency distribution of benefit premium levels. The amounts would be tabulated in ascending order in the 1-dimensional frequency distribution. The amount opposite a cumulative frequency of k per cent would be the benefit premium (P^B) at the k per cent confidence level.

Having determined P^B so as to be in a positive position k per cent of the time when the policy terminates, we may wish to examine the size of the positive and negative values of $[P^B(4) - (5)]$; note that the values of $[P^B(4) - (5)]$ represent accumulated values as of the end of the policy year of termination, not present values as of issue of the policy. Under most circumstances the determination of the benefit premium by the above method should be acceptable; however, if positive values of $[P^B(4) - (5)]$ to be experienced k per cent of the time were \$1, and if the negative values to be experienced $(100 - k)$ per cent of the time were \$1,000,000,000, then obviously this would be an inadequate benefit premium, even though the k per cent confidence level is satisfied.

The benefit premium (P^B), otherwise calculated, probably should be increased by the net annual stop-loss premium at the P^B stop-loss level, in order to provide for the possibility of needing larger benefit premiums, in spite of the fact that the k per cent confidence level has been met by the benefit premium (P^B). This net annual stop-loss premium can be calculated in a straightforward fashion from the 1-dimensional frequency distribution of (5)/(4) described above.

BENEFIT RESERVES

Once the benefit premium has been established for a policy, one can proceed to calculate benefit reserves at any valuation date, given the new-money rate in the year preceding the valuation date and given the old-money rate—the initial value of (2) below—being earned on existing assets as of the valuation date. The procedure described above for use in determining benefit premiums would apply, except that the variables would be as follows:

(1) New-money rate for the current policy year;
(2) Old-money rate for (4) below, that is, an average interest rate being earned on (4) during the current policy year;
(3) Old-money rate for (5) below, that is, an average interest rate being earned on (5) during the current policy;
(4) Value of $1 of invested assets as of the valuation date, accumulated at generated interest rates to the end of the current policy year;
(5) Benefit costs less benefit premiums, such net amounts from the valuation date to the end of the current policy year being accumulated at generated interest rates to the end of the current policy year.

The old-money rate (2) is initially the old-money rate being earned on existing assets as of the valuation date. (Perhaps the value of (2) as of the valuation date should be generated from actual new-money rates and rollover rates for the policy years between issue and the valuation date, together with experienced mortality and lapse rates during the same period.) Once again, the actual numerical processing could be done using a convolution approach.

Having obtained the desired 5-dimensional frequency distribution, we proceed to transform it into a 1-dimensional frequency distribution, where the amount is (5)/(4) and the frequency is the same as in the 5-dimensional frequency distribution. Thus we would have a 1-dimensional frequency distribution of present values of benefit costs less benefit premiums. The amount would be tabulated in ascending order in the 1-dimensional frequency distribution. The amount opposite a cumulative frequency of k per cent would be the benefit reserve $({}_tV^{\beta}_{[x]})$ at the k per cent confidence level.

It may be appropriate to add to the benefit reserve $({}_tV^{\beta}_{[x]})$, otherwise calculated, the net single stop-loss premium at a stop-loss level equal to ${}_tV^{\beta}_{[x]}$, in order to provide for the possibility of large losses being incurred, in spite of the k per cent confidence level having been met by both the benefit premium and the benefit reserves. This net single stop-loss premium can be calculated in a straightforward fashion from the 1-dimensional frequency distribution of (5)/(4) derived in the preceding paragraph.

EXPENSE PREMIUMS AND EXPENSE RESERVES

The procedure outlined above for use in determining benefit premiums would be followed to determine expense premiums, except that the fifth variable would be

(5) Expense costs accumulated at generated interest rates to the end of the current policy year.

A frequency distribution of expense rates ($e_{[x]+t}$) for each policy year would be required.

Similarly, having established the expense premium for a policy, one can proceed to calculate expense reserves at any valuation date by using the procedure previously outlined for benefit reserves, except that the fifth variable would be

(5) Expense costs less expense premiums, such net amounts from the valuation date to the end of the current policy year being accumulated at generated interest rates to the end of the current policy year.

RESERVES AND THE RUIN PROBLEM

The method outlined above enables the actuary to determine delta-ized reserves on any given policy, without reference to any other policy in the portfolio of policies or any other line of business. This approach retains a huge advantage over alternative approaches which treat the determination of delta-ized reserves as a ruin problem in the traditional sense. Using the ruin approach, either an entire line of business or all lines of business treated together would have to be considered in making the calculations.

Admittedly, in the approach suggested, frequency distributions of mortality rates and of lapse rates must be derived either empirically or theoretically (or both), to represent experience which can be expected from selected portfolios of policies. Once these frequency distributions have been developed, however, one can focus on one policy at a time.

PREMIUMS AND RESERVES FOR OTHER LINES OF BUSINESS

Methods comparable to the one outlined in this paper can be developed for individual accident and health policies, for individual disability income policies (with or without cash values), for variable life, group life, group health, and other types of policies. However, the specific method to be applied within a given line of business should be designed carefully to reflect the characteristics of the particular line of business.

COMPARISON WITH GAAP RESERVES

It would be impractical to perform the type of analysis described in the body of this paper for every single policy in a company's portfolio of individual policies. However, research could be undertaken which would help the actuary (and other interested persons) acquire a feel for the size of the delta-ized premiums and reserves required to meet various confidence levels for different types of policies and different sets of assumptions. The results of this research might be only a set of bench marks against which the practicing actuary could compare GAAP reserves calculated by using assumptions based on his own judgment. But even this amount of research would seem to go a long way toward helping him in this difficult area.

A MARKOV PROCESS

The multidimensional random walks described in the above sections can be considered to be Markov processes, where each state is defined by the values assumed by the five defined variables. The transition probabilities are defined implicitly by the random walk procedure, including the frequency distributions of mortality rates, lapse rates, new-money rates, and expense rates. (A Markov chain is a Markov process which involves transition probabilities which do not vary with duration; thus we are not dealing here with Markov chains.)

For ease of description in the above sections, three of the random variables (mortality rates, lapse rates, and expense rates) have been considered to be independent from policy year to policy year. The random variable of new-money rates has been assumed to be at least partially dependent upon the new-money rate experienced in the immediately preceding year. The random variables of mortality rates, lapse rates, expense rates, and new-money rates have been assumed to be independent of each other. The five defined variables

(1) New-money rates,
(2) Old-money rate for (4) below,
(3) Old-money rate for (5) below,
(4) Accumulated value of either a single $1 or a series of $1 per year,
(5) Accumulated value of either cash outgo or cash outgo minus cash income,

are severely interdependent, and this interdependence has been built into the process as described in the above sections.

One simple modification of the process described in the above sections would be to combine benefit premiums and expense premiums into a single calculation process and, similarly, to combine benefit reserves and expense reserves into a single calculation process. This seems a logical step to take since the same random variables of mortality and lapse rates are involved in both the benefit and expense sides; that is, treating benefits and expenses independently is theoretically improper. If the random variable of expense rates is independent from policy year to policy year, then the process can still be defined in terms of the five defined variables;

on the other hand, if expense rates are assumed, as with new-money rates, to be dependent to some extent on the expense rate experienced in the preceding policy year, then a sixth defined variable would be "expense rate."

Dependent relationships between mortality rates, lapse rates, expense rates, interest rates (new-money rates or old-money rates), and so on, can be built into the process; the degree of difficulty will depend on the form which such relationships take. Especially easy are those which relate one of the variables to another of the variables in the same policy year; variables in this connection include both the five defined variables and the random variables such as mortality rates, lapse rates, and expense rates. To the extent that the new- and old-money rates and accumulated monetary values reflect past experience, it is easy to build in dependencies of current random variables on past experience. More difficult are dependencies on past experience where we have no variable currently and suitably reflecting such past experience. In fact, if dependencies cannot be defined in terms of current variables (existing or yet to be defined), we no longer can consider the procedure to be a Markov process, and we are then dealing with a much more difficult structure.

DISCUSSION OF PRECEDING PAPER

DAVID G. HALMSTAD

I found Mr. Bailey's note extremely interesting, since it deals with two of my favorite subjects: the collective/individual computations debate and the use of simulation. I will restrict my remarks to these areas and leave any specific "delta-izing" or GAAP comments to others.

I trust that those who use the results of any simulation conducted by the method suggested by Mr. Bailey will realize that the method is not the traditional actuarial premium and reserve calculation method. Mr. Bailey recognizes this in his brief comment that the use of his method "retains a huge advantage over alternative approaches which treat the determination of delta-ized reserves as a ruin problem in the traditional sense." We should realize that such traditional actuarial calculations are, in the fundamentals, collective in nature.

This distinction is seen most easily if one concentrates on the simulation procedure used to obtain the "benefit premium," P^B. Stripping such a simulation of the possibility of a distribution of interest rates (i.e., assuming a certain interest rate), forgetting the possibility that the "true" underlying mortality rate for an individual might be different from that indicated by a class "expected" mortality rate, and neglecting considerations of lapsation and ancillary benefits such as expenses, we can reduce the complexity of the problem to the classic actuarial problem: What premium do we charge an individual?

My reading of this note suggests that, under the simple case outlined above (simple interest and mortality), the distribution of simulated results for a whole life premium for an individual aged x at issue will be (col. 5) $\div$ (col. 4), where, with probability equal to $d_{x+t-1} \div l_x$, column 4 = $(1+i)^t \ddot{a}_{\overline{t}|}$, and column 5 = $(1+i)^t A^{1}_{x:\overline{t}|}$. This distribution is illustrated numerically in Table 1 of this discussion for age 25 on the 1958 CSO Table with 5 per cent interest. In the illustration, it will be noted, the mean of the above distribution is \$5.19 per \$1,000, whereas the actuarial net premium is \$7.54. In fact, if my understanding of Mr. Bailey's scheme is correct, it would be impossible to simulate any "individual" premium as large as the actuarial net; the largest possible "individual" premium that could occur would be the \$6.72 at durations 67–69.

Naturally there is an obvious problem here, and the solution to it lies

TABLE 1

ILLUSTRATION OF AGE 25 WHOLE LIFE DISTRIBUTION STATUS

```
TABLE '58CSO'
INTEREST  0.05
      W
100
      ACCUM←1.05*⍳W-25
      COL4←ACCUM×ANDT ⍳W-25
      COL5←ACCUM×AX1N 25 BY ⍳W-25
      PREM←1000×COL5÷COL4
      PROB←(DX 24+⍳W-25)÷LX 25

      X25←(⍳W-25),PROB,COL4,COL5,[1.5]PREM

      5 BLOCK 'I5,F15.8,3F12.5' ΔFMT X25[,0 20 65 70∘.+⍳5;]
```

DUR	PROB.	COL 4	COL 5	PREM
1	0.00193000	1.05000	0.00193	1.83810
2	0.00195622	2.15250	0.00398	1.85028
3	0.00198227	3.31012	0.00616	1.86220
4	0.00201809	4.52563	0.00849	1.87607
5	0.00206360	5.80191	0.01098	1.89223
21	0.00505576	37.50521	0.09349	2.49269
22	0.00547989	40.43048	0.10364	2.56349
23	0.00594321	43.50200	0.11477	2.63824
24	0.00645324	46.72710	0.12696	2.71706
25	0.00700773	50.11345	0.14032	2.79997
66	0.01115427	504.66981	3.38842	6.71413
67	0.00927486	530.95330	3.56711	6.71832
68	0.00756919	558.55096	3.75304	6.71924
69	0.00604461	587.52851	3.94673	6.71752
70	0.00470218	617.95494	4.14877	6.71372
71	0.00356407	649.90268	4.35978	6.70835
72	0.00263691	683.44782	4.58040	6.70190
73	0.00192737	718.67021	4.81135	6.69480
74	0.00134884	755.65372	5.05327	6.68728
75	0.00066993	794.48640	5.30660	6.67928

```
      (⌊/PREM),    (⌈/PREM),    PROB+.×PREM
1.8381   6.71924   5.19166

      1000×APX 25
7.53727
      1000×(M 25)÷N 25
7.53727
```

in the accumulation of the reserve for the policy. Since this is the question Mr. Bailey addresses, and I have not found a way of including reserve increases, I will drop the illustration. For those interested in continuing along these lines, I urge them to consider what happens in the case of an endowment benefit; as described, the simulation method treats those who die in the given term (except in the last year of it) just as it treats those who purchase a term policy.

Instead of examining the point further, I would like to remind all of us of the statistical truth that, if we are dealing with random variates X and Y, in general $E[X] \div E[Y] \neq E[X \div Y]$. For those interested in pursuing the distribution of actuarial functions, including premiums, I suggest starting with the references indicated at the end of the discussion of Messrs. Fibiger and Kellison's recent paper (*TSA*, XXIII, 150). I have found the Pollard and Pollard reference particularly pertinent.

If we assume, for the time being, that we are interested in a distribution of individual premiums, there remains the problem of conducting the simulation. Here we must use extreme care. Sidney Benjamin has shown quite clearly ("Simulating Mortality Fluctuation," *Transactions of the International Congress of Actuaries* [London/Edinburgh], III [1964], 478–501) that a modicum of analytical work saves much random "grinding of wheels," and my own experience has confirmed this repeatedly (e.g., *TSA*, XXI, D115–D116). In the present case the method used is essentially "brute force," and the savings from a little bit of analysis will be large.

One should, I think, design any simulation with the end product firmly in mind. For Mr. Bailey's case, the end product is (in the premium simulation) producing one value ${}^kP^B$ for each value of k studied; the number of such k's is likely to be small. These ${}^kP^B$ values supposedly are higher than k per cent of the other simulated premiums, in order to be k per cent "confident." Since I am a bit unclear about the premium distribution desired for this note, I cannot be certain about the veracity of the following analysis, but I have tried to use "common sense" in the argument.

The common-sense supposition would be that the longer a simulated policy persists, the larger the simulated premium generally will be (for most level benefits, and with increasing mortality). If this is so, we will be doing much repetitive bookwork for those who lapse or die "early," and our feeling should be that this is perhaps unnecessary. In fact, such a feeling would lead us to realize that we repeat the bookwork (albeit with differing interest, mortality, and lapse assumptions) for the early part of each simulation. Wouldn't it be desirable to be able to continue each simulation to reach the critical area of interest—the later durations where the ${}^{k}P^{B}$ values are being calculated anyway?

If we ask ourselves a question like that, we will realize that, by using a mixture of analysis and simulation, we gain more accurate simulations more cheaply by modifying the "brute force" approach. Since the simulation essentially is meant to mix possible interest, mortality, and lapse streams of rates rather than to simulate a collective viewed over a period of time (for a total claim distribution), we can remove the extraneous simulation of mortality and lapse *chance* fluctuations. With this viewpoint, I would suggest the following alternative scheme for the simulation of P^{B}:

1. Choose a stream of new-money rates, possibly correlated, sufficient to carry the issue age to the end of the benefit period.
2. Choose a stream of "underlying" mortality rates (possibly differing from the expected, or average, mortality for the class and possibly correlated heavily from one year to the next) of the same length as the interest rates.
3. Choose an "underlying" lapse rate stream just as for mortality.
4. If desired, choose an "experienced" expense stream and any similar variables of interest.
5. Using the streams indicated above, do the bookkeeping (old-money, new-money, etc.) either as outlined by Mr. Bailey or with whatever GAAP accounting one may use. Do such bookkeeping for the entire stream of values, and give each value the probability indicated by the "underlying" mortality and lapse probabilities.
6. Return to step 1 if necessary.

This format almost always will return a sample point—properly weighted of course—in the area of final interest; its cost is roughly the reciprocal of the expected life of a policy. The work that is done to obtain a value above would have been done under Mr. Bailey's scheme, but we have collected a whole stream of observations, stripped of the irrelevant chance fluctuations that merely cloud the effects of the varying mixture of interest, mortality, lapse, and other factors. What we really have done is to apply Mr. Benjamin's Method III (the "forced death") principle, simply interjecting a small amount of analysis.

I must stress that the suggestion above is merely on the *simulation* aspects of Mr. Bailey's proposal, as it now stands. If called upon to conduct such an investigation myself, I would start from a very different basis. I would study the possible short-term cycle, long-term cycle, and trend elements of the interest, mortality, and other variables (lapses and expenses, for example), but on an overall class expected rather than an individual basis. I would then proceed to simulate as indicated above, *up to step 5.* My new step 5 would simply calculate the usual gross premium, along with GAAP reserves, from the simulated streams of variables. This essentially produces one point of the simulated results, and the usual laying out of such results and choosing a level "k per cent safe" can be done.

Even this proposal, however, probably is unnecessary. It is my belief that—by using convolutions and other analytic tools with fast Fourier transforms and heavy numerical analysis—the simulation that remains can also be avoided.

I would like to comment on one additional point. As I read this report, I was struck with the fact that Mr. Bailey proposes to simulate for each individual, each year's mortality rate around the tabular mortality rate, and that he also proposes that "the random variable of new-money rates has been assumed to be at least partially dependent upon the new-money rate experienced in the immediately preceding year." I find these two assumptions startling: as a "random walker," I would propose that new-money rates could be independent of one another, and I would think that if we are talking about truly "individual" simulations, the underlying mortality and lapse rates for each run would exhibit strong dependencies. However, the possibility of a "distribution of mortality rates" does raise a final question to Mr. Bailey: How should one conduct the study to determine the distribution of mortality rates within a particular underwriting class? I would find such a statistical algorithm fascinating.

CECIL J. NESBITT:

The author has tackled a difficult problem but unfortunately has not developed the underlying mathematics with sufficient clarity for me to follow, and I expect that many other readers have had similar difficulty. Presumably the author has been able to verify his ideas computationally, but he has not described those ideas in such a way as to convey mathematical understanding.

I made several tries at reconstructing the author's ideas, without much success. One small result did emerge, and I will describe it briefly. A simplified model is considered, with interest fixed at a given rate and with mortality according to a given table. Lapses are not taken into account. The insurance considered is whole life insurance for sum insured 1 issued at age x with net level annual premium P_x.

For this model the basic random variable is T, denoting the policy year of death. This random variable has probability function $P(T = t) = {}_{t-1|}q_x$, $t = 1, 2, \ldots$. The annual premium, $P_{\overline{t|}} = 1/\ddot{s}_{\overline{t|}}$, will be exactly sufficient with probability ${}_{t-1|}q_x$ and will be at least sufficient with probability $0.01k = {}_tp_x$. One might then think of the expected value $E[P_{\overline{T|}}]$, namely,

$$P_x^* = \sum_{t=1}^{\infty} P_{\overline{t|}} \ {}_{t-1|}q_x ,$$

and wonder how P_x^* is related to the usual net annual premium P_x.

By a slight rewriting of formula (10) of David H. Berne's paper "Net Premiums Viewed as Averages of Compound Interest Functions" (*TSA*, XIII, 215), one has

$$P_x = \sum_{h=1}^{\infty} P_{\overline{h|}} \frac{\ddot{a}_{\overline{h|}}}{\ddot{a}_x} \ {}_{h-1|}q_x ,$$

which exhibits P_x as $E[P_{\overline{H|}}]$, where H has probability function $P(H = h) = (\ddot{a}_{\overline{h|}}/\ddot{a}_x) \ {}_{h-1|}q_x$. Then the variance, Var $[P_{\overline{H|}}]$, is given by

$$\begin{aligned}
\text{Var}\,[P_{\overline{H|}}] &= \sum_h (P_{\overline{h|}})^2 \frac{\ddot{a}_{\overline{h|}}}{\ddot{a}_x} \ {}_{h-1|}q_x - (P_x)^2 \\
&= \sum_h P_{\overline{h|}} \frac{v^h}{\ddot{a}_x} \ {}_{h-1|}q_x - (P_x)^2 \\
&= \sum_h P_{\overline{h|}} \frac{1 - d\ddot{a}_{\overline{h|}}}{\ddot{a}_x} \ {}_{h-1|}q_x - (P_x)^2 \\
&= \frac{1}{\ddot{a}_x} P_x^* - dP_x - (P_x)^2 \\
&= \frac{1}{\ddot{a}_x} [P_x^* - P_x] .
\end{aligned}$$

Thus

$$P_x^* = P_x + \ddot{a}_x \,\text{Var}\,[P_{\overline{H|}}] .$$

This little crumb of mathematics is one thing suggested to me by the author's approach. However, this is purely in terms of an individual policy, and groups of policies require consideration too.

I believe that the Society and the author would have been better served if this actuarial note had been expressed in more rigorous terms.

(AUTHOR'S REVIEW OF DISCUSSION)

WILLIAM A. BAILEY:

Introduction

When actuaries produce or consider reserves, they seldom quantify their confidence in the adequacy of such reserves. Perhaps they recall the adage "Fools rush in where angels fear to tread." On the other hand, some research with stochastic models may provide some insights of value to the practicing actuary.

The fact that Mr. Halmstad and Mr. Nesbitt have had some difficulty in understanding the paper may be due in part to the ambiguity of the italicized portion of the following statement: "that is, the benefit reserve under a policy is to be sufficient, together with future benefit premiums under the policy, to provide for *the policy's share of the cost of benefits* over the lifetime of the policy k per cent of the time." Mr. Halmstad assumes that death at the end of the tth policy year requires that policy to provide $(1+i)^t A^{1}_{x:\overline{t}|}$, whereas Mr. Nesbitt assumes that death at the end of the tth policy year requires that policy to provide 1.00. The corresponding function contemplated by the method of the paper is

$$\sum_{r=1}^{t} [q_{x+r-1}(1+i)^{t-r}] .$$

In each of these three cases the frequency corresponding to these amounts is $_{t-1|}q_x$. Thus, expressed in square brackets as univariate frequency distributions we have, under Mr. Halmstad's assumption,

$$\left[\frac{(1+i)^t A^{1}_{x:\overline{t}|}}{\ddot{s}_{\overline{t}|}} = \frac{A^{1}_{x:\overline{t}|}}{\ddot{a}_{\overline{t}|}} ,\ {}_{t-1|}q_x\right] ;$$

under Mr. Nesbitt's,

$$\left[\frac{1}{\ddot{s}_{\overline{t}|}} = \frac{v^t}{\ddot{a}_{\overline{t}|}} ,\ {}_{t-1|}q_x\right] ;$$

and under Mr. Bailey's,

$$\left[\frac{\sum_{r=1}^{t}((1+i)^{t-r}q_{x+r-1})}{\ddot{s}_{\overline{t}|}} = \frac{\sum_{r=1}^{t} v^r q_{x+r-1}}{\ddot{a}_{\overline{t}|}} ,\ {}_{t-1|}q_x\right] .$$

The relationships

$$P_x = \frac{A_x = \sum_{t=1}^{w-x}(v^t\ {}_{t-1|}q_x) = \sum_{t=1}^{w-x}\left[\left(\sum_{r=1}^{t} v^r q_{x+r-1}\right)\ {}_{t-1|}q_x\right]}{\ddot{a}_x = \sum_{t=1}^{w-x}(\ddot{a}_{\overline{t}|}\ {}_{t-1|}q_x) = \sum_{t=1}^{w-x}(\ddot{a}_{\overline{t}|}\ {}_{t-1|}q_x)}$$

suggest that, although either Mr. Nesbitt's or my suggested frequency distribution (or both) may be appropriate, Mr. Halmstad's is probably not, because

$$\sum_{t=1}^{w-x}(A^{1}_{x:\overline{t}|}\ {}_{t-1|}q_x) \neq A_x .$$

Mr. Halmstad's "reminder" is pertinent here; namely, that the quotient of the means of two random variables is not (usually) equal to the mean of the quotient of the two random variables; thus

$$\sum_{t=1}^{w-x} \left(\frac{v^t}{\ddot{a}_{\overline{t|}}} \; {}_{t-1|}q_x \right) \neq P_x$$

and

$$\sum_{t=1}^{w-x} \left[\frac{\sum_{r=1}^{t} (q_{x+r-1} v^r)}{\ddot{a}_{\overline{t|}}} \; {}_{t-1|}q_x \right] \neq P_x \, .$$

However, the objective of calculating a premium (or reserve) at a k per cent confidence level does not require the mean of the constructed frequency distribution thereof to equal P_x (or ${}_tV_x$).

So far in this discussion of frequency distributions of levels of net premiums, we have been assuming a given fixed interest rate, and a given mortality rate at each attained age (or for each age at issue/policy year combination). The following section treats the situation, contemplated in the paper, in which not only are new-money rates treated stochastically, but also mortality and lapse are treated stochastically in a somewhat novel way.

A Convolution Technique to Facilitate the Method Suggested in the Paper

In this section we shall define special convolution operators which will enable us to perform the type of calculations contemplated, if not completely described, in the paper. The following two specific clarifications have been made:

1. Mortality costs are distinguished from, although related to, mortality rates; similarly, lapse costs are distinguished from, although related to, lapse rates.
2. Calculation of the rates of interest being earned on invested assets involves the weighting of old and new assets by the old rate of interest and the new-money rate, respectively, and also involves the rollover rate (r).

Assume that the matrix

$$[x_i, y_i, z_i, t_i, u_i, p_i]$$

represents a discrete 5-dimensional frequency distribution, where i refers to the number of the row (or line); x_i, y_i, z_i, t_i, and u_i are real numbers; and p_i is the probability or frequency of occurrence of the particular combination of x_i, y_i, z_i, t_i, and u_i. We shall take the liberty of referring to such a matrix as a frequency distribution even when the sum of the frequencies (i.e., $\Sigma_{i=1}^{n} p_i$, where n is the number of rows or lines) is less than unity.

Verbal descriptions of the variables in the matrices shown below are as follows:

$x_i^{(1)}$ = New-money rate for previous policy year ;
$y_i^{(1)}$ = Old-money rate for assets $t_i^{(1)}$;
$z_i^{(1)}$ = Old-money rate for assets $u_i^{(1)}$;
$t_i^{(1)}$ = Value of $1 received at the beginning of each previous policy year and accumulated at generated interest rates to the beginning of the current policy year ($t_i^{(1)} = 0$ at time of issue) ;
$u_i^{(1)}$ = Benefit costs, accumulated at generated interest rates to the beginning of the current policy year ($u_i^{(1)} = 0$ at time of issue) ;
$p_i^{(1)}$ = Probability or frequency of occurrence of the particular combination of $x_i^{(1)}$, $y_i^{(1)}$, $z_i^{(1)}$, $t_i^{(1)}$, $u_i^{(1)}$ ($p_i^{(1)} = 1.0$ at time of issue) .

(Note that $t_i^{(1)}$ is an "interest only" type of function.)

$x_j^{(2)}$ = New-money rate for the current policy year ;
$p_j^{(2)}$ = Probability or frequency of the occurrence of the value $x_j^{(2)}$ given a value of $x_i^{(1)}$ for the previous year .

(Note that the values of $x_j^{(2)}$ are independent of any particular policy year.)

$x_k^{(3)}$ = This policy's share of the mortality cost for the current policy year ;
$y_k^{(3)}$ = Mortality rate for the current policy year ;
$p_k^{(3)}$ = Probability or frequency of occurrence of the particular combination $x_k^{(3)}$ and $y_k^{(3)}$.

$x_l^{(4)}$ = New-money rate for current policy year ;
$y_l^{(4)}$ = Old-money rate for assets $t_l^{(4)}$;
$z_l^{(4)}$ = Old-money rate for assets $u_l^{(4)}$;
$t_l^{(4)}$ = Value of $1 received at the beginning of each previous policy year and accumulated at generated interest rates to the end of the current policy year ;
$u_l^{(4)}$ = Benefit costs, accumulated at generated interest rates to the end of the current policy year ;
$p_l^{(4)}$ = Probability or frequency of occurrence of the particular combination of $x_l^{(4)}$, $y_l^{(4)}$, $z_l^{(4)}$, $t_l^{(4)}$, $u_l^{(4)}$.

(Note that $t_l^{(4)}$ is an "interest only" type of function.)

$x_m^{(5)}$ = This policy's share of the surrender cost for current policy year ;
$y_m^{(5)}$ = Lapse rate for current policy year ;
$p_m^{(5)}$ = Probability or frequency of occurrence of the particular combination of $x_m^{(5)}$ and $y_m^{(5)}$.

The first convolution operator (designated as Ⓒ1) is a trinary operator, which carries the values of the variables from the beginning to the end of a policy year.

$$\text{Ⓒ1}\{[x_i^{(1)}, y_i^{(1)}, z_i^{(1)}, t_i^{(1)}, u_i^{(1)}, p_i^{(1)}],\ [x_j^{(2)}, p_j^{(2)} = \text{Prob}\ \{y_j^{(2)} \mid x_i^{(1)}\}],$$

$$[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}]\}$$

$$= [x_l^{(4)}, y_l^{(4)}, z_l^{(4)}, t_l^{(4)}, u_l^{(4)}, p_l^{(4)}],$$

where

$$x_l^{(4)} = x_j^{(2)};$$

$$t_l^{(4)} = t_i^{(1)}(1-r)(1+y_i^{(1)}) + t_i^{(1)}r(1+x_j^{(2)}) + 1\cdot(1+x_j^{(2)})$$

(r = rollover rate assumed operative at the beginning of the policy year);

$$y_l^{(4)} = \frac{[t_i^{(1)}(1-r)y_i^{(1)} + t_i^{(1)}rx_j^{(2)} + 1\cdot x_j^{(2)}](1+x_j^{(2)})}{t_l^{(4)}};$$

$$u_l^{(4)} = u_i^{(1)}(1-r)(1+z_i^{(1)}) + u_i^{(1)}r(1+x_j^{(2)}) + x_k^{(3)};$$

$$z_l^{(4)} = \frac{[u_i^{(1)}(1-r)z_i^{(1)} + u_i^{(1)}rx_j^{(2)}](1+x_j^{(2)}) + x_k^{(3)}x_j^{(2)}}{u_l^{(4)}};$$

and

$$p_l^{(4)} = p_i^{(1)}p_j^{(2)}p_k^{(3)}y_k^{(3)} \qquad \text{for terminations by death}$$

$$= p_i^{(1)}p_j^{(2)}p_k^{(3)}(1-y_k^{(3)}) \qquad \text{for survivors},$$

where i assumes each integer value from 1 to the number of lines in the first matrix, and, for each such value of i, j assumes each integer value from 1 to the number of lines in the second matrix;[1] and, for each pair of values (i, j), k assumes each integer value from 1 to the number of lines in the third matrix. The superscripts (1), (2), (3), and (4) merely indicate that the value is from the first, second, third, or fourth matrix, respectively. (To the extent that the volume of calculations implied by this or other definitions in this discussion would be inordinate, suitable meshes can be superimposed on the coordinate axes and sufficiently small probabilities ignored; control can be maintained by calculating various moments both before and after imposition of the meshes.)

Mr. Halmstad and Mr. Nesbitt assumed that

$$[x_j^{(2)}, p_j^{(2)}] = [0.05, 1.0].$$

Mr. Halmstad assumed (incorrectly, I believe) that

$$[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}] = [1\cdot {}_{t-1|}q_x, q_{x+t-1}, 1.0],$$

[1] There would be one "second matrix" for each value of $x_i^{(1)}$.

whereas Mr. Nesbitt assumed that

$$[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}] = [1.00, q_{x+t-1}, 1.0] \qquad \text{where } p_l^{(4)} = p_i^{(1)} p_j^{(2)} y_k^{(3)}$$

$$= [0.00, q_{x+t-1}, 1.0] \qquad \text{where } p_l^{(4)} = p_i^{(1)} p_j^{(2)} (1 - y_k^{(3)}) ,$$

and the corresponding assumption of the paper might have been that

$$[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}] = [1 \cdot q_{x+t-1}, q_{x+t-1}, 1.0] .$$

Each of these constitutes a one-line frequency distribution, whereas *the method of the paper provides for (and really expects) empirical or constructed multiline frequency distributions.*

Let the resulting 5-dimensional frequency distribution (5-d f.d.) for terminations by death in the tth policy year be represented by D_t, and the corresponding 5-d f.d. for survivors be represented by $L_t^{-\epsilon}$.

The second convolution operation (designated c2) is a binary operator which carries the values of the variables from the end of a policy year to the beginning of the next policy year, in the sense that lapses[2] are taken into account.

$$L_t^{-\epsilon} \text{ c2 } [x_m^{(5)}, y_m^{(5)}, p_m^{(5)}] = [x_l^{(4)}, y_l^{(4)}, z_l^{(4)}, t_l^{(4)}, u_l^{(4)}, p_l^{(4)}] \text{ c2 } [x_m^{(5)}, y_m^{(5)}, p_m^{(5)}]$$

$$= [x_n^{(6)}, y_n^{(6)}, z_n^{(6)}, t_n^{(6)}, u_n^{(6)}, p_n^{(6)}] ,$$

where

$$x_n^{(6)} = x_l^{(4)} ,$$

$$t_n^{(6)} = t_l^{(4)} ,$$

$$y_n^{(6)} = y_l^{(4)} ,$$

$$u_n^{(6)} = u_l^{(4)} + x_m^{(5)} ,$$

$$z_n^{(6)} = \frac{u_l^{(4)} z_l^{(4)} (1 + x_l^{(4)}) + x_m^{(5)} x_l^{(4)}}{u_n^{(6)}} ,$$

and

$$p_n^{(6)} = p_l^{(4)} p_m^{(5)} y_m^{(5)} \qquad \text{for terminations by lapse}$$

$$= p_l^{(4)} p_m^{(5)} (1 - y_m^{(5)}) \qquad \text{for survivors} ,$$

where l assumes each integer value from 1 to the number of lines in the first matrix; and, for each such value of l, m assumes each integer value from 1 to the number of lines in the second matrix. The superscripts (4),

[2] If lapse rates and mortality rates are not treated as independent random variables, then the convolution operator c2 would not be used; rather, the definition of the convolution operator c1 would be recast to handle the dependent relationship between lapse rates and mortality rates.

(5), and (6) merely identify whether the value is from the first, second, or third matrix, respectively.

Let the resulting 5-d f.d. for terminations by lapse (nonrenewal) at the beginning of the $(t+1)$st policy year be represented by W_t, and the corresponding 5-d f.d. for survivors by $L_t^{+\epsilon}$. As we proceed to perform the convolutions (ⓒ1 and ⓒ2) for the next policy year, the values of the variables in $L_t^{+\epsilon}$,

$$x_n^{(6)},\ y_n^{(6)},\ z_n^{(6)},\ t_n^{(6)},\ u_n^{(6)},\ p_n^{(6)}\ ,$$

become

$$x_i^{(1)},\ y_i^{(1)},\ z_i^{(1)},\ t_i^{(1)},\ u_i^{(1)},\ p_i^{(1)}\ ,$$

respectively; that is, the recursive nature of the process comes into play. Merging (concatenating) $D1, D2, \ldots, W1, W2, \ldots$, we have a 5-d f.d. which can be transformed into a univariate frequency distribution $[u/t, p]$ of benefit premiums. Tabulating the results in ascending order of amounts and calculating the implied cumulative frequencies, the amount opposite a cumulative frequency of k per cent would be the benefit premium at the k per cent confidence level.

The corresponding 5-d f.d.'s needed to calculate a 1-d f.d. of reserves are developed in an analogous fashion, where, in the 5-d f.d.,

$t_i^{(1)}$ = Value of \$1 of invested assets as of the valuation date, accumulated at generated interest rates to the end of the previous policy year;

$u_i^{(1)}$ = Benefit costs less benefit premiums, such net amounts from the valuation date to the beginning of the current policy year being accumulated at generated interest rates to the beginning of the current policy year;

and

$t_i^{(4)}$ = Value of \$1 of invested assets as of the valuation date accumulated at generated interest rates to the end of the current policy year;

$u_i^{(4)}$ = Benefit costs less benefit premiums, such net amounts from the valuation date to the end of the current policy year being accumulated at generated interest rates to the end of the current policy year.

A 2-Dimensional (Bivariate) Extension of the Frequency Distribution $[1/\ddot{s}_{\overline{t}|},\ {}_{t-1|}q_x]$

As Mr. Nesbitt noted, the frequency distribution $[1/\ddot{s}_{\overline{t}|} = v^t/\ddot{a}_{\overline{t}|},\ {}_{t-1|}q_x]$ essentially considers a portfolio consisting of one single policy, whereas "groups of policies require consideration too." To extend this univariate discrete frequency distribution to handle multipolicy portfolios, we shall find it convenient to define two operators on discrete bivariate frequency distributions. For this purpose, assume that the matrix

$$[x_i, y_i, p_i]$$

represents a discrete bivariate frequency distribution, where i refers to the number of the row (or line); x_i and y_i are real numbers; and p_i is the probability or frequency of occurrence of the particular combination of x_i and y_i.

The binary operator is "Convolute for Sums," designated by $\oplus$:

$$[x_i^{(1)}, y_i^{(1)}, p_i^{(1)}] \oplus [x_j^{(2)}, y_j^{(2)}, p_j^{(2)}] = [x_i^{(1)} + x_j^{(2)}, y_i^{(1)} + y_j^{(2)}, p_i^{(1)} p_j^{(2)}] ,$$

where i assumes each integer value from 1 to the number of lines in the first matrix; and, for each such value of i, j assumes each integer value from 1 to the number of lines in the second matrix; thus the resulting matrix is obtained by calculating the triplet of values $(x_i^{(1)} + x_j^{(2)}, y_i^{(1)} + y_j^{(2)}, p_i p_j)$ for each combination of i and j. The superscripts (1) and (2) merely indicate whether the value originates from the first or the second matrix.

The unary operator is "Transform," designated by $\rightarrow$:

$$[x_i, y_i, p_i] \xrightarrow{x/y \rightarrow z} [z_i, p_i] ,$$

Now consider the following discrete bivariate frequency distribution:

$$B_0 = [v^t, \ddot{a}_{\overline{t|}}, {}_{t-1|}q_x] ,$$

where $t = 1, 2, \ldots, \omega - x$ and represents both a policy year and the line (row) in the matrix. Then define

$$B_n = \underbrace{B_0 \oplus B_0 \oplus \ldots \oplus B_0}_{n \text{ times}} ,$$

which is well defined since the Convolute for Sums operator $\oplus$ is associative (as can be easily verified).

Transform B_n $(= [x, y, p])$ from a bivariate (2-dimensional) frequency distribution into a univariate (1-dimensional) frequency distribution C_n $(= [z, p])$:

$$B_n \xrightarrow{x/y \rightarrow z} C_n ,$$

and sort the rows of C_n into order by z. Calculating the implied cumulative frequency, we have a univariate frequency distribution of net premiums for a portfolio of n identical ordinary life policies issued at age x. Table 1 of this discussion shows mean values, standard deviations, and nearest percentiles of C_n, for some selected values of n and the following assump-

TABLE 1

NET PREMIUM PERCENTILES

(Per $1,000 of Insurance)

NUMBER (n) OF POLICIES	MEAN*	STANDARD DEVIATION*	NEAREST PERCENTILE														
			.0001	.001	.01	.10	.20	.30	.40	.50	.60	.70	.80	.90	.99	.999	.9999
1.......	12.894	51.188	1.26	1.32	1.54	2.43	2.99	3.49	4.09	5.07	5.96	7.45	9.94	17.42	140.02	464.58	952.38
2.......	8.651	9.727	1.70	1.70	2.16	3.15	3.68	4.19	4.72	5.79	6.83	7.89	10.50	15.82	47.03	84.66	227.04
4.......	8.082	5.009	1.79	2.22	2.70	3.72	4.23	4.75	5.80	6.33	7.38	8.45	10.02	14.26	25.92	41.76	59.18
16......	7.787	2.198	3.37	3.74	4.27	5.34	5.91	6.39	6.88	7.38	7.96	8.61	9.44	10.69	14.47	17.94	21.41
64......	7.723	1.069	4.82	5.15	5.62	6.42	6.80	7.10	7.36	7.62	7.90	8.20	8.58	9.13	10.58	11.80	12.91
256.....	7.708	0.531	6.00	6.25	6.57	7.04	7.25	7.41	7.55	7.68	7.82	7.96	8.15	8.40	9.04	9.54	9.97
1,024...	7.704	0.265	6.79	6.93	7.11	7.37	7.48	7.56	7.63	7.70	7.76	7.84	7.92	8.04	8.34	8.57	8.76
4,096...	7.703	0.132	7.23	7.30	7.40	7.53	7.59	7.63	7.67	7.70	7.74	7.77	7.81	7.87	8.02	8.12	8.21
16,394..	7.703	0.066	7.46	7.50	7.55	7.62	7.65	7.67	7.69	7.70	7.72	7.74	7.76	7.79	7.86	7.91	7.95
65,536	7.703	0.033	7.58	7.60	7.63	7.66	7.67	7.69	7.69	7.70	7.71	7.72	7.73	7.75	7.78	7.81	7.83

* Note both that the mean approaches $1{,}000P_x = 7.70$ and that the standard deviation approaches zero, in each case as n approaches ∞.

tions: $x = 25$; interest rate of 5 per cent; and probabilities of death according to the 1958 CSO Mortality Table (age last birthday).

If we are given a fixed interest rate and a fixed probability of death for each issue age/policy year combination, and if the benefit premium has been established, we can proceed to calculate frequency distributions of benefit reserves, starting with the discrete univariate frequency distributions:

$$D_0 = |\{v^s - P\ddot{a}_{\overline{s}|}, {}_{s-1}|q_{x+t}\}_{s=1}^{w-x-t}| .$$

Using a univariate Convolute for Sums operator $\oplus$,[3] we could then produce

$$D_n = \underbrace{D_0 \oplus D_0 \oplus \ldots \oplus D_0}_{n \text{ times}} .$$

D_n would be sorted into order by reserve amount and the implied cumulative frequencies calculated, and the result would be a univariate frequency distribution of aggregate reserves for a portfolio of n ordinary life policies issued at age x. Given a valuation net premium for each policy, a separate D_0 could be constructed for each such policy. The univariate frequency distributions could be convoluted for sums ($\oplus$) to obtain frequency distributions of aggregate reserves for a whole portfolio of dissimilar policies.

We could build in a frequency distribution of mortality tables (e.g., taking various percentages of a given mortality table) by performing the above calculations for each mortality table separately and then multiplying each of the resulting frequencies by the probability attached to that particular mortality table. This would be treating the policies as partially dependent with respect to mortality. Similarly, we could consider a frequency distribution of interest rates, perform the above calculations for each interest rate separately, and then multiply each of the resulting frequencies by the probability attached to that particular interest rate. (The interest rates would not have to be level by policy year.)

Further Remarks

The questions and objections raised early in Mr. Halmstad's discussion are answered by simply changing the function in his column 5 from $(1 + i)^t A^{1}_{x:\overline{t}|}$ to $\Sigma_{r=1}^{t}\ [q_{x+r-1}(1 + i)^{t-r}]$. His reference to treatment of endowment policies would be significant only if we did not follow each policy to its final disappearance from the portfolio, which we do.

Mr. Halmstad is critical of the efficiency of the Monte Carlo simulation

[3] $[x_i^{(1)}, p_i^{(1)}] \oplus [x_j^{(2)}, p_j^{(2)}] = [x_i^{(1)} + x_j^{(2)}, p_i^{(1)} p_j^{(2)}]$, where i assumes each integer value from 1 to the number of lines in the first matrix; and, for each value of i, j assumes each integer value from 1 to the number of lines in the second matrix.

procedures outlined in the paper. I used the Monte Carlo approach only as a method for describing what would, in effect, be accomplished by the convolution technique described in this discussion.

Mr. Halmstad's intuitive ideas about streams of new-money rates, mortality rates, lapse rates, and so on, seem quite close, in concept, to the convolution approaches described above. However, use of "fast Fourier transforms and heavy numerical analysis" are unnecessary. In fact, use of fast Fourier transforms would probably require that the frequency f (corresponding to an amount x) be expressible as a mathematical function of x, whereas the convolution techniques described above are not restricted in this way.

With regard to Mr. Halmstad's final three points:

1. My reference to assuming new-money rates to be at least partially dependent upon the new-money rate experienced in the immediately preceding year was merely to indicate that such a dependency could easily be built into the process; the assumption that new-money rates are independent from year to year would be even easier to handle.
2. Near the end of the paper I indicated that dependent relationships between mortality rates, lapse rates, and so on, can be built into the process.
3. With respect to distributions of mortality rates within a particular underwriting class, references [1] and [2] below may be of interest, since each considers the analysis of mortality by classes.

My thanks and appreciation to Mr. Halmstad and Mr. Nesbitt for trying to understand the paper and for giving us the benefit of their thoughts.

Questions for Further Discussion

1. Under what, if any, circumstances is it appropriate to Convolute for Sums ($\oplus$) either of the following bivariate frequency distributions?

$$[v^t, \ddot{a}_{\overline{t|}}, {}_{t-1}|q_x] ; \tag{1a}$$

$$\left[\sum_{r=1}^{t} (v^r q_{x+r-1}), \ddot{a}_{\overline{t|}}, {}_{t-1}|q_x\right] . \tag{1b}$$

2. Under what, if any, circumstances are either of the following assumptions for $[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}]$ appropriate in connection with the convolution operator (CI) defined previously in this discussion?

$$\begin{array}{ll} [1.00, q_{x+t-1}, 1.0] & \text{where } p_l^{(4)} = p_i^{(1)} p_j^{(2)} p_k^{(3)} y_k^{(3)} , \\ [0.00, q_{x+t-1}, 1.0] & \text{where } p_l^{(4)} = p_i^{(1)} p_j^{(2)} p_k^{(3)} (1 - y_k^{(3)}) \end{array} \tag{2a}$$

$$[1 \cdot q_{x+t-1}, q_{x+t-1}, 1.0] . \tag{2b}$$

3. Should the values of $x_k^{(3)}$ and $y_k^{(3)}$ in the frequency distribution $[x_k^{(3)}, y_k^{(3)}, p_k^{(3)}]$ be constructed from mortality rates based on amounts of insurance and numbers of policies (or numbers of lives), respectively?

4. Must the net valuation premiums be set at the same k per cent confidence level used for reserve purposes? Further, should net valuation premiums be set on some mean-value basis, without reference to frequency distributions, even though reserves are approached using frequency distributions?

5. Does use of the frequency distribution $[1/\ddot{s}_{\overline{t}|}, {}_{t-1|}q_x]$ (or its 2-d extension) imply a closed portfolio?

6. Does the approach described in the paper (and amplified in this discussion) imply an open portfolio?

7. Is it possible to use a convolution approach to construct a frequency distribution of net valuation premiums for a portfolio of dissimilar policies?

REFERENCES

1. LEVINSON, LOUIS. "A Theory of Mortality Classes," *TSA*, XI, 46.
2. ZIOCK, RICHARD. "Gross Premiums for Term Insurance with Varying Benefits and Premiums," *TSA*, XXII, 19.