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MATHEMATICAL ANALYSIS OF "THE LIFE INSURANCE COMPANY INCOME TAX ACT OF 1959" REVISITED

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ABSTRACT

This paper examines two interrelated aspects of federal income taxation of life insurance companies. First, a common-sense approach to the Life Insurance Company Income Tax Act of 1959 will be developed. Second, marginal tax rates for Phase III companies will be developed, utilizing the symbols, terminology, format, and techniques described by Mr. John Fraser in a paper entitled "Mathematical Analysis of Phase 1 and Phase 2 of 'The Life Insurance Company Income Tax Act of 1959'" (*TSA*, XIV, 51).

The paper is divided into four sections: Section I, a common-sense description of the tax formula; Section II, a derivation of the various tax situations of Phase III; Section III, a mathematical analysis of Phase III; and Section IV, an illustration of how the results obtained may be used by life insurance companies.

I. THE TAX FORMULA

THE tax imposed under the current tax law is (1) a regular corporate tax of 48 per cent on "taxable income," less (2) a surtax exemption of 26 per cent of $25,000 (in most cases), plus (3) usually a tax of 30 per cent on the excess, if any, of net long-term capital gains over net short-term capital losses.

Traditional Life Insurance Federal Income Tax (FIT) Development

The above tax rates are applicable to life insurance companies as well as to regular corporations. However, substantial complications are added because of the unique definition of taxable income for life insurance companies, namely, the lesser of "taxable investment income" and "gain from operations," plus 50 per cent of the excess, if any, of gain from operations over taxable investment income, plus the amount, if any, subtracted from the "policyholders' surplus account" (PSA), utilizing the following definitions:

1. "Investment yield" is basically the company's statutory net investment income before FIT.

2. "Taxable investment income" equals investment yield less
   a) Nonpension reserve deduction.
   b) Pension reserve deduction.
   c) Interest-paid deduction.
   d) Tax-exempt income deduction.
   e) Small-business deduction.
3. "Gain from operations" equals statutory gain from operations before FIT and dividends to policyholders, less the following:
   a) Tax-exempt income deduction (this is different from the tax-exempt income deductions used in arriving at taxable investment income).
   b) Increase or decrease in the change in reserves because of a change in valuation method or making the preliminary term election.
   c) Small-business deduction.
   d) Special deduction for certain nonparticipating contracts, subject to limitation.
   e) Special deduction for certain accident and health (A&H) insurance and group life insurance, subject to limitation.
   f) Dividends to policyholders, subject to limitation.
   g) Any other minor changes which conceivably could modify the statutory gain from operations.

In the discussion that follows, no attempt will be made to deal with operations loss carry-over or carry-back, capital gains tax, preliminary term election, variable annuities, and similar subjects. However, in Section III capital gains will be treated.

Special Limitations

There are several limitations on tax return variables which complicate any simple definition of taxable income.

1. Dividends to policyholders and the special deductions for nonparticipating contracts and A&H/group life contracts are subject to the limitation that the sum of these three deductions must not exceed $250,000 plus the excess, if any, of gain from operations before application of these deductions over taxable investment income. This limitation applies to reduce first the amount of the nonparticipating deduction, next the A&H/group life deduction, and finally the dividends to policyholders.
2. The PSA is limited to the largest of the following:
   a) 15 per cent of life insurance reserves at the end of the taxable year,
   b) 25 per cent of the amount by which life insurance reserves at the end of the taxable year exceed life insurance reserves at the end of 1958, and
   c) 50 per cent of the net premiums and other considerations taken into account in the calculation of gain from operations for the current taxable year.

   If the PSA exceeds the largest of these limitations, the excess must be withdrawn and is currently taxed.

   To understand how the PSA can exceed these limitations, it is necessary to see how the PSA is incremented each year. The PSA was established with a zero balance on January 1, 1959. It is increased each year by

a) 50 per cent of the excess, if any, of the gain from operations over taxable investment income, plus

b) The sum of the special deductions for nonparticipating contracts and A&H/group life contracts, after application of the aforementioned limitations.

It is decreased each year by (these are the amounts subtracted from the PSA which enter into taxable income):

a) Any distributions to stockholders in excess of the year-end "shareholders' surplus account" (SSA) (for example, dividends to stockholders) plus the tax thereon, plus

b) Any involuntary withdrawals due to exceeding the limitations previously mentioned, plus

c) Any involuntary withdrawals due to termination as a life insurance company.

Thus, one-half the gain from underwriting, plus the special deductions, are not taxed currently; instead, tax is deferred thereon until such time as any of the three mentioned decreases from the PSA occurs.

It should be noted that the PSA is established by stock life insurance companies only. Hence a mutual company never incurs a Phase III tax, since there is no PSA from which to withdraw funds.

3. Dividends to stockholders are not limited; however, an additional tax will be imposed on a company if the distributions to stockholders are greater than the funds held in the SSA.

   The SSA was established with a zero balance on January 1, 1958. It is increased each year by

   a) Any subtractions from the PSA less the tax thereon, plus

   b) Taxable income for the year, excluding any amounts subtracted from the PSA, plus

   c) The excess, if any, of net long-term capital gains over net short-term capital losses, plus

   d) The full amount of tax-exempt interest and the dividends-received deduction (that is, 85 per cent of dividends received, etc.), plus

   e) The small-business deduction.

   It is decreased each year by:

   a) The tax liability on items *b* and *c* above, plus

   b) The distributions to stockholders.

Four Different Tax Situations

The effect on these special limitations is to create four distinct tax situations. However, before detailing these four situations, it is convenient to define the following quantities:

I = Taxable investment income;

D = Dividends to policyholders and special deductions for nonparticipating and A&H/group life contracts, before application of limitation;

D^P = Deductible portion of policyholder dividends;

D^O = Deductible portion of special deductions for nonparticipating and A&H/group life contracts;

$D' = D^P + D^O = D$ after application of limitation;

G = Gain from operations before deduction of items in D above;

W = Amount withdrawn from PSA (the sum of the three decreases previously mentioned);

P_t = PSA at end of year t, $t = 0$ being the end of the current year.

The following tabulation illustrates the results obtained under the traditional analysis of these four situations.

Situation	Taxable Income Equals
A. $G - I < 0$	$G - \$250{,}000$*
B. $0 < G - I < D - \$250{,}000$	$I - \$250{,}000$
C. $D - \$250{,}000 < G - I < D$	$G - D$
D. $D < G - I$	$\frac{1}{2}(I + G - D)$

* Use D instead of \$250,000 if $D <$ \$250,000. Note that if $D <$ \$250,000, Situations A and C are identical and Situation B cannot exist.

This analysis seems to indicate that similar companies may have completely dissimilar FIT bases. The divorcing of Phase III considerations from the normal taxation formulas leads to the notion that most companies should not be concerned with Phase III, and, even if they are, it is next to impossible to analyze the situation. Hence the almost universal belief that life insurance federal income taxation is something mystical, to be understood only by actuaries and experts in the taxation profession. This belief can be dispelled through utilization of the common-sense approach.

Common-Sense Approach to FIT Development

A regular corporation is taxed on its gain from operations. This is a reasonable method, considering that most corporations complete their transactions within a relatively short period of time, and hence the current year's gain from operations is considered a very close approximation to what actually happened during the year. This carries over into fire and casualty insurance companies, which are engaged in short-term insurance activities (many of their policies are renewable on an annual basis) and as such are taxed as regular corporations.

Life insurance companies, however, are engaged in the long-term insurance business, and because of the large reserves which must be accumulated to provide for future contingencies (which can only be estimated), a modification of the regular corporate tax structure is necessary. Because the current year's earnings should be considered only as an estimate of the true earnings for the year due to the potential radical fluctuations in mortality and interest rates, the federal government allows a life insurance company to accumulate a contingency fund to protect the policyholders, that is, the PSA. The increase in this fund from one year to the next is a deduction from the gain from operations, as is the case with any other reserve increase. Hence the funds set aside in the PSA are not taxed currently. The funds in the PSA are taxed only on the occurrence of one of the following events:

1. When the prescribed limitations on this contingency fund are exceeded. The 50 per cent of net premium limitation was established for the benefit of insurance companies heavily involved in group insurance and A&H insurance, where reserves do not grow appreciably. The 15 per cent of life reserves limit was established for the benefit of the older insurance companies involved in traditional types of business. The 25 per cent of the increase in life reserves limit was established for the benefit of the newer insurance companies. The company is taxed on the excess of the PSA over the applicable limitation.
2. When the insurance company's actions indicate that the fund is no longer needed. Such actions are paying stockholder dividends out of the PSA, voluntarily transferring funds from the PSA to the SSA, or termination as a life insurance company. Only 52 per cent of the amount transferred from the PSA is transferred to the SSA. The remaining 48 per cent is the tax payable due to the withdrawal from the PSA. Similarly, the amount of dividends paid to stockholders is only 52 per cent of the required withdrawal from the PSA. If a company does not qualify as a life insurance company (for tax purposes and for a period of two consecutive years), the entire PSA is taxed.

The previous discussion indicates that, for a life insurance company, taxable income equals

1. Gain from operations before making the special deductions for nonparticipating and A&H/group life contracts, less
2. Increase in the PSA after application of the limitations.

In order to perform an analysis of the tax situations, it is necessary to translate the preceding comments into symbols. Thus taxable income equals $G - D^P - (P_0 - P_{-1})$, where the change in the PSA is

$$\begin{aligned} P_0 = P_{-1} & \\ & + \tfrac{1}{2}(G - D' - I) \quad (\geq 0) \\ & + D^O \\ & - W\,, \end{aligned}$$

and D', the deductible portion of D, equals the lesser of

$$D \quad \text{and} \quad \begin{bmatrix} \$250{,}000 \text{ plus} \\ (G - I)(\geq 0) \end{bmatrix} .$$

Therefore, the basic tax for a life insurance company is

$$T_0 = 0.48[G - D^P - (P_0 - P_{-1})] .$$

Because any one year's gain from underwriting is considered as only an estimate, the IRS has declared that only 50 per cent will be taxed, while the other 50 per cent is set aside until it is determined that it is no longer necessary to hold these funds for the safety of the policyholders. Since mutual companies and stock companies issuing a substantial amount of participating business can decrease their tax liability by increasing dividends, it was necessary to limit the amount of dividends which could be deducted from income and at the same time to create special deductions (that is, nonparticipating and A&H/group life deductions) for the benefit of companies not heavily engaged in participating business. These special deductions and limitations thereon are the result of the government's attempt to equalize the FIT for all life insurance companies.

Now consider the previous four tax situations utilizing the common-sense formulas.

Situation A. $G - I < 0$:

$$D' = \text{Lesser of } \$250{,}000 \text{ and } D$$

and

$$P_0 - P_{-1} = D^O - W .$$

Hence

$$\begin{aligned} T_0^A &= 0.48(G - D^P - D^O + W) \\ &= 0.48(G - D' + W) \\ &= 0.48(G - D') \quad \text{where } W = 0 . \end{aligned}$$

Situation B. $0 < G - I < D - \$250{,}000$:

$$D' = G - I + \$250{,}000$$

and

$$P_0 - P_{-1} = D^O - W .$$

Hence

$$\begin{aligned} T_0^B &= 0.48(G - D^P - D^O + W) \\ &= 0.48(G - D' + W) \\ &= 0.48(G - G + I - \$250{,}000 + W) \\ &= 0.48(I - \$250{,}000 + W) \\ &= 0.48(I - \$250{,}000) \quad \text{where } W = 0 . \end{aligned}$$

Situation C. $D - \$250{,}000 < G - I < D$:

$$D' = D$$

and

$$P_0 - P_{-1} = D^O - W\,.$$

Hence

$$T_0^C = 0.48(G - D^P - D^O + W)$$

$$= 0.48(G - D + W)$$

$$= 0.48(G - D) \quad \text{where } W = 0\,.$$

Situation D. $D < G - I$:

$$D' = D$$

and

$$P_0 - P_{-1} = \tfrac{1}{2}(G - D - I) + D^O - W\,.$$

Hence

$$T_0^D = 0.48[G - D^P - \tfrac{1}{2}(G - D - I) - D^O + W]$$

$$= 0.48[\tfrac{1}{2}G + \tfrac{1}{2}D + \tfrac{1}{2}I - D + W]$$

$$= 0.48[\tfrac{1}{2}(G - D + I) + W]$$

$$= 0.48(\tfrac{1}{2})(G - D + I) \quad \text{where } W = 0\,.$$

If $D < \$250{,}000$, Situations A and C are the same and Situation B cannot exist.

This demonstrates that the common-sense approach will yield the same results as the traditional approach. But this common-sense approach also leads to a greater understanding of the FIT on life insurance companies. Consider again the equation

$$T = 0.48[G - D^P - (P_0 - P_{-1})]\,,$$

and assume for the moment that there are no voluntary withdrawals from the PSA and that loss carry-overs and carry-backs are excluded.

a) If $P_0 = P_{-1}$, then the tax rate is 48 per cent. This is the case where a company is in Phase III and does not have any growth in the variable which determines the limitation on the PSA (that is, premiums or reserves, whichever are applicable). This is also the case where a company is in Situation A, B, or C and $D^O = 0$.

b) If $P_0 = 0$, then the tax rate approaches infinity as $G - D^P$ approaches zero. It would become infinite in the case where a company ceases to be a life insurance company and has no positive income either.

c) If $I = 0$ and $D^O = 0$, then $P_0 - P_{-1} = \tfrac{1}{2}(G - D^P)$; hence the tax rate is 24 per cent. However, if $D^O > 0$, the tax rate approaches 0 per cent as D^O approaches $G - D^P$.

Hence, under specialized conditions the tax rate can vary from zero to infinity.

Many Situation A, B, and C companies will have tax rates in the 40–48 per cent range depending on the size of D^O, and many Situation D companies will have tax rates ranging from 24 to 35 per cent depending on the size of I. A Phase III company with no growth (as above) will have a tax rate of 48 per cent; with moderate growth, 40–48 per cent; and with negative growth, greater than 48 per cent.

II. SEVEN TAX SITUATIONS OF PHASE III

If the situation in which a life insurance company terminates as either an insurance company or a life company is disregarded as being of little interest to the vast majority of life insurance companies, Phase III then gives rise to seven distinct tax situations. The first four situations arise as the result of voluntary withdrawals from the PSA, either by distribution to stockholders or by voluntary transfers of funds from the PSA to the SSA. This implies that W is not dependent on any other tax variable. The final three situations are due to involuntary withdrawals from the PSA due to the PSA's exceeding the prescribed limitations. This implies that W is dependent on one or more of the basic tax variables.

Three terms need to be defined before the tax situations under Phase III are derived (Sec. III defines these variables in more detail):

C = Net premiums and other considerations taken into account in the calculation of gain from operations for the current taxable year;

V^T = Total life reserves at the end of the current taxable year;

$V^{T(58)}$ = Total life reserves at the end of 1958.

Situation A. $G - I < 0$. As previously shown,

$$T_0^A = 0.48(G - D' + W) ,$$

where D' is the lesser of D and \$250,000.

Situation B. $0 < G - I < D - \$250{,}000$. As previously shown,

$$T_0^B = 0.48(I - \$250{,}000 + W) .$$

Situation C. $D - \$250{,}000 < G - I < D$. As previously shown,

$$T_0^C = 0.48(G - D + W) .$$

Situation D. $D < G - I$. As previously shown,

$$T_0^D = 0.48[\tfrac{1}{2}(G - D + I) + W] .$$

Situation E. $P_0 = \frac{1}{2}C$ (limitation c):

$$T_0^E = 0.48[G - D^P - (P_0 - P_{-1})]$$
$$= 0.48[G - D^P - (\tfrac{1}{2}C - P_{-1})]$$
$$= 0.48(G - D^P - \tfrac{1}{2}C + P_{-1})\ .^1$$

Situation F. $P_0 = 0.15\ V^T$ (limitation a):

$$T_0^F = 0.48[G - D^P - (P_0 - P_{-1})$$
$$= 0.48[G - D^P - (0.15\,V^T - P_{-1})]$$
$$= 0.48(G - D^P - 0.15\,V^T + P_{-1})\ .^1$$

Situation G. $P_0 = 0.25(V^T - V^{T(58)})$ (limitation b):

$$T_0^G = 0.48[G - D^P - (P_0 - P_{-1})]$$
$$= 0.48[G - D^P - (0.25\,V^T - 0.25\,V^{T(58)} - P_{-1})]$$
$$= 0.48(G - D^P - 0.25\,V^T + 0.25\,V^{T(58)} + P_{-1})\ .^1$$

An obvious question would be, "Aren't there really four situations in each of Situations E, F, and G, that is, where $G - I < 0$, $D < G - I$, $0 < G - I < D - \$250{,}000$, $D - \$250{,}000 < G - I < D$?" The answer is no, if the assumption is made that D^P is independent of G and I (see the following section for the rationale behind this assumption). These four conditions all lead to the same result in Situations E, F, and G.

For example, suppose that $0 < G - I < D - \$250{,}000$ and $P_0 = \frac{1}{2}C$.

$$D' = G - I + \$250{,}000 = D^P + D^O$$

and

$$W = P_{-1} - \tfrac{1}{2}C + D^O\ .$$

Hence

$$T_0^E = 0.48(I - \$250{,}000 + W)$$
$$= 0.48(I - \$250{,}000 + P_{-1} - \tfrac{1}{2}C + D^O)\ .$$

But

$$D^O = G - I + \$250{,}000 - D^P\ ,$$

so

$$T_0^E = 0.48(I - \$250{,}000 + P_{-1} - \tfrac{1}{2}C + G - I + \$250{,}000 - D^P)$$
$$= 0.48(P_{-1} - \tfrac{1}{2}C + G - D^P)\ .$$

This kind of analysis can be done for each set of conditions under Situations E, F, and G.

[1] Note that P_0 is no longer a function of I and D^O, hence $T_0^{E,F,G}$ are not functions of I and D^O.

Assumption Associated with D^P

In the situation where $0 < G - I < D - \$250{,}000$ and the company is in Phase III (Situation E, F, or G), D^P may or may not be dependent on G and I, since $D' = D^P + D^O = G - I + \$250{,}000$. However, the likelihood of a situation where D^P is dependent on G and I is very remote for two reasons. First, it is doubtful that there will be many companies that are in Phase III (Situations E, F, and G) with the condition $0 < G - I < D - \$250{,}000$, because the only addition to the PSA under this condition is D^O, which will definitely be limited in magnitude each year. Second, D^P is the last of three special deductions to be limited or reduced; the nonparticipating and A&H/group life deductions will be reduced or eliminated first.

Under the condition $G - I < 0$, D^P will be equal to dividends to policyholders or $250,000, whichever is less. Each individual company surely will know which of these amounts will be applicable in any given year, unless they are in a transitional period.

III. MATHEMATICAL ANALYSIS OF PHASE III

Definitions

In order to calculate marginal tax rates for the seven situations of Phase III, it is first necessary to define several variables in addition to those shown on pages 265–66 and 270 of this paper. Many of these definitions are identical with those in Fraser's paper; however, a few additional variables have been added.

I^{NT} = Tax-exempt investment yield in current year, including tax-exempt portion of stock dividends and partially tax-exempt interest;

I^T = Fully taxable investment yield in current year, including the portion of stock dividends and partially tax-exempt interest not included in I^{NT};

$h = I^T/(I^T + I^{NT})$ = Ratio of fully taxable investment yield to total investment yield in current year;

V_k^{NP} = Mean nonpension reserves valued at rate t_k in current taxable year (including qualified reserves not included in pension reserves because of "grade-in" period), adjusted to a tax basis by eliminating deficiency reserves and so on, but before application of "10 for 1" rule (assumed to be all qualified life reserves);

$V^{NP} = \Sigma_{k=1}^{n} V_k^{NP}$ = Total mean nonpension reserves in current taxable year assuming n different valuation rates t_k for nonpension reserves;

V_k^P = Mean pension reserves valued at rate t_k in current taxable year (excluding amounts included in V_k^{NP} during "grade-in" period), adjusted to a tax basis;

V^P = $\Sigma_{k=1}^{m} V_k^P$ = Total mean pension reserves in current taxable year assuming m different valuation rates t_k for pension reserves;

V^T = $V_0^L + V_0^O$ = Total life reserves at the end of the current taxable year;

V_t^O = Total pension reserves at the end of year t, $t = 0$ being the end of the current year;

$V^{T(58)}$ = Total life reserves at the end of 1958;

V_t^L = Total nonpension reserves at the end of year t, $t = 0$ being the end of the current year;

i^{NP} = Average valuation interest rate on total nonpension reserves in current year; note that

$$i^{NP} = \frac{1}{V^{NP}} \sum_{k=1}^{n} t_k V_k^{NP} ;$$

i^P = Average valuation interest rate on total pension reserves in current year; note that

$$i^P = \frac{1}{V^P} \sum_{k=1}^{m} t_k V_k^P ;$$

B' = "Interest paid" deduction for current year with respect to contracts and supplemental funds not involving life contingencies;

B'' = "Interest paid" deduction for current year with respect to interest on indebtedness and other items which are deductible in determining taxable investment income but are not included in the "share of investment yield set aside for policyholders" used in determining the gain from operations;

G' = Gains for current taxable year indicated by G exclusive of investment yield $(I^T + I^{NT})$, before deduction of interest paid B' (which is part of the payments and reserves increases on contracts and supplemental funds not involving life contingencies), before deduction of interest paid B'', and before making the tax-exempt and small-business deductions; thus G' equals (*a*) premiums and other operating income (except for investment yield) less (*b*) claims, insurance expenses, reserve increases, and other allowable deductions (except for B', B'', D, and the tax-exempt and small-business deductions);

F = Foreign tax credit in current year;

C = Gross premiums and other considerations on insurance and annuity contracts, less return premiums and premiums and other considerations arising out of reinsurance ceded taken into account in the calculation of gain from operations for the current taxable year;

O = All other income and expense items taken into account in the calculation of gain from operations for the current taxable year, such that $G' = C - O$;

$r = (i^{NP}V^{NP} + i^PV^P + B')/(I^T + I^{NT})$;

S = Small-business deduction.

Analysis of Tax in Situations A, B, C, and D of Phase III

These are the four situations in which the withdrawals from the PSA (that is, W) are not dependent on any other tax variable. Hence all the marginal tax rates developed by Fraser (adjusted to a 48 per cent tax rate) may continue to be used; however, one additional rate (in each situation) must be used as follows:

$$m_A^W = m_B^W = m_C^W = m_D^W = 0.48 .$$

This allows a company that voluntarily withdraws funds from the PSA to utilize its existing marginal tax rates, with but a single modification.

Analysis of Tax in Situation E of Phase III

This is the situation in which $P_0 = \frac{1}{2}C$, that is, the PSA is forced to equal one-half the net premiums and considerations for the current taxable year. We have seen that

$$T_0^E = 0.48(G - D^P - \tfrac{1}{2}C + P_{-1}) .$$

But

$$G = G' + I^T - B' - B'' - S + [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^PV^P + B')$$

and

$$G' = C - O .$$

Hence

$$\begin{aligned} T_0^E &= 0.48\{C - O + I^T - B' - B'' - S \\ &\quad + [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^PV^P + B') \\ &\quad - D^P - \tfrac{1}{2}C + P_{-1}\} \\ &= 0.48\{P_{-1} + \tfrac{1}{2}C - O - D^P + I^T - B' - B'' - S \\ &\quad + [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^PV^P + B')\} . \end{aligned}$$

The partial derivatives with respect to the basic tax variables are shown below.

$$\frac{\partial T_0^E}{\partial P_{-1}} = 0.48 = m_E^{P_{-1}} = \text{Marginal tax rate on } P_{-1} \text{ in Situation E},$$

$$\frac{\partial T_0^E}{\partial C} = 0.24 = m_E^C, \quad \frac{\partial T_0^E}{\partial O} = -0.48 = m_E^O, \quad \frac{\partial T_0^E}{\partial D^P} = -0.48 = m_E^{D^P},$$

$$\frac{\partial T_0^E}{\partial S} = -0.48 = m_E^S, \quad \frac{\partial T_0^E}{\partial B''} = -0.48 = m_E^{B''},$$

$$\frac{\partial T_0^E}{\partial V^{NP}} = 0.48 i^{NP}[I^{NT}/(I^T + I^{NT})] = 0.48 i^{NP}(1 - h) = m_E^{VP},$$

$$\frac{\partial T_0^E}{\partial V^P} = 0.48 i^P[I^{NT}/(I^T + I^{NT})] = 0.48 i^P(1 - h) = m_E^P,$$

$$\frac{\partial T_0^E}{\partial B'} = 0.48\{-1 + [I^{NT}/(I^T + I^{NT})]\} = -0.48h = m_E^{B'},$$

$$\begin{aligned}\frac{\partial T_0^E}{\partial I^T} &= 0.48\{1 - [I^{NT}/(I^T + I^{NT})^2](i^{NP}V^{NP} + i^P V^P + B')\} \\ &= 0.48\{1 - [(1 - h)/(I^T + I^{NT})](i^{NP}V^{NP} + i^P V^P + B')\} \\ &= 0.48[1 - (1 - h)r] = m_E^T,\end{aligned}$$

$$\begin{aligned}\frac{\partial T_0^E}{\partial I^{NT}} &= 0.48\{[1/(I^T + I^{NT})] - [I^{NT}/(I^T + I^{NT})^2]\} \\ &\quad \times (i^{NP}V^{NP} + i^P V^P + B') = 0.48hr = m_E^{NT}.\end{aligned}$$

Analysis of Tax in Situation F of Phase III

This is the situation in which $P_0 = 0.15V^T$, that is, the PSA is forced to equal 15 per cent of the life insurance reserves at the end of the current taxable year. We have seen that

$$T_0^F = 0.48(G - D^P - 0.15V^T + P_{-1}).$$

Hence

$$\begin{aligned}T_0^F = 0.48\{&C - O + I^T - B' - B'' - S \\ &+ [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^P V^P + B') \\ &- D^P - 0.15V_0^L - 0.15V_0^O + P_{-1}\}.\end{aligned}$$

The partial derivatives with respect to the basic tax variables are shown below.

$$\frac{\partial T_0^F}{\partial P_{-1}} = 0.48 = m_F^{P_{-1}}, \quad \frac{\partial T_0^F}{\partial C} = 0.48 = m_F^C,$$

$$\frac{\partial T_0^F}{\partial O} = -0.48 = m_F^O, \quad \frac{\partial T_0^F}{\partial D^P} = -0.48 = m_F^{D^P},$$

$$\frac{\partial T_0^F}{\partial S} = -0.48 = m_F^S, \quad \frac{\partial T_0^F}{\partial B''} = -0.48 = m_F^{B''},$$

$$\frac{\partial T_0^F}{\partial V^{NP}} = 0.48 i^{NP}[I^{NT}/(I^T + I^{NT})] = 0.48 i^{NP}(1 - h) = m_F^{VP},$$

$$\frac{\partial T_0^F}{\sigma V^P} = 0.48 i^P[I^{NT}/(I^T + I^{NT})]$$
$$= 0.48 i^P(1 - h) = m_F^P,$$

$$\frac{\partial T_0^F}{\partial B'} = 0.48\{-1 + [I^{NT}/(I^T + I^{NT})]\} = -0.48h = m_F^{B'},$$

$$\frac{\partial T_0^F}{\partial I^T} = 0.48\{1 - [I^{NT}/(I^T + I^{NT})^2](i^{NP}V^{NP} + i^P V^P + B')\}$$
$$= 0.48[1 - (1 - h)r] = m_F^T,$$

$$\frac{\partial T_0^F}{\partial I^{NT}} = 0.48\{[1/(I^T + I^{NT})] - [I^{NT}/(I^T + I^{NT})^2]\}$$
$$\times (i^{NP}V^{NP} + i^P V^P + B') = 0.48hr = m_F^{NT},$$

$$\frac{\partial T_0^F}{\partial V_0^L} = -0.72 = m_F^L, \quad \frac{\partial T_0^F}{\partial V_0^O} = -0.72 = m_F^V.$$

Analysis of Tax in Situation G of Phase III

This is the situation in which $P_0 = 0.25(V^T - V^{T(58)})$, that is, the PSA is forced to equal 25 per cent of the increase in year-end life insurance reserves since 1958. We have seen that

$$T_0^G = 0.48(G - D^P - 0.25V^T + 0.25V^{T(58)} + P_{-1}).$$

Hence

$$T_0^G = 0.48\{C - O + I^T - B' - B'' - S$$
$$+ [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^P V^P + B')$$
$$- D^P - 0.25V_0^L - 0.25V_0^O + 0.25V^{T(58)} + P_{-1}\}.$$

The partial derivatives with respect to the basic tax variables are shown below.

$$\frac{\partial T_0^G}{\partial P_{-1}} = 0.48 = m_G^{P-1}, \quad \frac{\partial T_0^G}{\partial C} = 0.48 = m_G^C,$$

$$\frac{\partial T_0^G}{\partial O} = -0.48 = m_G^O, \quad \frac{\partial T_0^G}{\partial D^P} = -0.48 = m_G^{D^P},$$

$$\frac{\partial T_0^G}{\partial S} = -0.48 = m_G^S, \quad \frac{\partial T_0^G}{\partial B''} = -0.48 = m_G^{B''},$$

$$\frac{\partial T_0^G}{\partial V^{NP}} = 0.48 i^{NP}[I^{NT}/(I^T + I^{NT})] = 0.48 i^{NP}(1 - h) = m_G^{NP},$$

$$\frac{\partial T_0^G}{\partial V^{T(58)}} = 0.48(0.25) = 0.12 = m_G^{T(58)},$$

$$\frac{\partial T_0^G}{\partial V^P} = 0.48 i^P[I^{NT}/(I^T + I^{NT})] = 0.48 i^P(1 - h) = m_G^P,$$

$$\frac{\partial T_0^G}{\partial B'} = 0.48\{[I^{NT}/(I^T + I^{NT})] - 1\} = -0.48h = m_G^{B'},$$

$$\frac{\partial T_0^G}{\partial I^T} = 0.48\{1 - [I^{NT}/(I^T + I^{NT})^2](i^{NP}V^{NP} + i^P V^P + B')\}$$
$$= 0.48[1 - (1 - h)r] = m_G^T,$$

$$\frac{\partial T_0^G}{\partial I^{NT}} = 0.48\{[1/(I^T + I^{NT})] - [I^{NT}/(I^T + I^{NT})^2]\}$$
$$\times (i^{NP}V^{NP} + i^P V^P + B') = 0.48hr = m_G^{NT},$$

$$\frac{\partial T_0^G}{\partial V_0^L} = -0.12 = m_G^L, \quad \frac{\partial T_0^G}{\partial V_0^O} = -0.12 = m_G^V.$$

Long-Term Capital Gains

Since the "alternate" tax calculation normally will produce a lower aggregate FIT liability, most insurance companies should use a marginal tax rate of 0.30 for the excess, if any, of long-term capital gains over short-term capital losses. However, for those companies which have a higher tax liability under the alternate computation, the marginal tax rate used should be 0.48. (This is due to inclusion of long-term gains in both taxable investment income and gain from operations.)

Surtax Exemption

The marginal tax rate for the surtax exemption is -0.26 unless the multiple surtax exemption is elected, in which case the marginal tax rate is -0.20. The normal surtax exemption is \$25,000, and the multiple surtax exemption is lower depending on the number of life insurance companies in the group.

Changes in Tax Situation

The marginal tax rates as developed do not anticipate any changes from one situation to another. The before-and-after technique is recommended in borderline situations where it is felt that a change in variables might cause a change in the tax situation.

An Observation on Tax Variables

In Phase III the variables C, O, V^T, and $V^{T(58)}$ become more important in determining marginal changes, because of the limitations of the PSA. However, in the current analysis and in Fraser's analysis, an obvious refinement has been left out. This deals with a new definition of G'. In all tax situations G' is dependent on premiums, nonpension reserve changes, pension reserve changes, and all other expenses. Hence, rather than using G', it would be more appropriate to use

$$C - E - (V_0^L - V_{-1}^L) - (V_0^O - V_{-1}^O) ,$$

where

$$V_0^{NP} = (V_0^L + V_{-1}^L)/2 \quad \text{and} \quad V_0^P = (V_0^O + V_{-1}^O)/2 ,$$

and

$$E = O - (V_0^L - V_{-1}^L) - (V_0^O - V_{-1}^O) .$$

This would also necessitate using

$$(V_0^L + V_{-1}^L)/2 \quad \text{and} \quad (V_0^O + V_{-1}^O)/2$$

instead of V_0^{NP} and V_0^P, respectively. Although this refinement might have some theoretical interest, it is hardly worthwhile from a practical point of view. Without this refinement, a company is still able to determine factors of equivalence for investments and to determine the change in tax liability given a change in the tax variables.

Properties of Tax Function

In Fraser's paper and the discussion which followed, the homogeneity of the tax function was mentioned. This property has led to the ease of utilization or marginal tax rates. The tax equation is homogeneous of degree 1 and has the property that

$$x_1 f_1 + x_2 f_2 + \ldots + x_n f_n = f(x_1, x_2, \ldots, x_n) .$$

It was also pointed out that f_i was homogeneous of degree zero, which means that its values remain unchanged if the x_i's change proportionately. But what happens if the x_i's do not change proportionately?

One obvious test of the stability of marginal tax rates would be to perform before-and-after tests, applying a variety of increments or decrements to the company's current tax variables. By trying enough combinations, the company could determine which changes in the tax variables will change the marginal tax rates the most and in which direction. However, one of the reasons for using marginal tax rates is to avoid doing before-and-after tests.

A better and more systematic method is to find an equation in the basic tax variables which will express the change in a marginal tax rate. This can be accomplished by using Taylor's theorem for as many variables as necessary. For example, Taylor's theorem in one variable is

$$f(x + h) = f(x) + hf'(x) + (h^2/2!)f''(x) + \dots .$$

Taylor's theorem in n variables is

$$\begin{aligned} f(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n) \\ = f(x_1, x_2, \dots, x_n)[1 + (h_1\partial/\partial x_1 + h_2\partial/\partial x_2 + \dots + h_n\partial/\partial x_n) \\ + (h_1\partial/\partial x_1 + h_2\partial/\partial x_2 + \dots + h_n\partial/\partial x_n)^2/2! \\ + (h_1\partial/\partial x_1 + h_2\partial/\partial x_2 + \dots + h_n\partial/\partial x_n)^3/3! + \dots] . \end{aligned}$$

As an example of what can be done, let us again consider Company A. Recall that the tax equation is

$$\begin{aligned} T_0^E = 0.48\{P_{-1} + \tfrac{1}{2}C - O - D^P + I^T - B' - B'' - S \\ + [I^{NT}/(I^T + I^{NT})](i^{NP}V^{NP} + i^P V^P + B')\} . \end{aligned}$$

The marginal tax rates $m_E^{P_{-1}}$, m_E^C, m_E^O, $m_E^{D^P}$, m_E^S, $m_E^{B''}$ all are constant and hence will never change. However, m_E^{NP}, m_E^P, $m_E^{B'}$, m_E^T, and m_E^{NT} will all change as the basic tax variables change.

1. The rate m_E^{NP} is a function of I^T and I^{NT} only. Hence, where $f^{NP}(I^T, I^{NT}) = m_E^{NP} = f^{NP}$, it follows that

$$\begin{aligned} f^{NP}(I^T + h_T, I^{NT} + h_{NT}) \\ = f^{NP}\Big[1 + \Big(h_T \frac{\partial}{\partial I^T} + h_{NT} \frac{\partial}{\partial I^{NT}}\Big) \\ + \frac{1}{2}\Big(h_T^2 \frac{\partial^2}{\partial I^{T\,2}} + 2h_T h_{NT} \frac{\partial^2}{\partial I^T \partial I^{NT}} + h_{NT}^2 \frac{\partial^2}{\partial I^{NT\,2}}\Big) + \dots\Big] . \end{aligned}$$

2. The rates m_E^P and $m_E^{B'}$ are also functions of I^T and I^{NT} only; hence the "new" marginal tax rates can be expressed as a function of the "old" marginal tax rates and the basic tax variables in exactly the same manner as in paragraph 1.

3. The rate m_E^{NT} is a function of I^T, I^{NT}, V^{NP}, V^P, and B'. Hence, where $f^{NT}(I^T, I^{NT}, V^{NP}, V^P, B') = m_E^{NT} = f^{NT}$, it follows that

$$
\begin{aligned}
&f^{NT}(I^T + h_T, I^{NT} + h_{NT}, V^{NP} + h_{NP}, V^P + h_P, B' + h_{B'}) \\
&= f^{NT}\Big[1 + \Big(h_T \frac{\partial}{\partial I^T} + h_{NT} \frac{\partial}{\partial I^{NT}} + h_{NP} \frac{\partial}{\partial V^{NP}} + h_P \frac{\partial}{\partial V^P} + h_{B'} \frac{\partial}{\partial B'}\Big) \\
&\quad + \frac{1}{2}\Big(h_T^2 \frac{\partial^2}{\partial I^{T\,2}} + h_{NT}^2 \frac{\partial^2}{\partial I^{NT\,2}} + h_{NP}^2 \frac{\partial^2}{\partial V^{NP\,2}} \\
&\qquad + h_P^2 \frac{\partial^2}{\partial V^{P\,2}} + h_{B'}^2 \frac{\partial^2}{\partial B'^{\,2}}\Big) \\
&\quad + h_T h_{NT} \frac{\partial^2}{\partial I^T \partial I^{NT}} + h_T h_{NP} \frac{\partial^2}{\partial I^T \partial V^{NP}} + h_T h_P \frac{\partial^2}{\partial I^T \partial V^P} \\
&\quad + h_T h_{B'} \frac{\partial^2}{\partial I^T \partial B'} + h_{NT} h_{NP} \frac{\partial^2}{\partial I^{NT} \partial V^{NP}} + h_{NT} h_P \frac{\partial^2}{\partial I^{NT} \partial V^P} \\
&\quad + h_{NT} h_{B'} \frac{\partial^2}{\partial I^{NT} \partial B'} + h_{NP} h_P \frac{\partial^2}{\partial V^{NP} \partial V^P} + h_{NP} h_{B'} \frac{\partial^2}{\partial V^{NP} \partial B'} \\
&\quad + h_P h_{B'} \frac{\partial^2}{\partial V^P \partial B'} + \ldots\Big].
\end{aligned}
$$

4. The rate m_E^T is also a function of I^T, I^{NT}, V^{NP}, V^P, and B'; hence the "new" marginal tax rate can be expressed as a function of the "old" marginal tax rate and the basic tax variables in exactly the same manner as in paragraph 3.

Please note that the smaller the change in the basic tax variables, the better this approximation will be. Although the stability of the marginal tax rates can be analyzed by mathematical techniques beyond the scope of this paper, the practicalities of the situation will lead most companies to utilize the before-and-after techniques to test for stability (if, indeed, any tests are made at all).

IV. MARGINAL TAX RATE USES FOR PHASE III COMPANIES

Consider Company A currently in Situation E and expecting to remain in Situation E indefinitely. The following are the basic data for Company A:

1. Mean assets (A)	\$11,400,000
2. Fully taxable investment yield (I^T)	190,000
3. Tax-exempt investment yield (I^{NT})	200,000
4. Total investment yield	390,000
5. Mean nonpension reserves at 3 per cent (assumed to be all life reserves) (V^{NP})	2,600,000
6. Mean pension reserves at 3 per cent (V^P)	0
7. Interest paid:	
Not involving (B')	15,000
On indebtedness, etc. (B'')	10,000
8. Premiums less certain deductions (G')	5,350,000
Premiums (C)	12,100,000
Certain deductions (O)	6,750,000
9. PSA at beginning of year (P_{-1})	4,670,000
10. Foreign tax credit (F)	100
11. Dividends to policyholders (D^P)	30,000
12. Other deductions (D^o)	242,000
13. Long-term capital gains (L)	75,000
14. Small-business deduction (S)	25,000
15. Surtax exemption	25,000

In Section III the following quantities were defined:

$$h = I^T/(I^T + I^{NT}) ,$$

$$r = (i^{NP}V^{NP} + i^P V^P + B')/(I^T + I^{NT}) .$$

Hence, for Company A,

$$h = \$190{,}000/\$390{,}000 = 0.4871795 ,$$

$$r = (\$78{,}000 + \$0 + \$15{,}000)/\$390{,}000 = 0.2384615 ,$$

where $i^{NP} = 0.03$. Therefore, the marginal tax rates for Company A are as follows:

$$m_E^{P_{-1}} = 0.48 , \quad m_E^C = 0.24 , \quad m_E^O = -0.48 ,$$

$$m_E^{D^P} = -0.48 , \quad m_E^S = -0.48 , \quad m_E^{B''} = -0.48 ,$$

$$m_E^{B'} = -0.48h = -0.2338462 ,$$

$$m_E^{NP} = 0.48i^{NP}(1 - h) = 0.0073846 ,$$

$$m_E^P = 0.48i^P(1 - h) = 0.0073846 ,$$

$$m_E^T = 0.48[1 - (1 - h)r] = 0.4213018 ,$$

$$m_E^{NT} = 0.48hr = 0.0557633 .$$

Using these marginal tax rates, a mathematical model of Company A's tax situation can be established (see Table 1). This model can now be used to measure the tax effect of changing any of the above basic tax variables. It should be noted that these tax variables may differ from annual statement figures; hence appropriate adjustments are in order before these marginal tax rates are applied. Fraser has cited numerous examples of how marginal tax rates can be used, and this paper accordingly will apply only a few of those examples to Company A in order that the reader may see the difference in the tax considerations for a Situation D and for a Situation E company. A slight inconsistency will develop, although not a material one, because Fraser used a 52 per cent tax rate while this paper uses 48 per cent.

Example 1 (Fraser, P. 84, Example 1)

Company A receives a large group annuity premium which affects financial operations as follows:

Tax Variable	Change	
Fully taxable investment yield	Higher by $	27,500
Tax-exempt investment yield	Higher by	21,250
Mean 3 per cent pension reserves	Higher by	1,000,000
Premiums less certain deductions (G')	Lower by	40,000
Assume premiums (C)*	Higher by	200,000
Assume deductions (O)†	Higher by	240,000

* Premiums in the year received will be the reserves received plus premiums received. For this purpose assume that the reserves received were $900,000.

† Deductions in the first year will be reserves received plus normal deductions.

The tax effect on Company A in the current taxable year is computed below.

Tax Variable		Tax Change	
Fully taxable investment yield	$ 27,500×	(0.4213018)=	$ 11,586
Tax-exempt investment yield	21,250×	(0.0557633)=	1,185
3% pension reserves	1,000,000×	(0.0073846)=	7,385
Premiums—first year	1,100,000×	(0.2400000)=	264,000
Deductions—first year	1,140,000×	(−0.4800000)=	−547,200
Change in tax in current year			$−263,044

Fraser's Company Z had an increase in the current tax of $794 because of the above transactions; however, Company A has a current-year tax savings (deferral might be a better word, depending on the probability of keeping this block of business for a long period of time) of $263,044. Fraser was correct in saying, "What is one company's meat, may very well be another company's poison."

It can be shown easily that in renewal years Company A will still save (defer) $47,044, in each year, if the above variables remain constant;

TABLE 1

MATHEMATICAL MODEL OF TAX OF COMPANY A

	Amount of Item (1)	Marginal Tax Rate (2)	Contribution of (1) to Tax [(1)×(2)] (3)
Assets (A)	$11,400,000	0 %	$ 0
Fully taxable yield (I^T)	190,000	42.13018	+ 80,047*
Tax-exempt yield (I^{NT})	200,000	5.57633	+ 11,153*
Nonpension reserves (V^{NP})	2,600,000	0.73846	+ 19,200
Pension reserves (V^P)	0	0.73846	0
Interest paid:			
(B')	15,000	− 23.38462	− 3,508
(B'')	10,000	− 48.00000	− 4,800
Premiums (C)	12,100,000	24.00000	+2,904,000
Certain deductions (O)	6,750,000	− 48.00000	−3,240,000
PSA—beginning (P_{-1})	4,670,000	48.00000	+2,241,600
Dividends to policyholders (D^P)	30,000	− 48.00000	− 14,400
Foreign tax credit (F)	100	−100.00000	− 100
Long-term capital gains (L)	75,000	30.00000	+ 22,500
Small-business deduction (S)	25,000	− 48.00000	− 12,000
Surtax exemption	25,000	− 26.00000	− 6,500
Net tax			$ 1,997,192

* The total of these two items must equal 48 per cent of I^T.

if they increase, there will be an even greater savings (deferral), but Company Z is saving about $140 in each renewal year.

Example 2 (*Fraser*, *P. 86*, *Example 3*)

Company A is considering qualifying its retirement plan for employees. The reserves are currently $1,000,000 (all figures will be one-tenth of Fraser's, since Company A does not have $10,000,000 of nonpension reserves currently) and are valued at 3 per cent. Qualification would increase deductible investment expenses by $5,000 and deductible insurance expenses by $25,000.

The tax effect on Company A in the current taxable year is computed below.

Change	Tax Effect
Due to lower fully taxable investment yield	$− 5,000× (0.4213018) =$− 2,107
Due to change in status of reserves:	
3% nonpension out	−1,000,000× (0.0073846) = − 7,385
3% pension in	+1,000,000× (0.0073846) = + 7,385
Due to higher insurance expenses	+ 25,000×(−0.4800000) = −12,000
Change in tax in current year	$−14,107

Fraser's Company Z experienced a present value of net change in taxes of $-9,643.

Example 3 (Fraser, P. 93)

Fraser calculated Company Z's factors of equivalence for wholly tax-exempt securities and for 85 per cent tax-exempt stocks; the results were 78.1 and 80.8 per cent, respectively.

For Company A the factor of equivalence for wholly tax-exempt securities is derived as follows:

$$X(1 - m_E^{NT}) = Y(1 - m_E^T) ,$$

$$X(1 - 0.0557633) = Y(1 - 0.4213018) .$$

Therefore,

$$X/Y = 0.5786982/0.9442367 = 61.3 \text{ per cent} .$$

The factor of equivalence for 85 per cent tax-exempt stocks is derived as follows:

$$X(1 - 0.15m_E^T - 0.85m_E^{NT}) = Y(1 - m_E^T) ,$$

$$X(1 - 0.0631953 - 0.0473988) = Y(1 - 0.4213018) .$$

Therefore,

$$X/Y = 0.5786982/0.8894059 = 65.1 \text{ per cent} .$$

TABLE 2

COMPARISON OF MARGINAL TAX RATES IN SITUATIONS E AND D

Tax Variables	Situation E P_{-1} = \$4,670,000	Situation D P_{-1} = \$2,670,000
Assets (A)	0 %	0.01277%
Fully taxable investment yield (I^T)	42.13018	41.44126
Tax-exempt investment yield (I^{NT})	5.57633	5.50280
Mean nonpension reserves (V^{NP})	0.73846	0.01846
Mean pension reserves (V^P)	0.73846	− 0.03065
Interest paid:		
(B')	− 23.38462	− 23.38462
(B'')	− 48.00000	− 35.69231
Premiums (C)	24.00000	24.00000
Certain deductions (O)	− 48.00000	− 24.00000
PSA at beginning of year (P_{-1})	48.00000	0
Foreign tax credit (F)	−100.00000	−100.00000
Dividends to policyholders (D^P)	− 48.00000	− 24.00000
Other deductions (D^o)	0	− 24.00000
Long-term capital gains (L)	30.00000	30.00000
Small-business deductions (S)	− 48.00000	− 48.00000
Surtax exemption	− 26.00000	− 26.00000
Total tax	*\$1,997,192*	*\$1,307,223*

Hence, wholly and partially tax-exempt securities are more beneficial to Company A than to Company Z, from the point of view of tax exemption.

Example 4

An interesting question is, "What would be the marginal tax rates for Company A if the PSA at the beginning of the current taxable year were such as to put Company A back into Situation D, say $2,670,000?" By using Fraser's marginal tax rates (adjusted to a 48 per cent tax) for a Situation D company where $i^c \geq i^a$ (assume $i^a = 0.03$ and $i^c = 0.0342$), the values shown in Table 2 obtain.

Table 2 indicates that factors of equivalence would be approximately the same for this company even if it were in Situation D. It also indicates that the emphasis placed on acquiring premiums at no gain or even a moderate loss of income to the company is lost in Situation D.