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A MULTIRISK STOCHASTIC PROCESS

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ABSTRACT

The subject of this paper is a mathematical model for insurance company risks formed by a linear combination of four stochastic processes. The first process models the evolution of claim patterns, recognizing both the random number of claims in a time period and the random nature of the claims. The other three processes serve as models for random deviations from assumptions about investment performance, operating expenses, and lapse expenses.

The paper has four purposes. First, we wish to share with readers refinements in the risk model published in four recent papers. Second, we wish to improve the model further to consider deviations in assumptions that have allowed for inflation. The third purpose is to illustrate the model with more realistic and detailed examples than were considered previously. The fourth purpose is to furnish tables and suggestions that readers can use in applying the multirisk model to planning projects for their companies.

I. THE MULTIRISK PROCESS

ASSUME that $\{X_i\}$ is a sequence of independent, identically distributed random variables with a common distribution function $P(x)$. Assume that $E(X_i) = p_1 > 0$. The X_i's represent the claims, and p_1 is the expected value of an individual claim. Consider a stochastic process $\{N(\tau),\ \tau \geq 0\}$ that models the random numbers of claims in time. We no longer require this to be a Poisson process. Instead, we assume that $\{N(\tau),\ \tau \geq 0\}$ is a nonnegative, integer-valued stochastic process, independent of the $\{X_i\}$, with $N(0) = 0$. Thus, the claims process can be described as

$$C(\tau) = \sum_{i=1}^{N(\tau)} X_i\,, \quad \tau > 0\,,$$

where τ is calendar time.

Let $\{I(\tau), 0 \leq \tau < \infty\}$, $\{O(\tau), 0 \leq \tau < \infty\}$, and $\{L(\tau), 0 \leq \tau < \infty\}$ be stochastic processes modeling random deviations from actuarial as-

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sumptions about investment performance, operating expenses, and lapse expenses. As explained in [3], we assume that each process enjoys the Gaussian and Markovian properties. This means that all finite-dimensional distributions are multivariate normal, allowing both adverse and favorable deviations in a symmetrical manner about the mean functions. We will assume that

$$E\{I(\tau)\} = E\{O(\tau)\} = E\{L(\tau)\} = 0\,, \quad 0 \leq \tau < \infty\,.$$

In general, future values for the three processes are dependent on the past results. It is difficult for a mathematical model to recognize those dependencies completely. However, if we assume that, as a first approximation, probabilities about future events depend only on the present values, our models are Markovian stochastic processes. As explained in [3], the Wiener stochastic process is the most widely studied and used process with the Gaussian and Markovian properties. However, its variance function is unbounded with evolving time. This was the main reason for using another Gaussian Markov process called the Ornstein-Uhlenbeck process.

Our collective multirisk process is defined to be

$$R(\tau) = C(\tau) - I(\tau) + O(\tau) + L(\tau)\,, \quad 0 \leq \tau < \infty\,, \tag{1}$$

where τ is calendar time. We use $-I(\tau)$ rather than $+I(\tau)$ because adverse investment results would be valued negatively.

Actuaries would be interested in the expected values and the variances of the $R(\tau)$ process for use in determining gross premiums.

$$\begin{aligned} E\{R(\tau)\} &= E\{C(\tau)\} - E\{I(\tau)\} + E\{O(\tau)\} + E\{L(\tau)\} \\ &= p_1 E\{N(\tau)\}\,, \quad 0 < \tau < \infty\,. \end{aligned} \tag{2}$$

To obtain Var $\{R(\tau)\}$, we must describe the other three processes more completely, utilizing results that have evolved in the five previously cited papers.

For the Ornstein-Uhlenbeck process, the transition density function $(s < t)$ is

$$\begin{aligned} p(x, s; y, t) &= \frac{\partial}{\partial y} P\{X(t) < y \,|\, X(s) = x\} \\ &= [2\pi A(s, t)]^{-1/2} \exp\left[\!\left[-\frac{\{y - x\exp[-\beta(t-s)]\}^2}{2A(s, t)} \right]\!\right], \end{aligned} \tag{3}$$

where $A(s, t) = \sigma^2\{1 - \exp[-2\beta(t-s)]\}$, and $\sigma^2 > 0$, $\beta > 0$. The variance function equals σ^2 for all $t \geq 0$, and the covariance $E\{X(s)X(t)\} = \sigma^2 \exp[-\beta(t-s)]$ for $s < t$.

We shall assume that the four processes are independent. Denoting the variances of the three Ornstein-Uhlenbeck processes by σ_I^2, σ_O^2, and σ_L^2, we obtain the result

$$\begin{aligned} \text{Var}\,\{R(\tau)\} &= \text{Var}\,\{C(\tau)\} + \sigma_I^2 + \sigma_O^2 + \sigma_L^2 \\ &= p_2 E\{N(\tau)\} + \sigma_I^2 + \sigma_O^2 + \sigma_L^2\,, \quad 0 \le \tau < \infty\,, \end{aligned} \tag{4}$$

where $p_2 = E\{X_i^2\}$.

It would be convenient to have some knowledge of the probability structure of the multirisk process. Assume that the initial reserve is denoted by u and that (for simplicity) premium income flows in at a steady rate. Then the actuary is interested in a gross premium G such that the probability that the greatest difference between $R(\tau)$ and τG at any point in time is greater than u is appropriately small, say 0.001.

Each sample function $w(\tau)$ of the $C(\tau)$ process is of the form

$$\begin{aligned} w(\tau) &= 0\,, \quad 0 < \tau < t_1 \\ &= a_i\,, \quad \sum_{j=1}^{i} \tau_j \le \tau < \sum_{j=1}^{i+1} \tau_j\,, \quad i = 1, 2, \ldots \end{aligned}$$

where $\tau_i > 0$ for $i = 1, 2, \ldots$; $-\infty < a_i < \infty$ for $i = 1, 2, \ldots$; $a_{i+1} - a_i \neq 0$ for $i = 1, 2, \ldots$; and $\Sigma_{i=1}^{\infty} \tau_i = +\infty$. This last condition allows at most a finite number of discontinuities in any finite interval. Consider a positive constant λ such that $p_1 + \lambda < G$. A sample path of $C(\tau) - \tau(p_1 + \lambda)$ would have the appearance of the diagram in figure 1.

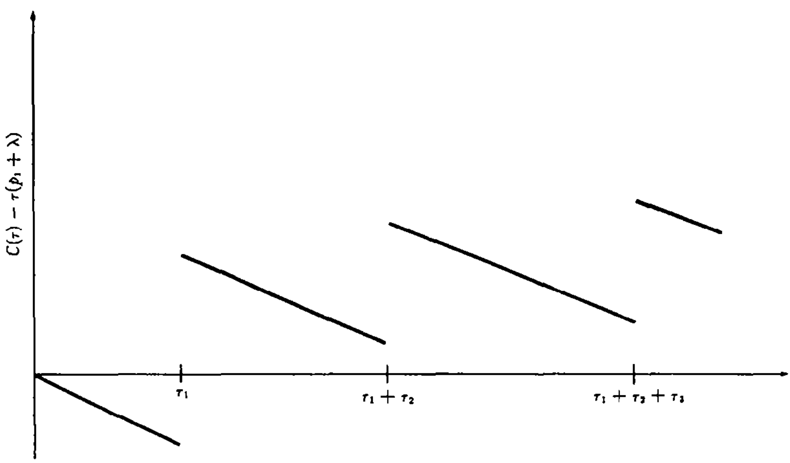

FIG. 1

Let $S(\tau) = -I(\tau) + O(\tau) + L(\tau),\ 0 \le \tau < \infty$, where each of the three processes has $\beta = 1$. Theorem 1 of Part I of [2] proves that $\{S(\tau), 0 \le \tau < \infty\}$ is also an Ornstein-Uhlenbeck (O.U.) process with $\beta = 1$. Let $\sigma_I^2 + \sigma_O^2 + \sigma_L^2 = K^2$ for $K > 0$. As explained on page 89 of [4], the sample functions of an O.U. process are continuous. As discussed on pages 578 and 588 of [3], the conditional mean function

$$E\{S(t) \mid S(s) = x\} = xe^{-\beta(t-s)} \quad \text{for } \beta > 0 .$$

This implies a drift downward if the present position $S(s)$ is positive and a drift upward if it is negative. For our purpose, "position" refers to a deviation from the expected value. A minigraph of the $S(\tau)$ process thus would have the characteristics shown in figure 2. A sample path of $S(\tau)$, $0 \le \tau \le T$, could have the appearance of the curve illustrated in figure 3.

Let u be an initial allowance for adverse claims, and AK a provision for deviations from assumptions about investment performance, operating expenses, and lapse expenses. Risk managers are concerned with probabilities of the following type:

$$P\{\max_{0\le\tau\le T} [C(\tau) - \tau(p_1 + \lambda)] > u ,\ \max_{0\le\tau\le T} S(\tau) \ge AK \mid S(0) = 0\} .$$

In words, this is the probability that, in a future time interval $0 \le \tau \le T$, the $C(\tau)$ process exceeds and the $S(\tau)$ process exceeds or equals allowable values. The assumed independence of the four processes allows us to factor the above probability into the following product:

$$P\{\max_{0\le\tau\le T} [C(\tau) - \tau(p_1 + \lambda)] > u\} P\{\max_{0\le\tau\le T} S(\tau) \ge AK \mid S(0) = 0\}$$

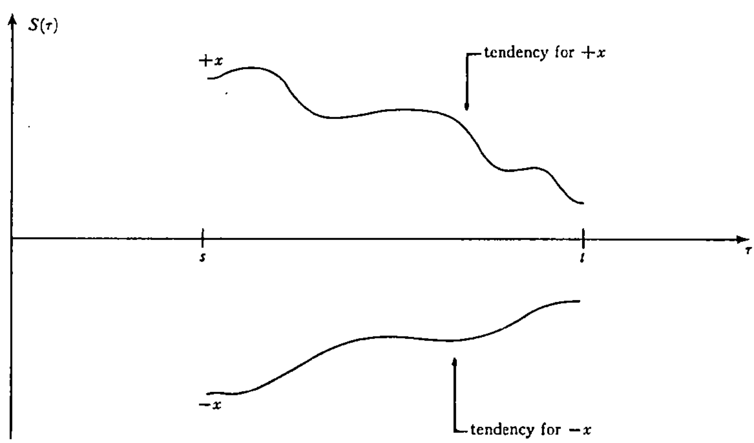

FIG. 2

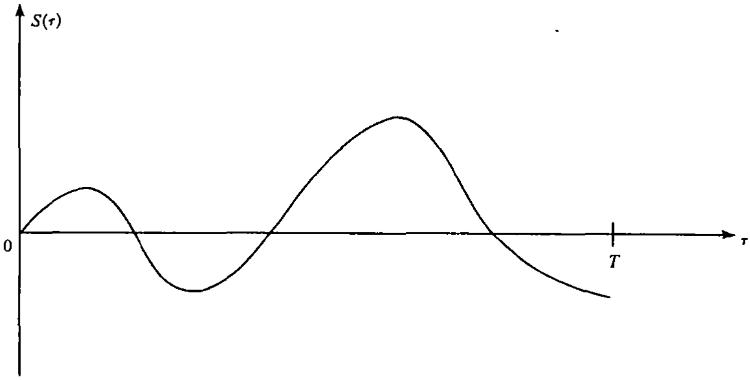

FIG. 3

The actuary is interested in determining a gross premium such that this product is appropriately small, say 0.001.

II. DISCUSSION OF PREVIOUS RESULTS, AND OTHER REFERENCES

One of the purposes of the five previous papers on this subject was to develop a more realistic model and to illustrate it. Section III of [3] was concerned partially with providing for inflation in premiums. This will be treated more extensively in a later section of this paper. Theorems 2 and 3 of [2] obtained techniques for calculating

$$P\{\max_{0\leq\tau\leq T} S(\tau) > AK \mid S(0) = 0\} .$$

The third section of that paper illustrated the product probabilities, where the first factors (the $\psi(u, T)$'s) were drawn from [25] for various claim distributions. It was assumed that $\{N(\tau), 0 \leq \tau < \infty\}$ was a Poisson process and that operational time was in effect, that is, $E\{N(\tau)\} = \tau$. The Poisson restriction was removed in Part II of [2], and a theorem was proved that allowed the determination of O.U. probabilities for more realistic values of the parameter β. The elimination of the Poisson restriction allowed the examples to utilize twice as many tables from [25]. Some of those numerical values will be repeated in a later section of this paper.

The simulation of $R(\tau)$ sample paths with resulting approximate probabilities was performed in [5]. The Monte Carlo technique for the $C(\tau)$ process was based largely on [22], while the simulation technique for the $S(\tau)$ process was patterned after [12]. As a by-product of implementing the latter method, it was observed that the correlation between $S(\tau)$ observations separated by one unit of time is $e^{-\beta}$. Observations of the $S(\tau)$ process can be used to estimate $e^{-\beta}$. Thus, $e^{-\beta}$ can be regarded as the theoretical autocorrelation function for lag $k = 1$ in equation (5) of [19]. An observed series can be used to calculate the sample autocorrelation coefficient with lag 1:

$$r_1 = \frac{\sum_{i=2}^{n} (Z_i - \bar{Z})(Z_{i-1} - \bar{Z})}{\sum_{i=1}^{n} (Z_i - \bar{Z})^2},$$

where n is the number of observations and $\bar{Z} = \Sigma_{i=1}^{n} Z_i/n$. This is formula (1) of [19] with $k = 1$. The sample paths for $\{C(\tau) + S(\tau) - \tau(p_1 + \lambda), 0 \leq \tau \leq T\}$ were simulated, and a count was kept of those that exceeded the quantity u. If n such financial histories (sample paths) are simulated and k exceed u in the time interval $[0, T]$, the estimate of the desired probability is k/n with an estimated standard error of $[(k/n)(1 - k/n)/n]^{1/2}$. Five examples with different claim distributions and claim time distributions were considered, and tables of results were included.

The fact that this paper has been phrased in terms of calendar time rather than operational time is a reflection of some of the results of [6]. Operational time is very convenient for the claims process, since it measures time by the expected number of events during the period, but it is not natural for the O.U. processes. This became apparent in the examples of Parts I and II of [2] in which $T = 100$ or 1,000, and the probabilities of excessive deviations for the $S(\tau)$ process became inappropriately large. For the $S(\tau)$ process, calendar time is more appropriate, and logical choices of T are 1, 2, . . . , 10. One of the purposes of [6] was to derive techniques and tables of O.U. distributions for such choices of T. The second purpose was to reconsider the examples in [2] and make them more realistic by an improved consideration of the time variables.

The multirisk model is directed toward the same problem as SOFASIM [24]. SOFASIM is a computer model simulating a stock life insurance company and is a tool for the study of adverse deviations. John C. Wooddy, chairman of the Committee on Theory of Risk, had a major role in its development, refinement, and dissemination to members of the Society of Actuaries [26]. SOFASIM is a very thorough and practical solution to the problem. The multirisk model should provide readers with a global view of the problem and a stimulus for viewing the problem in different manners.

Richard W. Ziock's paper [27] relates to the process $\{I(\tau), 0 \leq \tau < \infty\}$. The purposes of that paper were to present and justify a time series model for interest rates, and hence to quantify the interest rate delta. The time series model presented (p. 5) is

$$y_{t+1} = y_t + a_t , \quad \text{or} \quad \Delta y_t = a_t ,$$

"where a_t is white noise: a random selection from some known distribution which may or may not vary with time." As indicated on page 18 of [4], a frequent model for white noise is the O.U. process. On page 8 of [27] is the sentence: "If we are standing at the origin t and choose to forecast the value of y_{t+1}, the variance of the forecast is the variance of a_t which is independent of time and equal to σ^2." This reinforces the choice of the O.U. process rather than the Wiener process for $\{I(\tau), 0 \leq \tau < \infty\}$ (see p. 577 of [3]).

Obviously, Richard G. Horn's paper [14] is still an important reference. In addition, references [1], [8], [9], [11], [13], and [18] contributed to the authors' understanding of the subject and may prove useful to readers of this paper. The list is not exhaustive, and other references may be more valuable for some readers.

III. PROVISIONS FOR INFLATION

As stated previously, Section III of [3] was devoted partially to allowing for inflationary assumptions. For example, assume that an insurance line expects 100 claims per year, that is, operational time $t = 100$ corresponds to calendar time $\tau = 1$. Let us assume that the actuary expects an inflationary trend of 4 percent of the average claim p_1. This can be handled very easily by simply viewing part of the security loading λ as $0.04p_1$. Thus the collective premium for one year would be $100(p_1 + \lambda)$, or $100[p_1 + 0.04p_1 + (\lambda - 0.04p_1)]$. In other words, no extra mathematics is needed to handle a provision for inflation in claims.

It is not that simple with a provision for inflation in deviations from assumptions about investment performance, operating expenses, and lapse expenses. Our previous O.U. distributions

$$P\{\max_{0 \leq \tau \leq T} S(\tau) \geq AK \mid S(0) = 0\}$$

have concerned themselves with deviations above a constant boundary AK. We now wish to replace AK by a function $AKe^{\delta\tau}$, $\delta > 0, 0 \leq \tau \leq T$, and derive values for the O.U. distributions

$$D(T, A, K, \delta) = P\{\max_{0 \leq \tau \leq T} [S(\tau) - AKe^{\delta\tau}] > 0 \mid S(0) = 0\}. \quad (5)$$

This will measure probabilities of deviations exceeding a safety margin that explicitly provides for inflation.

Let $E(T, A, K, R) = D(T, A, K, \delta)$, where $D(T, A, K, \delta)$ is defined by equation (5) and R is the annual rate of inflation defined by $1 + R = e^{\delta}$. Tables 1–20 were developed from the results in Section IV and the

TABLE 1

$R = 0.0300$; $\delta = 0.02955880$; $T = 1$

A	N/M					
	6/0	7/1	8/2	9/3	10/4	11/5
0.50........	0.760464	0.760412	0.760403	0.760401	0.760401	0.760400
1.00........	0.443515	0.443484	0.443473	0.443470	0.443469	0.443469
2.00........	0.062152	0.062139	0.062135	0.062135	0.062135	0.062135
3.00........	0.002572	0.002571	0.002571	0.002571	0.002571	0.002571

CPU time, 191.4700 seconds

TABLE 2

$R = 0.0400$; $\delta = 0.03922071$; $T = 1$

A	N/M					
	6/0	7/1	8/2	9/3	10/4	11/5
0.50........	0.759290	0.759237	0.759228	0.759226	0.759225	0.759225
1.00........	0.440245	0.440213	0.440202	0.440198	0.440197	0.440197
2.00........	0.059968	0.059954	0.059951	0.059951	0.059951	0.059951
3.00........	0.002354	0.002354	0.002353	0.002353	0.002353	0.002353

CPU time, 188.4050 seconds

TABLE 3

$R = 0.0500$; $\delta = 0.04879016$; $T = 1$

A	N/M					
	6/0	7/1	8/2	9/3	10/4	11/5
0.50........	0.758117	0.758064	0.758055	0.758053	0.758052	0.758052
1.00........	0.436992	0.436959	0.436948	0.436945	0.436944	0.436944
2.00........	0.057853	0.057840	0.057836	0.057836	0.057836	0.057836
3.00........	0.002154	0.002154	0.002153	0.002153	0.002153	0.002153

CPU time, 187.0690 seconds

TABLE 4

$R = 0.0600$; $\delta = 0.05826891$; $T = 1$

A	N/M					
	6/0	7/1	8/2	9/3	10/4	11/5
0.50........	0.756947	0.756893	0.756883	0.756881	0.756880	0.756880
1.00........	0.433758	0.433724	0.433713	0.433709	0.433708	0.433708
2.00........	0.055806	0.055792	0.055789	0.055789	0.055789	0.055789
3.00........	0.001971	0.001970	0.001970	0.001970	0.001970	0.001970

CPU time, 187.9120 seconds

TABLE 5

$R = 0.0700$; $\delta = 0.06765865$; $T = 1$

A	N/M					
	6/0	7/1	8/2	9/3	10/4	11/5
0.50........	0.755778	0.755723	0.755713	0.755711	0.755711	0.755710
1.00........	0.430541	0.430507	0.430496	0.430492	0.430491	0.430491
2.00........	0.053825	0.053811	0.053808	0.053807	0.053807	0.053807
3.00........	0.001802	0.001801	0.001801	0.001801	0.001801	0.001801

CPU time, 181.3300 seconds

TABLE 6

$R = 0.0300$; $\delta = 0.02955880$; $T = 5$

A	N/M					
	12/8	14/10	16/12	18/14	20/16	22/18
0.50........	0.975376	0.979820	0.980858	0.980911	0.980907	0.980906
1.00........	0.854065	0.865124	0.865831	0.865702	0.865693	0.865693
2.00........	0.292032	0.293196	0.292024	0.291938	0.291935	0.291935
3.00........	0.024142	0.023274	0.023153	0.023147	0.023147	0.023147

CPU time, 127.2290 seconds

TABLE 7

$R = 0.0400$; $\delta = 0.03922071$; $T = 5$

A	N/M					
	12/8	14/10	16/12	18/14	20/16	22/18
0.50........	0.974033	0.978743	0.979841	0.979898	0.979893	0.979892
1.00........	0.843876	0.855734	0.856490	0.856351	0.856341	0.856340
2.00........	0.266116	0.267303	0.266089	0.266000	0.265997	0.265997
3.00........	0.019085	0.018258	0.018142	0.018135	0.018135	0.018135

CPU time, 127.7010 seconds

TABLE 8

$R = 0.0500$; $\delta = 0.04879016$; $T = 5$

A	N/M					
	12/8	14/10	16/12	18/14	20/16	22/18
0.50........	0.972604	0.977596	0.978760	0.978820	0.978814	0.978814
1.00........	0.833115	0.845816	0.846625	0.846475	0.846464	0.846463
2.00........	0.241838	0.243046	0.241793	0.241701	0.241698	0.241698
3.00........	0.015180	0.014395	0.014283	0.014277	0.014277	0.014277

CPU time, 128.1170 seconds

TABLE 9

$R = 0.0600$; $\delta = 0.05826891$; $T = 5$

A	N/M					
	12/8	14/10	16/12	18/14	20/16	22/18
0.50........	0.971083	0.976376	0.977610	0.977673	0.977667	0.977667
1.00........	0.821783	0.835374	0.836237	0.836076	0.836064	0.836064
2.00........	0.219325	0.220552	0.219264	0.219169	0.219166	0.219166
3.00........	0.012178	0.011435	0.011327	0.011321	0.011321	0.011321

CPU time, 128.4790 seconds

TABLE 10

$R = 0.0700$; $\delta = 0.06765865$; $T = 5$

A	N/M					
	12/8	14/10	16/12	18/14	20/16	22/18
0.50........	0.969464	0.975079	0.976386	0.976453	0.976447	0.976447
1.00........	0.809889	0.824416	0.825336	0.825163	0.825150	0.825150
2.00........	0.198653	0.199897	0.198577	0.198479	0.198476	0.198476
3.00........	0.009870	0.009168	0.009064	0.009059	0.009059	0.009059

CPU time, 128.8030 seconds

TABLE 11

$R = 0.0300$; $\delta = 0.02955880$; $T = 7$

A	N/M					
	18/14	20/16	22/18	24/20	26/22	28/24
0.50........	0.992596	0.993811	0.994067	0.994075	0.994074	0.994074
1.00........	0.921482	0.926528	0.926704	0.926652	0.926648	0.926648
2.00........	0.354407	0.354344	0.353514	0.353456	0.353455	0.353455
3.00........	0.028061	0.027261	0.027173	0.027169	0.027169	0.027169

CPU time, 243.7040 seconds

TABLE 12

$R = 0.0400$; $\delta = 0.03922071$; $T = 7$

A	N/M					
	18/14	20/16	22/18	24/20	26/22	28/24
0.50........	0.991834	0.993180	0.993464	0.993473	0.993472	0.993472
1.00........	0.911163	0.916886	0.917084	0.917025	0.917020	0.917020
2.00........	0.314336	0.314261	0.313380	0.313318	0.313317	0.313317
3.00........	0.020979	0.020214	0.020129	0.020125	0.020125	0.020125

CPU time, 244.4640 seconds

TABLE 13

$R = 0.0500$; $\delta = 0.04879016$; $T = 7$

A	N/M					
	18/14	20/16	22/18	24/20	26/22	28/24
0.50........	0.990979	0.992473	0.992788	0.992798	0.992797	0.992797
1.00........	0.899694	0.906170	0.906393	0.906325	0.906320	0.906320
2.00........	0.277791	0.277704	0.276777	0.276712	0.276710	0.276710
3.00........	0.016007	0.015276	0.015195	0.015191	0.015191	0.015191

CPU time, 245.2280 seconds

TABLE 14

$R = 0.0600$; $\delta = 0.05826891$; $T = 7$

A	N/M					
	18/14	20/16	22/18	24/20	26/22	28/24
0.50........	0.990019	0.991680	0.992029	0.992041	0.992040	0.992039
1.00........	0.887032	0.894342	0.894592	0.894515	0.894509	0.894509
2.00........	0.245156	0.245061	0.244093	0.244024	0.244023	0.244023
3.00........	0.012488	0.011793	0.011714	0.011711	0.011710	0.011710

CPU time, 249.9180 seconds

TABLE 15

$R = 0.0700$; $\delta = 0.06765865$; $T = 7$

A	N/M					
	18/14	20/16	22/18	24/20	26/22	28/24
0.50........	0.988942	0.990790	0.991179	0.991192	0.991190	0.991190
1.00........	0.873166	0.881390	0.881670	0.881583	0.881576	0.881576
2.00........	0.216549	0.216446	0.215442	0.215371	0.215370	0.215370
3.00...... ..	0.009951	0.009292	0.009216	0.009213	0.009213	0.009213

CPU time, 254.4850 seconds

TABLE 16

$R = 0.0300$; $\delta = 0.02955880$; $T = 10$

A	*N/M*					
	26/22	28/24	30/26	32/28	34/30	36/32
0.50........	0.998475	0.998796	0.998884	0.998892	0.998892	0.998892
1.00........	0.963344	0.966934	0.967377	0.967329	0.967325	0.967326
2.00........	0.408850	0.412455	0.411070	0.410948	0.410942	0.410941
3.00........	0.030715	0.029858	0.029640	0.029627	0.029627	0.029626

CPU time, 468.9470 seconds

TABLE 17

$R = 0.0400$; $\delta = 0.03922071$; $T = 10$

A	*N/M*					
	26/22	28/24	30/26	32/28	34/30	36/32
0.50........	0.998144	0.998536	0.998645	0.998654	0.998654	0.998654
1.00........	0.953497	0.958060	0.958622	0.958561	0.958556	0.958556
2.00........	0.348417	0.352342	0.350817	0.350683	0.350676	0.350675
3.00........	0.021935	0.021124	0.020912	0.020899	0.020899	0.020898

CPU time, 481.3370 seconds

TABLE 18

$R = 0.0500$; $\delta = 0.04879016$; $T = 10$

A	*N/M*					
	26/22	28/24	30/26	32/28	34/30	36/32
0.50........	0.997730	0.998212	0.998345	0.998357	0.998356	0.998356
1.00........	0.941410	0.947169	0.947878	0.947801	0.947795	0.947795
2.00........	0.296430	0.300619	0.298975	0.298830	0.298823	0.298821
3.00........	0.016379	0.015616	0.015412	0.015399	0.015399	0.015399

CPU time, 488.0750 seconds

TABLE 19

$R = 0.0600$; $\delta = 0.05826891$; $T = 10$

A	N/M					
	26/22	28/24	30/26	32/28	34/30	36/32
0.50........	0.997211	0.997806	0.997970	0.997985	0.997984	0.997984
1.00........	0.926859	0.934060	0.934947	0.934850	0.934842	0.934842
2.00........	0.253383	0.257779	0.256037	0.255883	0.255876	0.255874
3.00........	0.012683	0.011967	0.011771	0.011759	0.011759	0.011758

CPU time, 507.2670 seconds

TABLE 20

$R = 0.0700$; $\delta = 0.06765865$; $T = 10$

A	N/M					
	26/22	28/24	30/26	32/28	34/30	36/32
0.50........	0.996562	0.997299	0.997501	0.997519	0.997518	0.997518
1.00........	0.909735	0.918636	0.919732	0.919611	0.919602	0.919602
2.00........	0.218542	0.223093	0.221275	0.221113	0.221106	0.221103
3.00........	0.010090	0.009422	0.009234	0.009222	0.009222	0.009222

CPU time, 532.2070 seconds

Appendix. The implementation of the algorithm is also described in the Appendix. In all cases, the O.U. parameter β is equal to 1. Seldom would data produce $\beta = 1$, but a method for handling this problem is discussed and illustrated in Section V.

These twenty tables provide values for an O.U. stochastic process exceeding an upper boundary that grows with time and that depends on an assumed inflation rate. The key parameters for each table are R (assumed annual rate of inflation) and T (number of calendar years in the future observation period). These parameters are indicated at the top of each table.

In Section IV we will resume a consideration of the theory needed to handle real data. We will obtain equations (6) and (7), which are also concerned with O.U. stochastic processes exceeding various upper boundaries that reflect provisions for adverse deviations.

The proofs of theorems needed to obtain values for the O.U. distributions in equations (5) and (7) are given in the Appendix. They are presented in detail partially because the O.U. process may prove useful in different actuarial applications. The basis for this hope is the fact that the authors of [16] utilized the O.U. process in discussing a continuous time model for credibility formulas. Assume that $P(t)$ is the premium density at time t, $S(t)$ is the aggregate of the claims at time t, and $c > 0$. Then $P(t)$ satisfies the stochastic differential equation

$$dP(t) = c[dS(t) - P(t)dt] .$$

This is of the same form as

$$dU(t) = -bU(t)dt + dW(t) , \quad b > 0 ,$$

where $U(t)$ is an O.U. process and $W(t)$ is a Wiener process. (The Wiener process is discussed in many textbooks and papers; it is a Gaussian Markov process with mean function $E\{W(t)\} \equiv 0$ and covariance function $E\{W(s)W(t)\} = \min(s, t)$.) Formal integration of these equations leads to the analogous results

$$P(t) = e^{-ct}P(0) + c\int_0^t e^{-c(t-u)}dS(u)$$

$$U(t) = e^{-bt}U(0) + \int_0^t e^{b(t-s)}dW(s) .$$

Although [16] did not use distributions for the O.U. process, it does suggest that the results and methods of proof contained in the Appendix may be useful in other actuarial applications.

IV. PRELIMINARY RESULTS FOR EXAMPLES

The reader will need additional theory to handle problems with real data. These results are put in a separate section for ready reference in future applications.

First, it would be unusual if real data allowed the actuary to conclude that the β's for the $I(\tau)$, $L(\tau)$, and $O(\tau)$ processes were the same. This precludes using the idea that $S(\tau) = -I(\tau) + L(\tau) + O(\tau)$ is also an O.U. process with the same β. The $\{S(\tau), 0 \leq \tau < \infty\}$ process is still a Gaussian stochastic process with $E\{S(\tau)\} = 0$, $\text{Var}\{S(\tau)\} = \sigma_I^2 + \sigma_L^2 + \sigma_O^2$, for $0 \leq \tau < \infty$. However, Covar $\{S(s), S(t)\}$ cannot be expressed in the factorable form (see p. 89 of [4]); therefore, the $S(\tau)$ process is not Markovian and is not an O.U. process. Nevertheless, meaningful probability expressions can be set up. Thus, the independence of the four processes allows us to write the probability of excessive adverse deviations in calendar time $[0, T]$ for all four processes as the product of four individual probabilities. In symbols,

$$
\begin{aligned}
&P\{\max_{0\le\tau\le T}[C(\tau)-\tau(p_1+\lambda)]>u\,,\ \max_{0\le\tau\le T}[-I(\tau)-f(\tau)]\ge 0\,,\\
&\quad\max_{0\le\tau\le T}[L(\tau)-g(\tau)]\ge 0\,,\ \max_{0\le\tau\le T}[O(\tau)-h(\tau)]\ge 0\\
&\quad|\,I(0)=L(0)=O(\tau)=0\}\\
&\quad=P\{\max_{0\le\tau\le T}[C(\tau)-\tau(p_1+\lambda)]>u\}\\
&\qquad\times P\{\max_{0\le\tau\le T}[-I(\tau)-f(\tau)]\ge 0\,|\,I(0)=0\}\\
&\qquad\times P\{\max_{0\le\tau\le T}[L(\tau)-g(\tau)]\ge 0\,|\,L(0)=0\}\\
&\qquad\times P\{\max_{0\le\tau\le T}[O(\tau)-h(\tau)]>0\,|\,O(0)=0\}\,.
\end{aligned}
\tag{6}
$$

Important examples with equal boundary functions are $f(\tau) = g(\tau) = h(\tau) = A\sigma$ or $A\sigma e^{\delta\tau}$ for $\tau \ge 0$. Reference [6] used $g(\tau) \equiv A\sigma$. In order to consider inflation, we now will consider $g(\tau) = A\sigma e^{\delta\tau}$ for $\tau \ge 0$.

Second, it would be impossible to tabulate O.U. probabilities for all reasonable choices of β. In many cases, this problem can be handled by the theorem that follows.

Let $\{X(\tau),\ 0 \le \tau < \infty\}$ be an O.U. process with transition density function given by equation (3), and let

$$F(T) = P\{\operatorname*{supremum}_{0\le\tau\le T}[X(\tau) - A\sigma e^{\delta\tau}] \ge 0\,|\,X(0) = 0\}\,. \tag{7}$$

For practical purposes, the supreme of each sample function can be thought of as the maximum value over the time interval.

THEOREM 1. *Let* $\{Y(\tau),\ 0 \le \tau < \infty\}$ *be a second O.U. process with variance and covariance parameters of* 1, *that is,* $\sigma^2 = \beta = 1$ *in equation (3). Then*

$$F(T) = P\{\operatorname*{supremum}_{0\le\tau\le \beta T}[Y(\tau) - Ae^{\delta\tau}] \ge 0\,|\,Y(0) = 0\}\,. \tag{8}$$

Proof: This follows from Theorem 2 of Part II of [2].

V. EXAMPLES OF DATA, AND MULTIRISK PROCESSES

Fundamental to the multirisk model is the use of data to determine the four processes. We will show some fairly realistic data to illustrate the procedures and formulas. Assumptions about investment performance, lapse expenses, and operating expenses can be expressed in monetary terms or relative percentage figures. It would be natural to view investment performance in percentage terms.

Let us examine lapse expenses in greater detail. Assume that there are

100,000 policies in a portfolio. This ensemble will be viewed as a stationary population in which 20,000 policies are in their first two policy years and 80,000 are in their third or later policy years. Assume that a loss of c_1 occurs if a policy is lapsed in the first two years, and a loss of c_2 occurs if a policy is lapsed in the third or later years. The random variable L representing aggregate lapse loss can be viewed as follows:

$$L = \sum_{i=1}^{20,000} X_i + \sum_{i=1}^{80,000} Y_i .$$

The random variables $\{X_i\}$ and $\{Y_i\}$ are assumed to be independent. The probabilities for these random variables will be determined from the *1974 Life Insurance Fact Book* figures supplied by E. J. Moorhead in his discussion of the Brzezinski paper ([8], p. 299). Thus, each X_i equals c_1 with probability 0.195 and zero with probability 0.805, and each $Y_i = c_2$ with probability 0.045 and zero with probability 0.955.

$$\begin{aligned} E(L) &= \sum_{i=1}^{20,000} E(X_i) + \sum_{i=1}^{80,000} E(Y_i) \\ &= 20,000(0.195c_1) + 80,000(0.045c_2) . \end{aligned}$$

By the assumed independence,

$$\text{Var}(L) = 20,000(0.195c_1)(0.805c_1) + 80,000(0.045c_2)(0.955c_2) .$$

Although the random variables do not have the same distribution, the *central limit theorem* still applies (see, for example, [10]), and L is a normal (Gaussian) random variable. A logical delta for lapses would be an appropriate multiple of *standard deviation* (L). However, it would be just as meaningful for the actuary to consider relative deviations of the form $[L_i - E(L)]/\sigma_L$, and that is the way in which our data are presented. Similarly, operating expense deviations will be presented in the standardized form $[O_i - E(O)]/\sigma_O$.

Table 21 displays observed deviations from assumptions about investment performance, lapse expenses, and operating expenses at fifteen six-month intervals. For summary purposes, $E(I(\tau))$, $E(L(\tau))$, and $E(O(\tau))$ represent our assumptions. The deviations at time i are

$$-I(i) = -[I_i - E(I(i))] ; \quad L(i) = [L_i - E(L(i))]/\sigma_L ;$$

$$O(i) = [O_i - E(O(i))]/\sigma_O .$$

An adverse deviation occurs if $I_i < E(I(i))$. For consistency, and in order to use our probability distributions about maximum values, we multiply through by minus one. The approximate equality of the theoretical and sample autocorrelation coefficients with lag 1, that is, $e^{-\beta}$ and r, permits the approximation $\beta = -\ln_e r$.

Assume that the actuary is considering provisions for adverse deviations of $A\sigma_I$, $A\sigma_L$, and $A\sigma_O$ for $A = 2$ or 3 and inflation at an annual rate of R (instantaneous rate of δ). He or she is then interested in

$$P\{\max_{0\leq\tau\leq T}[-I(\tau) - A\sigma_I e^{\delta\tau}] > 0\} ;$$

$$P\{\max_{0\leq\tau\leq T}[L(\tau) - A\sigma_L e^{\delta\tau}] > 0\} ;$$

$$P\{\max_{0\leq\tau\leq T}[O(\tau) - A\sigma_O e^{\delta\tau}] \geq 0\} .$$

Each is a probability of excessive adverse deviations.

The provision for adverse deviations grows from $A\sigma$ to $A\sigma e^{T\delta}$, with an aggregate provision of

$$\int_0^T A\sigma e^{\delta\tau} d\tau = A\sigma(e^{T\delta} - 1)/\delta .$$

A reasonable procedure for the annual premium would be to add $A\sigma(e^{T\delta} - 1)/(T\delta)$ as a provision for adverse deviations and inflation.

TABLE 21

OBSERVED DEVIATIONS FROM ASSUMPTIONS

Time (i)	Deviations		
	$-I(i)$	$L(i)$	$O(i)$
1	0.00159	0.857	−0.213
2	0.00134	−0.143	−0.288
3	−0.00463	−0.743	0.244
4	0.00678	0.230	2.507
5	0.00798	0.444	2.092
6	−0.00645	0.439	1.962
7	−0.00652	−0.087	1.979
8	−0.00768	0.428	1.394
9	−0.00011	0.177	0.818
10	−0.00180	−0.877	1.116
11	−0.01733	1.967	0.026
12	−0.00671	1.989	−0.351
13	−0.01145	1.698	0.791
14	−0.01975	1.591	0.115
15	−0.01850	1.193	0.899
Sample mean ($\bar{Z}$)	−0.00555	0.611	0.873
$\Sigma_{i=1}^{15}(Z_i-\bar{Z})^2$	10.286×10^{-4}	11.807	12.619
$\Sigma_{i=2}^{15}(Z_i-\bar{Z})(Z_{i-1}-\bar{Z})$	4.773×10^{-4}	5.401	6.995
Sample r, lag of 1	0.464	0.457	0.554
Corresponding beta (approximately)	0.768	0.782	0.590

Consider a planning horizon of $T = 10$ calendar years. Since our data produced sample autocorrelation coefficients r, and not true β's, we will round our β's to the nearest multiple of 0.01. We use relation (8) and linear interpolation. Thus the probabilities for $\beta = 0.77$ are found by interpolating linearly between values for $\beta = 1$, $T = 7$ and 10. The probabilities of excessive adverse deviation are shown in Table 22.

It is recognized that Tables 1–20 do not consider all choices for β and T, but there are sufficient tables to yield reasonable approximations. Furthermore, the Appendix has been written so that readers can prepare their own tables for different values of T and β.

The figures for $R = 0.00$ come from [6]. Linear extrapolation was used for $\beta = 0.59$, assuming equal differences for $\beta = 0.59$–0.60 and $\beta = 0.60$–0.61. Smaller values of β can be handled with the tables in [17]. The tables in [6] give probabilities of *not* exceeding $A\sigma$, so these probabilities must be subtracted from 1. It should be reassuring to readers to see the steady decrease in the probabilities of excessive adverse deviations as the assumed rate R grows from 0.00 to 0.07. It should also be observed that the probabilities drop markedly as R goes from 0.00 to 0.03.

Probabilities for the claims process will be drawn from [25]. Assume

TABLE 22

PROBABILITIES OF EXCESSIVE ADVERSE DEVIATION

$A = 2$

	R					
	0.00	0.03	0.04	0.05	0.06	0.07
$I(\tau)$ probability, $\beta=0.77$...	0.519705	0.366868	0.322034	0.281869	0.246788	0.216708
$L(\tau)$ probability, $\beta=0.78$...	0.524446	0.368785	0.323279	0.282606	0.247183	0.216899
$O(\tau)$ probability, $\beta=0.59$..	0.429037	0.319619	0.287291	0.257453	0.230352	0.206078
Product................	0.116937	0.043243	0.029909	0.020508	0.014052	0.009686

$A = 3$

	R					
	0.00	0.03	0.04	0.05	0.06	0.07
$I(\tau)$ probability $\beta=0.77$...	0.077742	0.027742	0.020305	0.015240	0.011721	0.009215
$L(\tau)$ probability, $\beta=0.78$..	0.078819	0.027824	0.020331	0.015246	0.011723	0.009215
$O(\tau)$ probability, $\beta=0.59$..	0.058199	0.024957	0.019031	0.014688	0.011496	0.009128
Product................	0.000357	0.000019	0.000008	0.000003	0.000002	0.000001

that $N(t)$ has a Poisson distribution with $E\{N(t)\} = t$. This is equivalent to assuming that the interoccurrence time between claims has a distribution $1 - e^{-t}$, $t \geq 0$. For the purpose of illustration, we assume that calendar time τ and operational time t are related by the simple equation $100\tau = t$. Assume that the claim distribution $P(x)$ equals

$$\sum_{i=1}^{5} a_i[1 - \exp(-b_i x)]$$

for a_i's and b_i's given on page 148 of [25]. We will use the values $\lambda = 0.10$ and $u = 100$. Since $P(x)$ has a mean p_1 of 1, $\lambda = 0.10$ means 10 percent of the average claim, and $u = 100$ means 100 monetary units. The excessive claims probability

$$P\{\max_{0 \leq \tau \leq T} [C(\tau) - \tau(p_1 + \lambda)] > u\}$$

equals 0.19972. By multiplying this number by the product probabilities we obtain the table that follows. Each number is a probability that, in a future calendar time interval $0 \leq \tau \leq 10$, the $C(\tau)$ process exceeds allowable values and the $I(\tau)$, $L(\tau)$, and $O(\tau)$ processes exceed or equal allowable values.

A	*R*					
	0.00	0.03	0.04	0.05	0.06	0.07
2.	0.023355	0.008636	0.005973	0.004096	0.002806	0.001934
3.	0.000071	0.000004	0.000002	0.000001	0.000000	0.000000

As in Table 8 of [25], we assume next that the interclaim time distribution function $F(t)$ equals $1 - 0.25e^{-0.4t} - 0.75e^{-2t}$, $t \geq 0$. The claim distribution is the same as in the previous example. We again use the values $\lambda = 0.10$, $A = 2$ or 3, $u = 100$, and $T = 10$. The excessive claims probability equals 0.20929. Since the O.U. probabilities stay the same, the product probabilities would equal those in the table multiplied by 0.20929/0.19972.

A Pareto distribution has been used for claims in [22], [23], [7], and [25]. We will use values of excessive claims probabilities from Tables 5 and 6 of [25]. The individual claim distribution $P(x)$ is equal to $1 - (1 + 2x)^{-3/2}$, $x \geq 0$. Again, $p_1 = 1$. In view of the dangerousness of $P(x)$, we will enlarge λ to 0.30. Assume $A = 2$ or 3, $u = 100$, and $T = 10$. We assume first that the interclaim time distribution $F(t)$ equals

$1 - e^{-t}$, $t \geq 0$. The excessive claims probability then would be 0.11769, and the product probabilities could be arrayed as follows:

A	*R*					
	0.00	0.03	0.04	0.05	0.06	0.07
2.	0.013762	0.005089	0.003520	0.002414	0.001654	0.001140
3.	0.000042	0.000002	0.000001	0.000000	0.000000	0.000000

We now assume that

$$F(t) = 1 - 0.25e^{-0.4t} - 0.75e^{-2t}, \quad t \geq 0 .$$

The claims process probability would be 0.12251. Again the O.U. probabilities would remain the same, and product probabilities would equal those in the above table multiplied by 0.12251/0.11769.

For each of the above examples, the aggregate annual net premium is $100p_1$, and the aggregate annual provision for claim deviations is 100λ. The annual provision for deviations in investment performance would consist of decreasing the assumed interest rate by $A\sigma_I(e^{10\delta} - 1)/(10\delta)$. For our data, the square root of the unbiased estimate of σ_I^2 is 0.00857. If $A = 2$ and $R = 0.03$, $A\sigma_I(e^{10\delta} - 1)/(10\delta) \doteq 0.020$. The aggregate annual provisions for deviations in lapse expenses and operating expenses are $A\sigma_i(e^{10\delta} - 1)/(10\delta)$, where $\sigma_1 = \sigma_L$ and $\sigma_2 = \sigma_O$. Thus the aggregate annual provision for deviations from the four assumptions is $100\lambda + \Sigma_{i=1}^{2} A\sigma_i(e^{10\delta} - 1)/(10\delta)$, coupled with an assumed lower investment earnings rate. A method for distributing the provision equitably among the members of the portfolio is discussed on pages 585–86 of [3].

Two comments should be made about the illustrative data. The presence of strings of plus values and strings of minus values is not inconsistent with the normality assumptions. An examination of tables of normal random numbers will confirm this. Also, pages 73–76 and 84–88 of [10] pertain to comparable strings. Second, it may be bothersome to use a σ_I generated by a favorable string of deviations, as was this case, but the same σ_I applies to the next set of data, which may *not* be favorable.

Many more examples could be constructed. Thus, we have used only a few of the values in Tables 1–20. Other values could prove useful if we changed the relation between operational time and calendar time from $t = 100\tau$ to $t = 50\tau$, 200τ, etc. Moreover, equation (6) allows us to use different A's and R's for the $I(\tau)$, $L(\tau)$, and $O(\tau)$ processes, thus allowing various levels of conservatism and provisions for the effects of inflation. For simplicity, we used the same A's and R's in forming our product probabilities, but that was not necessary. Also, excessive claims probabilities are available in some of the other references listed at the end of this paper, and in other papers. It is hoped that readers will consider examples of interest to their own companies. This is one of the reasons for including the rather extensive set of tables.

VI. ACKNOWLEDGMENTS

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APPENDIX

The same assumptions as in Section IV will be made. We wish to determine the $F(T)$ given by equation (7). Significant use of result (8) will be made.

THEOREM 2. *Let* $\{W(\tau), \tau \geq 0\}$ *be the Wiener process with* $E\{W(\tau)\} = 0$ *for all* $\tau \geq 0$, *and* Covar $\{W(s), W(t)\}$ = minimum (s, t). *Then*

$$F(T) = P\{\underset{0\leq x\leq 1}{\text{supremum}} [(x + (e^{2\beta T} - 1)^{-1})^{-1/2}W(x) - A(1 + x(e^{2\beta T} - 1))^{\delta/2}] \geq 0\} .$$

Proof: We will transform O.U. probabilities into probabilities concerning the Wiener process. From page 138 of [21],

$$\mathrm{L}\{Y(\tau), 0 \leq \tau < \infty\} = \mathrm{L}\{e^{-\tau}W(e^{2\tau}), 0 \leq \tau < \infty\} ,$$

where L stands for probability law. Thus

$$\begin{aligned}\mathrm{L}\{Y(\tau), \tau > 0 \mid Y(0) = 0\} &= \mathrm{L}\{e^{-\tau}W(e^{2\tau}), \tau \geq 0 \mid W(1) = 0\} \\ &= \mathrm{L}\{e^{-\tau}[W(e^{2\tau}) - W(1)], \tau > 0 \mid W(1) = 0\} \\ &= \mathrm{L}\{e^{-\tau}W(e^{2\tau} - 1), \tau > 0\} ,\end{aligned}$$

since the distributions of increments are stationary in time and $W(0) = 0$; see page 94 of [21]. Therefore, one obtains

$$F(T) = P\{\underset{0 \leq \tau \leq \beta T}{\text{supremum}} [e^{-\tau}W(e^{2\tau} - 1) - Ae^{\delta\tau}] \geq 0\} .$$

We first transform τ to $\beta\Delta$, so that

$$F(T) = P\{\underset{0 \leq \Delta \leq T}{\text{supremum}} [e^{-\beta\Delta}W(e^{2\beta\Delta} - 1) - Ae^{\delta\beta\Delta}] \geq 0\} .$$

Then let $x = (e^{2\beta\Delta} - 1)/(e^{2\beta T} - 1)$. This yields

$$\begin{aligned}F(T) = P\{\underset{0 \leq x \leq 1}{\text{supremum}} [&(1 + x(e^{2\beta T} - 1))^{-1/2}W((e^{2\beta T} - 1)x) \\ &- A(1 + x(e^{2\beta T} - 1))^{\delta/2}] > 0\} .\end{aligned}$$

This probability now will be reduced by using the following property: if $W(u)$, $u \geq 0$ is a Wiener process, then distributions for the process $\{W(u), u \geq 0\}$ and the process $\{\Theta^{1/2}W(u/\Theta), u \geq 0\}$ are the same. Thus $F(T)$ becomes

$$\begin{aligned}P\{\underset{0 \leq x \leq 1}{\text{supremum}} [&(x + (e^{2\beta T} - 1)^{-1})^{-1/2}W(x) \\ &- A(1 + x(e^{2\beta T} - 1))^{\delta/2}] \geq 0\} .\end{aligned}$$

THEOREM 3. *Subject to the previous assumptions, $F(T) = G(1)$, where the function $G(t)$, $t \geq 0$, is the solution of the integral equation*

$$\int_0^t \psi\{[f(t) - f(s)]/(t - s)^{1/2}\} dG(s) = \psi[f(t)/t^{1/2}] , \qquad (9)$$

where $f(x) = A[1 + x(e^{2\beta T} - 1)]^{\delta/2}[x + (e^{2\beta T} - 1)^{-1}]^{1/2}$ *and*

$$\psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2) du .$$

Proof: By Theorems 1 and 2, the original O.U. probability $F(T)$ in equation (7) equals the last Wiener probability in Theorem 2. With $f(x)$ as specified, that probability can be rewritten as

$$P\{\underset{0 \leq x \leq 1}{\text{supremum}} [W(x) - f(x)] > 0\}$$

and can be determined through Theorem 1 and section 4 of [20]. Since $\beta T > 0$, $f(x)$ is continuous on [0, 1], and $f(0) > 0$. With the time end-point of 1 replaced by t, the Wiener probability is denoted by $G(t)$ and satisfies the integral equation

$$G(t) = \psi[f(t)/t^{1/2}] + \int_0^t \Phi\{[f(t) - f(s)]/(t-s)^{1/2}\} dG(s) ,$$

where $\Phi(x) = 1 - \psi(x)$. Because $f(0) > 0$ implies that $G(0) = 0, G(t) = \int_0^t dG(s)$, and the integral equation may be rewritten as in the conclusion of Theorem 3.

Numerical Results

The theorem's form of the integral equation is suggested by section 4 of [20]. Divide the interval [0, 1] into subintervals of size $h = 2^{-n}$. The approximating set of equations is

$$\begin{aligned} &\sum_{j=1}^{k} \psi\left\{\left[f(kh) - f\left((2j-1)\frac{h}{2}\right)\right]\left[(2(k-j)+1)\frac{h}{2}\right]^{-1/2}\right\} \\ &\quad \times [G(jh) - G((j-1)h)] \qquad (10) \\ &\qquad = \psi[f(kh)(kh)^{-1/2}] , \quad k = 1, 2, \ldots, 2^n , \end{aligned}$$

and the $G(jh)$'s in the sum are approximate values of "real" $G(jh)$ values. Equation (10) uses the definition of a Stieltjes integral as discussed in various textbooks (see, e.g., pp. 21–22 of [4]). Since $G(0) = 0$, we can use equation (10) to find $G(jh)$ as follows:

$$\begin{aligned} G(kh) = G((k-1)h) + \Big[\!\Big[&\psi[f(kh)(kh)^{-1/2}] \\ &- \left\{\sum_{j=1}^{k-1} \psi\{[f(kh) - f(jh - h/2)][kh - (jh - h/2)]^{-1/2}\}\right. \\ &\left.\times [G(jh) - G((j-1)h)]\right\} \Big]\!\Big] \qquad (11) \\ &\times \Big[\!\Big[\psi\{[f(kh) - f(kh - h/2)](h/2)^{-1/2}\}\Big]\!\Big]^{-1} \\ &\qquad \text{for } k = 1, 2, \ldots, 2^n . \end{aligned}$$

The set of tables was produced using basically this method.

In [6], $g(\tau) \equiv A\sigma$, and hence $f(x) = A[x + (e^{2\beta T} - 1)^{-1}]^{1/2}$. In order to consider inflation, we now are interested in the probability that an O.U. process exceeds or equals $A\sigma e^{\delta\tau}$ for some time τ_0 in $[0, T]$. With $g(\tau) = A\sigma e^{\delta\tau}$, $\delta \geq 0$,

$$f(x) = A[1 + x(e^{2\beta T} - 1)]^{\delta/2\beta}[x + (e^{2\beta T} - 1)^{-1}]^{1/2} .$$

Implementation of the Algorithm

The numerical approximating of $G(1)$ using equation (10), $f(x) = A[1 + x(e^{2\beta T} - 1)]^{\delta/2}[x + (e^{2\beta T} - 1)^{-1}]^{1/2}$, $A = 0.5, 1.0, 2.0, 3.0$, and $\beta T = 1, 5, 7, 10$ leads to difficulties. Essentially, n must be around 15 in order to obtain reasonable approximations of $G(1)$. Computational experience using the FORTRAN language on the DEC system-10 indicated that more than 150 hours of computing time would be required, with results that would be uncertain.

A study of the behavior of $G(x)$ for $A = 0.5$ and $\beta T = 1, 5, 7, 10$ indicated that $G(1)$ is near 1, and that for x near zero, say 2^{-20}, $G(x) \approx 0.94$. Therefore, $G(x)$ was calculated using small h for x near zero and larger h as we move away from zero. The following method with varying step size, which gave favorable results in reasonable computing time, was implemented.

Assume that we have fixed integers n and p, such that $n > p > 0$. Let $m = n - p$ and $h = h(0) = 2^{-n}$. We find $G(1)$ by using $m + 1$ different step sizes of $2^{-n}, 2^{-n+1}, \ldots, 2^{-n+m}$. By dividing the interval $[0, 1]$ into subintervals with endpoints $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$, the approximating set of equations (10) becomes

$$\sum_{j=1}^{k} \psi\{[f(x_k) - f((x_j + x_{j-1})/2)][x_k - (x_j + x_{j-1})/2]^{-1/2}\}$$
$$\times [G(x_j) - G(x_{j-1})] = \psi[f(x_k)x_k^{-1/2}], \qquad k = 1, 2, \ldots, N$$

The subintervals are chosen in the following way:

$[0, 2^{-m}]$ is divided into 2^{n-m} subintervals of size 2^{-n},

$[2^{-m}, 2^{-m+1}]$ is divided into 2^{n-m-1} subintervals of size 2^{-n+1},

$[2^{-m+1}, 2^{-m+2}]$ is divided into 2^{n-m-1} subintervals of size 2^{-n+2},

. . . .

$[2^{-1}, 1]$ is divided into 2^{n-m-1} subintervals of size 2^{-n+m}.

Thus, we have $N = 2^{n-m}(1 + m/2)$, and we require $2^{n-m}(1 + m/2)$ evaluations of $G(x)$ in order to approximate $G(1)$.

By this method the tables in Section III were produced. These results were produced with 88.22 minutes of central processor time on a DEC system-10. The time required for each table is shown in the table. The principal parameter βT dictated the values of m and n so that convergence was demonstrated as n increased. With the aid of relation (8), the tables in Section III use merely the parameter T rather than βT. Various other tables were computed for each T in order to determine reasonable range values for m and n. In particular, tables were computed for each T with the indicated values of n and the various values of $n - m$ so that there was little change as $n - m$ increased.

All computing was done in FORTRAN on the DEC system-10 computer at Ball State University. The function $\psi(x)$ was approximated by use of the error function DERF. Since DERF was not available in the DEC system-10 library, the error function of the IBM System/360 FORTRAN IV library was coded in FORTRAN for use in this application [15].

DISCUSSION OF PRECEDING PAPER

HARRY H. PANJER AND DAVID R. BELLHOUSE:*

Professors Beekman and Fuelling are to be commended for their thorough treatment of the theory and of the corresponding simulation. We shall address ourselves not to any of the detailed calculations but, more generally, to the model that was used and to the numerous assumptions that were made in order to simplify the mathematical and statistical results. Finally we shall present some alternative models and corresponding results.

Simulation is a tool with which the actuary has great familiarity. Virtually all aspects of life insurance company operations have been analyzed with the use of simulation methods. Although simulation has a great many disadvantages (the most important being the large amount of computer time required to carry out a simulation to a high degree of accuracy), it has some significant advantages over analytic methods. First, the actuary who uses simulation methods need not have knowledge of high-powered mathematical and statistical results. Indeed, the actuary need only specify a model that he considers most appropriate for the *real* situation under study, pass the specifications to a computer programmer, and wait for the results. Typically the actuary will obtain a number of sets of simulation results, each based on a different set of assumptions. The actuary then can study the sensitivity of the results to the various assumptions. Simulation, then, is a "black box" that gives a set of numerical results as output for a set of numerical input. The actuary studies the relationship between the input and the output.

Some actuarial problems lend themselves to analytic solutions. When problems are solved analytically, models of the "black box" are chosen and analyzed by the use of mathematical techniques. The emphasis is on the development of an understanding of the workings of the black box and of the way in which any set of input is processed to produce a set of output. Analytic methods are used to prove theorems, thereby producing a greater understanding of the model being studied.

Frequently, the real situation being studied is felt to be too complex to be dealt with analytically. In such situations assumptions may be made that simplify the analysis. One then must examine the analytic results in the light of the simplifying assumptions.

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Professors Beekman and Fuelling have in the first part of the paper made a number of simplifying assumptions to obtain an expression for the probability of the occurrence of a certain risk. The primary results are the numerical values of this probability given in the large number of tables in the paper. The numerical results were obtained by using simulation methods. It is our feeling that, if simulation is to be used, either the model should be made as realistic as possible (that is, the number of simplifying assumptions should be minimized) or the effect of the various assumptions should be studied. Neither is done in this paper. The authors introduce a simplified model and then proceed to carry out simulations based on this model.

We now examine these simplifying assumptions and have the following observations:

1. The processes $I(\tau)$, $O(\tau)$, and $L(\tau)$ were assumed to be independent of one another. It is well known to any practical actuary that high interest rates signal inflation, which in turn results in high expense rates. Thus interest rates and expenses are highly correlated with each other. The assumption that the processes are mutually independent allows the variances to be added in the authors' equation (4). A high correlation between these three processes could have a significant impact on the numerical results obtained.
2. We would like to question the Markovian property that the three interest and expense processes are assumed to have. We have studied a large number of annual investment yield series and found many of them to be non-Markovian. We believe that this assumption also may have a significant impact on the numerical results obtained by the authors using simulation.
3. We believe that the assumption that β has the same value for each process also may be overly restrictive. It allows the process $S(t)$ to be of the same type, which simplifies the analysis but may have a significant impact on the numerical results.
4. We question the appropriateness of calculating probabilities of the type calculated by the author in Section III of the paper. Inflation in claims will be highly correlated with inflation in expenses and in investment yields. It might be more appropriate to allow the barrier to be a stochastic process also. Probabilities of the type calculated in the authors' equation (5) then involve four, not three, stochastic processes and a constant boundary.
5. We question the use of a continuous time model. Insurance companies deal almost exclusively in annual, semiannual, or quarterly bases. The investment returns that would be appropriate for calculations are actual returns over the annual, semiannual, or quarterly period. These actual returns are themselves averages of the continuous time process. A "spike" at a single point in time could cause the process $S(\tau)$ to exceed the bound $AKe^{\delta\tau}$. In practice, however, when the investment returns are averaged over the annual, semiannual, or quarterly periods, the rate of return may be such

that the bound is not exceeded. The periodic statements tell the real story to the company. On the basis of these arguments, we believe that it would be more useful to let $S(\tau)$ and its component processes be discrete processes that reflect values obtained over the period in question. Similarly, the probabilities that are important are of the same type as equation (5) of the authors' paper, but the maximum need only be over a finite set of values of the time variable τ. For example, if annual results are of interest and a ten-year period is used, only ten values need be generated and compared with the bound. This simplifies greatly the simulation problem.

We now present a class of alternative models that address each of the five criticisms of the authors' model. The models are the time-series models of [1]. Using the notation of Miller and Hickman [4] to present the models, let $\dot{Z}_{1t}$, $\dot{Z}_{2t}$, and $\dot{Z}_{3t}$ denote $I(t)$, $O(t)$, and $L(t)$, respectively, $t = 1, 2, 3, \ldots, T$. The first-order autoregressive process $AR(1)$ as described in [4] is of the form $\dot{Z}_{it} = \phi_{i1}\dot{Z}_{i,t-1} + a_{it}$, where a_{it}, $t = 1, 2, \ldots, T$, are independently and identically distributed normal variables with zero mean and variance σ_{ii}. Conditioning on $\dot{Z}_{i,0} = 0$ yields $E(\dot{Z}_{it}) = 0$ and $E(\dot{Z}_{it}\dot{Z}_{i,t-u}) = \sigma_{ii}\phi_{i1}^{u}$. On letting $\beta_i = -\ln \phi_{1i}$, we obtain the correlation structure of the Ornstein-Uhlenbeck (O.U.) process (but in discrete time). Note also that if in general $\dot{Z}_{is} = x$, then $E(\dot{Z}_{it}|\dot{Z}_{is} = x) = x\phi_{i1}^{t-s} = xe^{-\beta_i(t-s)}$, $\beta_i > 0$, as in the authors' paper.

At this point we consider the correlations between the processes $I(t)$, $O(t)$, and $L(t)$. For any time t, assume that the vector (a_{1t}, a_{2t}, a_{3t}) is a multivariate normal variable with zero means and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix},$$

where σ_{ij} represents the covariance between a_{it} and a_{jt} at any time t. We also assume that the vectors $(a_{1t}, a_{2t}, a_{3t})'$ and $(a_{1s}, a_{2s}, a_{3s})'$ are independent of each other for $t \neq s$. This model is a direct generalization of the model of Beekman and Fuelling. If we set $\sigma_{ij} = 0$, $i \neq j$, and $\beta_1 = \beta_2 = \beta_3 = \beta$, we have (in discrete time) the three O.U. processes in the paper.

It is now easy to see how to generalize this to more complex time-series models. The second-order autoregressive model $AR(2)$ is of the form

$$\dot{Z}_{it} = \phi_{i1}\dot{Z}_{i,t-1} + \phi_{i2}\dot{Z}_{i,t-2} + a_{it}.$$

This is a process that is non-Markovian, since the current value depends on two prior values. Other time-series models can be generated in a similar fashion. By using the O.U. model, the authors have restricted consideration to models of a single type. The use of time-series models allows the three processes to be of different types, with their cross-covariances given by the matrix Σ.

The probabilities

$$P\{\max_{t=1,2,\ldots,T}(-\dot{Z}_{1t}+\dot{Z}_{2t}+\dot{Z}_{3t})>ke^{\delta t}\}$$

can be simulated quite easily using the following algorithm:

Step 1: Generate three independent standard normal variates, say $\tilde{a}_{1t}$, $\tilde{a}_{2t}$, and $\tilde{a}_{3t}$. This may be done using the Box-Müller [2] transformation (see [5] for some cautionary remarks) or the method of Marsaglia and Bray [3].

Step 2: Calculate

$$\begin{pmatrix} a_{1t} \\ a_{2t} \\ a_{3t} \end{pmatrix} = \Sigma^{1/2}\begin{pmatrix} \tilde{a}_{1t} \\ \tilde{a}_{2t} \\ \tilde{a}_{3t} \end{pmatrix},$$

where $\Sigma^{1/2}\,\Sigma^{1/2}=\Sigma$. The determination of $\Sigma^{1/2}$ will be considered later.

Step 3: Calculate $\dot{Z}_{it}$, $i=1, 2, 3$, for the time-series models used; for example,

$$\dot{Z}_{it}=\phi_{i1}\dot{Z}_{i,t-1}+a_{it}$$

or

$$\dot{Z}_{it}=\phi_{i1}\dot{Z}_{i,t-1}+\phi_{i2}\dot{Z}_{i,t-2}+a_{it}\,,$$

where $\dot{Z}_{i0}=\dot{Z}_{i,-1}=0$.

Step 4: Calculate $-\dot{Z}_{1t}+\dot{Z}_{2t}+\dot{Z}_{3t}$ and compare it with the bound $ke^{\delta t}$. If it exceeds the bound, stop the simulation trial; if not, continue to the next time period.

Step 5: Count the number of trials for which the simulation trial was stopped (that is, the bound was exceeded) and compare with the total number of trials.

The matrix $\Sigma^{1/2}$ is of the form QD, where

D is a diagonal matrix with entries $\lambda_i^{1/2}$;
λ_i is the ith eigenvalue of Σ;
the ith column of Q is π_i, the eigenvector associated with the ith eigenvalue.

Eigenvalues and eigenvectors can be found for a general $n\times n$ symmetric matrix using standard mathematical and statistical computer packages. However, for a 3×3 matrix the eigenvalues and eigenvectors may be found directly. The eigenvalues are the solution of

$$|\Sigma-\lambda I|=0\,,$$

which yields

$$C_3\lambda^3+C_2\lambda^2+C_1\lambda+C_0=0\,;$$

here

$$C_3 = -1\,;$$
$$C_2 = \sigma_{33} + \sigma_{22} + \sigma_{11}\,;$$
$$C_1 = \sigma_{13}^2 + \sigma_{12}^2 + \sigma_{23}^2 - \sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{33} - \sigma_{11}\sigma_{22}\,;$$
$$C_0 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{13}^2\sigma_{22} - \sigma_{23}^2\sigma_{11} - \sigma_{12}^2\sigma_{33}\,.$$

The cubic can be solved numerically to find the three eigenvalues. If $\pi = (\pi_1, \pi_2, \pi_3)'$ is the eigenvector associated with any eigenvalue λ, then $(\Sigma - \lambda I)\pi = 0$. The first two equations of this system of simultaneous equations yield

$$(\sigma_{11} - \lambda)\pi_1 + \sigma_{12}\pi_2 + \sigma_{13}\pi_3 = 0\,,$$
$$\sigma_{12}\pi_1 + (\sigma_{22} - \lambda)\pi_2 + \sigma_{23}\pi_3 = 0\,,$$

or

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} = \frac{-\pi_3}{[(\sigma_{11} - \lambda)(\sigma_{22} - \lambda) - \sigma_{12}^2]}\begin{bmatrix}\sigma_{22} - \lambda & -\sigma_{12}\\ -\sigma_{12} & \sigma_{11} - \lambda\end{bmatrix}\begin{pmatrix}\sigma_{13}\\ \sigma_{23}\end{pmatrix}.$$

The third equation,

$$\sigma_{13}\pi_1 + \sigma_{23}\pi_2 + (\sigma_{33} - \lambda)\pi_3 = 0\,,$$

is satisfied automatically. Since $\Sigma\, \pi_i^2 = 1$, we set $\tilde{\pi}_3 = 1$, solve for $\tilde{\pi}_1$ and $\tilde{\pi}_2$, and obtain the π_i's from

$$\pi_i = \tilde{\pi}_i/\sqrt{(\tilde{\pi}_1^2 + \tilde{\pi}_2^2 + \tilde{\pi}_3^2)}\,.$$

This can be done for each of the three eigenvalues, and $\Sigma^{1/2}$ can be calculated as QD.

Up to this point we have not addressed the question of stochastic inflation, which would result in a stochastic barrier. If we let $\dot{Z}_{4t}$ be a corresponding stochastic process representing the boundary, the probabilities will involve four stochastic processes that are mutually correlated. The same algorithm (modified to four variables) can be used for the simulation.

Any company considering this approach would have to estimate the elements in Σ and the coefficients in the time-series models on the basis of its own data. There are standard statistical procedures available to handle the estimation. Calculation of the probabilities by simulation then can be carried out using these estimates.

We would like to thank the authors for an extremely stimulating paper. It caused us to have many hours of fruitful discourse, resulting in the alternate model presented in this discussion.

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(AUTHORS' REVIEW OF DISCUSSION)

JOHN A. BEEKMAN AND CLINTON P. FUELLING:

The authors are grateful that Professors Panjer and Bellhouse have provided a stimulating discussion. There are some good suggestions contained in their remarks, and we will focus on them after several preliminary comments.

Professors Panjer and Bellhouse state that the paper used simulation. However, the tables in the paper were derived by approximating numerically the solution of an integral equation. The authors used a large amount of computer time in the simulations reported in reference [5] of the paper. One of the main purposes of the present paper was to provide enough basic tables so that users of the model would not have to spend computer time in simulations.

Professors Panjer and Bellhouse provide five criticisms of the assumptions used in the model. Although four of these have value, the third criticism is not valid. Tables 1–20 are all for β-values of 1, but those tables are only basic tools in the application of the model. Beneath Table 20 is the sentence, "Seldom would data produce $\beta = 1$, but a method for handling this problem is discussed and illustrated in Section V." Section IV cautions the reader that it would be unusual if real data allowed the actuary to conclude that the three β-values were the same, but utilizes the independence of the four processes in rewriting a four-process probability as a product of four individual process probabilities in equation (6). Theorem 1 of that section provides the theory for handling probabilities where $\beta \neq 1$. Section V provides further theory for handling β-values not equal to 1. Furthermore, the illustrative data produced different beta values for the investment, operating expense, and lapse expense stochastic processes, as shown in Tables 21 and 22. Tables 1–20 and equation (8) of Section IV provide readers with enough values and theory to approximate most of the probability values needed in risk managers' use of the model.

The authors appreciate the concerns expressed in the other four criticisms. However, a few comments could be made to balance the discussion. Reference [3] provided some results applicable when the stochastic processes are not independent, especially on pages 578–79 and formula (3) on page 575. It was acknowledged on page 577 of [3] that the Markovian property is an approximation. However, the choice of the particular Markov process, namely, the Ornstein-Uhlenbeck process, had several advantages. Its conditional mean function helps to model phenomena that react to offset excessive movements in any one direction, which is true of many economic phenomena. It also tied in with time-series analysis, as mentioned in Section II of the paper. Professors Panjer and Bellhouse suggest a stochastic barrier for the multirisk process, and we agree that this would be an interesting project. A discrete time model has some advantages, but much of risk theory is done on a continuous time basis and has proved useful.

The authors feel that the time-series models proposed by Professors Panjer and Bellhouse are useful alternative models for risk managers considering the fourfold risk problem. Their suggestion that stochastic inflation would result in a stochastic barrier is a challenging research idea. We hope that they pursue the idea further.

In summary, the authors appreciate the time and effort given by Professors Panjer and Bellhouse to studying the multirisk problem and to advancing possible solutions for risk managers to use.