TRANSACTIONS OF SOCIETY OF ACTUARIES
1974 VOL. 26 PT. 1 NO. 76

RELATIONSHIPS AMONG THE FULLY CONTINUOUS, THE DISCOUNTED CONTINUOUS, AND THE SEMICONTINUOUS RESERVE BASES FOR ORDINARY LIFE INSURANCE

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ABSTRACT

There seems to be much confusion because of the existence of several distinct yet interrelated "continuous" reserve bases for ordinary life insurance. This paper develops explicit relationships connecting these reserve bases, and, in the course of this development, derives several new and interesting actuarial relationships, all of which can be generalized to the case of $m$thly premium payments.

INTRODUCTION

THE "continuous" reserve basis for ordinary life insurance, as described in Jordan's textbook,[1] results from a whole life insurance payable at the moment of death and purchased by continuous premiums. Published "continuous" mean reserves,[2] however, are not calculated using that system directly, but instead employ a "discounted continuous" reserve basis, which involves discrete payment of premium and also a premium refund feature. In addition, there is a third form of "continuous" reserve basis, the semicontinuous, which is similar to the "discounted continuous" basis, but does not involve a premium refund feature.

The existence of these three distinct yet interrelated "continuous" reserve bases, as well as the lack of standard nomenclature by which to clearly differentiate among them, has been a source of much confusion, as indicated by the following quotation from a recent *Transactions* article:[3] "Several times in the past few years, since our adoption of the continuous functions basis for reserve determination, I have been asked to explain

[1] C. W. Jordan, Jr., *Life Contingencies* (Chicago: Society of Actuaries, 1967), p. 113.

[2] Society of Actuaries, *Monetary Tables Based on 1958 CSO Mortality Table and Interest, Continuous Functions*, Vol. II: *Premiums and Reserves by Net Level Premium Method* (1961).

[3] LaVerne W. Cain, Discussion of paper by Lauer, *TSA*, XIX (1967), 140.

how continuous functions provide for a premium refund at death. Also I have noted, as Lauer has, some confusion because of the existence of two net premiums associated with the use of continuous functions. This confusion has existed even among actuarial students and Fellows."

It is the purpose of this paper to develop explicit relationships connecting these three reserve bases, in the interest of dispelling such confusion. In the course of this development, several new and interesting actuarial relationships are derived.

Let us define at the outset the three "continuous" reserve bases that we seek to relate.

1. The *fully continuous* reserve basis, described on page 113 of Jordan's textbook,[4] involves a premium of $\bar{P}(\bar{A}_x)$, payable continuously throughout the year, and a death benefit of 1, payable at the moment of death.
2. The *semicontinuous* reserve basis, not directly discussed by Jordan, involves an annual premium of $P(\bar{A}_x)$, payable at the beginning of the year, and a death benefit of 1, payable at the moment of death.
3. The *discounted continuous* reserve basis, not directly discussed by Jordan but, as will be seen, underlying the published continuous mean reserves, involves an annual premium of $(d/\delta)\bar{P}(\bar{A}_x)$, payable at the beginning of the year, a death benefit of 1, payable at the moment of death, and a premium refund benefit of $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$, also payable at the moment of death. Here s represents the moment of death as measured from the beginning of the year of death, and thus $0 < s \leq 1$.

From compound interest theory it is clear that a premium of (d/δ) $\bar{P}(\bar{A}_x)$, or $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$, paid at the beginning of the year is equivalent in each year, except the year of death, to a premium of $\bar{P}(\bar{A}_x)$ paid continuously throughout the year.

The appropriateness of choosing $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$ for the amount of the premium refund benefit at the moment of death in order to maintain this equivalence between the premiums paid (less any premium refund) under the discounted continuous basis and the premiums paid under the fully continuous basis has been demonstrated by John M. Boermeester in a paper dating back to 1949.[5]

[4] Jordan refers to reserves on this basis simply as continuous reserves. I have introduced here the modifier "fully," and in definitions 2 and 3, respectively, I have introduced the modifiers "semi" and "discounted," in order to clearly distinguish among the three bases.

[5] John M. Boermeester, "Actuarial Note: Certain Implications Which Arise when the Assumption Is Made that Premiums Are Paid Continuously and Death Benefits Are Paid at the Moment of Death," *TASA*, L (1949), 73.

For convenience, and because this specific choice of premium refund benefit is central to our discussion, I have given below the reasoning supporting it.

Under the fully continuous basis, the present value at the beginning of the year of death of premium paid in that year is $\bar{P}(\bar{A}_x)\bar{a}_{\overline{s}|}$, assuming death occurs at moment s. Under the discounted continuous basis a premium of $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$ is paid at the beginning of the year of death. The value of the "overpayment" of premium at the beginning of the year, therefore, is $\bar{P}(\bar{A}_x)(\bar{a}_{\overline{1}|} - \bar{a}_{\overline{s}|})$. Hence the value of the "overpayment" of premium at the moment of death is $\bar{P}(\bar{A}_x)(\bar{a}_{\overline{1}|} - \bar{a}_{\overline{s}|})(1 + i)^s$, or, equivalently, $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$, which is therefore chosen as the appropriate premium refund.

Let us now determine the present value of this premium refund benefit. The present value at issue of the premium refund benefit in policy year $t + 1$ is

$$v^t \; {}_tp_x \int_0^1 v^s \; {}_sp_{x+t}\mu_{x+t+s}[\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}]ds \; .$$

The present value at issue of the premium refund benefit in all policy years under a whole life policy is therefore

$$\sum_{t=0}^{\omega} v^t \; {}_tp_x \int_0^1 v^s \; {}_sp_{x+t}\mu_{x+t+s}[\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}]ds \; .$$

Factoring out $\bar{P}(\bar{A}_x)$ and substituting $(1 - v^{1-s})/\delta$ for $\bar{a}_{\overline{1-s}|}$, we have

$$\bar{P}(\bar{A}_x)\sum_{t=0}^{\omega} v^t \; {}_tp_x \int_0^1 v^s \; {}_sp_{x+t}\mu_{x+t+s}\left(\frac{1 - v^{1-s}}{\delta}\right) ds \; .$$

Factoring out $1/\delta$ and multiplying, we obtain

$$\frac{\bar{P}(\bar{A}_x)}{\delta}\sum_{t=0}^{\omega} v^t \; {}_tp_x\left(\int_0^1 v^s \; {}_sp_{x+t}\mu_{x+t+s}ds - \int_0^1 v \; {}_sp_{x+t}\mu_{x+t+s}ds\right) .$$

Substituting values for the two integrals, we have, more briefly,

$$\frac{\bar{P}(\bar{A}_x)}{\delta}\sum_{t=0}^{\omega} v^t \; {}_tp_x(\bar{A}^{\;1}_{x+t:\overline{1}|} - A^{\;1}_{x+t:\overline{1}|}) \; .$$

This gives us finally, as the present value of the premium refund benefit (call it $\bar{A}_x^{\mathrm{PR}}$),

$$\bar{A}_x^{\mathrm{PR}} = \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_x - A_x) \; . \qquad (1)$$

This simple expression can be established by general reasoning as follows: $\bar{P}(\bar{A}_x)/\delta$ represents the present value of a continuous perpetuity of

$\bar{P}(\bar{A}_x)$ per year. $[\bar{P}(\bar{A}_x)/\delta]\bar{A}_x$ represents the present value of a death benefit of $[\bar{P}(\bar{A}_x)/\delta]$, payable at the moment of death. Therefore, it follows that $[\bar{P}(\bar{A}_x)/\delta]\bar{A}_x$ represents the present value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ per year, beginning at the moment of death.

Similarly, $[\bar{P}(\bar{A}_x)/\delta]A_x$ is the present value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ per year, beginning at the end of the year of death. The difference between them, $[\bar{P}(\bar{A}_x)/\delta]\ (\bar{A}_x - A_x)$, therefore, is the present value of a continuous annuity-certain of $\bar{P}(\bar{A}_x)$ per year, beginning at the moment of death and ending at the end of the year of death. But the value of this annuity-certain at the moment of death is $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$, which is recognized as precisely the value of the premium refund benefit at the moment of death. Hence equation (1) is established.

Let us now return to the analysis of the premium refund benefit. If the present value or net single premium for the premium refund benefit is

$$\bar{A}_x^{\text{PR}} = \frac{\bar{P}(\bar{A}_x)}{\delta}\,(\bar{A}_x - A_x)\,,$$

then clearly the net annual premium for the premium refund benefit is

$$P(\bar{A}_x^{\text{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta\ddot{a}_x}\,(\bar{A}_x - A_x)\,. \tag{2}$$

Or, incorporating the $\ddot{a}_x$ in the parentheses, we have

$$P(\bar{A}_x^{\text{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}\,[P(\bar{A}_x) - P_x]\,. \tag{3}$$

As described previously, under the discounted continuous basis there is an annual premium of $(d/\delta)\bar{P}(\bar{A}_x)$, and there are two benefits: a death benefit of 1, payable at the moment of death, and a premium refund benefit, also payable at the moment of death.

We have seen that an annual premium of $[\bar{P}(\bar{A}_x)/\delta]\ [P(\bar{A}_x) - P_x]$ provides for the premium refund benefit. Therefore, the remainder of the annual premium under the discounted continuous basis, namely,

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}\,[P(\bar{A}_x) - P_x]\,,$$

must provide for the death benefit of 1, payable at the moment of death. But, since this remainder is itself an annual premium, providing a death benefit of 1 payable at the moment of death, it must therefore be equal to $P(\bar{A}_x)$. That is,

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}\,[P(\bar{A}_x) - P_x] = P(\bar{A}_x)\,. \tag{4}$$

It is easy to prove this algebraically, as follows: We begin with the identity

$$\frac{d}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right) - \frac{\bar{A}_x}{\ddot{a}_x \delta \bar{a}_x}(d\ddot{a}_x - \delta\bar{a}_x) = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{5}$$

Adding and subtracting 1 in the parentheses and rearranging, we obtain

$$\frac{d}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right) - \frac{\bar{A}_x}{\ddot{a}_x \delta \bar{a}_x}[(1 - \delta\bar{a}_x) - (1 - d\ddot{a}_x)] = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{6}$$

Recognizing that $\bar{A}_x = 1 - \delta\bar{a}_x$ and $A_x = 1 - d\ddot{a}_x$, we have

$$\frac{d}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right) - \frac{\bar{A}_x}{\ddot{a}_x \delta \bar{a}_x}(\bar{A}_x - A_x) = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{7}$$

Incorporating the $\ddot{a}_x$ in the parentheses, and making the substitutions $\bar{P}(\bar{A}_x) = \bar{A}_x/\bar{a}_x$, $P(\bar{A}_x) = \bar{A}_x/\ddot{a}_x$, and $P_x = A_x/\ddot{a}_x$, we obtain the desired relationship:

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x] = P(\bar{A}_x). \tag{8}$$

Considered as a separate insurance, the premium refund benefit $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$, and its associated net annual premium $[\bar{P}(\bar{A}_x)/\delta]\,[P(\bar{A}_x) - P_x]$, determine an annual reserve.

The existence of this reserve has been alluded to previously in the literature.[6] Let us now determine an explicit expression for this reserve. Since the net single premium for the premium refund benefit is $[\bar{P}(\bar{A}_x)/\delta]\,(\bar{A}_x - A_x)$ and the net annual premium for the premium refund benefit is $[\bar{P}(\bar{A}_x)/\delta]\,[P(\bar{A}_x) - P_x]$, it follows that the tth-year terminal reserve for the premium refund benefit is

$${}_tV(\bar{A}_x^{\text{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x]. \tag{9}$$

Equation (9) can be established directly as follows: On the basis of the general-reasoning derivation of equation (1), the present value of the premium refund benefit at duration t is

$${}_t\bar{A}_x^{\text{PR}} = \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_{x+t} - A_{x+t}). \tag{10}$$

[6] J. Alan Lauer, "Apportionable Basis for Net Premiums and Reserves," *TSA*, XIX (1967), 22; John C. Fraser, Walter N. Miller, and Charles M. Sternhell, "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance," *TSA*, XXI (1969), 451.

Hence the tth-year prospective terminal reserve for the premium refund benefit is

$$ {}_tV(\bar{A}_x^{\mathrm{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_{x+t} - A_{x+t}) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x]\ddot{a}_{x+t} . \quad (11) $$

Factoring out $\bar{P}(\bar{A}_x)/\delta$ and rearranging, we have

$$ {}_tV(\bar{A}_x^{\mathrm{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}\{[\bar{A}_{x+t} - P(\bar{A}_x)\ddot{a}_{x+t}] - [A_{x+t} - P_x\ddot{a}_{x+t}]\} . \quad (12) $$

But the expression in the first pair of brackets equals ${}_tV(\bar{A}_x)$, and the expression in the second pair of brackets equals ${}_tV_x$. Thus we have confirmed the reserve equation:

$$ {}_tV(\bar{A}_x^{\mathrm{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] . \quad (13) $$

The reserve corresponding to the death benefit of 1, payable at the moment of death, and its associated net annual premium of $P(\bar{A}_x)$, is of course ${}_tV(\bar{A}_x)$.

Thus the discounted continuous reserve basis may be viewed as a combination of two completely separate benefits, each having its own premium and its own annual reserves: (1) the premium refund benefit, which has the net annual premium $P(\bar{A}_x^{\mathrm{PR}})$, equal to $[\bar{P}(\bar{A}_x)/\delta]\,[P(\bar{A}_x) - P_x]$, and the terminal reserve ${}_tV(\bar{A}_x^{\mathrm{PR}})$, equal to $[\bar{P}(\bar{A}_x)/\delta]\,[{}_tV(\bar{A}_x) - {}_tV_x]$, and (2) the death benefit of 1, payable at the moment of death, which has the net annual premium

$$ \frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x] , $$

or $P(\bar{A}_x)$, and the terminal reserve ${}_tV(\bar{A}_x)$.

The value of the death benefit of 1, payable at the moment of death, is the same under the discounted continuous basis and the fully continuous basis. Also, recall that the specific premium refund benefit of $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$ was chosen to make the value of the premiums (less the value of the premium refund benefit) under the discounted continuous basis equal to the value of the premiums under the fully continuous basis. Therefore, the discounted continuous terminal reserve will be equal to the fully continuous terminal reserve. But, by our analysis above, the discounted continuous terminal reserve equals the sum of the premium refund benefit reserve and the death benefit reserve. Thus we will have

$$ {}_tV(\bar{A}_x^{\mathrm{PR}}) + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x) . \quad (14) $$

Or, substituting the expression for ${}_tV(\bar{A}_x^{\text{PR}})$ from equation (9), we obtain

$$\frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)\,. \tag{15}$$

Equation (15) may be established directly as follows: Let us begin with the identity

$$\begin{aligned}\bar{A}_x\left(\frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t} + \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t}& \\ = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t}\,.&\end{aligned} \tag{16}$$

Factoring out $\bar{A}_x/\delta\bar{a}_x$ from the first two terms, we have

$$\begin{aligned}\frac{1}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right)\left[\left(\frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right)\delta\bar{a}_x - \delta\bar{a}_{x+t}\right] + \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t}& \\ = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t}\,.&\end{aligned} \tag{17}$$

Substituting $1 - \bar{A}_x$ for $\delta\bar{a}_x$ and $\bar{A}_{x+t} - 1$ for $-\delta\bar{a}_{x+t}$ in equation (17) produces

$$\begin{aligned}\frac{1}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right)\left[\frac{\ddot{a}_{x+t}}{\ddot{a}_x}(1 - \bar{A}_x) + \bar{A}_{x+t} - 1\right] + \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t}& \\ = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t}\,.&\end{aligned} \tag{18}$$

Multiplying and rearranging inside the brackets in equation (18), we obtain

$$\begin{aligned}\frac{1}{\delta}\left(\frac{\bar{A}_x}{\bar{a}_x}\right)\left\{\left[\bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t}\right] - \left[1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right]\right\} + \bar{A}_{x+t}& \\ - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t} = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t}\,.&\end{aligned} \tag{19}$$

Substituting

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}\,, \qquad {}_tV(\bar{A}_x) = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\ddot{a}_x}\right)\ddot{a}_{x+t}\,,$$

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\,, \qquad {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \left(\frac{\bar{A}_x}{\bar{a}_x}\right)\bar{a}_{x+t}$$

in equation (19) results in the desired relationship:

$$\frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)\,. \tag{20}$$

Using the usual mean reserve formula (the mean of successive terminal reserves plus the amount of net premium paid for any period beyond the date of valuation)[1] and making use of equation (20), we have the following mean reserve expressions for the several reserve bases (including the curtate, for comparison) that we have been considering:

Reserve Basis	tth-Year Mean Reserve
Fully continuous	$\frac{1}{2}[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x)]$
Semicontinuous	$\frac{1}{2}[{}_{t-1}V(\bar{A}_x) + {}_tV(\bar{A}_x) + P(\bar{A}_x)]$
Discounted continuous	$\frac{1}{2}[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x) + (d/\delta)\bar{P}(\bar{A}_x)]$
Curtate	$\frac{1}{2}({}_{t-1}V_x + {}_tV_x + P_x)$

Since published continuous mean reserves[2] are calculated from the formula $\frac{1}{2}[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x) + (d/\delta)\bar{P}(\bar{A}_x)]$, it is clear that the published continuous mean reserves are simply the mean reserves for the discounted continuous reserve basis.

Finally, it should be noted that the equivalence between the discounted continuous basis and the fully continuous basis with respect to the value of premiums (less any premium refund) and with respect to terminal reserves does not depend upon the annual payment of premiums in the former basis. The verbal and algebraic reasoning presented throughout the paper has been general in nature, so that the expressions and relationships derived can be generalized to the case of $m$thly payment of premiums under the discounted continuous basis.

What is contemplated under this generalized discounted continuous basis is an annual premium of $(d^{(m)}/\delta)\bar{P}(\bar{A}_x)$, payable $m$thly, that is, the actual amount of each premium payment is $[(d^{(m)}/\delta)\bar{P}(\bar{A}_x)]/m$; a death benefit of 1, payable at the moment of death; and a premium refund benefit of $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m-s}|}$, also payable at the moment of death. Here s represents the moment of death as measured from the beginning of that $m$thly part of a year in which death occurs, and thus $0 < s \leq 1/m$.

The analogy with the discounted continuous basis involving annual premiums, discussed throughout the paper, is clear. The generalized expressions and relationships arising from the above basis, together with their annual premium analogues, previously derived, are shown in Table 1.

TABLE 1

DISCOUNTED CONTINUOUS RESERVE BASIS

Item	Annual Premiums	$m$thly Premiums
1. Discounted continuous annual premium	$\frac{d}{\delta}\bar{P}(\bar{A}_x)$, payable annually	$\frac{d^{(m)}}{\delta}\bar{P}(\bar{A}_x)$, payable $m$thly
2. Premium refund benefit	$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}\rceil}$, $0 < s \leq 1$, where s is the moment of death as measured from the beginning of the year of death	$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m-s}\rceil}$, $0 < s \leq 1/m$, where s is the moment of death as measured from the beginning of that $m$thly part of a year in which death occurs
3. Net single premium (present value) for the premium refund benefit	$\bar{A}_x^{\text{PR}} = \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_x - A_x)$	$\bar{A}_x^{\text{PR},m} = \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_x - A_x^{(m)})$
4. Net annual premium for the premium refund benefit	$P(\bar{A}_x^{\text{PR}}) = \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x]$ payable annually	$P^{(m)}(\bar{A}_x^{\text{PR},m}) = \frac{\bar{P}(\bar{A}_x)}{\delta}[P^{(m)}(\bar{A}_x) - P^{(m)}(A_x^{(m)})]$, payable $m$thly
5. Remainder of discounted continuous annual premium (semicontinuous annual premium)	$\frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x] = P(\bar{A}_x)$, payable annually	$\frac{d^{(m)}}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P^{(m)}(\bar{A}_x) - P^{(m)}(A_x^{(m)})]$ $= P^{(m)}(\bar{A}_x)$, payable $m$thly

TABLE 1—*Continued*

Item	Annual Premiums	*m*thly Premiums
6. Premium refund terminal reserve	${}_tV(\bar{A}_x^{\mathrm{PR}}) = \dfrac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x]$	${}_tV^{(m)}(\bar{A}_x^{\mathrm{PR},m}) = \dfrac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV^{(m)}(\bar{A}_x) - {}_tV^{(m)}(A_x^{(m)})]$
7. Equivalence of discounted continuous terminal reserve and fully continuous terminal reserve	${}_tV(\bar{A}_x^{\mathrm{PR}}) + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)$	${}_tV^{(m)}(\bar{A}_x^{\mathrm{PR},m}) + {}_tV^{(m)}(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)$
8. Relationship connecting semicontinuous, curtate, and fully continuous terminal reserves (substitution of item 6 in item 7)	$\dfrac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)$	$\dfrac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV^{(m)}(\bar{A}_x) - {}_tV^{(m)}(A_x^{(m)})] + {}_tV^{(m)}(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)$
9. Discounted continuous *t*th-year mean reserve	$\dfrac{1}{2}\left[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x) + \dfrac{d}{\delta}\bar{P}(\bar{A}_x)\right]$	$\dfrac{1}{2}\left[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x) + \dfrac{1}{m}\dfrac{d^{(m)}}{\delta}\bar{P}(\bar{A}_x)\right]$ if m is odd, and $\frac{1}{2}[{}_{t-1}\bar{V}(\bar{A}_x) + {}_t\bar{V}(\bar{A}_x)]$ if m is even, under the assumption that the premium then due (i.e., due at time $t - \frac{1}{2}$) is unpaid

DISCUSSION OF PRECEDING PAPER

J. ALAN LAUER:

This paper provides a significant clarification of what is often referred to as "the" continuous reserve basis. Mr. Scher points out that there are really three "continuous" reserve bases, which are distinct but closely related. He also points out that a lack of standard nomenclature has been a source of much confusion. This question of standard nomenclature is one that might be considered by the Committee on Standard Notation and Nomenclature. Mr. Scher's suggestions in this regard are worthy candidates for official sanction.

The refund benefit for a whole life policy is defined in the paper as $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$, where s represents the moment of death as measured from the beginning of the year of death, and $0 < s \leq 1$. This benefit has been defined elsewhere in various ways, and it seems worthwhile to point out the following easily established identities.

$$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|} = \frac{d}{\delta}\bar{P}(\bar{A}_x)\ddot{a}_{\overline{1-s}|} = \frac{d}{\delta}\bar{P}(\bar{A}_x)\,\frac{\bar{a}_{\overline{1-s}|}}{\bar{a}_{\overline{1}|}}\,. \tag{1}$$

Mr. Scher refers briefly to my paper in Volume XIX of the *Transactions* on the apportionable basis for reserves. In that paper I described the apportionable annuity due, and my review of the discussion of that paper indicated that there are several possible definitions, not identical, of the amount to be refunded at death. Table 1 on page 154 of Volume XIX shows that, when the amount to be refunded at death is defined as

$$\frac{1 - v^{1/m-s}}{d^{(m)}}\,, \quad \text{that is,} \quad \ddot{a}^{(m)}_{\overline{1/m-s}|}\,,$$

the formula for the apportionable annuity due is

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right)\frac{\delta}{d^{(m)}}\,\bar{A}^{1}_{x:\overline{n}|}\,. \tag{2}$$

This can be restated as

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{d^{(m)}}\,\bar{A}^{1}_{x:\overline{n}|} + \frac{1}{d^{(m)}}\,\frac{\delta}{i^{(m)}}\,\bar{A}^{1}_{x:\overline{n}|}\,. \tag{3}$$

Formula (2) is based on the assumption of uniform distribution of deaths. Using the same assumption,

$$\bar{A}^{1}_{x:\overline{n}|} = \frac{i^{(m)}}{\delta}\,A^{(m)1}_{x:\overline{n}|}\,, \tag{4}$$

and

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{d^{(m)}}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right) . \tag{5}$$

Let us define $PVFB_x$ as the present value at age x of some future benefit (not taking into account any premium refund at death), ${}_nP^{\{m\}}$ as the net apportionable yearly premium payable m times per year for n years for the same benefit, and ${}_n\bar{P}$ as the net fully continuous yearly premium payable continuously for n years for the same benefit. Then

$$PVFB_x = {}_nP^{\{m\}}\ddot{a}^{\{m\}}_{x:\overline{n}|} ; \tag{6}$$

$$PVFB_x = {}_nP^{\{m\}}\ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{d^{(m)}}\,{}_nP^{\{m\}}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right) ; \tag{7}$$

$${}_nP^{\{m\}}\ddot{a}^{(m)}_{x:\overline{n}|} = PVFB_x + \frac{1}{d^{(m)}}\,{}_nP^{\{m\}}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right) . \tag{8}$$

It is evident that

$$\frac{1}{d^{(m)}}\left[{}_nP^{\{m\}}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right)\right]$$

is the present value at issue of the premium refund benefit (when the benefit itself is defined as ${}_nP^{\{m\}}\ddot{a}^{(m)}_{\overline{1/m-s}|}$). It has been demonstrated elsewhere that

$${}_nP^{\{m\}} = \frac{d^{(m)}}{\delta}\,{}_n\bar{P} , \tag{9}$$

when the premium refund benefit is defined as in formula (1) above (with appropriate modification when $m \neq 1$).

This leads us to conclude that

$$\bar{A}^{\mathrm{PR},m} = \frac{1}{d^{(m)}}\frac{d^{(m)}}{\delta}\,{}_n\bar{P}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right) , \tag{10}$$

$$\bar{A}^{\mathrm{PR},m} = \frac{1}{\delta}\,{}_n\bar{P}\left(\bar{A}^{\;1}_{x:\overline{n}|} - A^{(m)\,1}_{x:\overline{n}|}\right) , \tag{11}$$

But equation (11) is the same formula that was developed by Mr. Scher (except that eq. [11] has been generalized to provide for limited payments). This demonstrates that the development in the paper is fully consistent with the apportionable basis if the premium refund benefit for the apportionable basis is defined as in formula (1) above.

If discounted continuous reserves involving a refund benefit based on annual premiums are carried for all policies, regardless of premium mode, it might be thought that reserves would be overstated because they provide for an excessive refund benefit. As a practical matter, such reserves are correct because policies with premiums payable more frequently than annually usually provide a nondeduction benefit. The difference between the reserve for the premium refund benefit based on the annual premium and the reserve for the premium refund benefit based on the modal premium is a close approximation to the reserve for the nondeduction benefit.

An interesting aspect of Mr. Scher's development is that it separates discounted continuous net premiums into semicontinuous net premiums for the death benefit of the face amount and for the premium refund benefit, and similarly for discounted continuous reserves. This makes it a simple matter to substitute gross premiums for net premiums in the formula for the premium refund benefit, which may be justified on the grounds that the actual amount refunded is based on the gross premium rather than on the net premium. (The purpose of this discussion is to show how this could be done, and not to consider the relative merits of such a procedure.)

Letting $m = 1$ in formula (10) above, and substituting G for $(d/\delta)\ {}_n\bar{P}$, where G is the gross annual premium, we arrive at

$$\bar{A}^{\mathrm{GPR}} = \frac{1}{d}\, G(\bar{A}^{\,1}_{x:\overline{n}|} - A^{\,1}_{x:\overline{n}|}) \,. \tag{12}$$

This is easily extended to net premiums and reserves.

STEPHEN G. KELLISON:

Mr. Scher should be congratulated on a most lucid paper clarifying and relating various "continuous" bases for net premiums and reserves. It has been my experience that students often are confused by the various continuous bases encountered in practical work and by the lack of identity between published values involving continuous functions and the text material for Part 4 of the actuarial examinations. Mr. Scher's paper should prove useful in eliminating misconceptions in this area.

Mr. Scher extends his analysis to fractional premiums on policies with claims payable at the moment of death. Net premiums and reserves on such policies have not always been treated consistently in the actuarial literature.

Formula (4.14) on page 86 of Jordan's *Life Contingencies* gives the following approximation for the fractional net premium on an ordinary life policy with claims payable at the end of the year of death:

$$P_x^{(m)} \doteq P_x + \frac{m-1}{2m}\, P_x^{(m)} d + \frac{m-1}{2m}\, P_x^{(m)} P_x \,. \tag{1}$$

Thus $P_x^{(m)}$ is approximately equal to P_x with two adjustments: (1) an adjustment of $[(m-1)/2m]P_x^{(m)}d$ to compensate for the loss of interest during each policy year on fractional premiums paid during that policy year and (2) an adjustment of $[(m-1)/2m]P_x^{(m)}P_x$ to compensate for the loss of premium in the year of death.

An analogous formula can be derived for policies with claims payable at the moment of death:

$$\begin{aligned} P^{(m)}(\bar{A}_x) &= \frac{\bar{A}_x}{\ddot{a}_x^{(m)}} \\ &\doteq \frac{\bar{A}_x}{\ddot{a}_x - (m-1)/2m} \\ &= \frac{P(\bar{A}_x)}{1-(m-1)/2m\ddot{a}_x} \\ &= \frac{P(\bar{A}_x)}{1-[(m-1)/2m](d+P_x)} \,. \end{aligned}$$

Thus a formula analogous to formula (1) is

$$P^{(m)}(\bar{A}_x) \doteq P(\bar{A}_x) + \frac{m-1}{2m}P^{(m)}(\bar{A}_x)d + \frac{m-1}{2m}P^{(m)}(\bar{A}_x)P_x \,. \qquad (2)$$

Formula (2) is interesting because the net premium in the last term, P_x, is based on claims payable at the end of the year of death instead of at the moment of death. Intuitively, if the formula had been written down without an algebraic derivation, $P(\bar{A}_x)$ might have had more appeal than P_x.

If the limit of formula (2) is taken, we have

$$\begin{aligned} \lim_{m\to\infty} P^{(m)}(\bar{A}_x) &= \bar{P}(\bar{A}_x) \\ &\doteq \frac{P(\bar{A}_x)}{1-\frac{1}{2}(d+P_x)} \\ &\doteq \frac{i}{\delta}\frac{P_x}{1-\frac{1}{2}(d+P_x)} \\ &= \frac{i}{\delta}\bar{P}_x \,, \end{aligned}$$

which is consistent with formula (4.21) on page 87 of Jordan. This derivation would seem to lend additional weight to the conclusion that P_x, instead of $P(\bar{A}_x)$, should appear in the last term of formula (2).

Analogous results also hold for reserves. Formula (5.21) on page 108 of

Jordan gives the following formula for the reserve on an ordinary life policy with fractional net premiums and with claims payable at the end of the year of death

$$ {}_tV_x^{(m)} \doteq {}_tV_x + \frac{m-1}{2m} P_x^{(m)}\ {}_tV_x\,, \tag{3} $$

which is consistent with the net premium given in formula (1).

The analogous formula for policies with claims payable at the moment of death consistent with the net premium given in formula (2) is

$$ {}_tV^{(m)}(\bar{A}_x) \doteq {}_tV(\bar{A}_x) + \frac{m-1}{2m} P^{(m)}(\bar{A}_x)\ {}_tV_x\,. \tag{4} $$

Again we have the interesting and unexpected result that ${}_tV_x$, instead of ${}_tV(\bar{A}_x)$, appears in the last term of formula (4).

If the limit of formula (4) is taken, we have

$$ \begin{aligned} \lim_{m\to\infty} {}_tV^{(m)}(\bar{A}_x) &= {}_t\bar{V}(\bar{A}_x) \\ &\doteq {}_tV(\bar{A}_x) + \tfrac{1}{2}\bar{P}(\bar{A}_x)\ {}_tV_x \\ &\doteq \frac{i}{\delta}\ {}_tV_x + \tfrac{1}{2}\bar{P}(\bar{A}_x)\ {}_tV_x \\ &= \frac{e^\delta - 1}{\delta}\ {}_tV_x + \tfrac{1}{2}\bar{P}(\bar{A}_x)\ {}_tV_x \\ &\doteq \left(1+\frac{\delta}{2}\right){}_tV_x + \tfrac{1}{2}\bar{P}(\bar{A}_x)\ {}_tV_x \end{aligned} \tag{5} $$

by truncating the series expansion of $(e^\delta - 1)/\delta$ after two terms.

It is interesting to note that formula (5) is identical with formula (5.30) on page 115 of Jordan. However, formula (5.30) was derived by Jordan in an entirely different manner. This provides an interesting crosscheck that ${}_tV_x$, instead of ${}_tV(\bar{A}_x)$, should appear in the last term of formula (4).

The above results have not always been recognized in the literature. For example, exercise 15(c) on pages 94–95 of Jordan reads as follows:

15. Express each of the following by a single symbol, and give formulas for evaluating each in terms of commutation functions (using standard approximations where necessary):
    (c) the net annual premium payable monthly on the apportionable basis for 10 years to provide a 20 year endowment of 1 on (35) with death benefit payable at the moment of death.

The answer given is

$$ {}_{10}P^{\{12\}}(\bar{A}_{35:\overline{20|}}) \doteq \frac{\bar{M}_{35} - \bar{M}_{55} + D_{55}}{(1 - \frac{11}{24}d)(N_{35} - N_{45}) - \frac{1}{2}(\bar{M}_{35} - M_{45})}\,. $$

A better answer would replace $\bar{M}$'s by M's in the denominator to be consistent with the results developed above.

As an example of a problem involving reserves, Part 4 Study Note 40-20-73, chapter v, question 13, page 15, reads as follows:

Obtain an expression for the net level premium reserve at the end of ten years and eight months for a 25 year endowment policy issued at age x with premiums payable quarterly. Assume that claims are paid at the moment of death and that deferred fractional premiums are not deducted at death. Your expression should contain only interest functions, net level premium reserves at integral durations and net annual premiums for insurance benefits payable at the end of the year of death.

The answer given is

$$
{}_{10\frac{2}{3}}V^{(4)}(\bar{A}_{x:\overline{25|}}) \doteq \frac{i}{\delta}\left(\tfrac{1}{3}\ {}_{10}V^{1}_{x:\overline{25|}} + \tfrac{2}{3}\ {}_{11}V^{1}_{x:\overline{25|}}\right) + \tfrac{1}{3}\ {}_{10}V_{x:\overline{25|}}^{\ \ \ \ 1} + \tfrac{2}{3}\ {}_{11}V_{x:\overline{25|}}^{\ \ \ \ 1}
$$

$$
+ \left[\frac{(i/\delta)P^{1}_{x:\overline{25|}} + P_{x:\overline{25|}}^{\ \ \ \ 1}}{1 - \tfrac{3}{8}(P^{1}_{x:\overline{25|}} + d)}\right]\left[\left(\tfrac{1}{8}\ {}_{10}V^{1}_{x:\overline{25|}} + \tfrac{1}{4}\ {}_{11}V^{1}_{x:\overline{25|}}\right) + \tfrac{1}{12}\right],
$$

which is correct.

However, in the same study note, chapter v, question 21, page 17, reads as follows:

A 20-year endowment policy in the face amount of \$1,000 was issued at age 30 with premiums payable quarterly. The death claim is payable immediately upon receipt of proof of death. In the event of death, quarterly premiums for the current policy year not yet due at date of death are not deducted.

Derive an expression for the total theoretical reserve under this policy at the end of 15 years and 5 months, with premiums paid for 15 years and 6 months, including the reserve for possible loss of premiums in the year of death. Show how your expression could be evaluated from a table of temporary annuities due and the interest factors v and d.

The answer given is

$$
1{,}000\{\tfrac{7}{12}\ {}_{15}V(\bar{A}_{30:\overline{20|}}) + \tfrac{5}{12}\ {}_{16}V(\bar{A}_{30:\overline{20|}})
$$

$$
+ \tfrac{3}{8}P^{(4)}(\bar{A}_{30:\overline{20|}})\,[\tfrac{7}{12}\ {}_{15}V(\bar{A}^{1}_{30:\overline{20|}}) + \tfrac{5}{12}\ {}_{16}V(\bar{A}^{1}_{30:\overline{20|}})] + \tfrac{1}{12}P^{(4)}(\bar{A}_{30:\overline{20|}})\}\ .
$$

A better answer would replace ${}_{15}V(\bar{A}^{1}_{30:\overline{20|}})$ and ${}_{16}V(\bar{A}^{1}_{30:\overline{20|}})$ with ${}_{15}V^{1}_{30:\overline{20|}}$ and ${}_{16}V^{1}_{30:\overline{20|}}$, respectively, to be consistent with the results developed above.

(AUTHOR'S REVIEW OF DISCUSSION)

EDWARD SCHER:

I would like to thank Mr. Lauer and Professor Kellison for their discussions of the paper.

Mr. Lauer develops an expression in the case where premiums are payable $m$thly for the present value of the premium refund benefit, using a form of refund that is based on $\ddot{a}^{(m)}_{\overline{1/m-s}|}$ rather than on $\bar{a}_{\overline{1/m-s}|}$, as in the paper. Of course, one has a certain amount of freedom to define the form of premium refund benefit which one wishes to use, provided that the premium itself is then defined consistently. The reason for the choice in the paper of basing the refund on $\bar{a}_{\overline{1-s}|}$, or on $\bar{a}_{\overline{1/m-s}|}$ in the case of premiums payable $m$thly, is that the consequence of using this customary form of refund, which has a long history, as noted in the paper, is that the resulting terminal reserves for this discounted continuous reserve basis are conveniently identical with the terminal reserves for the fully continuous basis.

Mr. Lauer notes that my analysis of the discounted continuous reserve basis into its two distinct and separable components facilitates the use, if one so desires, of a premium refund based on the gross premium instead of the net premium.

Professor Kellison derives the semicontinuous analogue of the familiar relationship (eq. [4.14] on page 86 of Jordan's textbook) that equates (approximately) a fractional premium to the corresponding annual premium plus an adjustment for the annual loss of interest on the deferred portions of the fractional premium and an adjustment for the average loss in fractional premium in the year of death.

Professor Kellison notes that the premium involved in the term for the second type of adjustment mentioned above is based on claims payable at the end of the year of death instead of at the moment of death. Similarly, in the corresponding equation for reserves (Kellison's eq. [4]), the reserve involved in the same adjustment item is a curtate reserve rather than a reserve involving any continuous functions. He comments on the perhaps unexpected nature of this result.

I think that the main reason for one's expectation of finding only insurances payable at the moment of death in the final term of his equation (2), for example, and one's surprise that this does not occur, is that each of the other premium symbols in equation (2) involves only this type of insurance, as indicated by the presence throughout of the $\bar{A}_x$ symbol. However, if one examines his derivation of equation (2), it is clear that the method is general and could have been carried through for any other type of benefit, such as a deferred annuity, for example, or indeed could have been carried through in a completely general fashion for an unspecified benefit. In that case, the resulting equation corresponding to his equation (2) would be

$$P^{(m)}(\ ) \doteq P(\ ) + \frac{m-1}{2m} P^{(m)}(\ )d + \frac{m-1}{2m} P^{(m)}(\ )P_x\,, \qquad (1)$$

where the sets of parentheses have been left blank to indicate that the benefit involved is unspecified, emphasizing that the relationship exhibited among the premiums is independent of the particular benefit. Here there is no spate of continuous functions to mislead one, and the appearance of the symbol P_x in the final term seems quite consistent with the other curtate symbols on the right-hand side of the equation. Similar comments pertain to his equation (4).

Professor Kellison cites two specific examples, one from Jordan's textbook and one from a study note for Part 4, that bear out his statement that net premiums and reserves on policies with fractional premiums and claims payable at the moment of death have not always been treated consistently. I agree with the corrections he suggests in both cases.

It is interesting that two of the equations derived in the paper enable us to rank by size the premiums and reserves corresponding to the discounted continuous, semicontinuous, and curtate reserve bases.

Thus, starting with equation (4) of the paper, we have

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) - \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x] = P(\bar{A}_x)\,. \qquad (2)$$

Transposing terms, we have

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) - P(\bar{A}_x) = \frac{\bar{P}(\bar{A}_x)}{\delta}[P(\bar{A}_x) - P_x]\,. \qquad (3)$$

Since $\bar{P}(\bar{A}_x)/\delta$ is positive, it follows that the expressions $(d/\delta)\bar{P}(\bar{A}_x) - P(\bar{A}_x)$ and $P(\bar{A}_x) - P_x$ must have the same sign. But, clearly, $\bar{A}_x \geq A_x$. Hence $P(\bar{A}_x) \geq P_x$, and, therefore, $(d/\delta)\bar{P}(\bar{A}_x) \geq P(\bar{A}_x)$.

Thus the ordering of premiums for the three reserve bases is

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) \geq P(\bar{A}_x) \geq P_x\,. \qquad (4)$$

Similarly, starting with equation (15) of the paper, we have

$$\frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] + {}_tV(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)\,. \qquad (5)$$

Transposing, we have

$$\frac{\bar{P}(\bar{A}_x)}{\delta}[{}_tV(\bar{A}_x) - {}_tV_x] = {}_t\bar{V}(\bar{A}_x) - {}_tV(\bar{A}_x) \,. \tag{6}$$

If we reason as before, since $\bar{P}(\bar{A}_x)/\delta$ is positive, it follows that the expressions ${}_tV(\bar{A}_x) - {}_tV_x$ and ${}_t\bar{V}(\bar{A}_x) - {}_tV(\bar{A}_x)$ must have the same sign. Under the assumption of a uniform distribution of deaths, $\bar{A}_x = (i/\delta)A_x$, and from this relationship it is easy to show that ${}_tV(\bar{A}_x) = (i/\delta)\ {}_tV_x$. Hence ${}_tV(\bar{A}_x) \geq {}_tV_x$, and, therefore, ${}_t\bar{V}(\bar{A}_x) \geq {}_tV(\bar{A}_x)$.

Thus the ordering of reserves for the three reserve bases is

$${}_t\bar{V}(\bar{A}_x) \geq {}_tV(\bar{A}_x) \geq {}_tV_x \,. \tag{7}$$

Note that ${}_t\bar{V}(\bar{A}_x)$ is the terminal reserve for both the fully continuous and discounted continuous reserve bases, and note also that the assumption of a uniform distribution of deaths has been used to derive inequalities (7) above.

The derivation of the rankings of the premiums and reserves for the three reserve bases based on equations (4) and (15) of the paper was pointed out to me by Larry Warren, a student of the Society.