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ASSET SHARE MATHEMATICS

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ABSTRACT

This paper presents a new insight into the mathematical structure of asset share–type calculations. Asset shares are interpreted as accumulations of insurance cash flows, which, along with their related investment income, are treated as Stieltjes integrals. The result is a logical, systematic, and general method of approaching the insurance and investment cash-flow elements.

The techniques are developed within the context of individual life insurance asset shares, but may be applied to a much broader range of situations.

I. INTRODUCTION

THE concept of an asset share is familiar to actuaries. Traditionally, an asset share is defined as "the estimated amount attributable to an individual [unit of coverage] if the accumulated net funds of a class of a large number of identical policies . . . is divided at some time t among all the [remaining units of coverage on a pro rata basis]."[1] The particular policy characteristics and experience assumptions used reflect those factors that the actuary feels are relevant to the purpose of the calculation.

The basic asset share calculation is comparatively simple in both theory and practice, requiring only a set of experience assumptions, a rudimentary knowledge of algebra, a calculator, and a fourteen-column worksheet. Asset shares have been used since the nineteenth century for a wide variety of purposes, including calculating and testing premium rates, setting nonforfeiture values, establishing dividend scales, testing solvency, and making projections of countless types.

Early actuaries, with limited resources for complex calculations, made either subjective or approximate adjustments when introducing an unusual factor into an asset share. The increased availability and utilization of high-speed computers have led to a vast number of mathematical refinements to the asset share calculation, as recent students of the Society of Actuaries Fellowship examinations can confirm readily. How-

[1] Charles H. Page, "Asset Shares and Model Offices," Society of Actuaries, Study Note 89-22-70, p. 1.

ever, there has been little reexamination of the basic mathematical nature of the asset share. The asset share itself remains an exercise in algebra (albeit an increasingly complicated one), and it deserves to be examined from a more sophisticated viewpoint.

II. ANALYSIS OF A SIMPLE ASSET SHARE FORMULA

Consider the following elementary asset share formula for an annual premium policy with a face amount of \$1,000:

$$A_t = \frac{1}{p_{t-1}} \Big\{ [A_{t-1} + GP(1 - E_t^p) - E_t^\$](1+i) - q_{t-1}^d\Big[1{,}000\left(1 + \frac{i}{2}\right)\Big] - q_{t-1}^w CV_t \Big\}, \tag{1}$$

where

t = Policy year;
A_t = Asset share per \$1,000 unit of coverage in force at the end of policy year t;
GP = Gross premium;
CV_t = Cash value available at the end of policy year t;
E_t^p = Percent-of-premium expense rate in policy year t;
$E_t^\$$ = Dollars-per-unit expense in policy year t;
i = Interest rate;
q_{t-1}^d = Probability of entrant to policy year t terminating because of death during policy year t;
q_{t-1}^w = Probability of entrant to policy year t terminating because of withdrawal during policy year t;
$p_{t-1} = 1 - q_{t-1}^d - q_{t-1}^w$
= Probability of entrant to policy year t entering policy year $t+1$ in force.

Also, let

l_0 = Number of units of the policy initially issued at time $t = 0$;
$l_t = l_{t-1} - d_{t-1}^d - d_{t-1}^w$
= Number of units surviving to policy duration t;
$d_{t-1}^d = l_{t-1}q_{t-1}^d$
= Number of units terminating during policy year t because of death;
$d_{t-1}^w = l_{t-1}q_{t-1}^w$
= Number of units terminating during policy year t because of withdrawal.

Concept of Asset Fund

Formula (1) gives the asset share per unit in force. For reasons that will be discussed soon, it is preferable to deal in terms of F_t, the asset fund per l_0 initially issued units, accumulated at interest to duration t. Thus,

$$F_t = l_t A_t \tag{2}$$

$$= \{F_{t-1} + l_{t-1}[GP(1 - E_t^p) - E_t^\$]\}(1 + i) - 1{,}000 d_{t-1}^d\left(1 + \frac{i}{2}\right) - d_{t-1}^w CV_t \,. \tag{3}$$

After the asset fund has been computed at duration t, conversion to the traditional asset share per unit in force may be effected, of course, by dividing the asset fund by the units surviving at the same duration (l_t).

Conceptually, the asset fund represents a shift from the policyholder's point of view to the insurer's point of view. The asset share prorates funds among the policies so that each gets its share; the asset fund does not, thereby measuring the accumulated funds held by the insurer. The proration can produce misleading results. For example, consider two plans with premiums set so that the asset shares exceed the respective maturity values by \$10. If both mature at the same policy duration and projected mortality and lapse experience is identical except in the first year, when one has a 10 percent lapse rate while the other has a 40 percent lapse rate, then the first policy is expected to add 50 percent more to company surplus than the second, although their asset shares are identical at maturity. The difference between these policies would be conspicuous using asset funds.

It is not suggested that the asset fund is superior to the asset share. Rather, the asset fund and asset share are complementary, alternative ways of viewing the development of funds held by an insurer. Each has advantages. One advantage of the asset fund is that it does not require a normalization process at the end of each policy year. As a result, the asset fund may be viewed as an accumulation of cash flows. The impact of a cash flow on the asset fund is independent of subsequent persistency; it affects the asset fund by its dollar amount plus accumulated interest. This is the advantage that will be explored in this paper.

Treatment of Interest

One aspect of asset share methodology that has been accorded little attention is the treatment of interest. In the formula under consideration, the asset fund at the end of the previous year plus the premium after expenses earn a full year's interest. The benefit cash flows are charged interest depending upon when they are paid during the year. Only the death benefit is assumed to be paid at a nonintegral policy duration. For the death benefit interest element, $-\frac{1}{2}i(1{,}000d^d_{t-1})$ is a typical formulation and generally has been considered a reasonable and convenient approximation. A more precise expression for this interest component, based on the assumption that deaths are distributed uniformly over the year, is

$$-\int_0^1 [(1+i)^{1-s}-1]1{,}000d^d_{t-1}ds = -\left(\frac{i}{\delta}-1\right)1{,}000d^d_{t-1}\,.$$

Alternatively, a midyear death assumption results in

$$-[(1+i)^{1/2}-1]1{,}000d^d_{t-1}\,.$$

The simpler $-\frac{1}{2}i(1{,}000d^d_{t-1})$ approximates both well and also happens to be conservative.

When cash-flow elements other than death benefits need to be introduced at nonintegral policy durations, the actuary generally has to use improvised methods for treating interest. An alternative that simplifies the handling of interest is based on the following assumption:

INTEREST ASSUMPTION: Cash flows within a policy year earn simple interest from their respective dates of incidence to the end of the policy year. The accumulated amount at the end of a policy year earns interest thereafter at the regular compounded annual rate.

That is, a cash flow c at moment s within policy year t accumulates to $[1+(1-s)i]c$ at the end of policy year t. The corresponding accumulation under the standard compound interest formulation is $(1+i)^{1-s}c$. Reconsidering the assumption of a uniform distribution of deaths, the interest component simplifies to

$$-\int_0^1 (1-s)i(1{,}000d^d_{t-1})ds = -\tfrac{1}{2}i(1{,}000d^d_{t-1})\,.$$

In fact, the midyear death assumption also simplifies to become exactly the "approximation."

Returning now to the asset fund formula (3), the terms can be rearranged to provide additional insight:

$$
\begin{aligned}
F_t = {} & F_{t-1} && \text{(4a)}\\
& + iF_{t-1} && \text{(4b)}\\
& + l_{t-1}[GP(1-E^p_t)-E^\$_t] - 1{,}000d^d_{t-1} - d^w_{t-1}CV_t && \text{(4c)}\\
& + i\{l_{t-1}[GP(1-E^p_t)-E^\$_t] - \tfrac{1}{2}(1{,}000d^d_{t-1})\}\,. && \text{(4d)}
\end{aligned}
$$

The end-of-year asset fund consists of the previous year-end's asset fund (4a), plus interest on that fund (4b), plus the year's net insurance cash flows (4c), plus interest on the year's net insurance cash flows, with due regard for their incidence (4d). The interest element associated with the current year's net insurance cash flows (4d) may be interpreted as an application of the Interest Assumption to each of the insurance cash-flow categories (premium, expenses, death benefits, and withdrawal benefits).

III. A STIELTJES INTEGRAL INTERPRETATION

The asset fund formula given in expressions (4a)–(4d) demonstrates that if the insurance cash flows (4c) in each year are known, along with their contribution to investment income (4d), then the asset fund is known. Each policy year's insurance cash flows and related interest can be calculated in a much more general way. The first step is to partition the insurance cash flows into a finite number of distinct categories. In the case of the *simple* asset share consider five such categories, as follows:

1. Premium income $= GPl_{t-1}$;
2. Percent-of-premium expense $= -E_t^p GPl_{t-1}$;
3. Dollars-per-unit expense $= -E_t^\$ l_{t-1}$;
4. Death benefits $= -1{,}000 d_{t-1}^d$;
5. Withdrawal benefits $= -CV_t d_{t-1}^w$.

This partition is a matter of taste and convenience; items 1–3 could have been combined into a single category to give an "effective premium." The requisite characteristic is that the cash flows of a given category be generated by the same events.

In each category, the cash flow is a product of an amount and a rate of payment. For example, premium income is the product of GP and l_{t-1}; death benefits are the product of $-1{,}000$ and d_{t-1}^d. If it is assumed that deaths are distributed uniformly over the policy year, the product $-1{,}000 d_{t-1}^d$ is actually the sum of momentary cash flows over the policy year, that is,

$$\int_0^1 -1{,}000\, d(d_{t-1}^d s) = -1{,}000 d_{t-1}^d .$$

Considering premium income over the policy year, note that annual premiums are paid only at the beginning of the policy year. Defining a step function

$$\begin{aligned} g(s) &= 0\,, \qquad s \le 0 \\ &= l_{t-1}\,, \quad s > 0\,, \end{aligned}$$

premium income also may be expressed in an integral form,

$$\int_0^1 GP dg(s) = GP l_{t-1} .$$

The other cash-flow categories may be expressed similarly.

For any given category of cash flow, we are looking at a Stieltjes integral. Given functions $f(s)$ and $g(s)$ defined on the unit interval, the Stieltjes integral of "f over g" on the unit interval is denoted by

$$\int_0^1 f(s) dg(s) ,$$

and is defined as the following limit (if it exists):

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(s_i)[g(t_i) - g(t_{i-1})] ,$$

where $\Delta = (t_0, t_1, \ldots, t_n)$ is an arbitrary partition of the unit interval, s_i is an arbitrarily chosen point in $[t_{i-1}, t_i]$, and $\|\Delta\|$ is the largest interval $[t_{i-1}, t_i]$.[2] Generally, the Stieltjes integral exists if f is continuous and g is of bounded variation, or vice versa. In the cases considered in this paper, existence is clearly satisfied. In verbal terms, $f(s)$ is an "amount function," while $dg(s)$ is an "incidence function." That is, $f(s)$ describes the amount of expected value of cash flow *if* the event upon which it is contingent occurs, while $dg(s)$ describes the expected incidence of events giving rise to cash flows.

Let $\{C_k\}$ be an arbitrary partition of the cash flows into a finite number of categories such that each category may be described as a Stieltjes integral over the policy year (unit interval),

$$C_k = \int_0^1 f_k(s) dg_k(s) .$$

The total of insurance cash flows during the policy year is simply the sum of the individual integrals:

$$\sum_{k=1}^{n} C_k = \sum_{k=1}^{n} \int_0^1 f_k(s) dg_k(s) .$$

Finally, consider I_k, the investment income associated with cash flow C_k. A cash flow $f_k(s)dg_k(s)$ at moment s earns $i(1 - s)f_k(s)dg_k(s)$ during

[2] Angus E. Taylor, *General Theory of Functions and Integration* (New York: Blaisdell Publishing Co., 1965).

the balance of the year. Thus, category k generates

$$I_k = \int_0^1 i(1 - s) f_k(s) dg_k(s)$$

$$= i\left[C_k - \int_0^1 s f_k(s) dg_k(s)\right] .$$

By letting T_k denote the "first normalized moment of f_k over g_k," we have

$$T_k = \frac{\int_0^1 s f_k(s) dg_k(s)}{C_k} , \qquad C_k \neq 0$$

$$= 0 , \qquad C_k = 0 ;$$

then $I_k = iC_k(1 - T_k)$.

Conceptually, T_k is the weighted average duration of incidence of cash-flow category k. The total investment income generated by the policy year's insurance cash flows is

$$\sum_{k=1}^{n} I_k = i \sum_{k=1}^{n} C_k(1 - T_k) .$$

Restating formulas (4a)–(4d) in general terms, the asset fund formula becomes

$$F_t = F_{t-1} + iF_{t-1} + \sum_{k=1}^{n} C_k + i \sum_{k=1}^{n} C_k(1 - T_k) , \tag{5}$$

where C_k and T_k are computed for policy year t. This generalized formula can be used for any type of insurance or annuity.

IV. AN EXAMPLE

To demonstrate the Stieltjes technique, consider the plan of coverage underlying the *simple* asset share, modified so that m premiums $(GP^{(m)}/m)$ are payable each policy year. Assume that the d_{t-1}^{w} withdrawals occurring in policy year t are distributed equally over eligible withdrawal dates (off-premium due date withdrawals ignored), and that withdrawals within the policy year receive the interpolated cash value. Also, assume that deaths are distributed uniformly over the policy year. The amount functions, incidence functions, cash flows, and average durations of incidence of the five cash-flow categories now may be developed. For convenience, $dg_k(s)$ will be given rather than $g_k(s)$.

1. Premium income:

$$f_1(s) = GP^{(m)}/m \,,$$

$$\begin{aligned} dg_1(s) &= l_{t-1+s}\,, \qquad s = 0/m, 1/m, \ldots, (m-1)/m \\ &= 0\,, \qquad \text{otherwise}\,. \end{aligned}$$

Note that $l_{t-1+s} = l_{t-1} - sd^d_{t-1} - u(s)d^w_{t-1}$ provided that $u(s) = (k-1)/m$, where k is the smallest integer such that $s < k/m$. Calculating C_1 and T_1, we have

$$\begin{aligned} C_1 &= \int_0^1 f_1(s)dg_1(s) \\ &= \frac{1}{m}\,GP^{(m)}\sum_{k=0}^{m-1} l_{t-1+k/m} \\ &= GP^{(m)}\Big[l_{t-1} - \frac{(m-1)}{2m}\,d^d_{t-1} - \frac{(m-1)}{2m}\,d^w_{t-1}\Big]\,; \end{aligned}$$

$$\begin{aligned} C_1T_1 &= \int_0^1 sf_1(s)dg_1(s) \\ &= \frac{1}{m}\,GP^{(m)}\sum_{k=0}^{m-1}\frac{k}{m}\,l_{t-1+k/m} \\ &= GP^{(m)}\Big[\frac{m-1}{2m}\,l_{t-1} - \frac{(m-1)(2m-1)}{6m^2}\,d^d_{t-1} \\ &\qquad\qquad - \frac{(m-1)(2m-1)}{6m^2}\,d^w_{t-1}\Big]; \end{aligned}$$

$$\begin{aligned} T_1 &= \Big[\frac{m-1}{2m}\,l_{t-1} - \frac{(m-1)(2m-1)}{6m^2}\,d^d_{t-1} \\ &\quad - \frac{(m-1)(2m-1)}{6m^2}\,d^w_{t-1}\Big] \\ &\qquad \times \Big(l_{t-1} - \frac{m-1}{2m}\,d^d_{t-1} - \frac{m-1}{2m}\,d^w_{t-1}\Big)^{-1}. \end{aligned}$$

2. Percent-of-premium expense, assumed to be incurred at the time premium income is received:

$$f_2(s) = -\,E^p_t GP^{(m)}/m\,, \qquad dg_2(s) = dg_1(s)\,.$$

Hence

$$C_2 = -\,E^p_t C_1\,, \qquad T_2 = T_1\,.$$

3. Dollars-per-unit expense, assumed to be incurred entirely at the beginning of the year:

$$f_3(s) = -E_t^{\$},$$

$$\begin{aligned} dg_3(s) &= l_{t-1}, & s &= 0 \\ &= 0, & & \text{otherwise}. \end{aligned}$$

Then

$$C_3 = -E_t^{\$} l_{t-1}, \quad T_3 = 0.$$

4. Death benefits, assumed to be unaffected by withdrawals within the policy year:

$$f_4(s) = -1{,}000, \quad dg_4(s) = d(d_{t-1}^{d} s).$$

Hence

$$C_4 = -1{,}000 d_{t-1}^{d}, \quad T_4 = \tfrac{1}{2}.$$

5. Withdrawal benefits:

$$f_5(s) = -[CV_{t-1} + s(CV_t - CV_{t-1})],$$

$$\begin{aligned} dg_5(s) &= \frac{1}{m} d_{t-1}^{w}, & s &= \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} \\ &= 0, & & \text{otherwise}. \end{aligned}$$

Then

$$\begin{aligned} C_5 &= \int_0^1 f_5(s) dg_5(s) \\ &= -\sum_{k=1}^{m} \left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]\frac{1}{m} d_{t-1}^{w} \\ &= -\left[CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})\right] d_{t-1}^{w}; \end{aligned}$$

$$\begin{aligned} C_5 T_5 &= \int_0^1 s f_5(s) dg_5(s) \\ &= -\sum_{k=1}^{m} \frac{k}{m}\left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]\frac{1}{m} d_{t-1}^{w} \\ &= -\left[\frac{m+1}{2m} CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})\right] d_{t-1}^{w}; \end{aligned}$$

$$\begin{aligned} T_5 &= \left[\frac{m+1}{2m} CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})\right] \\ &\quad \times \left[CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})\right]^{-1} \\ &\qquad (\text{if } CV_{t-1} = CV_t = 0, \quad T_5 = 0). \end{aligned}$$

These cash-flow categories are not intended to represent a contemporary set of asset share elements. However, these techniques easily can be used to reflect many more factors. The purpose of this cash-flow element analysis is to demonstrate the Stieltjes techniques. Even in this simple example, manual calculation of the premium income and withdrawal benefit pieces would be unreasonable. The same calculations, however, can be performed very simply once programmed on a computer.

The assumption of a uniform distribution of withdrawals over eligible withdrawal dates can be modified to an arbitrary distribution without complicating matters greatly. Let $H(s)$ be the portion of withdrawals during policy year t occurring by duration s. Retaining the prohibition against withdrawals at other than premium due dates, $H(s)$ is a monotonically nondecreasing step function such that it is 0 for $s \leq 0$ and 1 for $s \geq 1$, and is discontinuous only at $s = k/m$ for $k = 1, \ldots, m$. Also, let $h(s) = dH(s)$. Note that $h(s)$ is zero everywhere except for $s = 1/m, 2/m, \ldots, m/m$, and that $\int_0^1 h(s)ds = 1$. Thus, $l_{t-1+s} = l_{t-1} - sd_{t-1}^d - H(s)d_{t-1}^w$.

Using this more general representation of l_{t-1+s}, the premium income and withdrawal benefit cash flows may be reevaluated.

1*. Premium income:

$$C_1 = \int_0^1 f_1(s)dg_1(s)$$

$$= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} l_{t-1+k/m}$$

$$= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} \left[l_{t-1} - \frac{k}{m} d_{t-1}^d - H(k/m) d_{t-1}^w \right]$$

$$= GP^{(m)} \left[l_{t-1} - \frac{m-1}{2m} d_{t-1}^d - \sum_{k=0}^{m-1} \frac{1}{m} H(k/m) d_{t-1}^w \right].$$

Similarly,

$$T_1 = \left[\frac{m-1}{2m} l_{t-1} - \frac{(m-1)(2m-1)}{6m^2} d_{t-1}^d - \sum_{k=0}^{m-1} \frac{k}{m^2} H(k/m) d_{t-1}^w \right]$$

$$\times \left[l_{t-1} - \frac{m-1}{2m} d_{t-1}^d - \sum_{k=0}^{m-1} \frac{1}{m} H(k/m) d_{t-1}^w \right]^{-1}.$$

5*. Withdrawal benefits:

$$dg_5(s) = d[H(s)d_{t-1}^w \,.$$

Hence,

$$
\begin{aligned}
C_5 &= \int_0^1 f_5(s)\,dg_5(s) \\
&= -\int_0^1 [CV_{t-1} + s(CV_t - CV_{t-1})]\,d[H(s)d_{t-1}^{w}] \\
&= -\sum_{k=1}^{m}\left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]h(k/m)d_{t-1}^{w} \\
&= -\sum_{k=1}^{m}\left[h(k/m)CV_{t-1} + \frac{k}{m}h(k/m)(CV_t - CV_{t-1})\right]d_{t-1}^{w},
\end{aligned}
$$

and

$$
\begin{aligned}
T_5 = {} & \left\{\sum_{k=1}^{m}\left[\frac{k}{m}h(k/m)CV_{t-1} + \left(\frac{k}{m}\right)^2 h(k/m)(CV_t - CV_{t-1})\right]\right\} \\
& \times \left\{\sum_{k=1}^{m}\left[h(k/m)CV_{t-1} + \frac{k}{m}h(k/m)(CV_t - CV_{t-1})\right]\right\}^{-1}.
\end{aligned}
$$

Note that 1* and 5* in fact reduce to 1 and 5, respectively, when withdrawals are assumed to occur equally at each eligible withdrawal date. A sample calculation of an asset share using the asset fund techniques and the five cash-flow categories whose integrals have been evaluated here is shown in Table 2 of the Appendix.

V. CALENDAR-YEAR ASSET SHARES

Very little has been published about asset shares measured over other than policy-year intervals. There are many good reasons for this. For example, in calculating asset shares for rate-making it is the individual plan-age cell that is under consideration; hence, it is appropriate to measure the asset share from policy anniversary to policy anniversary. In addition, there is comparatively little difference between, say, the asset share measured at the twentieth duration and an intermediate asset share measured between the nineteenth and twentieth policy durations. Even when the use of a calendar-year asset share is clearly appropriate (in modeling applications, for example), actuaries have continued to use aggregations of policy-year asset shares. Practical problems also arise: What is a calendar-year asset share? Is it an asset share for a policy whose issue date is June 30 or July 1? If so, when calculating the asset share at December 31, have one or two semiannual premiums been received? Cash flows within a policy year tend to be skewed so that positive cash flows occur near the beginning of the year and negative cash flows occur near the end; hence, an interpolated cash flow cannot be used arbitrarily. Fortunately, Stieltjes integration techniques provide an answer.

How is the calendar year-end to be interpreted with respect to an asset share? The June 30/July 1 issue-date assumption produces a less than satisfactory result. In Figure 1, the first year of a policy issued in year y is designated as AB. Assume that policies are issued uniformly over calendar year y, that is, l_0dz units are issued at moment z within calendar year y. Then, in Figure 2 the first policy year of the issues of calendar year y becomes the area $ABCD$. Note that line BD is the end of calendar year y, while BC is the end of the first policy year. If the cash-flow amounts within ABD and their average durations of incidence within ABD are determined, the policy year's cash flows will have been split into pieces that can be used to generate calendar-year asset shares.

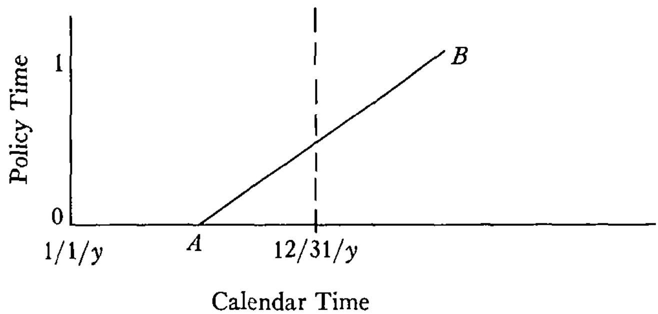

FIG. 1

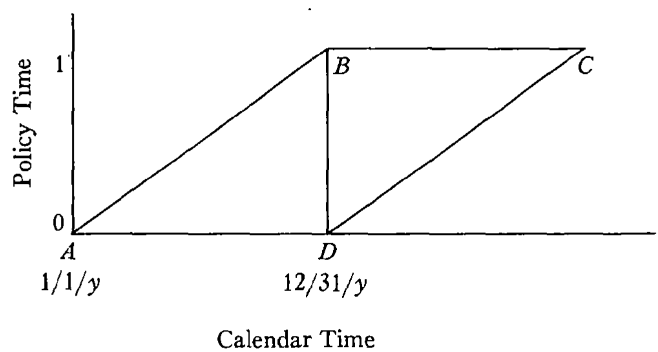

FIG. 2

A policy issued at moment z within calendar year y will remain in calendar year y until policy duration $1 - z$. Considering an arbitrary cash-flow category, the cash flow generated in area ABD is

$$C' = \int_0^1 \int_0^{1-z} f(s)dg(s)dz \,.$$

Reversing the order of integration and solving,

$$\begin{aligned} C' &= \int_0^1 f(s) \int_0^{1-s} dz dg(s) \\ &= \int_0^1 (1 - s) f(s)dg(s) \\ &= C(1 - T) \,. \end{aligned}$$

In other words, the portion of the cash flows occurring in calendar year y is $1 - T$, where T is intuitively the average duration of their incidence within the policy year. For example, an annual premium policy will have $T = 0$ for premium income, so $1 - T = 1$, indicating that all premium income occurs in year y.

Next, what is the investment income generated by cash flows within calendar year y? Using the same arbitrary cash-flow category, we have

$$\begin{aligned} I' &= \int_0^1 \int_0^{1-z} i(1 - z - s) f(s)dg(s)dz \\ &= i\int_0^1 f(s) \int_0^{1-s} (1 - z - s)dzdg(s) \\ &= i\int_0^1 f(s) \frac{(1 - s)^2}{2} dg(s) \\ &= iC[\tfrac{1}{2}(1 - 2T + M)] \,, \end{aligned}$$

where

$$M = \int_0^1 s^2 f(s)dg(s) \Big/ \int_0^1 f(s)dg(s)$$

is the "second normalized moment of f over g." As with T, M is measured over the *policy* year. Using again the example of the premium income associated with an annual premium policy, we obtain $M = 0$. As expected, $I' = \frac{1}{2}iC$.

Using either integration or algebra, we find that the cash flow and interest in calendar year $y + 1$ are $C'' = CT$ and $I'' = iC[\frac{1}{2}(2T - M)]$.

Let $\bar{F}_t$ denote the asset fund measured at the calendar year-end occurring during policy year t. Then the generalized accumulation formula corresponding to formula (5) is

$$\bar{F}_t = \bar{F}_{t-1} + i\bar{F}_{t-1}$$

$$+ \sum_{k=1}^{n} \left[{}_tC_k(1 - {}_tT_k) + {}_{t-1}C_k \ {}_{t-1}T_k \right] \tag{6}$$

$$+ i\sum_{k=1}^{n} \left({}_tC_k \frac{1 - 2{}_tT_k + {}_tM_k}{2} + {}_{t-1}C_k \frac{2{}_{t-1}T_k - {}_{t-1}M_k}{2} \right).$$

Calendar-year calculations are of greatest utility for modeling applications. In addition, a calendar-year asset share can be computed. In order to convert $\bar{F}_t$ to an asset share $\bar{A}_t$ per unit in force, $\bar{l}_t$, the mean number of units of coverage in force at calendar year-end, must be computed from

$$\bar{l}_t = \int_0^1 l_{t-1+s} ds \, .$$

Then $\bar{A}_t$ is calculated as

$$\bar{A}_t = \bar{F}_t / \bar{l}_t \, . \tag{7}$$

Calendar-year reserves per unit in force also must go through a normalizing process. For example, if for the policy in question $V_{t-1}^{(m)}$ and $V_t^{(m)}$ are the consecutive terminal reserves and $P^{(m)}$ is the net premium, the reserve per l_0 initial units is

$$V_t^* = \int_0^1 \{ l_{t-1+s}[\tfrac{1}{2}(V_{t-1} + V_t + P^{(m)})] - l_{t-1+s}w(s)P^{(m)} \} ds \, , \tag{8}$$

where $w(s) = (m - k)/m$ given that k is the largest integer such that $(k - 1)/m < s$. The term $l_{t-1+s}w(s)P^{(m)}$ represents the deferred net valuation premium, which is subtracted from the reserve. The reserve per unit in force thus becomes

$$\bar{V}_t = V_t^* / \bar{l}_t \, . \tag{9}$$

To summarize, calendar-year asset funds can be calculated directly from the *policy-year* cash-flow amounts and first and second normalized moments. This serendipitous result follows directly from the Stieltjes integration interpretation of cash flows. Asset shares can be computed in turn by dividing by the mean number of units in force at calendar year-end. Similarly, calendar-year reserves are calculated from an integral.

VI. AN EXAMPLE REVISITED

To extend the cash-flow formulas cited previously to the calendar-year case, it is necessary only to add "third moment" functions. The more general withdrawal distribution will be used.

1. Premium income:

$$C_1M_1 = \int_0^1 s^2 f(s)\,dg(s)$$

$$= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} \left(\frac{k}{m}\right)^2 l_{t-1+k/m}$$

$$= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} \left(\frac{k}{m}\right)^2 \left[l_{t-1} - \frac{k}{m} d_{t-1}^d - H(k/m) d_{t-1}^w\right]$$

$$= GP^{(m)} \left[\frac{(m-1)(2m-1)}{6m^2} l_{t-1} - \left(\frac{m-1}{2m}\right)^2 d_{t-1}^d - \sum_{k=0}^{m-1} \frac{1}{m} \left(\frac{k}{m}\right)^2 H(k/m) d_{t-1}^w\right].$$

2. Percent-of-premium expense:

$$M_2 = M_1 .$$

3. Dollars-per-unit expense:

$$M_3 = 0 .$$

4. Death benefits:

$$C_4M_4 = \int_0^1 - s^2 (1{,}000)\, d(d_{t-1}^d s)$$

$$= -\tfrac{1}{3}(1{,}000 d_{t-1}^d) .$$

Hence $M_4 = \frac{1}{3}$.

5. Withdrawal benefits:

$$C_5M_5 = \int_0^1 - [CV_{t-1} + s(CV_t - CV_{t-1})]\, d[H(s) d_{t-1}^w]$$

$$= - \sum_{k=1}^{m} \left(\frac{k}{m}\right)^2 \left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right] h(k/m) d_{t-1}^w .$$

The mean number of units in force is

$$\bar{l}_t = \int_0^1 l_{t-1+s}\, ds$$

$$= l_{t-1} - \tfrac{1}{2} d_{t-1}^d - \frac{1}{m} \sum_{k=0}^{m-1} H(k/m) d_{t-1}^w .$$

The mean reserve per $\bar{l}_t$ units in force is

$$V_t^* = \int_0^1 \{l_{t-1+s}[\tfrac{1}{2}(V_{t-1} + V_t + P^{(m)})] - l_{t-1+s}w(s)P^{(m)}\}\,ds$$

$$= \tfrac{1}{2}(V_{t-1} + V_t + P^{(m)})\bar{l}_t$$

$$- P^{(m)}\Big[\frac{m-1}{2m}\,l_{t-1} - \frac{(2m-1)(m-1)}{12m^2}\,d_{t-1}^d$$

$$- \sum_{k=0}^{m-1} \frac{m-1-k}{m^2}\,H(k/m)d_{t-1}^w\Big].$$

In Table 3 of the Appendix, the numerical example is extended to the calendar-year case.

VII. CONCLUSION

The expected cash flows of virtually any insurance product may be described in terms of amount functions and incidence functions. In some cases, it is necessary to substitute an expected value of claim payments function for the amount function. An example would be a waiver of premium benefit, where a disabled life annuity may be used in place of the amount function; the incidence function, of course, would be the rate of disablement. Once the amount and incidence functions have been determined, C, T, and M can be determined readily. Once C, T, and M are available for each category of related cash flows, the asset share, asset fund, and model-office applications follow easily.

The apparent precision of these formulas should not be allowed to obscure the inherent volatility of the experience under insurance contracts. Prospective asset shares, for example, are always *estimates* of future results. To the extent that future experience matches the experience assumptions, the historical asset shares at future points in time will correspond to the prospective asset shares. The question of the predictive utility of prospective asset shares is a general one, not inherently related to the use of Stieltjes techniques. A danger is that the complexity of the formulas produced by Stieltjes techniques may lend an aura of spurious precision to the resulting asset shares.

An important result of applying Stieltjes techniques to rate making is the differentiability between the asset shares of various plans. The asset shares of similar coverages should vary in a logical manner. The more precisely the amount and incidence functions are defined, the more refined becomes the differentiation in impact to both the policyholder and the company. The degree of refinement sought will vary from application to application. The constraints within which the actuary must work also will differ, depending on the purpose for which the calculations are being made. The Stieltjes methodology is equally applicable for prospective and retrospective calculations and can be utilized for other purposes, such as the calculation of GAAP unit reserve factors. Stieltjes techniques also can be adapted to calendar-year issue assumptions other than uniform and to new-money investment methods.

In any case, the techniques that have been presented will enable actuaries to make their calculations in an easier and more systematic manner. In addition, the mathematical structure underlying the asset share hopefully has been made clearer.

APPENDIX

To demonstrate the Stieltjes techniques, a whole life policy issued at age 35 on the semiannual mode is used. The semiannual mode has been chosen so that manual verification of the results is feasible. The premium, claim, expense, and interest assumptions are not intended to reflect current experience.

Assumptions

1. *Gross premium:* $16.00 per unit, payable in two semiannual installments.
2. *Percent-of-premium expenses:* 102 percent in policy year 1, 9.5 percent in policy years 2–10, and 4.5 percent in policy years 11–65, incurred on the premium due date.
3. *Dollars-per-unit expenses:* $12.00 in policy year 1 and $0.50 in policy years 2–65, incurred at the beginning of each policy year.
4. *Death benefit:* $1,000; claims distributed uniformly over the policy year.
5. *Cash values:* Minimum cash values based on 3½ percent interest and curtate functions; interpolated value payable at midyear.
6. *Statutory reserves:* Net level fractional premium reserves based on 3½ percent interest and immediate payment of death claims; $P^{(2)} = 15.48563$.
7. *Mortality:* Modification of 1965–70 intercompany experience.
8. *Withdrawals:* Linton B rates; withdrawals distributed uniformly over eligible termination dates within each policy year except the first, when two-thirds are assumed to occur at midyear.
9. *Interest:* 5.5 percent

Table 1 displays the plan data and experience assumptions in a more complete and graphic form. Table 2 presents the development of policy-year asset funds and asset shares. F_t is calculated by formula (5) and converted to A_t by dividing by l_t. Table 3 develops calendar-year asset funds, asset shares, and reserves. $\bar{F}_t$ is computed by formula (6) and converted to $\bar{A}_t$ by dividing by $\bar{l}_t$. V_t^* is computed by formula (8) and converted to $\bar{V}_t$ by dividing by $\bar{l}_t$.

TABLE 1

PLAN AND EXPERIENCE DATA

t	l_{t-1}	q^d_{t-1}	q^w_{t-1}	d^d_{t-1}	d^w_{t-1}	$GP^{(m)}$	E^p_t	$E^\$_t$	i	Death Benefit	CV_t	$V_t^{(m)}$
1	1,000.000	0.00084	0.200	0.840	200.000	16.00	102.00	12.00	5.50	1,000	0	13.36
2	799.160	0.00101	0.120	0.807	95.899	16.00	9.50	0.50	5.50	1,000	0	27.10
3	702.454	0.00117	0.100	0.822	70.245	16.00	9.50	0.50	5.50	1,000	11	41.19
4	631.387	0.00134	0.088	0.846	55.562	16.00	9.50	0.50	5.50	1,000	25	55.62
5	574.979	0.00150	0.080	0.862	45.998	16.00	9.50	0.50	5.50	1,000	40	70.37
6	528.119	0.00170	0.072	0.898	38.025	16.00	9.50	0.50	5.50	1,000	55	85.42
7	489.196	0.00196	0.064	0.959	31.309	16.00	9.50	0.50	5.50	1,000	71	100.77
8	456.928	0.00221	0.058	1.010	26.502	16.00	9.50	0.50	5.50	1,000	87	116.41
9	429.416	0.00249	0.054	1.069	23.188	16.00	9.50	0.50	5.50	1,000	103	132.35
10	405.159	0.00285	0.050	1.155	20.258	16.00	9.50	0.50	5.50	1,000	119	148.58
11	383.746	0.00323	0.048	1.239	18.420	16.00	4.50	0.50	5.50	1,000	136	165.10
12	364.087	0.00371	0.046	1.351	16.748	16.00	4.50	0.50	5.50	1,000	153	181.87
13	345.988	0.00422	0.044	1.460	15.223	16.00	4.50	0.50	5.50	1,000	170	198.90
14	329.305	0.00480	0.042	1.581	13.831	16.00	4.50	0.50	5.50	1,000	187	216.16
15	313.893	0.00540	0.040	1.695	12.556	16.00	4.50	0.50	5.50	1,000	205	233.63
16	299.642	0.00617	0.040	1.849	11.986	16.00	4.50	0.50	5.50	1,000	223	251.29
17	285.807	0.00684	0.040	1.955	11.432	16.00	4.50	0.50	5.50	1,000	241	269.13
18	272.420	0.00750	0.040	2.043	10.897	16.00	4.50	0.50	5.50	1,000	259	287.13
19	259.480	0.00823	0.040	2.136	10.379	16.00	4.50	0.50	5.50	1,000	277	305.27
20	246.965	0.00905	0.040	2.235	9.879	16.00	4.50	0.50	5.50	1,000	296	323.55

TABLE 2

POLICY-YEAR ASSET FUNDS AND ASSET SHARES

t	C_1	C_2	C_3	C_4	C_5	T_1	T_2	T_3	T_4	T_5	F_t	A_t
1.	14,930	−15,229	−12,000	− 840	0	0.23208	0.23208	0.0	0.50000	0.00000	−13,835	−17.31
2.	12,400	− 1,178	− 400	− 807	0	0.24220	0.24220	0.0	0.50000	0.00000	− 4,157	− 5.92
3.	10,955	− 1,041	− 351	− 822	− 580	0.24351	0.24351	0.0	0.50000	0.83333	4,141	6.56
4.	9,877	− 938	− 316	− 846	−1,195	0.24429	0.24429	0.0	0.50000	0.79070	11,268	19.60
5.	9,012	− 856	− 287	− 862	−1,667	0.24480	0.24480	0.0	0.50000	0.77586	17,506	33.15
6.	8,294	− 788	− 264	− 898	−1,949	0.24531	0.24531	0.0	0.50000	0.76829	23,111	47.24
7.	7,698	− 731	− 245	− 959	−2,098	0.24581	0.24581	0.0	0.50000	0.76493	28,269	61.87
8.	7,201	− 684	− 228	−1,010	−2,200	0.24618	0.24618	0.0	0.50000	0.76205	33,104	77.09
9.	6,774	− 643	− 215	−1,069	−2,296	0.24642	0.24642	0.0	0.50000	0.76010	37,658	92.95
10.	6,397	− 608	− 203	−1,155	−2,330	0.24665	0.24665	0.0	0.50000	0.75870	41,996	109.44
11.	6,061	− 273	− 192	−1,239	−2,427	0.24676	0.24676	0.0	0.50000	0.75806	46,399	127.44
12.	5,753	− 259	− 182	−1,351	−2,491	0.24685	0.24685	0.0	0.50000	0.75714	50,568	146.16
13.	5,469	− 246	− 173	−1,460	−2,523	0.24695	0.24695	0.0	0.50000	0.75641	54,549	165.65
14.	5,207	− 234	− 165	−1,581	−2,528	0.24704	0.24704	0.0	0.50000	0.75581	58,368	185.95
15.	4,965	− 223	− 157	−1,695	−2,517	0.24713	0.24713	0.0	0.50000	0.75561	62,059	207.11
16.	4,739	− 213	− 150	−1,849	−2,619	0.24708	0.24708	0.0	0.50000	0.75515	65,473	229.08
17.	4,519	− 203	− 143	−1,955	−2,704	0.24704	0.24704	0.0	0.50000	0.75476	68,669	252.07
18.	4,307	− 194	− 136	−2,043	−2,773	0.24700	0.24700	0.0	0.50000	0.75442	71,676	276.23
19.	4,102	− 185	− 130	−2,136	−2,828	0.24695	0.24695	0.0	0.50000	0.75413	74,499	301.66
20.	3,903	− 176	− 123	−2,235	−2,877	0.24690	0.24690	0.0	0.50000	0.75408	77,136	328.45

TABLE 3

CALENDAR-YEAR ASSET FUNDS AND ASSET SHARES

t	M_1	M_2	M_3	M_4	M_5	$\bar{F}_t$	V_t^*	$\bar{l}_t$	$\bar{A}_t$	$\bar{V}_t$
1	0.11604	0.11604	0.0	0.33333	0.00000	−12,993	9,585	932.913	−13.93	10.27
2	0.12110	0.12110	0.0	0.33333	0.00000	− 6,337	18,580	774.782	− 8.18	23.98
3	0.12176	0.12176	0.0	0.33333	0.75000	2,522	25,953	684.482	3.68	37.92
4	0.12214	0.12214	0.0	0.33333	0.68605	10,152	32,204	617.074	16.45	52.19
5	0.12240	0.12240	0.0	0.33333	0.66379	16,760	37,604	563.049	29.77	66.79
6	0.12265	0.12265	0.0	0.33333	0.65244	22,589	42,331	518.164	43.59	81.69
7	0.12290	0.12290	0.0	0.33333	0.64739	27,874	46,599	480.889	57.96	96.90
8	0.12309	0.12309	0.0	0.33333	0.64307	32,784	50,558	449.797	72.89	112.40
9	0.12321	0.12321	0.0	0.33333	0.64015	37,407	54,238	423.084	88.42	128.20
10	0.12333	0.12333	0.0	0.33333	0.63804	41,777	57,644	399.517	104.57	144.28
11	0.12338	0.12338	0.0	0.33333	0.63710	46,175	60,814	378.521	121.99	160.66
12	0.12343	0.12343	0.0	0.33333	0.63571	50,406	63,693	359.224	140.32	177.31
13	0.12347	0.12347	0.0	0.33333	0.63462	54,423	66,313	341.452	159.39	194.21
14	0.12352	0.12352	0.0	0.33333	0.63372	58,263	68,703	325.057	179.24	211.36
15	0.12356	0.12356	0.0	0.33333	0.63342	61,960	70,882	309.906	199.93	228.72
16	0.12354	0.12354	0.0	0.33333	0.63272	65,487	72,832	295.721	221.45	246.29
17	0.12352	0.12352	0.0	0.33333	0.63214	68,764	74,450	281.971	243.87	264.03
18	0.12350	0.12350	0.0	0.33333	0.63163	71,835	75,754	268.674	267.37	281.96
19	0.12347	0.12347	0.0	0.33333	0.63119	74,717	76,751	255.817	292.07	300.02
20	0.12345	0.12345	0.0	0.33333	0.63112	77,415	77,451	243.378	318.09	318.23

DISCUSSION OF PRECEDING PAPER

PIERRE C. CHOUINARD:

I read this paper with great interest. It presents a revolutionary approach to the calculation of asset shares—revolutionary in the sense that, once the cash flows, ${}_tC_K$, and their average durations, ${}_tT_K$, have been determined, evaluation of the asset fund turns out to be a problem of interest only; ${}_tC_K$ and ${}_tT_K$ take care of the contingencies.

This discussion has the following two purposes: first, to introduce some "distribution of issues" assumptions in the model, and, second, to present the results I have obtained in calculating calendar-year asset shares in a different manner.

Distribution of Issues

All formulas of Sections V and VI were derived assuming a uniform distribution of issues during a calendar year. As stated in Mr. Huffman's conclusion, the model need not be confined to this hypothesis.

A generalized distribution of issues could be incorporated in the model in the following way:

$$C' = \int_0^1 \int_0^{1-s} f(s)r(z)dg(s)dz\,,$$

$$I' = i\int_0^1 \int_0^{1-s} (1 - z - s)f(s)r(z)dg(s)dz\,,$$

where C' is an arbitrary cash-flow category generated in area ABD of Figure 2 of the paper, I' is the investment income generated by this cash flow in the same area ABD, and $r(z)$ is the annualized proportion of total policies issued at moment z during the year of issue. In actuarial terms, $r(z)$ is nothing more than the force of issue at time z. Now let $R(z)$ be the cumulative portion of policies issued by time z, such that $R(0) = 0$ and $R(1) = 1$. Thus the differential of $R(z)$, $dR(z)$, represents the infinitesimal portion of total issues sold at moment z, that is, $r(z)dz$. In the case of a uniform distribution of issues,

$$R(z) = z \quad \text{and} \quad r(z)dz = dz\,.$$

However, other distribution assumptions also could be used. For example, suppose that sales increase continuously at an annualized rate δ. The issue functions $r(z)$ and $R(z)$ then can be shown to take the following forms:

$$R(z) = \frac{e^{\delta z} - 1}{e^{\delta} - 1}, \quad r(z) = \frac{\delta e^{\delta z}}{e^{\delta} - 1}.$$

If we replace e^{δ} by $(1 + j)$, where j is the effective rate of sales increase during the year, we obtain

$$R(z) = \frac{(1+j)^z - 1}{j}, \quad r(z) = \frac{\delta(1+j)^z}{j}.$$

C' and I' then become

$$C' = \int_0^1 f(s) \int_0^{1-s} \frac{\delta}{j}(1+j)^z dz dg(s)$$

$$= \frac{\delta}{j} \int_0^1 \bar{S}^{j}_{\overline{1-s|}} f(s) dg(s)$$

$$= \frac{\delta}{j} NC,$$

where

$$N = \int_0^1 \bar{S}^{j}_{\overline{1-s|}} f(s) dg(s) \Big/ \int_0^1 f(s) dg(s)$$

and $\bar{S}^{j}_{\overline{1-s|}}$ is the usual continuous forborne annuity-certain, at rate j, and

$$I' = i \int_0^1 f(s) \int_0^{1-s} (1 - z - s) \frac{\delta}{j}(1+j)^z dz dg(s)$$

$$= i \int_0^1 f(s) \left[(1-s)\bar{S}^{j}_{\overline{1-s|}} - (\bar{D}\bar{S})^{j}_{\overline{1-s|}}\right] \frac{\delta}{j} dg(s)$$

$$= i \int_0^1 f(s) \left[\frac{\bar{S}^{j}_{\overline{1-s|}} - (1-s)}{j}\right] dg(s)$$

$$= iC\left(\frac{N - 1 + T}{j}\right).$$

One could verify that

$$\lim_{j \to 0} \frac{\bar{S}^{j}_{\overline{1-s|}} - (1-s)}{j} = \frac{(1-s)^2}{2}.$$

Another distribution of issues that could be assumed is one based on the fact that some companies experience an increase in sales during particular months. Let us suppose that two-thirds of the total annual sales of a company occur uniformly between July 1 and December 31. Furthermore, let us also assume a uniform distribution of sales during the first six months. In this case, $r(z)$ would be a two-step constant function

$$\begin{aligned} r(z) &= \tfrac{2}{3}, \quad 0 \le z \le \tfrac{1}{2} \\ &= \tfrac{4}{3}, \quad \tfrac{1}{2} < z \le 1, \end{aligned}$$

which has been derived from

$$\int_0^{1/2} r(z)dz = \tfrac{1}{3} \quad \text{and} \quad \int_{1/2}^{1} r(z)dz = \tfrac{2}{3}.$$

Thus C' and I' take the following forms:

$$C' = \int_0^{1/2}\int_0^{1-z} \tfrac{2}{3}f(s)dg(s)dz + \int_{1/2}^{1}\int_0^{1-z} \tfrac{4}{3}f(s)dg(s)dz;$$

$$I' = i\int_0^{1/2}\int_0^{1-z} \tfrac{2}{3}(1-z-s)f(s)dg(s)dz + i\int_{1/2}^{1}\int_0^{1-z} \tfrac{4}{3}(1-z-s)f(s)dg(s)dz.$$

In general, if a company does not experience a uniform distribution of issues, it should try to fit a polynomial function $P(z)$ to the cumulative sales values at various points in time. $P(z)/P(1) = R(z)$ then would approximate the cumulative proportion of policies sold by time z, and $r(z)$ would be the derivative of $R(z)$. The general expressions for C' and I' would involve the determination of moments of "f over g" of third and higher degrees.

Calendar-Year Asset Shares Revisited

In Section V, Mr. Huffman states: "How is the calendar year-end to be interpreted with respect to an asset share? The June 30/July 1 issue-date assumption produces a less than satisfactory result." This implies that calendar-year asset shares cannot be derived with the traditional equation of the type (4a), (4b), (4c), and (4d) without producing large errors. Although I consider the Stieltjes technique as giving the best results, I want to demonstrate here that it is very possible to derive a calendar-year asset share in the traditional manner and nevertheless obtain satisfactory results. I will consider the same case as in the appendix to the paper, with the same basic assumptions except that, instead of assuming a uniform distribution of issues, I will use the June 30/July 1 issue-date assumption. In the other words, I will consider a single cohort of policies all issued exactly at midyear. The tth calendar-year asset fund then becomes the policy-year asset fund at duration $t - \frac{1}{2}$.

To calculate the fund midway between integral policy durations $t - 1$ and t, I will take the fund at the end of policy year $t - 1$ (the fund traditionally evaluated by life insurance companies) and accumulate it for half a year.

This is how the cash flows appear during the last six months of year y under our assumptions:

1. *Premium income:* $l_{t-1}(GP^{(2)}/2)$ will be received on June 30/July 1, y; $l_{t-1/2}(GP^{(2)}/2)$ will be received six months later, at the end of calendar year y.
2. *Percent-of-premium expenses:* The cash flows will be those of premium income times E_t^p, with the same timings.
3. *Dollars-per-unit expenses:* One cash flow of $l_{t-1}E_t^{\$}$ will be incurred on July 1, y.
4. *Death benefits:* Deaths are distributed uniformly between July 1, y, and June 30, $y + 1$, so that, on December 31, y, benefits of $\$1{,}000d_{t-1}^d/2$ will have been paid. On the average the benefit is paid on October 1, y.
5. *Withdrawal benefits:* $\frac{1}{2}d_{t-1}^w[CV_{t-1} + \frac{1}{2}(CV_t - CV_{t-1})]$ will be paid at the end of calendar year y; $\frac{1}{2}d_{t-1}^w CV_t$ will be paid on June 30, $y + 1$.

In the accumulation of the funds from July 1, y, to December 31, y, the only problem is the treatment of the cash flows occurring at the end of year y, namely, of the second cash flow of the premium income and of the first of the withdrawal benefits. Do they occur on December 31, y, or on January 1, $y + 1$? The solution is to consider these ambiguous cash flows as split into two parts as follows: one-half occurs one second before midnight of December 31, y, and the other half occurs one second after midnight of December 31, y. The calendar-year asset fund is evaluated between these two seconds, at midnight exactly.

We then get the following formula for $\bar{F}_t$:

$$
\begin{aligned}
\bar{F}_t = {} & \left\{F_{t-1} + l_{t-1}\left[\frac{GP^{(2)}}{2}(1 - E_t^p) - E_t^{\$}\right]\right\}\left(1 + \frac{i}{2}\right) \\
& + \tfrac{1}{2}(l_{t-1} - \tfrac{1}{2}d_{t-1}^d - \tfrac{1}{2}d_{t-1}^w)\,\frac{GP^{(2)}}{2}(1 - E_t^p) \qquad (1) \\
& - \tfrac{1}{2}d_{t-1}^d\left[1{,}000\left(1 + \frac{i}{4}\right)\right] - \tfrac{1}{4}d_{t-1}^w[CV_{t-1} + \tfrac{1}{2}(CV_t - CV_{t-1})]\,.
\end{aligned}
$$

This splitting procedure could be used for any cash flow occurring at the end of a calendar year.

According to the midnight evaluation assumption, those who will share in the fund are $l_{t-1} - \frac{1}{2}d^d_{t-1} - \frac{1}{4}d^w_{t-1}$, and not $l_{t-1} - \frac{1}{2}d^d_{t-1} - \frac{1}{2}d^w_{t-1}$(which equals $l_{t-1/2}$), since $\frac{1}{4}d^w_{t-1}$ withdrawals will occur one second after midnight and they consequently have the right to share in the fund. Note that $l_{t-1} - \frac{1}{2}d^d_{t-1} - \frac{1}{4}d^w_{t-1}$ gives exactly the $\bar{l}_t$ of Mr. Huffman, except for the first year, where $\frac{1}{4}$ must be replaced by $\frac{1}{3}$ (see assumption 8 in the appendix).

TABLE 1

COMPARISON OF STIELTJES TECHNIQUE AND TRADITIONAL METHOD

t	Policy-Year Asset Fund	Calendar-Year Asset Fund			Calendar-Year Asset Share			Percentage Difference {100× [(5)/(3)]}
		Stieltjes	Traditional	Difference	Stieltjes	Traditional	Difference	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1....	−13,835	−12,993	−12,992	+ 1	− 13.93	− 13.93	0.00	0.00
2....	− 4,157	− 6,337	− 6,372	−35	− 8.18	− 8.22	−0.04	0.55
3....	4,141	2,522	2,494	−28	3.68	3.64	−0.04	1.11
4....	11,268	10,152	10,132	−20	16.45	16.42	−0.03	0.20
5....	17,506	16,760	16,746	−14	29.77	29.74	−0.03	0.08
6....	23,111	22,589	22,579	−10	43.59	43.58	−0.01	0.04
7....	28,269	27,874	27,868	− 6	57.96	57.95	−0.01	0.02
8....	33,104	32,784	32,780	− 4	72.89	72.88	−0.01	0.01
9....	37,658	37,407	37,406	− 1	88.42	88.41	−0.01	0.00
10...	41,996	41,777	41,780	+ 3	104.57	104.58	+0.01	0.01
11...	46,399	46,175	46,179	+ 4	121.99	122.00	+0.01	0.01
12...	50,568	50,406	50,412	+ 6	140.32	140.34	+0.02	0.01
13...	54,549	54,423	54,432	+ 9	159.39	159.41	+0.02	0.02
14...	58,368	58,263	58,275	+12	179.24	179.28	+0.04	0.02
15...	62,059	61,960	61,973	+13	199.93	199.97	+0.04	0.02
16...	65,473	65,487	65,504	+17	221.45	221.50	+0.05	0.03
17...	68,669	68,764	68,782	+18	243.87	243.93	+0.06	0.03
18...	71,676	71,835	71,855	+20	267.37	267.44	+0.07	0.03
19...	74,499	74,717	74,740	+23	292.07	292.16	+0.09	0.03
20...	77,136	77,415	77,439	+24	318.09	318.19	+0.10	0.03

Table 1 of this discussion shows the results of such calculations for the twenty years of the illustration. It also shows a comparison of the calendar-year asset fund (and asset share) obtained by the two methods. The asset shares differ only by a few cents per unit. The percentage differences are negligible. In another illustration, the percentage differences obviously would not have been the same; they could have been smaller or larger.

I do not think that a mathematical explanation of the differences (absolute or relative) would be worthwhile, because too many factors are involved. Nevertheless, the reader could verify that the five categories of cash flows generated by the two approaches are almost equal when not identical. This verification is accomplished by comparing, at each duration, $(1 - {}_tT_k)\ {}_tC_k$ of formula (6) of the paper with the cash flows mentioned above and evidenced by formula (1) of this discussion.

The same identity cannot be found in the investment income categories when comparing $i_tC_k(1 - 2_tT_k + {}_tM_k)/2$ of formula (6) with the investment incomes generated by formula (1). For example, the withdrawal benefits do not generate any interest in formula (1), since they are paid at the end of calendar year y. In formula (6), however, they produce a small negative income of approximately $\frac{1}{16}i_tC_5$. As the withdrawal benefits occurring at duration t in year y are approximately equal to $\frac{1}{4}{}_tC_5$, we conclude readily that they are paid on the average one-fourth of a year before December 31, under the Stieltjes technique. Looking now at the death benefits, the traditional method forces them to occur one-fourth of a year before December 31; by the Stieltjes technique, they would occur one-third of a year before December 31, since $C_4' = \frac{1}{2}{}_tC_4$ and $I_4' = \frac{1}{6}i_tC_4$ in formula (6). The dollars-per-unit expenses generate the same investment income under both approaches, namely, $\frac{1}{2}i_tC_3$. Finally, considering the premium income, C_1', and the percent-of-premium expenses, C_2', we deduce under the Stieltjes technique that they are received on the average five months before December 31, because $C_j' \approx \frac{3}{4}{}_tC_j$ and $I_j' \approx \frac{5}{16}i_tC_j$, where $j = 1, 2$. On the other hand, by the traditional method, these flows are received on the average four months before December 31.

In summary, I have not intended to show that the traditional approach is better. Neither have I wanted to show that the two methods are equivalent; they cannot be, because the assumption of a uniform distribution of issues is not identical with the June 30/July 1 issue-date assumption even if they sometimes produce the same results (as in the case of policy-year asset shares). In fact, I am content just to observe that the traditional method could approximate the "exact" value satisfactorily.

MARK D. J. EVANS:

Mr. Huffman has displayed a very enlightening approach to the calculation of asset shares and GAAP reserves. One may wish to explore the possibility of assuming compound interest during the policy year. The motivation for this is twofold. First, the formulas using compound interest during the policy year are no more involved than those using simple interest during the policy year. Second, the compound interest assumption is theoretically more correct.

The simple interest assumption may be justified on the basis that it is conservative. While it does overstate interest lost on death benefits, withdrawals, and other cash outflows not occurring upon policy anniversaries, it also overstates earnings on premiums not due on policy anniversaries. Thus there would be many situations, weekly industrial life insurance for example, where the simple interest assumption could overstate asset shares.

In order to facilitate the development of asset share formulas assuming compound interest during the policy year, Mr. Huffman's I's, reflecting interest earnings on cash flows during the policy year, will be eliminated. Mr. Huffman's C's will be redefined to include interest earnings during the policy year. Otherwise the symbols used in the following development are consistent with those defined by Mr. Huffman.

Redefine C_k as

$$C_k = \int_0^1 (1+i)^{1-s} f_k(s)\, dg_k(s) \; .$$

Formula (5) becomes

$$F_t = F_{t-1} + iF_{t-1} + \sum_{k=1}^{n} C_k \; .$$

Revising Mr. Huffman's policy-year example,

$$\begin{aligned} C_1 &= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} l_{t-1+k/m} (1+i)^{1-k/m} \\ &= GP^{(m)} \left\{ \frac{i}{d^{(m)}} l_{t-1} - (d^w_{t-1} + d^d_{t-1}) \left[\frac{i - d^{(m)} - id^{(m)}/m}{(d^{(m)})^2} \right] \right\} ; \end{aligned}$$

$$C_2 = -E^p_t C_1 \; ;$$

$$C_3 = -E^{\$}_t l_{t-1} (1+i) \; ;$$

$$\begin{aligned} C_4 &= \int_0^1 (1+i)^{1-s} (-1{,}000) d^d_{t-1} ds \\ &= -1{,}000 \frac{i}{\delta} d^d_{t-1} \; ; \end{aligned}$$

$$\begin{aligned} C_5 &= -\sum_{k=1}^{m} \left[CV_{t-1} + \frac{k}{m} (CV_t - CV_{t-1}) \right] \frac{1}{m} d^w_{t-1} (1+i)^{1-k/m} \\ &= -d^w_{t-1} v^{1/m} \left\{ CV_{t-1} \frac{i}{d^{(m)}} + (CV_t - CV_{t-1}) \left[\frac{i - d^{(m)}}{(d^{(m)})^2} \right] \right\} . \end{aligned}$$

Calendar-year asset shares can be developed easily assuming compound interest during the policy year. Except for the interest assumption during the policy year, the following development uses Mr. Huffman's assumptions.

Let ${}_tC_k^\alpha$ represent the cash flows in policy year t and calendar year t accumulated with compound interest to the end of calendar year t for a policy issued in calendar year 1. Similarly, let ${}_tC_k^\beta$ represent the cash flows in policy year t and calendar year $t+1$ accumulated with compound interest to the end of calendar year $t+1$ for a policy issued in calendar year 1.

General reasoning suggests that

$$ {}_tC_k^\alpha + {}_tC_k^\beta = \bar{s}_{\overline{1}|}\ {}_tS_k\,, $$

where ${}_tS_k$ represents the cash flow during policy year t due to source k. This formula may be proved mathematically using the results that follow.

Formula (6) becomes

$$ \bar{F}_t = \bar{F}_{t-1} + i\bar{F}_{t-1} + \sum_{k=1}^{n} ({}_tC_k^\alpha + {}_{t-1}C_k^\beta)\,, $$

where

$$ C^\alpha = \int_0^1 \int_0^{1-s} (1+i)^{1-s-z} f(s)\,dg(s)\,dz\,, $$

$$ C^\beta = \int_0^1 \int_{1-s}^{1} (1+i)^{2-s-z} f(s)\,dg(s)\,dz\,. $$

We have

$$ \begin{aligned} {}_tC_1^\alpha &= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} l_{t-1+k/m} \frac{(1+i)^{1-k/m}-1}{\delta} \\ &= \frac{1}{\delta}\,{}_tC_1 - \frac{1}{\delta} GP^{(m)}\Big[l_{t-1} - \frac{m-1}{2m}(d_{t-1}^w + d_{t-1}^d)\Big]\,, \end{aligned} $$

$$ \begin{aligned} {}_tC_1^\beta &= \frac{1}{m} GP^{(m)} \sum_{k=0}^{m-1} l_{t-1+k/m} \frac{(1+i)-(1+i)^{1-k/m}}{\delta} \\ &= \frac{(1+i)GP^{(m)}}{\delta}\Big[l_{t-1} - \frac{m-1}{2m}(d_{t-1}^w + d_{t-1}^d)\Big] - \frac{1}{\delta}\,{}_tC_1\,, \end{aligned} $$

$$ {}_tC_2^\alpha = -\Big\{\frac{1}{\delta}\,{}_tC_2 - \frac{1}{\delta} E_t^p GP^{(m)}\Big[l_{t-1} - \frac{m-1}{2m}(d_{t-1}^w + d_{t-1}^d)\Big]\Big\}\,, $$

$$ {}_tC_2^\beta = -\Big\{\frac{(1+i)}{\delta} E_t^p GP^{(m)}\Big[l_{t-1} - \frac{m-1}{2m}(d_{t-1}^w + d_{t-1}^d)\Big] - \frac{1}{\delta}\,{}_tC_2\Big\}\,, $$

$$ {}_tC_3^\alpha = -E_t^{\$} l_{t-1} \frac{i}{\delta}\,, $$

$$ {}_tC_3^\beta = 0\,, $$

$$ {}_tC_4^\alpha = \int_0^1\int_0^{1-z} (1+i)^{1-s-z}(-1{,}000d_{t-1}^d)\,ds\,dz $$

$$ = -1{,}000d_{t-1}^d\,\frac{i-\delta}{\delta^2}\,, $$

$$ {}_tC_4^\beta = \int_0^1\int_{1-z}^1 (1+i)^{2-s-z}(-1{,}000d_{t-1}^d)\,ds\,dz $$

$$ = -1{,}000d_{t-1}^d(1+i)\,\frac{\delta-d}{\delta^2}\,, $$

$$ {}_tC_5^\alpha = -\sum_{k=1}^m \left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]\frac{1}{m}\,d_{t-1}^w\,\frac{(1+i)^{1-k/m}-1}{\delta} $$

$$ = -\left\{\frac{1}{\delta}\,{}_tC_5 - \frac{1}{\delta}\,d_{t-1}^w\left[CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})\right]\right\}\,, $$

$$ {}^tC_5^\beta = -\sum_{k=1}^m \left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]\frac{1}{m}\,d_{t-1}^w\,\frac{(1+i)-(1+i)^{1-k/m}}{\delta} $$

$$ = -\left\{\frac{1+i}{\delta}\,d_{t-1}^w\left[CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})\right] - \frac{1}{\delta}\,{}_tC_5\right\}\,. $$

FRANK C. METZ:

My compliments to Mr. Huffman for presenting such a general and powerful approach to asset share mathematics. His paper is an extremely valuable and welcome addition to actuarial literature and most certainly should be included in the course of reading for Part 8. The ultimate utility of Mr. Huffman's approach is limited only by the ingenuity of the actuary applying it and by the reasonableness of the assumptions employed.

Several areas developed in the paper impressed me as being particularly interesting and valuable. The ability to allow for skewness of withdrawals and the ability to calculate calendar-year asset shares are noteworthy attributes of Mr. Huffman's equations. The latter attribute should be very useful in modeling applications. The separation of insurance cash flow and investment elements for each year is a novel and valuable feature that allows for the isolation of new-money cells in developing investment-year asset shares.

In order to illustrate how the general concepts presented in the paper can be applied under a different set of assumptions, I have attempted to recast the equations in Section IV of the paper using rates rather than probabilities and have allowed for skewness of the deaths as well as of the withdrawals. This modification of Mr. Huffman's equations is not an attempt to be more precise but is merely an illustration of how a particular actuary might apply the logic and methodology presented in the paper to a different set of circumstances. The recast equations are prefaced by the set of definitions that follows. The use of rates (denoted by J and H) rather than probabilities is for the sake of simplicity.

q'^{d}_{t-1} = Death rate for policy year t;
q'^{w}_{t-1} = Withdrawal rate for policy year t;
l_0 = Number of units of policy initially issued at time $t = 0$;
$l_t = l_{t-1}(1 - q'^{d}_{t-1})(1 - q'^{w}_{t-1})$
Number of units surviving to policy duration t;
$J(s)$ = Proportion of death rate that has taken effect by time s within a policy year, $0 \leq s \leq 1$;
$H(s)$ = Proportion of withdrawal rate that has taken effect by time s within a policy year, $0 \leq s \leq 1$;
$j(s) = dJ(s)/ds$;
$h(s) = dH(s)/ds$.

1. Premium income:

$$f_1(s) = GP^{(m)}/m\ ;$$

$$dg_1(s) = l_{t-1+s}\ , \qquad s = 0/m, \ldots, (m-1)/m\ ,$$

$$= 0 \qquad \text{otherwise}\ ;$$

$$C_1 = S_0^1 f_1(s)dg_1(s) = \frac{GP^{(m)}}{m}\sum_{k=0}^{m-1} l_{t-1+k/m}\ ;$$

$$l_{t-1+k/m} \doteq l_{t-1}\{1 - S_0^{k/m} j(r)q'^{d}_{t-1}[1 - H(r)q'^{w}_{t-1}]dr$$

$$-S_0^{k/m} h(r)q'^{w}_{t-1}[1 - J(r)q'^{d}_{t-1}]dr\}$$

$$= l_{t-1}[1 - J(k/m)q'^{d}_{t-1} - H(k/m)q'^{w}_{t-1}$$

$$+S_0^{k/m} j(r)q'^{d}_{t-1}H(r)q'^{w}_{t-1}dr$$

$$+S_0^{k/m} h(r)q'^{w}_{t-1}J(r)q'^{d}_{t-1}dr]$$

$$= l_{t-1}\Big[1 - J(k/m)q'^{d}_{t-1} - H(k/m)q'^{w}_{t-1}$$

$$+\sum_{i=0}^{k-1} H(i/m)S_{i/m}^{(i+1)/m}dJ(r)q_{t-1}^{\prime d}q_{t-1}^{\prime w}$$

$$+\sum_{i=1}^{k} J(i/m)S_{(i-1)/m}^{i/m}dH(r)q_{t-1}^{\prime w}q_{t-1}^{\prime d}\Big]$$

if withdrawals are assumed to occur at the end of $m$thly intervals, that is, $H(s) = H((k-1)/m)$, $(k-1)/m \leq s < k/m$.

Under the assumption that $H(k/m) = J(k/m) = k/m$,

$$l_{t-1+k/m} = l_{t-1}\left(1 - \frac{k}{m}q_{t-1}^{\prime d} - \frac{k}{m}q_{t-1}^{\prime w} + \frac{k^2}{m^2}q_{t-1}^{\prime d}q_{t-1}^{\prime w}\right) .$$

Thus

$$\frac{GP^{(m)}}{m}\sum_{k=0}^{m-1} l_{t-1+k/m} = C_1$$

$$\doteqdot GP^{(m)}l_{t-1}\Big[1 - \frac{m-1}{2m}q_{t-1}^{\prime d} - \frac{m-1}{2m}q_{t-1}^{\prime w}$$

$$+\frac{(m-1)(2m-1)}{6m^2}q_{t-1}^{\prime d}q_{t-1}^{\prime w}\Big] ;$$

$$C_1T_1 = S_0^1 sf_1(s)dg_1(s) = \frac{GP^{(m)}}{m}\sum_{k=0}^{m-1}\frac{k}{m}l_{t-1+k/m}$$

$$\doteqdot GP^{(m)}l_{t-1}\Big[\frac{m-1}{2m} - \frac{(m-1)(2m-1)}{6m^2}(q_{t-1}^{\prime d} + q_{t-1}^{\prime w})$$

$$+\frac{(m-1)^2}{4m^2}q_{t-1}^{\prime w}q_{t-1}^{\prime d}\Big] ;$$

$$T_1 \doteqdot \Big[\frac{m-1}{2m} - \frac{(m-1)(2m-1)}{6m^2}q_{t-1}^{\prime d} - \frac{(m-1)(2m-1)}{6m^2}q_{t-1}^{\prime w}$$

$$+\frac{(m-1)^2}{4m^2}q_{t-1}^{\prime d}q_{t-1}^{\prime w}\Big]\Big[1 - \frac{m-1}{2m}q_{t-1}^{\prime d} - \frac{m-1}{2m}q_{t-1}^{\prime w}$$

$$-\frac{(m-1)(2m-1)}{6m^2}q_{t-1}^{\prime d}q_{t-1}^{\prime w}\Big]^{-1} .$$

2. Percent-of-premium expense, assumed to be incurred at the time premium is received:

$$f_2(s) - E_t^P\frac{GP^{(m)}}{m} ; \qquad dg_2(s) = dg_1(s) .$$

Hence

$$C_2 = -E_t^P C_1 , \qquad T_2 = T_1 .$$

3. Dollars-per-unit expense, assumed to be incurred entirely at the beginning of the year:

$$f_3(s) - E_t^\$;$$

$$\begin{aligned} dg_3(s) &= l_{t-1}, \quad & s = 0 \\ &= 0 & \text{otherwise}. \end{aligned}$$

Then

$$C_3 = -E_t^\$ l_{t-1}, \quad T_3 = 0 .$$

4. Death benefits:

$$f_4(s) = -1{,}000 ;$$

$$dg_4(s) \doteqdot l_{t-1} j(s) q'^{d}_{t-1}[1 - H(s) q'^{w}_{t-1}] ds ;$$

$$\begin{aligned} H(s) &= 0, & 0 \le s < 1/m \\ &= H(1/m), & 1/m \le s < 2/m \\ &\cdot \quad \cdot & \cdot \\ &\cdot \quad \cdot & \cdot \\ &\cdot \quad \cdot & \cdot \\ &= H((m-1)/m), & (m-1)/m \le s < 1 ; \end{aligned}$$

$$\begin{aligned} C_4 &= S_0^1 f_4(s) dg_4(s) \\ &\doteqdot -1{,}000 l_{t-1} S_0^1 j(s) q'^{d}_{t-1}(1 - H(s) q'^{w}_{t-1}) ds \\ &= -1{,}000 l_{t-1} q'^{d}_{t-1}\left(1 - \sum_{k=0}^{m-1} \frac{k}{m} q'^{w}_{t-1} \frac{1}{m}\right). \end{aligned}$$

If we assume $J(s) = s$ and $H(k/m) = k/m$, then

$$C_4 \doteqdot -1{,}000 l_{t-1} q'^{d}_{t-1}\left(1 - \frac{m-1}{2m} q'^{w}_{t-1}\right),$$

and

$$\begin{aligned} C_4 T_4 &= S_0^1 s f_4(s) dg_4(s) \\ &\doteqdot -1{,}000 l_{t-1}(\tfrac{1}{2} q'^{d}_{t-1}) \left[1 - \frac{(4m+1)(m-1)}{6m^2} q'^{w}_{t-1}\right] ; \end{aligned}$$

$$T_4 \doteqdot \frac{1}{2}\left[1 - \frac{(4m+1)(m-1)}{6m^2} q'^{w}_{t-1}\right]\left(1 - \frac{m-1}{2m} q'^{w}_{t-1}\right)^{-1} .$$

5. Withdrawal benefits:

$$f_5(s) = -[CV_{t-1} + s(CV_t - CV_{t-1})] ;$$

$$dg_5(s) \doteqdot l_{t-1} h(s) q'^{w}_{t-1}(1 - J(s) q'^{d}_{t-1}) ds ;$$

$$C_5 = S_0^1 f_5(s)dg_5(s)$$

$$\doteq -l_{t-1}S_0^1\{CV_{t-1} + s(CV_t - CV_{t-1})h(s)q'^{w}_{t-1}(1 - J(s)q'^{d}_{t-1})ds$$

$$= -l_{t-1}\sum_{k=1}^{m}\left[CV_{t-1} + \frac{k}{m}(CV_t - CV_{t-1})\right]$$

$$\times [H(k/m) - H((k-1)/m)]q'^{w}_{t-1}[1 - J(k/m)q'^{d}_{t-1}] .$$

Under the assumption that $H(k/m) = J(k/m) = k/m$,

$$C_5 \doteq -l_{t-1}q'^{w}_{t-1}\Big\{CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})$$

$$-q'^{d}_{t-1}\left[\frac{m+1}{2m}CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})\right]\Big\} ;$$

$$C_5T_5 = S_0^1 sf_5(s)dg_5(s)$$

$$\doteq -l_{t-1}q'^{w}_{t-1}\Big\{\frac{m+1}{2m}CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})$$

$$-q'^{d}_{t-1}\left[\frac{(m+1)(2m+1)}{6m^2}CV_{t-1} + \frac{(m+1)^2}{4m^2}(CV_t - CV_{t-1})\right]\Big\},$$

and

$$T_5 \doteq \Big\{\frac{m+1}{2m}CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})$$

$$-q'^{d}_{t-1}\left[\frac{(m+1)(2m+1)}{6m^2}CV_{t-1} + \frac{(m+1)^2}{4m^2}(CV_t - CV_{t-1})\right]\Big\}$$

$$\times\Big\{CV_{t-1} + \frac{m+1}{2m}(CV_t - CV_{t-1})$$

$$-q'^{d}_{t-1}\left[\frac{m+1}{2m}CV_{t-1} + \frac{(m+1)(2m+1)}{6m^2}(CV_t - CV_{t-1})\right]\Big\}^{-1} .$$

ROBERT R. REITANO:

Mr. Huffman has presented an interesting model for asset share funds that seemingly has more potential as a vehicle for theoretical investigations than as a device for numerical evaluation. Since neither the theory nor the application of Stieltjes integration has received much attention in the *Transactions* in the past, it would have been worthwhile if the author had explored some of their simpler properties and had been more rigorous in the development of his examples.

The purpose of this discussion is to fill in this "gap" (my opinion) and to suggest alternative approaches to the handling of interest that can be used to approximate the fund, F_n, as well as to construct upper and lower bounds for its exact value.

Stieltjes Integration

By definition,

$$\int_a^b f(s)dg = \lim_{|\Delta|\to 0} \sum_{i=1}^{m} f(s_i')[g(s_i) - g(s_{i-1})] \tag{1}$$

if the limit exists and is uniquely determined, where $s_0 = a, s_1, \ldots, s_m = b$ is a partition of the interval $[a, b]$; s_i' is any point in the ith interval; and $|\Delta|$ is the length of the largest interval determined by the partition.

Although this is an unwieldy definition, it turns out, as Mr. Huffman notes, that the integral will exist if $f(s)$ is continuous and $g(s)$ is of bounded variation. This means that there is a constant c such that, given any partition of $[a, b]$, the sum of $|g(s_i) - g(s_{i-1})|$ is less than c. Many functions encountered in applied mathematics are of bounded variation. For example, any function with a continuous derivative is of bounded variation on every closed interval $[a, b]$. The usual example of a function that is not of bounded variation is

$$\begin{aligned} g(s) &= 0\,, && s = 0 \\ &= \sin(1/s)\,, && 0 < s \leq 1\,, \end{aligned}$$

which oscillates wildly near zero.

Very often it is possible to calculate the value of a Stieltjes integral by reducing it to a Riemann integral or a sum of Riemann integrals. In order to avoid questions of existence, we will assume throughout this section that $f(s)$ is a continuous function. Also, for the sake of brevity, the statement that "$g(s)$ has a continuous derivative on $[a, b]$" will mean that $g(s)$ is defined on a slightly larger interval $(a - \epsilon, b + \epsilon)$ and has a derivative that is a continuous function on $[a, b]$.

As a first example of this reduction, we have the result that if $g(s)$ has a continuous derivative on $[a, b]$, then

$$\int_a^b f(s)dg = \int_a^b f(s)g'(s)ds\,. \tag{2}$$

This is because the mean-value theorem can be applied in equation (1) to obtain $g(s_i) - g(s_{i-1}) = g'(s_i'')(s_i - s_{i-1})$, where s_i'' is in the ith interval. Mr. Huffman's death benefit cash flow is of this type. There, $g_4(s)$ equals the amount of unit death benefits claimed by duration s and is given by

$g_4(s) = cs$, where c is the expected number of deaths during one year at the given age.

If $g(s)$ is piecewise continuously differentiable, the corresponding Stieltjes integral will become a sum of integrals as in equation (2), with a few adjustments. The derivation of these adjustments is handled most easily by considering the simplest such $g(s)$, which is a step function. To this end, let $t_0 = a, t_1, \ldots, t_n = b$ be a partition of $[a, b]$, and let

$$\begin{aligned} g(s) &= c_1 , \quad t_0 \le s \le t_1 \\ &= c_i , \quad t_{i-1} < s \le t_i , \quad i = 2, \ldots, n , \end{aligned}$$

where the c_i's are constants. For this kind of function, the terms of the sum in equation (1) are evaluated easily. To see this, let the partition $s_1, \ldots, s_m$ be given as in (1), and consider the value of $f(s_i')[g(s_i) - g(s_{i-1})]$. If s_{i-1} and s_i are under the same "step," this value clearly is zero. On the other hand, if they are under adjacent steps, the value of this term is $f(s_i')\ (c_{j+1} - c_j)$ for some j. In the limit, it is clear that these points s_i' will be forced to converge to t_j and, since $f(s)$ is continuous, $f(s_i')$ will converge to $f(t_j)$; hence,

$$\int_a^b f(s)dg = \sum_{j=1}^{n-1} f(t_j)(c_{j+1} - c_j) . \tag{3}$$

As an example, consider the premium cash flow in Section IV of Mr. Huffman's paper. Since $g_1(s)$ represents the volume of unit premiums received by duration s, we have

$$\begin{aligned} g_1(s) &= c_1 , \quad s = 0 \\ &= c_i , \quad (i-2)/m < s \le (i-1)/m , \\ &\qquad\qquad i = 2, \ldots, m, m+1 , \end{aligned} \tag{4}$$

where $c_1 = 0$ and $c_i = \Sigma\, l_{t-1+(j-1)/m}$, the sum being from $j = 1$ to $j = i - 1$. Applying equation (3) with $g_1(s)$ as given in (4) and $f_1(s) = GP^{(m)}/m$ yields the expected premium cash flow.

Withdrawal benefits are handled similarly, except that $c_1 = 0$ on $[0, 1/m]$.

Strictly speaking, it is incorrect to try to evaluate dg in general for functions of this type by means of the intuitive formula $dg = g'(s)ds$. Within each step, $g(s)$ is differentiable, so $dg = g'(s)ds = 0$. However, across the "jumps," dg has no representation in terms of functions and ds. Actually, dg has an interpretation as a special kind of measure, called a distribution (in the sense of Laurent Schwartz), but for the purpose of asset shares this interpretation would be somewhat artificial at this stage and will not be pursued.

Finally, the case where $g(s)$ is an arbitrary piecewise continuously differentiable function is easily handled. Let $t_0, \ldots, t_n$ be a partition of $[a, b]$, and let

$$\begin{aligned} g(s) &= g_1(s)\,, \quad t_0 \le s \le t_1 \\ &= g_i(s)\,, \quad t_{i-1} < s \le t_i, \quad i = 2, \ldots, n\,, \end{aligned}$$

where $g_i(s)$ has a continuous derivative on $[t_{i-1}, t_i]$. Combining the reasoning behind formulas (2) and (3), we obtain

$$\int_a^b f(s)dg = \sum_{j=1}^{n-1} f(t_j)[g_{j+1}(t_j) - g_j(t_j)] + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(s)g_j'(s)ds\,. \tag{5}$$

Equation (5) is valid as long as $g(a) = g_1(a)$ and $g(b) = g_n(b)$, whereas $g(t_j)$ for $j = 1, \ldots, n-1$ can be defined as either $g_j(t_j)$ or $g_{j+1}(t_j)$. A similar statement holds for equation (3).

Approximating Compound Interest

Let G_t be the value of the sum of all tth-year cash flows at the end of the tth year. Then the fund, F_n, can be expressed as

$$F_n = F_{n-1}(1 + i) + G_n\,. \tag{6}$$

Since G_t is the sum of both positive and negative cash flows, let $G_t = G_t^+ - G_t^-$, where both G_t^+ and G_t^- are sums. For example,

$$G_t^+ = \sum_{k=1}^{1} \int_0^1 (1 + i)^{1-s} f_{t,k}(s)dg_{t,k}\,, \tag{7a}$$

$$G_t^- = \sum_{j=1}^{m} \int_0^1 (1 + i)^{1-s} \bar{f}_{t,j}(s)d\bar{g}_{t,j}\,, \tag{7b}$$

where $j = 1$ may represent death claims, $k = 1$ may represent premiums received, and so on.

The most common approximation to compound interest, and the one that was used by Mr. Huffman, is simple interest, that is,

$$(1 + i)^{1-s} \doteq 1 + i - is\,. \tag{8}$$

As a function of s, this approximation overstates the true value everywhere except at $s = 0$ and $s = 1$. Hence, using this interest approximation in asset share calculations will overstate both the positive and negative cash flows, and, since some of this overstatement will cancel after subtraction, one expects a reasonably accurate result. Of course, the problem with this approach is that it is difficult to determine how "reasonably accurate" the approximation is, even to the extent of whether the error is positive or negative.

Bounds for the fund F_n can be developed by using both an overstating and an understating approximation to $h(s) = (1+i)^{1-s}$. Then overstating G_t^+ and understating G_t^- will produce an upper bound for F_n, and reversing the approximations will produce a lower bound.

1. LINEAR APPROXIMATIONS

It should be obvious that of all linear upper bounds to $h(s) = (1+i)^{1-s}$ on [0, 1] the approximation in formula (8) is the best possible. Let us call this approximation $h_1^{\max}(s)$.

For a good linear lower bound it is certainly necessary that the line be tangent to $h(s)$ at some point $t \in (0, 1)$; hence,

$$h_t^{\min}(s) = (1+i)^{1-t}[1 - \delta(s-t)] . \tag{9}$$

The best fit will be produced by choosing that line in equation (9) with maximum integral over [0, 1]. If expression (9) is integrated with respect to s over [0, 1], the resulting function of t will be maximized when $t = \frac{1}{2}$. Hence, the best possible linear lower bound is given by

$$h_1^{\min}(s) = e^{\delta/2}[(1+\delta/2) - \delta s] . \tag{10}$$

Thus, for any cash flow with $f(s)dg \geq 0$,

$$\int_0^1 (1+i)^{1-s} f(s)dg \leq (1+i)C - iD , \tag{11a}$$

$$\int_0^1 (1+i)^{1-s} f(s)dg \geq e^{\delta/2}[(1+\delta/2)C - \delta D] , \tag{11b}$$

where

$$C = \int_0^1 f(s)dg , \quad D = \int_0^1 sf(s)dg . \tag{12}$$

The notation here is slightly different from that presented in Mr. Huffman's paper in that the second integral in formula (12) was defined there as CT. This had the slight advantage that C could be factored from the resulting formulas but produces the distinct disadvantage of trying to define T when $C = 0$. Setting $T = 0$ when $C = 0$ has the erroneous implication that cash flows that net to zero always will have simple interest accumulations that also net to zero. The same type of problem occurs with the integral that he defines as CM, which for a later use will be defined as E, where

$$E = \int_0^1 s^2 f(s)dg .$$

Combining (7a), (7b), (11a), and (11b), we obtain

$$G_t^{\min} \le G_t \le G_t^{\max}\,, \tag{13}$$

where

$$G_t^{\max} = [(1+i)C_t^+ - iD_t^+] - e^{\delta/2}[(1+\delta/2)C_t^- - \delta D_t^-]\,, \tag{14a}$$

$$G_t^{\min} = e^{\delta/2}[(1+\delta/2)C_t^+ - \delta D_t^+] - [(1+i)C_t^- - iD_t^-]\,, \tag{14b}$$

and

$$C_t^+ = \sum_{k=1}^{l} \int_0^1 f_{t,k}(s)dg_{t,k} \qquad \text{(positive cash flows)}\,,$$

$$D_t^- = \sum_{j=1}^{m} \int_0^1 s\bar{f}_{t,j}(s)d\bar{g}_{t,j} \qquad \text{(negative cash flows)}\,, \quad \text{etc.}$$

Combining (14a), (14b), and (6), we have

$$F_n^{\min} \le F_n \le F_n^{\max}\,, \tag{15}$$

where $F_n^{\min}$ ($F_n^{\max}$) is defined by equation (6) using $G_t^{\min}$ ($G_t^{\max}$) instead of G_t.

From a programming point of view, it would be easier first to accumulate C_t^+, C_t^-, D_t^+, and D_t^- separately at the appropriate interest rate from $t = 1$ to $t = n$. Then the bounds for F_n could be expressed as linear combinations of these values, that is,

$$F_n^{\max} = [a(AC)_n^+ + b(AD)_n^+] - [c(AC)_n^- + d(AD)_n^-]\,, \tag{16a}$$

$$F_n^{\min} = [c(AC)_n^+ + d(AD)_n^+] - [a(AC)_n^- + b(AD)_n^-]\,, \tag{16b}$$

where

$$a = 1 + i\,, \quad b = -i\,, \quad c = e^{\delta/2}(1+\delta/2)\,, \quad d = -\delta e^{\delta/2}\,,$$

$$(AC)_n^+ = \sum_{t=1}^{n} C_t^+(1+i)^{n-t}\,, \quad \text{etc.}$$

Applying (16a) and (16b) to Mr. Huffman's numerical example, the bounds produced are

$$77{,}032 \le F_{20} \le 77{,}212\,. \tag{17}$$

If only one linear approximation is to be used throughout, it makes sense to use a line with the same area as $h(s)$ on [0, 1]. Any such averaging line will have the following form:

$$h_1^{\text{av}}(s) = -c(s - \tfrac{1}{2}) + i/\delta\,, \tag{18}$$

where we restrict c to values greater than zero in order to have a decreasing function (as is $h(s)$). A reasonable goal for equation (18) is to choose c so that the maximum value of the error function, $\mathcal{E}(s) = |h(s) - h_1^{av}(s)|$, will be minimized.

Since $i/\delta > h(\frac{1}{2})$ it is clear that, near $s = \frac{1}{2}$, $h_1^{av}(s) > h(s)$ for any c. The errors at zero and 1, however, can be positive or negative. In any event, the absolute value of the larger of these two extreme errors will be minimized if the errors are equal. This produces $c = i$ and $\mathcal{E}(0) = \mathcal{E}(1) = 1 + i/2 - i/\delta$. For this choice of c, the maximum "interior" error in absolute value will occur at $\bar{s} = 1 - (1/\delta) \ln (i/\delta)$ (standard calculus technique) and will be equal to $\mathcal{E}(\bar{s}) = -i/2 + (i/\delta) \ln (i/\delta)$.

A somewhat lengthy calculation shows that $\mathcal{E}(\bar{s}) \leq \mathcal{E}(0)$ for any δ, which implies that any other choice of c will produce a maximum error larger than that produced for $c = i$. Hence, the "optimum" averaging line is given by

$$h_1^{av}(s) = -i(s - \tfrac{1}{2}) + i/\delta \,. \tag{19}$$

Using (19), we define G_t^{av} by

$$\begin{aligned} G_t^{av} &= (i/2 + i/\delta)(C_t^+ - C_t^-) - i(D_t^+ - D_t^-) \\ &= (i/2 + i/\delta)C_t - iD_t \,. \end{aligned} \tag{20}$$

Combining equations (6) and (20) produces F_n^{av}, which for Mr. Huffman's example and $n = 20$ produces $F_{20}^{av} = 77{,}118$.

2. QUADRATIC APPROXIMATIONS

By introducing quadratic approximations to $h(s)$, the bounds in formula (17) can be sharpened considerably. Since it is reasonable to demand that the approximating quadratics $h_2(s)$ will intersect $h(s)$ at $s = 0$ and $s = 1$, we begin with the general form of $h_2(s)$, which is

$$h_2(s) = bs(s - 1) - is + 1 + i \,. \tag{21}$$

By considering the derivatives of $h(s)$ and $h_2(s)$ at $s = 0$ and $s = 1$, it is simple to show that, if

$$i - \delta < b < \delta e^\delta - i \,, \tag{22}$$

then $h_2(s)$ also will intersect $h(s)$ at some point within the interval (0, 1). This is because, for such b, $h_2'(0) > h'(0)$ and $h_2'(1) > h'(1)$, which implies that $h_2(s)$ starts out from $s = 0$ above $h(s)$ and ends up at $s = 1$ from below $h(s)$.

To see this, let $f(s) = h_2(s) - h(s)$. Then, since $f'(s)$ is continuous, $f'(0) > 0$ implies that $f'(s) > 0$ on $(-\epsilon, \epsilon)$, and $f'(1) > 0$ implies that

$f'(s) > 0$ on $(1 - \epsilon, 1 + \epsilon)$, for some $\epsilon > 0$. Hence, since $f(0) = 0$ and and $f(1) = 0$, we have

$$f(s) = \int_0^s f'(t)dt > 0 \quad \text{on} \quad (0, \epsilon) ,$$

$$f(s) = -\int_s^1 f'(t)dt < 0 \quad \text{on} \quad (1 - \epsilon , 1) .$$

As one might expect, the extreme values of b in inequality (22) produce the closest fitting parabolas from above and below $h(s)$ out of the one-parameter family defined in equation (21). That is, setting $b = i - \delta$ will produce the best quadratic upper bound, $h_2^{\max}(s)$, and setting $b = \delta e^{\delta} - i$ will produce the best lower bound, $h_2^{\min}(s)$.

To see this, let

$$h_2^{\max}(s) = (i - \delta)s^2 - (2i - \delta)s + 1 + i , \tag{23a}$$

$$h_2^{\min}(s) = (\delta e^{\delta} - i)s^2 - \delta e^{\delta} s + 1 + i . \tag{23b}$$

One way of showing that $h_2^{\max}(s) > h(s)$ on $(0, 1)$ is to expand $h_2^{\max}(s) - h(s)$ as a Taylor series in δ about $\delta = 0$, keeping s fixed. The coefficients of the powers of δ will be polynomials in s that are strictly positive for $s \in (0, 1)$. The same technique applied to $h(s) - h_2^{\min}(s)$ will show that $h_2^{\min}(s) < h(s)$ on $(0, 1)$. These expansions also will show that the error in each of these approximations is $O(\delta^3)$.

Finally, if $b > \delta e^{\delta} - i$, then clearly $h_2(s) < h_2^{\min}(s)$. Similarly, if $b < i - \delta$, then $h_2(s) > h_2^{\max}(s)$. Hence, the quadratics defined in equations (23) have the minimal properties stated.

Other properties include the following:

a) $h_2^{\max}(s)$ is tangent to $h(s)$ at $s = 1$;

b) $h_2^{\min}(s)$ is tangent to $h(s)$ at $s = 0$;

c) If

$$\mathcal{E}_1 = \int_0^1 (h_2^{\max} - h)(s)ds , \qquad \mathcal{E}_2 = \int_0^1 (h - h_2^{\min})(s)ds ,$$

then

(1) $\mathcal{E}_i = \frac{1}{3}\frac{\delta^3}{4!} + O(\delta^4) , \quad i = 1, 2,$

(2) $\mathcal{E}_2 > \mathcal{E}_1$, and

(3) $\mathcal{E}_2 - \mathcal{E}_1 = \frac{1}{3}\frac{\delta^4}{5!} + O(\delta^5)$

The two inequalities (11) now can be written in terms of C, D, and E using (23a) and (23b), that is,

$$\int_0^1 (1+i)^{1-s} f(s)dg \leq (1+i)C - (2i-\delta)D + (i-\delta)E\,, \quad (24a)$$

$$\int_0^1 (1+i)^{1-s} f(s)dg \geq (1+i)C - \delta e^{\delta} D + (\delta e^{\delta} - i)E\,. \quad (24b)$$

From (24a) and (24b), G_t^{max} and G_t^{min} can be defined as in the linear case in (14a) and (14b), where E_t^+ and E_t^- will have the obvious meaning. Developing F_n^{max} and F_n^{min} as in (16a) and (16b) and applying these formulas to Mr. Huffman's example, the following bounds are developed:

$$77{,}116.85 \leq F_{20} \leq 77{,}118.31\,. \quad (25)$$

To obtain an averaging quadratic, it suffices to note that a function in (21) will have the correct area on [0, 1] if and only if $b = 6(1 - i/\delta) + 3i$. Defining $h_2^{av}(s)$ and G_t^{av} with this b, we have

$$h_2^{av}(s) = [6(1 - i/\delta) + 3i]s^2 - [6(1 - i/\delta) + 4i]s + 1 + i \quad (26)$$

and

$$G_t^{av} = (1+i)C_t - [6(1 - i/\delta) + 4i]D_t + [6(1 - i/\delta) + 3i]E_t\,. \quad (27)$$

The value of F_{20}^{av}, for example, using equations (27) and (6) becomes 77,117.57.

It is interesting to note that the value of F_{20}^{av} obtained by using the linear averaging approximation was 77,117.89. The accuracy of this approximation is due largely to the fact that the maximum error in $h_1^{av}(s)$ is $\delta^2/12 + O(\delta^3)$ and that a more "uniform" type of canceling takes place here than the canceling that takes place when simple interest is used. This uniformity is attributable to the fact that the averaging takes place over each cash flow separately (that is, overstated premiums during part of the year average with understated premiums during another part of the year, etc.) as compared with simple interest, which averages the overstatement of some cash flows against the understatement of different cash flows.

Of course, some of the accuracy of the linear averaging approximation also may be due to the particular example itself, in that an equivalent amount of exactness may not occur in other examples.

In applying the above approximations to calendar-year asset shares, it is recommended that the approximations not be made until after the evaluation of the inner integral. Using Mr. Huffman's notation from Section V,

$$
\begin{aligned}
I' + C' &= \int_0^1 \int_0^{1-s} (1+i)^{1-s-z} f(s)\,dg\,dz \\
&= \int_0^1 \int_0^{1-s} (1+i)^{1-s-z}\,dz\,f(s)\,dg \qquad (28)\\
&= \frac{1}{\delta} \int_0^1 [(1+i)^{1-s} - 1] f(s)\,dg \,.
\end{aligned}
$$

By delaying the approximation until the last step in (28), one is able to choose whether a linear or quadratic approximation will be used, thereby obtaining the most accuracy for the efforts involved. Approximating linearly in the first step would produce a quadratic in the last step, which will involve the same efforts of calculation as any quadratic approximation but generally with only "linear approximation" accuracy.

Other applications of the above approximations include continuous and discrete insurance and annuity premiums and reserves.

In any given application the choice of approximation, whether it is linear, quadratic, or of higher order, will depend on practical limitations inherent in developing the necessary factors (C, D, E, etc.) as well as on considerations of accuracy.

JAMES A. TILLEY:

The principal contribution of Mr. Huffman's paper is the systematic approach to asset share and model-office calculations. I feel, however, that the author has placed too much emphasis on the phrase "Stieltjes integral interpretation." Every actuary knows that asset funds are calculated by accumulating all cash-inflow and -outflow items with interest to an appropriate point in time. To perform the calculation, the actuary must make assumptions about both the amount and the incidence of each cash-flow item. If the cash flows occur at discrete points this accumulation can be expressed as a sum, and if the cash flows occur continuously the accumulation can be expressed as an integral. Even the discrete case can be represented as an integral if the "incidence function" is defined in terms of step functions with discontinuities at the occurrences of cash flow.

Mr. Huffman focuses on the importance of identifying the various cash-flow items and their policy-year distribution functions. Under the assumption of simple interest from the occurrence of cash flow to the end of the policy year, any financial variable can be expressed succinctly as a sum of terms involving at most the second moments of the various cash-flow distributions. As a minor technical point, the assumption of compound interest within a policy or calendar year leads to expressions involving *all* the moments of the cash-flow distribution. However, the higher-order moments do not result in significant corrections to the original calculation. In lieu of a moment expansion, integrals of the form

$$\int_0^1 (1+i)^{1-s} f(s)\,dg(s)$$

can be evaluated directly. However, this would result in the author's treatment losing much of its simplicity with little gain in accuracy.

It should be pointed out that the results of Mr. Huffman's paper can be obtained by traditional actuarial methods. In the calculation of policy-year asset funds, only the zero and first moments of the policy-year cash-flow distribution appear in the equations. Hence, the entire distribution for a particular cash-flow item can be replaced by a single spike of appropriate height (zero moment) at the appropriate duration (first moment). For example, deaths skewed slightly toward the end of the policy year can be lumped together at a mean fractional policy duration slightly after the middle of the policy year. This is the conventional approach to policy-year asset share calculations.

When it comes to computing calendar-year results, Mr. Huffman shows that investment income for a particular calendar year depends on the second moment of the policy-year cash-flow distribution, that is, on the dispersion of the distribution. Thus, when computing calendar-year results, it is *not* correct theoretically to replace a continuous distribution of cash flow by a single spike at a particular policy-year duration. The problem is that, in general, a policy year extends from one calendar year into the next. The end of the calendar year divides the policy year into two pieces: an "alpha" portion and a "delta" portion, borrowing standard notation from the theory of mortality table construction. The policy-year cash-flow item can be replaced by *two* spikes, one in each of the alpha and delta portions of the policy year, with each at the proper duration to produce exact simple interest for calendar years. This is the approach used in mortality table construction to ensure the proper exposure to risk of death.

(AUTHOR'S REVIEW OF DISCUSSION)

PEYTON J. HUFFMAN:

Many thanks to Messrs. Chouinard, Evans, Metz, Reitano, and Tilley for their valuable and diverse discussions.

Professor Chouinard's comments on the distribution of issues provide a good introduction to a knotty problem. Modeling applications often involve nonuniformly distributed issues. Indiscriminately assuming a uniform distribution of issues can lead to timing differences and misleading results.

The continuously increasing (decreasing) sales approach will work well for new sales forecasts, provided that "continuously increasing" is consistent with the expected sales pattern and provided that each year has a full year's sales. Difficulties arise in trying to adapt this approach to historical issue patterns. The step-function approach produces integrals that are not evaluated conveniently on a computer. The polynomial approach appears to be the most promising. Unfortunately, the quadratic $R(z)$, which requires only the addition of a third moment, will produce negative values somewhere on the unit interval whenever the average duration of the distribution under consideration is outside the range $[\frac{1}{3}, \frac{2}{3}]$.

A fourth candidate is the pair of functions $R(z) = z^k$ and $r(z) = kz^{k-1}$. The parameter k may be chosen so that the average duration of $r(z)/R(z)$ matches that of the distribution being approximated by setting $k = D/(1 - D)$, where D is the average duration of issue. The resulting integrals, unfortunately, are also difficult to evaluate. It appears to me that an approach to nonuniform distribution of issues that will be suitable for automated modeling applications will require an approximation of the cash flow (as well as the distribution of issues). At a minimum, the approximation would need to reproduce the cash flow's first and second moments. For example, the beta function,

$$B(p, q, t) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} t^{p-1}(1 - t)^{p-1} ,$$

may be so adapted by setting $p = T(1 - M)/(M - T^2)$ and $q = (1 - T)\,(1 - M)\,(M - T^2)$. Approximations, of course, must be tested thoroughly before being used.

Professor Chouinard also demonstrates that traditional nonannual mode calendar-year asset shares are consistent with Stieltjes asset shares/funds.

Mr. Evans applies the Stieltjes method using compound interest, rather than the interest assumption used in the paper. The resulting formulas are useful as a comparison with those in the paper. The formulas in the paper have two advantages. First, the withdrawals need not be distributed uniformly over eligible withdrawal dates. Second, the interest and insurance cash-flow elements are separated. The latter advantage is particularly useful for modeling applications where it is necessary only to calculate the insurance cash flows and moments of a plan-age cell once, even though it is used for several issue years. As Dr. Reitano demonstrates, the difference between compound and simple interest is small.

Mr. Metz demonstrates how an actuary might modify the paper's formulas to suit his particular circumstance. In practice, asset shares generally incorporate dozens of cash-flow categories. In applying the Stieltjes approach, the actuary can make modifications more easily to recognize the characteristics of the block of business under study.

Dr. Reitano's exposition of Stieltjes integration is a welcome addition. The Stieljes integral was introduced first in 1894 in T. J. Stieltjes' *Recherches sur les fractions continues*. It was a generalization of the Riemann and Darboux integrals and subsequently was itself generalized as the Lebesgue-Stieltjes integral. The Stieltjes integral carries with it considerable structure and is easy to apply.

In the section of his comments entitled "Approximating Compound Interest," Dr. Reitano describes an interest treatment that reduces the "error" introduced by using the interest assumption in the paper rather than the standard compound interest approach. It is extremely gratifying to see the mathematical structure of this paper used so elegantly. In addition, it is comforting to find that the resulting value of F_{20} (77,117.89) is close to the value of F_{20} shown in Table 2 of the paper (77,136).

As Dr. Tilley points out, many of the results of this paper can be obtained by traditional actuarial methods. These methods, however, tend to be ad hoc and informal in nature and usually are based on general reasoning. The Stieltjes integral formalizes the structure of the cash flows and provides the actuary with a systematic method of approaching any cash flow.

Dr. Tilley suggests that the cash flows of a given category within a policy or calendar year may be replaced by a single "spike" cash flow at the appropriate duration. For many purposes, this is acceptable. It should be borne in mind, however, that the cash flows actually do not occur at that moment. In the case of cash-value surrenders, the average duration of cash flow and the average duration of termination are not even the same. The key point to remember is that the first and second moments do not describe the cash flows fully. A useful alternative to Dr. Tilley's spike approach is to allocate each cash-flow category C to the beginning and end of the year in the proportions $1 - T$ and T. This approach is convenient when new-money methods are used. Investment transactions can be limited to integral policy (calendar) durations.

The objective of this paper is to share a more general and more mathematical approach to asset shares. The paper is fully compatible with existing methods. It provides a benchmark against which approximate methods may be measured.

The methods presented are thoroughly practical. They have been in use for over four years, during which they have been the basis of well over 100,000 asset shares, profit studies, and models of individual life, health, and pension coverages. Hopefully, others will find the Stieltjes asset share/fund equally useful.

In addition to the five discussants, I wish to thank the actuaries of the Continental Assurance Company, especially Sam Gutterman and Linda Bronstein, and my wife, Maria Huffman, for their help with this paper.