Recurrence Relations In Life Contingencies

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Abstract

Recurrence relations arise naturally in life contingencies for two reasons - the aggregate law of mortality, which relates the distribution of the future life times of two insureds, is one of them and the other is that most life insurance products can be constructed as a portfolio of two basic product types (for e.g., deferred one year term and pure endowment). We use the above simple idea and the extra freedom provided by the fractional age distribution to reveal a structure behind some common recursions. The structure has obvious pedagogical value. More interestingly, a software based on it has been implemented which facilitates actuarial computing. As one of its uses, it can be used in a classroom setting to help students gain additional insight into actuarial quantities.

1 Introduction

It is difficult to think of a course on Life Contingencies without recurrence relations. The prevalence of recursions is due to two basic results. First, the **aggregate law of mortality** (ALoM), which relates the distribution of the future life times of two insureds. ALoM facilitates recursions for actuarial present value of products traversing different issue ages. Both the continuous and discrete versions of it are stated in the Appendix. Second, that most life insurance products can be decomposed into a portfolio of utmost two basic product types, for e.g., deferred one year term and pure endowment, gives rise to relations between products with different terms. Of course, for the latter, that the present value and expectation operators are linear (and hence so is the actuarial present value operator) is crucial. Making explicit the use of the above two basic results in proving the recursive formulae and hence demonstrating that they are the raíson de être for the recursions is one of the principal goals of this pedagogical article.

Texts on Life Contingencies work with three versions of each insurance product type - sum assured payable at the moment of death, payable at the end of the m^{th} period containing the moment of death and payable at the end of the year of death. Even though the second version includes the last, as the case when m equals one, they are mentioned separately as the last version, apparently, does not require the specification of the fractional age distribution (FAD) unlike those with m greater than one. By the FAD we mean the conditional distribution of $T(x)$ minus $K(x)$ given $K(x)$. Since insurances of the latter two types are inconceivable nowadays, there is no value to be gained by treating them as separate products. We view the latter two types as special cases of the first, by

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altering the fractional age distribution. Apart from unification, it makes clear that latter two types are approximations to the first. In the case of annuities the value of separate versions is unarguable. In this case our proposal is just a mathematical nicety. Exposition of this unification is the second of our main goals.

Given that the term and whole life insurance can be naturally arranged in two dimensions implies that just two recurrences should lead to all the others. Moreover, the above unification implies that we can derive those for the insurances which pay at the end of the m^{th} period containing the moment of death from those that pay at the moment of death. These two observations lead to a host of recursions from just two. Presenting this structure is the third of our important objectives.

The function $\mathrm{a}(\cdot)$, defined as $\mathrm{a}(x) = \mathbb{E}(\mathrm{T}(x)|\mathrm{T}(x) < 1)$, appears in recurrences involving complete future life expectations. It has, in particular, two nice properties. First, the FAD enters into the computation only through it. Second, in the case that the FAD is independent of $\mathrm{K}(x)$, i.e. has the same distribution irrespective of the last age at death, then $\mathrm{a}(\cdot)$ is a constant. Note that the condition is sufficient but not necessary for $\mathrm{a}(\cdot)$ to be a constant. An example where the condition is satisfied is when we assume the uniform distribution on $(0, 1)$ for the fractional age. The natural analogue of $\mathrm{a}(\cdot)$ for actuarial present value of insurance products is $\mathbb{E}(\exp\{-\delta\mathrm{T}(x)\}|\mathrm{T}(x) < 1)$, which is the Laplace transform of the FAD. We will use $\mathrm{m}_\delta(\cdot)$ to denote it as a function of x, i.e.,

$$\mathrm{m}_\delta(x) = \mathbb{E}(\exp\{-\delta\mathrm{T}(x)\}|\mathrm{T}(x) < 1) \tag{1}$$

The letter m is chosen as the moment generating function is related to the Laplace transform and the letter l would cause conflict with existing notations. Hence, a software providing these two functions, for all the standard fractional age distributions, will make computation of actuarial present value of an insurance product paying at the moment of death as simple as that of its version paying at the end of the year of death. Such a software has been implemented on the Excel platform and is being used in a classroom setting to help students gain additional insight into actuarial quantities through computing. The software is available on request. This is the fourth on our list of goals for this article.

A simple relation between endowments and life expectations makes all the recursions of the former lead to ones for the latter. Moreover, as the present value random variable for endowments and annuities are linearly related, we can derive the recurrences for the annuities too from those for endowments. Finally that an endowment is a portfolio of a term and a pure endowment makes the recursions for term insurance lead to all of those for endowments, annuities and life expectations. This is the final objective of this article.

The use of floors and ceilings has been motivated by Shiu (1982). By it's nature a pedagogical article is bound to have some of its ideas presented before in some of the texts. This is the case here too; see for e.g. Bowers et al (1997). We have not made an effort to search the literature in order to trace the first exponent of every idea that has been repeated here.

Notations, unless specified otherwise, are those found in the text by Bowers et al (1997). A discrete uniform random variable over the integers between m and n, both inclusive, would be denoted by $\mathrm{DU}(m, n)$. Moreover, for any random variable X we shall denote by $\phi_X(\cdot)$ its moment generating function. Always, x will denote a non-negative integer. For a mortality table, ω will denote the age such that $\mathrm{q}_{\omega-1} = 1$.

2 Unified View of Insurance Products

A typical life contingencies course contains three versions of, for e.g., a whole life insurance. They vary in the moment at which the sum assured is paid to the beneficiary - payable at the moment of death, payable at the end of the m$^{\text{th}}$ subperiod of year containing the moment of of death and payable at the end of the year of death. Nowadays, the existence of the products of the latter two types is inconceivable. Nevertheless, the products of the latter two types serve as approximations to the first. Moreover, given that the FAD has to be specified for valuations of the first two types, the charm of the last has yet to fade away, especially vis a vis the regulators. In the spirit of approximations, we show that the last two versions can be thought of as the first but with different specifications of the FAD. Since the last type is included in the second as the case m equals one, we will work with only types one and two.

Following Shiu (1982), we will be using floor's and ceiling's in the following. We start with the expressions for the actuarial present values in terms of expectations of the present value random variables. They are,

$$\mathrm{A}_x^{(m)} = \mathbb{E}\left(\exp\left\{-\delta\left(\frac{\lceil \mathrm{T}(x) * m\rceil}{m}\right)\right\}\right), \qquad \forall m \geq 1 \tag{2}$$

and

$$\bar{\mathrm{A}}_x = \mathbb{E}(\exp\{-\delta \mathrm{T}(x)\}). \tag{3}$$

Using the basic properties of ceiling and floor, we have

$$\begin{aligned} \frac{\lceil \mathrm{T}(x) * m\rceil}{m} &= \lfloor \mathrm{T}(x)\rfloor + \left\lceil \frac{(\mathrm{T}(x) - \lfloor \mathrm{T}(x)\rfloor) * m}{m} \right\rceil \\ &= \mathrm{K}(x) + \left\lceil \frac{(\mathrm{T}(x) - \mathrm{K}(x)) * m}{m} \right\rceil. \end{aligned} \tag{4}$$

Recalling that the FAD is the conditional distribution of $\mathrm{T}(x) - \mathrm{K}(x)$ given K(x), the above implies that $\mathbb{E}(\exp\{-\delta \mathrm{T}(x)\})$ evaluated, instead, under the FAD given by the conditional distribution of

$$\left\lceil \frac{(\mathrm{T}(x) - \mathrm{K}(x)) * m}{m} \right\rceil \qquad \text{given} \qquad \mathrm{K}(x) \tag{5}$$

would result in $\mathbb{E}\left(\exp\left\{-\delta\left(\frac{\lceil \mathrm{T}(x)*m\rceil}{m}\right)\right\}\right)$ evaluated under the original FAD.

In the case of the uniform FAD, using Lemma 3, the above reduces to saying,

$$\mathbb{E}\left(\exp\left\{-\delta\left(\frac{\lceil \mathrm{T}(x) * m\rceil}{m}\right)\right\}\right)_{\text{under uniform FAD}} = \mathbb{E}(\exp\{-\delta \mathrm{T}(x)\})\ \text{under}\ \frac{\mathrm{DU}(1,m)}{m}\ \text{for the FAD}. \tag{6}$$

More generally, Lemma 3 implies that

$$\exp\left\{-\delta\left(\frac{\lceil \mathrm{T}(x) * m \rceil}{m}\right)\right\} \text{ under uniform FAD } \stackrel{\mathrm{d}}{=} \exp\left\{-\delta \mathrm{T}(x)\right\} \text{ under } \frac{\mathrm{DU}(1,m)}{m} \text{ for the FAD,} \tag{7}$$

which in turn implies that the product version that pays at the end of the m$^{\text{th}}$ subperiod of year containing the moment of death is financially identical to the version that pays at the moment of death, as both the present value random variables have the same distribution, albeit under different fractional age distributions.

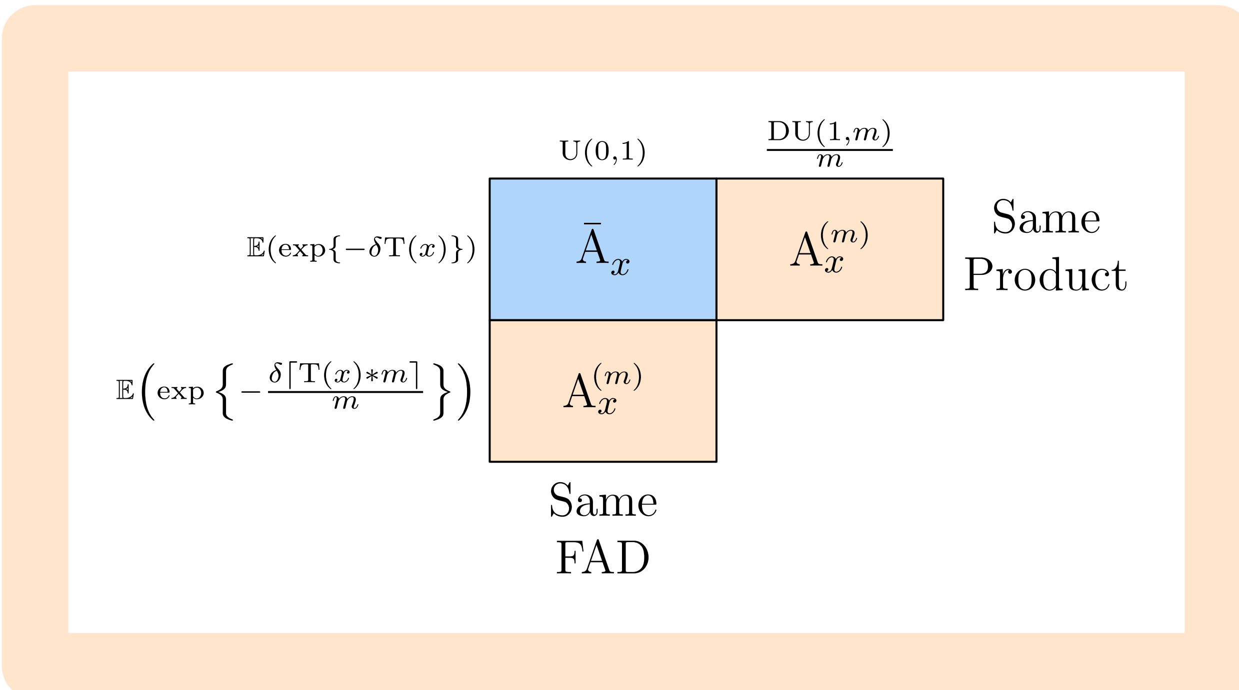

Figure 1 The Two Approaches

The above, summarized in Figure 1, holds for two reasons. First, that

$$\left\lfloor \frac{\lceil \mathrm{T}(x) * m \rceil}{m} \right\rfloor = \mathrm{K}(x) = \lfloor \mathrm{T}(x) \rfloor . \tag{8}$$

Second, that any two positive real random variables, having the same distribution for their integer parts, can possibly differ only in their conditional distributions of their fractional part given their integer part.

The conclusion of the above is that we can consider just one version of the whole life insurance - that which pays at the moment of death. Whole life insurance has been used for the above discussion for the sake of clarity. It is important to observe that any other insurance product, whose cashflows occur at a time point which is a function of the moment of death and whose sum assured is a function of the time of payment, could have replaced it. We end this section by summarizing the above using the theorem below.

Theorem 1 Let us consider an insurance product such that it pays at $\psi(\mathrm{T}(x))$ a sum assured of $\alpha(\psi(\mathrm{T}(x)))$. Further, let $\psi(\cdot)$ be such that

$$\lfloor \psi(\mathrm{T}(x)) \rfloor = \mathrm{K}(x), \qquad \text{with probability 1.} \tag{9}$$

Then, under the FAD denoted by $\mathrm{F}_{\mathrm{K}(x)}(\cdot)$,

$$\alpha(\psi(\mathrm{T}(x))) \exp\{-\delta\psi(\mathrm{T}(x))\} \tag{10}$$

has the same distribution as

$$\alpha(\mathrm{T}(x)) \exp\{-\delta \mathrm{T}(x)\} \tag{11}$$

but the latter under the FAD of $\mathrm{G}_{\mathrm{K}(x)}(\cdot)$. $\mathrm{G}_{\mathrm{K}(x)}(\cdot)$ is defined as the distribution of $\lfloor \psi(\mathrm{T}(x)) \rfloor - \mathrm{K}(x)$ conditioned on $\mathrm{K}(x)$, where the distribution of $\mathrm{T}(x) - \mathrm{K}(x)$ conditioned on $\mathrm{K}(x)$ is taken as the original FAD of $\mathrm{F}_{\mathrm{K}(x)}(\cdot)$.

Proof

$$\begin{aligned}
&\alpha(\psi(\mathrm{T}(x))) \exp\{-\delta\psi(\mathrm{T}(x))\} \qquad (\text{under FAD of } \mathrm{F}_{\mathrm{K}(x)}(\cdot))\\
\stackrel{\mathrm{d}}{=}&\alpha(\mathrm{K}(x) + (\psi(\mathrm{T}(x)) - \mathrm{K}(x))) \exp\{-\delta(\mathrm{K}(x) + (\psi(\mathrm{T}(x)) - \mathrm{K}(x)))\} \qquad (\text{under FAD of } \mathrm{F}_{\mathrm{K}(x)}(\cdot))\\
\stackrel{\mathrm{d}}{=}&\alpha(\mathrm{T}(x)) \exp\{-\delta(\mathrm{T}(x))\} \qquad (\text{under FAD of } \mathrm{G}_{\mathrm{K}(x)}(\cdot)) \qquad \S
\end{aligned}$$

3 Decomposition of Insurance Products

Life contingent cashflows, hence products, can be classified into two pure types and a hybrid. Cashflows which occur at a time point which is a function of the moment of death represent the first pure type. Those occurring at fixed time points and contingent on the insured being alive represent the other pure type. Cashflows of the hybrid type are composed of both the pure types; for e.g., cashflows of an endowment. We will call products of the first type as pure insurance products and the second type as pure annuity products. In the following, we will work only with products that have a constant sum assured between successive integral ages.

Due to the theorem of the previous section, among pure insurance products, we can restrict our attention to only those that pay at the moment of death. Hence, continuous deferred one year term (DOYT) serves as a basic product type for pure insurance products and pure endowments for pure annuity products. Hybrid products can be decomposed into a portfolio containing both DOYTs and pure endowments. For example, below we give a list of products whose portfolio representation is given in terms of their actuarial present values. In the case of the increasing and decreasing sum assured products it is important to maintain the same sum assured between successive integral ages, i.e. to consider products like $\left(\mathrm{I}\bar{\mathrm{A}}\right)^{1}_{x:\overline{n}|}$ and not $\left(\overline{\mathrm{IA}}\right)^{1}_{x:\overline{n}|}$.

$$\bar{\mathrm{A}}^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} {}_{k|1}\bar{\mathrm{A}}_x \qquad \text{n year Term } \surd$$

$$\bar{\mathrm{A}}_x = \sum_{k=0}^{\infty} {}_{k|1}\bar{\mathrm{A}}_x \qquad \text{Whole Life } \surd$$

$$\left(\mathrm{I}\bar{\mathrm{A}}\right)^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1}(k+1)_{k|1}\bar{\mathrm{A}}_x \qquad \text{Increasing Term } \surd$$

$$\left(\mathrm{D}\bar{\mathrm{A}}\right)^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)_{k|1}\bar{\mathrm{A}}_x \qquad \text{Decreasing Term } \surd$$

$$_{m|n}\bar{\mathrm{A}}_x = \sum_{k=m}^{m+n-1} {}_{k|1}\bar{\mathrm{A}}_x \qquad \text{m - Deferred n year Term } \surd$$

4 Recurrence - Term and Whole Life

This and the next section deal with the main topic of this article, recurrence relations. Recursions, specifically for term and whole life insurances form the content of this section. These would lead to recurrences for endowments, which in turn to those for annuities and life expectations. The recursions for endowments, annuities and life expectations are dealt with in the next following section.

An arrangement of all possible term and whole life insurances, along with the four type of valuation problems, is depicted in Figure 2. Being a two-dimensional arrangement, we should be able to derive all recurrences from just two - one traversing vertically and the other horizontally. The first principal result of this section affirms this. The other principal result shows that recurrences for insurances payable at the moment of death, lead to those for insurances payable at the end of the m$^{\text{th}}$ period containing the time of death.

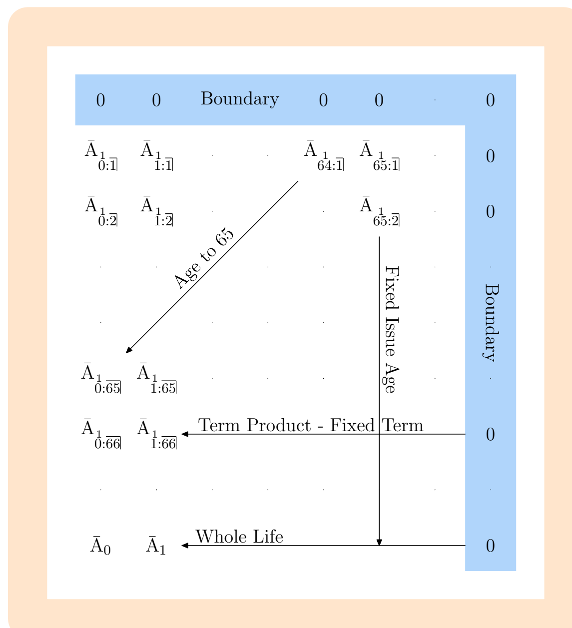

Figure 2 Valuation Problems : Term & Whole Life

The recurrences make use of the function $\mathrm{m}_\delta(\cdot)$, which we recall was defined in (1) as,

$$\mathrm{m}_\delta(x) = \mathbb{E}(\exp\{-\delta\mathrm{T}(x)\}|\mathrm{T}(x) < 1). \tag{12}$$

Hence,

$$\bar{\mathrm{A}}^{1}_{x:\overline{1}|} = \mathbb{E}\Big(\exp\{-\delta\mathrm{T}(x)\}\mathrm{I}_{\{\mathrm{T}(x)<1\}}\Big) = \mathrm{q}_x\mathbb{E}(\exp\{-\delta\mathrm{T}(x)\}|\mathrm{T}(x) < 1) = \mathrm{q}_x\mathrm{m}_\delta(x) \tag{13}$$

In the recursion formulae we use $\mathrm{m}_\delta(\cdot)$ instead of $\bar{\mathrm{A}}^{1}_{x:\overline{1}|}$ as the former is a constant when the FAD is independent of $\mathrm{K}(x)$ whereas the latter would still be a function of q_x. The choice is not only a matter of aesthetics but is important from the software design point of view.

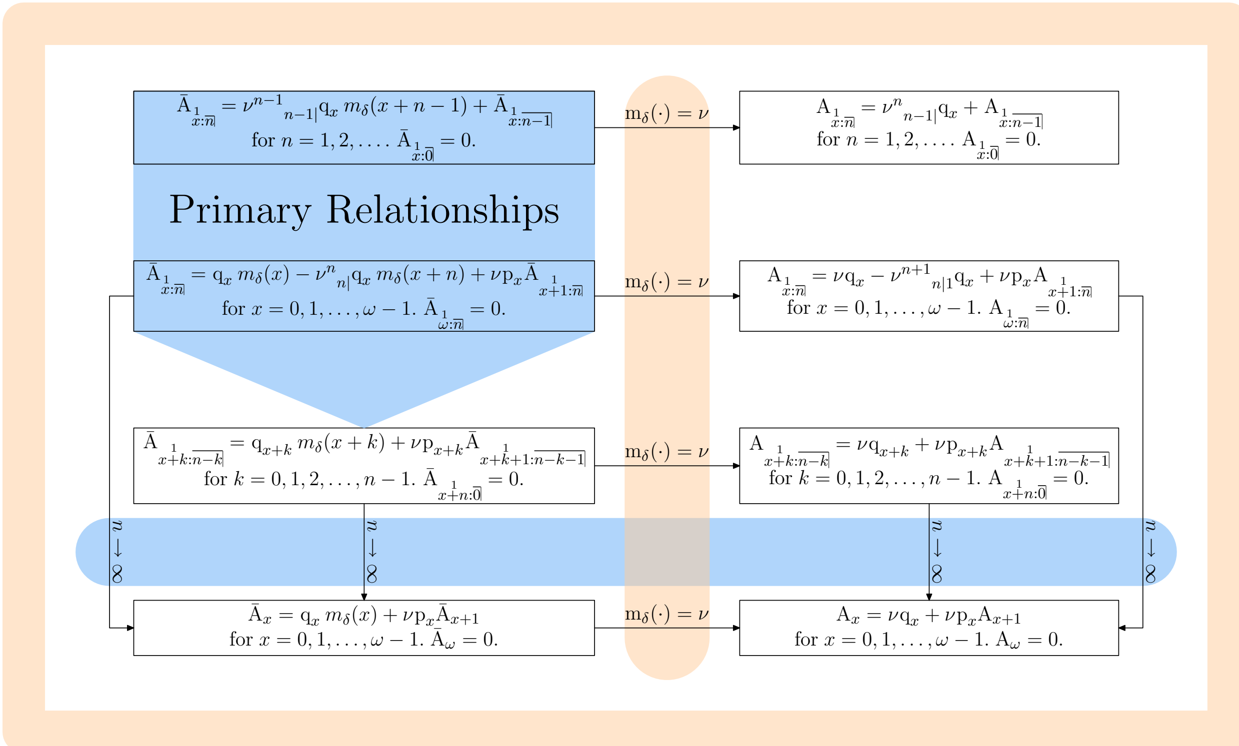

Figure 3 Term Insurances

The four recurrences of Figure 2 are in the first column of Figure 3. Their discrete versions are in the second column. The top two recurrences are designated as primary relationships as the others can be derived starting from them. The first of the primary recurrences is that travelling vertically down and the other is that going horizontally left. We start by giving a proof of these two primary recurrences.

Theorem 2

$$\bar{A}^{1}_{x:\overline{n}|} = \nu^{n-1}{}_{n-1|}q_x\, m(x+n-1) + \bar{A}^{1}_{x:\overline{n-1}|} \tag{14}$$

for $n = 1, 2, \ldots$. The boundary condition is given by $\bar{A}^{1}_{x:\overline{0}|} = 0$.

Proof We start by observing that, by the portfolio representation of term insurances, we have,

$$\bar{A}^{1}_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j|1}\bar{A}_x. \tag{15}$$

Hence,

$$\begin{aligned}\bar{A}^{1}_{x:\overline{n}|} &= \sum_{j=0}^{n-1} {}_{j|1}\bar{A}_x \\ &= \sum_{j=0}^{n-2} {}_{j|1}\bar{A}_x + {}_{n-1|1}\bar{A}_x = \nu^{n-1}{}_{n-1|}q_x\, m(x+n-1) + \bar{A}^{1}_{x:\overline{n-1}|}\end{aligned}$$

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Theorem 3

$$\bar{\mathrm{A}}^{\,1}_{x:\overline{n}|} = \mathrm{q}_x\, m(x) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n) + v\mathrm{p}_x \bar{\mathrm{A}}^{\,1}_{x+1:\overline{n}|} \tag{16}$$

for $x = 0, 1, \ldots, \omega - 1$. The boundary condition is given by $\bar{\mathrm{A}}^{\,1}_{\omega:\overline{n}|} = 0$.

Proof By definition of a term insurance, we have

$$\bar{\mathrm{A}}^{\,1}_{x:\overline{n}|} = \mathbb{E}\Big(\exp\{-\delta \mathrm{T}(x)\}\mathrm{I}_{\{\mathrm{T}(x)<n\}}\Big). \tag{17}$$

Hence, for $n \geq 1$, we have

$$\begin{aligned}
\bar{\mathrm{A}}^{\,1}_{x:\overline{n}|} &= \bar{\mathrm{A}}^{\,1}_{x:\overline{n+1}|} - {}_{n|1}\bar{\mathrm{A}}_x && \text{(Decomposition)}\\
&= \mathbb{E}\Big(\exp\{-\delta \mathrm{T}(x)\}\mathrm{I}_{\{\mathrm{T}(x)<n+1\}}\Big) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n)\\
&= \mathbb{E}\Big(\exp\{-\delta \mathrm{T}(x)\}\mathrm{I}_{\{\mathrm{T}(x)<n+1\}}|\mathrm{T}(x) < 1\Big)\Pr(\mathrm{T}(x) < 1)\\
&\quad + \mathbb{E}\Big(\exp\{-\delta \mathrm{T}(x)\}\mathrm{I}_{\{\mathrm{T}(x)<n\}}|\mathrm{T}(x) \geq 1\Big)\Pr(\mathrm{T}(x) \geq 1) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n)\\
&= \mathrm{q}_x \mathbb{E}(\exp\{-\delta \mathrm{T}(x)\}|\mathrm{T}(x) < 1)\\
&\quad + \mathrm{p}_x \mathbb{E}\Big(\exp\{-\delta(\mathrm{T}(x+1)+1)\}\mathrm{I}_{\{\mathrm{T}(x+1)+1<n+1\}}\Big) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n) && \text{(By ALoM)}\\
&= \mathrm{q}_x \mathrm{m}(x) + v\mathrm{p}_x \mathbb{E}\Big(\exp\{-\delta(\mathrm{T}(x+1))\}\mathrm{I}_{\{\mathrm{T}(x+1)<n\}}\Big) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n)\\
&= \mathrm{q}_x\, \mathrm{m}(x) - v^n\, {}_{n|}\mathrm{q}_x\, m(x+n) + v\mathrm{p}_x \bar{\mathrm{A}}^{\,1}_{x+1:\overline{n}|} && \S
\end{aligned}$$

The third relationship in the first column of Figure 3 can be derived using the above two, as shown below. This is depicted in Figure 4.

$$\begin{aligned}
\bar{\mathrm{A}}^{\,1}_{x+k:\overline{n-k}|} &= \mathrm{q}_{x+k}\mathrm{m}(x+k) - v^{n-k}\, {}_{n-k|}\mathrm{q}_{x+k}\mathrm{m}(x+n) + v\mathrm{p}_{x+k}\bar{\mathrm{A}}^{\,1}_{x+k+1:\overline{n-k}|} && \text{(Theorem 3)}\\
&= \mathrm{q}_{x+k}\mathrm{m}(x+k) - v^{n-k}\, {}_{n-k|}\mathrm{q}_{x+k}\mathrm{m}(x+n)\\
&\quad + v\mathrm{p}_{x+k}\left(v^{n-k-1}\, {}_{n-k-1|}\mathrm{q}_{x+k+1}\mathrm{m}(x+n) + \bar{\mathrm{A}}^{\,1}_{x+k+1:\overline{n-k-1}|}\right) && \text{(Theorem 2)}\\
&= \mathrm{q}_{x+k}\mathrm{m}(x+k) - v^{n-k}\, {}_{n-k|}\mathrm{q}_{x+k}\mathrm{m}(x+n) + v^{n-k}\, {}_{n-k|}\mathrm{q}_{x+k}\mathrm{m}(x+n) + v\mathrm{p}_{x+k}\bar{\mathrm{A}}^{\,1}_{x+k+1:\overline{n-k-1}|}\\
&= \mathrm{q}_{x+k}\mathrm{m}(x+k) + v\mathrm{p}_{x+k}\bar{\mathrm{A}}^{\,1}_{x+k+1:\overline{n-k-1}|}
\end{aligned}$$

In fact, it requires the use of only the ALoM unlike the use of both ALoM and the portfolio representation for the second. As any two of the first three relationships, together, give rise to the other, we could have as well presented the first and the third as the two primary relationships from which all the others derive. We chose the first two as they geometrically are more natural a choice even though, conceptually, the first and the third would have been more appropriate as they use only the portfolio representation and the ALoM, respectively.

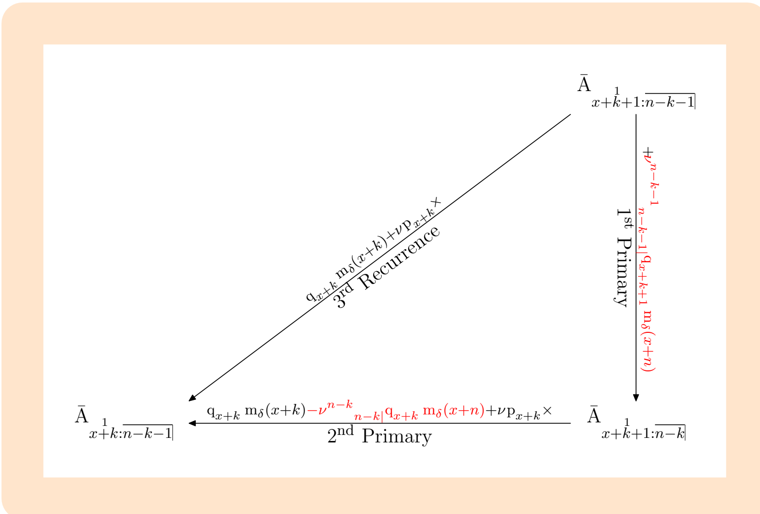

Figure 4 Combining the Two Primary Relationships

By either the dominated convergence theorem or the simple fact that for mortality tables there exists a finite terminal age ω, we have term insurances converging as $n \to \infty$ to whole life insurances. This is used to arrive at the last row of Figure 3.

The generality of the relationships in the first column of Figure 3, accorded to it by the theorem of section 3, imply that we can derive recurrences for any m$^{\text{th}}$ly payable insurance as special cases of it. For example, using the FAD of $\frac{\mathrm{DU}(1,m)}{m}$, we have

$$\mathrm{m}_\delta(\cdot) = \frac{d}{i^{(m)}}, \qquad \forall m \geq 1. \tag{18}$$

Using the above, in particular, the first recurrence gives us,

$$\mathrm{A}^{(m)}_{\overset{1}{x}:\overline{n}|} = \nu^{n-1}{}_{n-1|}q_x \left(\frac{d}{i^{(m)}}\right) + \mathrm{A}^{(m)}_{\overset{1}{x}:\overline{n-1}|}, \qquad \text{for } n = 1, 2, \ldots, \infty.\ \mathrm{A}^{(m)}_{\overset{1}{x}:\overline{0}|} = 0. \tag{19}$$

As an example of the above, the relationships in the case where the sum assured is payable at the end of the year of death are given in the second column of Figure 3.

5 Recurrence - Endowments, Annuities and Life Expectations

To arrive at recurrence relationships for endowments (Figure 6) from that of term insurances (Figure 3), we have to only add the appropriate pure endowment component on either side of the recurrence equation. Figure 5 gives the recurrences for pure endowments that is required to implement those for the endowments.

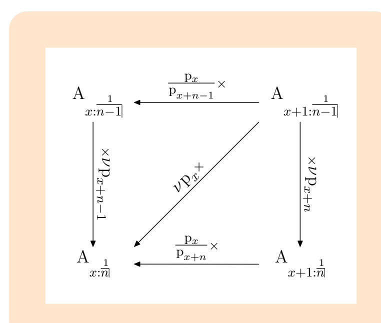

Figure 5 Pure Endowments

Endowments and annuities are closely related as the random variables representing present value of cashflows under each product are perfectly negatively correlated - one is a linear combination of the other. This, in particular, implies that their actuarial present values are linearly related. Using this linear relationship one can derive the recurrences for annuities from those for endowments.

In the case of the annuities, we would not have a relationship exactly analogous to the equivalence that exists in the case of a whole life payable at the moment of death under $\frac{\mathrm{DU}(1,m)}{m}$ FAD and one that pays at the end of the m$^{\text{th}}$ period containing the moment of death but under the U(0, 1) FAD (see Figure 1). Instead, we have the following, which is close enough;

$$\bar{\mathrm{a}}_x \text{ under } \frac{\mathrm{DU}(1, m)}{m} \quad = \quad \left(\frac{d^{(m)}}{\delta}\right) \ddot{\mathrm{a}}_x^{(m)} \text{ under } \mathrm{U}(0, 1). \tag{20}$$

The relationship of the above kind are valid for a n year deferred one year temporary life annuity and hence any other product which can be decomposed into a portfolio of such annuities.

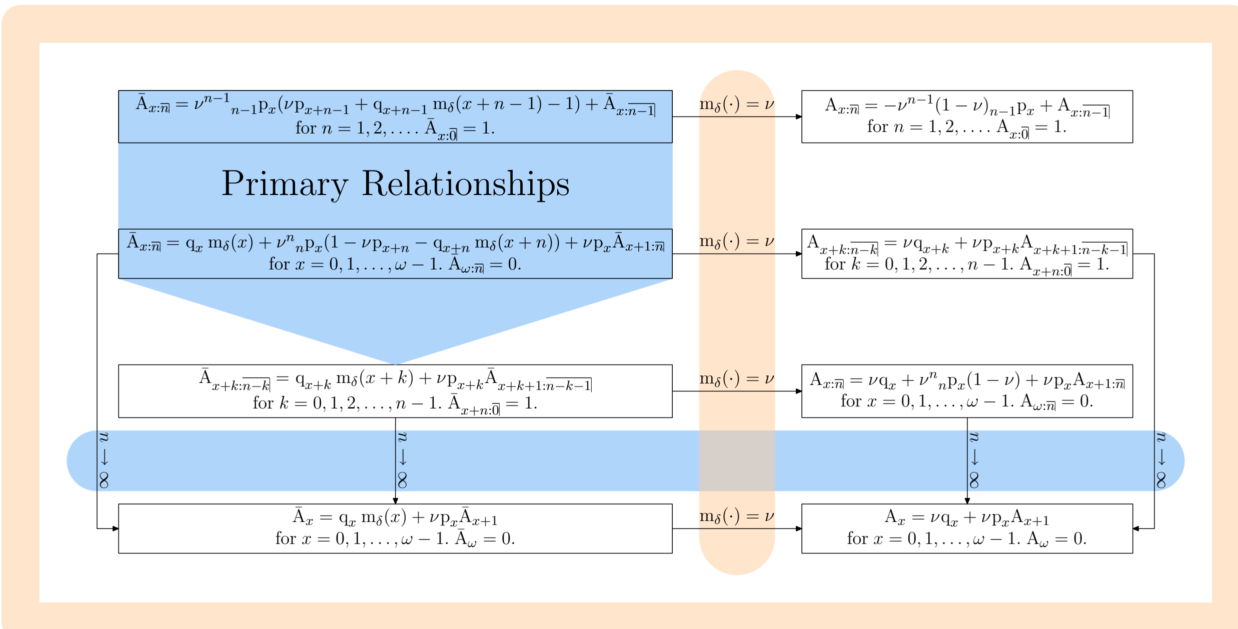

Figure 6 Endowments

For recursions involving life expectations we would need the following relationship between $\mathrm{m}_\delta(\cdot)$ and $\mathrm{a}(\cdot)$.

$$\left.\frac{\mathrm{d}}{\mathrm{d}\delta}\mathrm{m}_\delta(x)\right|_{\delta=0} = -\mathbb{E}(\mathrm{T}(x)|\mathrm{T}(x) < 1) = -\mathrm{a}(x). \tag{21}$$

Similar to the above are the following relationships that will be needed below.

$$\left.\frac{\mathrm{d}}{\mathrm{d}\delta}\bar{\mathrm{A}}_{x:\overline{n}|}\right|_{\delta=0} = -\mathring{\mathrm{e}}_{x:\overline{n}|} \tag{22}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\delta}\mathrm{A}_{x:\overline{n}|}\right|_{\delta=0} = -\left(\mathrm{e}_{x:\overline{n}|} + {}_n\mathrm{q}_x\right) \tag{23}$$

To arrive at recurrences for future life expectations from endowments, we have a choice between two routes. First, differentiating both sides of recurrences for endowments and using equations (22) and (23) along with (21), we convert these to relationships involving future life expectations; these are given in Figure 7. Second, by observing that analogous to equations (22) and (23), we have

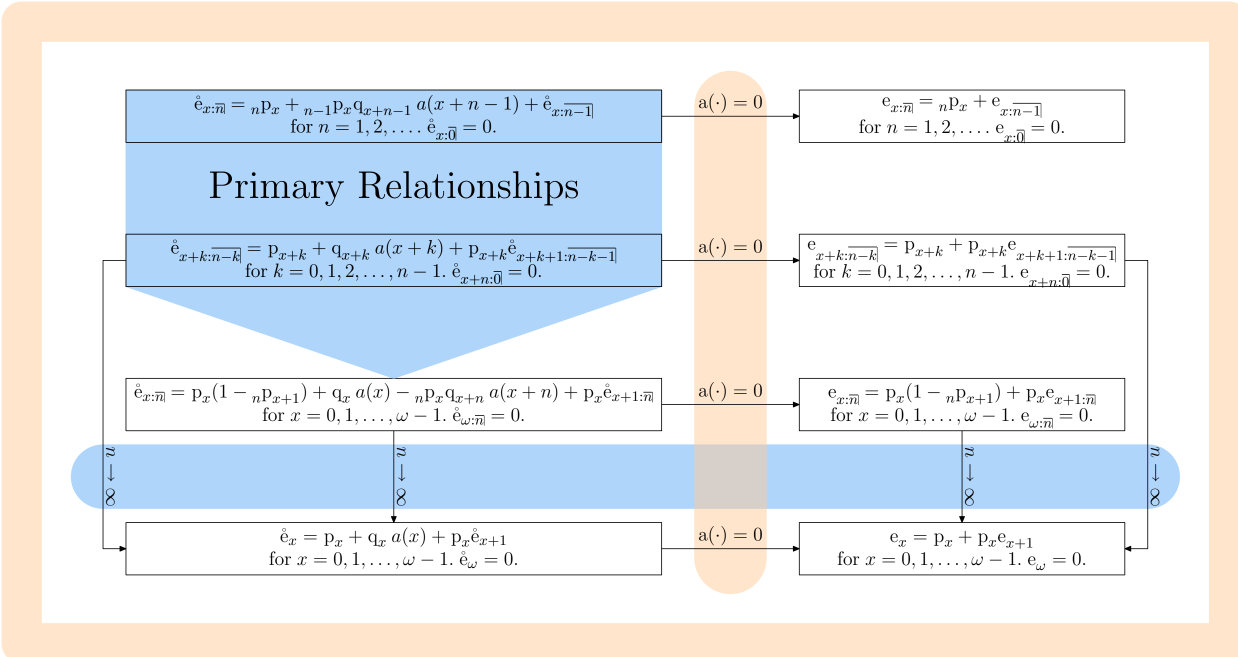

Figure 7 Life Expectations

$$\lim_{\delta \to 0} \bar{a}_{x:\overline{n}|} = -\mathring{e}_{x:\overline{n}|} \tag{24}$$

and

$$\lim_{\delta \to 0} \ddot{a}_{x:\overline{n}|} = -\left(e_{x:\overline{n}|} + {}_n q_x\right), \tag{25}$$

implies that recursions for annuities will lead to those for life expectations by taking limits as δ tends to zero. The above imply that $a(\cdot)$ is the analogue of $m_\delta(\cdot)$, for recursions involving life expectations.

To arrive at curtate life expectations from complete life expectation, we assign the degenerate distribution at zero (DU(0, 0)) as the FAD. $K(x)$ is equal to $T(x)$ with probability 1 under this assignment for FAD. In particular, the curate life expectation and complete life expectations coincide. Moreover, since $a(\cdot)$ being a constant zero characterizes the degenerate distribution at zero fractional age distribution and the FAD appears in the recurrence relationships only through it, replacing $a(\cdot)$ by zero and complete life expectations by corresponding curtate ones, yields recurrence relationships for curtate life expectations. This is shown in Figure 7.

6 SOFTWARE

The idea that computing leads to better insight made us search for means to facilitate computation of actuarial quantities. Students, typically at ease with computing actuarial quantities in the case of degenerate FADs, are diffident when it comes to non-degenerate FADs. The above results makes it clear that a software, providing readily to the user the functions $m_\delta(\cdot)$ and $a(\cdot)$, would make the non-degenerate case as easy as the degenerate case. Our search for a computing platform providing such functions and access to standard mortality tables led us to a Microsoft Excel based solution.

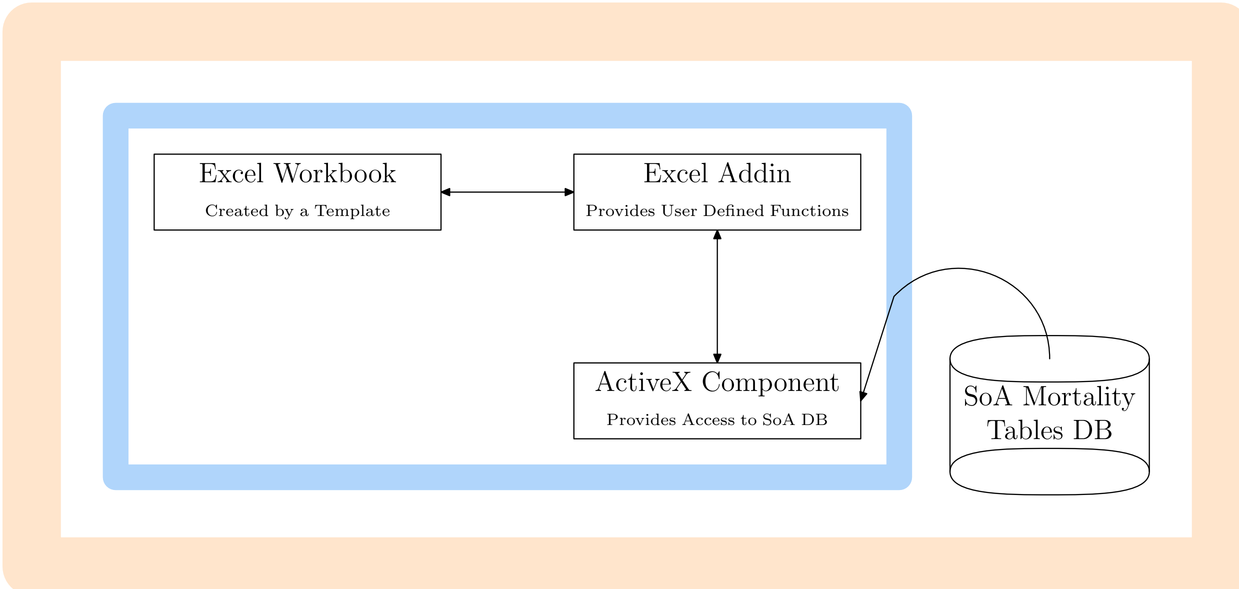

Figure 8 Software Design

The software is described Figure 8. There are three basic components - a template, an Excel addin and an ActiveX component. The Active X component facilitates access to the binary format mortality database of the Society of Actuaries. The database contains close to a thousand different mortality tables. The component is OOP based and can be accessed by other ActiveX enabled applications on the Windows operating system. The Excel addin apart from providing fractional age distribution (FAD) and interest rate dependent functions like $\mathrm{a}(\cdot)$ and $\mathrm{m}_\delta(\cdot)$ also eases communication from a workbook to the ActiveX component by providing a host of (*user defined*) functions. The choice of fractional age distributions includes all standard ones. Finally, the template avoids repetitive task of creating the same look and feel for a workbook accessing the addin, hence reducing errors and saving time.

7 CONCLUSION

In Figure 9 we show the interrelationship between the recursions for term, endowment, annuity and life expectation. This in particular implies that the two primary recursions for term lead to all the other recursions through simple operations. Above, we did not address the stability of these recurrences. The recursions with their stated boundary conditions can be shown to be stable - the proof of their stability, though, is out of the scope of this paper.

Two prominent life contingency recursions missing are that of the Fackler's formula and the Hattendorf's theorem. Reserves can be handled easily, as the reserve is the difference between the actuarial present value of the future benefits and the future premiums. This suggests, from the above recursions, the backward direction whereas the Fackler is a forward recursion formula. It is interesting to note that the Fackler is not stable in the forward direction. The Hattendorf's theorem gives a recursive relation for the variance of the present value random variable. Since the focus here has been on the actuarial present value of products, Hattendorf's theorem has not been treated here.

$$
\begin{tikzcd}
\text{Term Insurances} \arrow[r, "\text{Pure Endowments}"] & \text{Endowments} \arrow[dl, "\delta\bar{\mathrm{a}}_x+\bar{\mathrm{A}}_x=1"'] \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0}"] \\
\text{Annuities} \arrow[r, "\lim\limits_{\delta\to 0}"] & \text{Life Expectations}
\end{tikzcd}
$$

Figure 9 The Interrelationships

For a good source for fractional age distributions which are independent of $\mathrm{K}(x)$, see Willmot (1997). For these distributions, as mentioned above, both $\mathrm{m}_\delta(\cdot)$ and $\mathrm{a}(\cdot)$ are constants. In fact, the discussions by Shiu E. and Tiong S. that follow the article is also a pertinent reference for the usage of floors and ceilings in Life Contingencies.

Other insurance products like the continuously increasing term payable at the moment of death, even though lying outside the scope of this paper, can be treated along the same principles as above.

8 APPPENDIX

Here we state some results without proof: the results are either standard or their proofs follow easily from first principles. The first result is the discrete version of the Aggregate Law of Mortality (ALoM).

Lemma 1 The aggregate law of mortality states that

$$\mathrm{K}(x)|\mathrm{K}(x) \geq m \stackrel{\mathrm{d}}{=} \mathrm{K}(x+m)+m, \quad \forall x, m \geq 0. \tag{26}$$

holds when the distribution of $\{\mathrm{K}(x)\}_{x\geq 0}$ is derived from an aggregate mortality table.

A continuous version of ALoM also holds true under the condition that the FAD does not depend on the issue age. For e.g., it may depend on $\mathrm{K}(x)+x$, the last age at death, but not on the issue age x.

Lemma 2 The continuous version of the aggregate law of mortality states that

$$\mathrm{T}(x)|\mathrm{T}(x) \geq m \stackrel{\mathrm{d}}{=} \mathrm{T}(x+m) + m, \quad \forall x, m \geq 0, \tag{27}$$

when the FAD does not depend on the issue age and the distribution of $\{\mathrm{K}(x)\}_{x\geq 0}$ is derived from an aggregate mortality table.

ALoM is a misnomer as it holds for even the ultimate portion of a select & ultimate table and is a valuable tool in computations involving such tables.

Lemma 3 The following results hold true for two independent discrete and continuous uniform random variables.

i.

$$\lceil m * \mathrm{U}(0,1) \rceil \stackrel{\mathrm{d}}{=} \mathrm{DU}(1,m), \qquad \forall m \geq 1 \tag{28}$$

ii.

$$\phi_{\mathrm{DU}(m,n)}(t) = \begin{cases} \frac{\exp\{t(n+1)\} - \exp\{tm\}}{(n+1-m)(\exp\{t\}-1)}, & t \neq 0; \\ 1, & t = 0; \end{cases}. \tag{29}$$

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