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THE PRICING OF NONPARTICIPATING SINGLE PREMIUM IMMEDIATE ANNUITIES

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ABSTRACT

This paper develops pricing theory for single premium immediate annuities using Anderson's present value of book profits method. Federal income tax and contingency reserves are recognized explicitly in the expression for the annual book profits. The interest assumption is based on new-money rates and an investment-generation method of allocating net investment income, resulting in nonlinear dependence of the present value of book profits on the annuity rate. Newton-Raphson iteration is used to solve for the annuity rate that meets a given profit objective. Equations for after-tax portfolio net earned rates are derived as an aid to fitting the interest rate structure of the annuity rate basis. The theory is applied to the pricing of straight-life and guaranteed-ten nonqualified single premium immediate annuities for males and females aged 65 at issue.

Many papers have been written on the theory of pricing, several of which are included among the references listed at the end of this paper. Most of these cast equations and discuss assumptions as they apply to the pricing of whole life, limited life, term, or endowment policies. The purpose of this paper is to develop the theory underlying the pricing of single premium immediate annuities.

The paper basically is organized in three sections. Section I outlines important issues in the pricing of single premium immediate annuities. Section II derives the pricing theory using Anderson's [1] present value of book profits method. Sample pricings are presented in Section III. An Appendix has been included so that definitions of all symbols used in the paper can be found in one place.

I. GENERAL CONSIDERATIONS

In certain respects, setting single premium annuity rates is easier than setting rates for life insurance policies. There is seldom any consideration of equitable distribution of surplus to policyholders, since most single premium immediate annuities are offered on a nonparticipating basis. Since such annuities have no cash values, there is only a single decrement, mortality, with which to contend. One should not infer from these comments, however, that the task of developing single premium immediate annuity rates is straightforward.

The necessity of using conservative mortality assumptions in pricing annuities is well recognized. The following issues are important in deciding upon a mortality basis.

1. Annuitant mortality exhibits the effects of selection, particularly on contracts without a refund provision.
2. There are observed differences between mortality under contracts containing a refund feature (or guarantee of annuity payments for a specified period) and mortality under contracts not containing any such provision.
3. The level of mortality is higher under contracts written on a tax-qualified basis than under those written on a nonqualified basis.
4. In addition to a table based on current mortality experience, the pricing of annuities requires the development of an appropriate scale for projecting future improvement in mortality resulting from advances in medicine and geriatric care.
5. Few companies write a sufficiently large volume of single premium immediate annuity business to develop their own mortality tables. Even on an industry-wide basis, the volume of mortality experience under annuity contracts is far smaller than that under life insurance contracts, adding to the difficulty of establishing credible industry tables.

Competitive position is a very important factor in pricing single premium immediate annuities. The volume of business a company writes appears to be a sensitive function of the competitiveness of its rates. Since most companies offer only nonparticipating single premium immediate annuity contracts, a prospective purchaser often can base his decision on a single number—the amount of periodic income that can be purchased for a given amount of money. Thus, shopping is common, and the companies with the most competitive rates write most of the new business. Since a substantial volume of annuity business is placed through insurance brokers, it is important for a company to acquire a reputation of having competitive rates if it wishes to attract this type of business. Of course, it must be recognized that there is an initial statutory surplus strain associated with the writing of single premium immediate annuities. This strain is redressed year by year after issue, but the strain will persist if a large volume of new business continues to be written. Thus, a small or medium-sized company may be forced to "turn off the tap" by making its rates less competitive.

The manner in which investment assumptions are reflected in the pricing can affect significantly the competitiveness of the rates. It is standard practice in today's marketplace to use new-money interest rates in pricing single premium immediate annuities. It also is common to allocate investment income to blocks of annuity business on the basis of an investment-generation method rather than the portfolio method traditionally used in pricing life insurance. Companies following these practices will revise their annuity rates when new-money interest rates change by an amount large enough to result in a significant change in annuity prices. As might be anticipated from the previous comments, it is important that the actuary responsible for the pricing maintain close contact with the investment department, not only in monitoring the progress of interest rates but also in developing investment strategies consistent with the investment assumptions underlying the annuity rates. The companion paper in these *Transactions* is devoted to a discussion of the problem of determining investment assumptions consistent with the investment practices of the company.

II. THEORY OF PRICING

A. *Method*

The theory of pricing single premium immediate annuities as derived in this paper is based on Anderson's method [1]. The fundamental principle on which Anderson's method rests is that the capital investment in a block of business be repaid at a yield reflecting the risks inherent in the business. In stock companies, one usually speaks about the *capital investment made by the shareholders*, while in a mutual company one usually refers to *general surplus funds invested in a block of business*, since there are no shareholders. In the remainder of this paper, references to shareholders and to general surplus funds will be made interchangeably.

There are several reasons why capital may have to be invested in a block of business.

1. A new line of business may require initial capitalization to cover development costs. Repayment to general surplus funds should be made by several generations of policyholders, not by the first block of issues alone.
2. Capital may be needed to provide an adequate level of contingency reserves to the extent that these cannot be accumulated from premiums and investment earnings on the block of business.
3. Writing new business usually involves an initial surplus strain, since the accumulated assets are insufficient to cover the statutory reserves.

The development costs of a new line of business are not considered in this paper, but the second and third reasons cited above are explicitly recognized. In the pricing theory presented here, the assets specifically identified with a block of business are those accumulated from premiums and investment earnings after payment of commissions, expenses, and taxes. For purposes of presenting a statutory balance sheet, the accumulated assets at any year-end are earmarked first to provide the necessary statutory reserve, with the balance earmarked for the contingency reserve. The capital investment by shareholders in the block of business at year-end is taken to be the excess of the statutory reserve plus the contingency reserve over the accumulated assets.

The approach used in this paper is to regard the capital investment by shareholders not as funds explicitly held by the block of business but as an amount put "on the shelf" to back the block of business. While on the shelf, these general surplus funds are unavailable for alternative investment in other blocks of business, agency expansion, acquisition of electronic data processing equipment, and so on. Thus, the block of business should be expected to pay to the company's general unassigned surplus funds a fair rate of return for tying up the capital. This fair rate of return will include an adequate premium for risks inherent in the type of business supported by the funds set aside. There are three basic risks in single premium annuity business: mortality risk, expense risk, and investment risk. In addition, there is a general business risk that the product will not prove to be as marketable as originally supposed, or that the company will become insolvent as a result of difficulties in other lines of business.

In determining an appropriate rate of return to shareholders for use of capital, two approaches can be considered.

1. The pricing can be based on reasonable expected assumptions for mortality, expenses, and investment results, and a shareholders' rate of return that includes a composite risk premium for all the basic risks and the general business risk.
2. The pricing can be based on reasonable expected assumptions for mortality, expenses, and investment results, with specific additional provision for the accumulation of a contingency reserve against adverse fluctuations in these factors. The shareholders' rate of return then would include only the risk premium for the general business risk.

In either case, it may be the decision of management to have a very profitable (and necessarily less competitive) line of business; the shareholders' yield rate then would include a specific profit component in addition to the risk-free return and the risk premium.

Method 2 is more appealing theoretically, since the composite risk premium of method 1 effectively is broken down into specific formula contributions to a contingency reserve against each of the basic risks. Such an approach is desirable because the mortality, expense, and investment risks are all different in nature. For example, it is unclear a priori how a specific formula contribution to a contingency reserve covering the mortality risk translates into a specific percentage return on shareholders' invested funds. Contingency reserves against the mortality, expense, and investment risks are considered in Section II, B.

Anderson's original paper was based on the calculation of book profits before federal income tax or on the assumption that the net earned rate on assets was an after-tax rate. To account for both the level and the incidence of federal income tax, it is desirable to incorporate into the expression for the annual book profit a term based on the actual 1959 Insurance Company Income Tax Act. Such a procedure encounters the usual difficulties of allocating income tax to the major lines of business and within a line to various blocks of issues. Also, there always seems to be discussion as to whether the contingency reserves and the unassigned surplus funds are to pay their own income taxes or whether they are to be treated as part of the assets allocated to the block of business being priced, explicit provision for the federal income taxes thereon being made in the equation for the annual book profits. These points deserve careful discussion.

The approach adopted in this paper considers the contingency reserves to be accumulated from specific annual contributions by the block of issues. The contingency reserve is treated as a block of assets held with all other assets backing the block of business. Investment earnings on the contingency reserve are considered part of the investment earnings for the block of business. Thus, federal income taxes on investment earnings on the contingency reserve are treated as part of the federal income taxes on the block of business.

The investment earnings and federal income taxes on unassigned surplus funds are treated differently. To be precise, the shareholders make a *capital investment* in the block of business, not a *loan* to the block. The distinction arises because there is no obligation to repay a capital investment, but there is an obligation to repay a loan. Thus, a capital investment can be used to erase a statutory insolvency in a particular block of business without increasing the liabilities of the block, but a loan increases the assets and liabilities of the block by the same amount without curing the statutory insolvency. The distinction can be carried through to the treatment of federal income taxes. The return on a capital investment is in the form of dividends, while the return on a loan is in the form of contractual interest payments. Dividends usually are paid from *after-tax* earnings, but interest is paid from *pre-tax* earnings. Moreover, intracorporate dividends are wholly exempt from federal income tax, whereas interest received is fully taxable. This suggests that the shareholders' yield on the capital investment should be stated as an *after-tax* yield.

Many papers using Anderson's method of pricing give a detailed expression for the annual book profits without providing much explanation of the explicit form of the equations. Generally it is left to the reader's intuition or his ability with paper and pencil to verify the details. Considerable insight into the foundations of pricing theory and the connection between book profits and asset shares can be gained by deriving Anderson's method from balance sheet and income statement equations together with the definition of a yield rate on an investment. To this end, the following discussion departs momentarily from pricing theory specifically as concerns single premium immediate annuities and is directed to an analysis of Anderson's method.

Let S_t^U denote the unassigned surplus at the end of policy year t for a block of business. Let i_t' represent the portfolio net earned rate (before income tax) for policy year t on assets backing the block of business. The values of the cumulative capital investment by shareholders at the beginning and end of policy year t are $-S_{t-1}^U$ and $-S_t^U$, respectively. The investment earnings credited during year t to shareholders on the capital investment are $-i_t'S_{t-1}^U$. However, federal income taxes of $-0.48i_t'S_{t-1}^U$ are charged against these earnings, assuming a marginal tax rate of 48 percent. The additional capital investment required at the end of year t to bring the cumulative investment up to $-S_t^U$ is thus equal to $(1 + 0.52i_t')S_{t-1}^U - S_t^U$. In all these expressions, a negative investment by shareholders is regarded as a (positive) repayment to shareholders.

As stated above, it is appropriate to quote the shareholders' yield as an after-tax rate. It is convenient, however, to assume that shareholders are in a marginal tax bracket of 48 percent and to quote the shareholders' yield in year t as a pre-tax rate j_t. The shareholders' investment horizon is the first n policy years. The condition that the shareholders' pre-tax return for use of funds be $j_1, j_2, \ldots, j_n$ in years $1, 2, \ldots, n$, respectively, is the usual yield equation

$$\sum_{t=1}^{n} F_t(j)[(1 + 0.52i_t')S_{t-1}^U - S_t^U] = 0\,, \tag{1}$$

where

$$F_t(j) = \prod_{s=1}^{t} \left(\frac{1}{1 + 0.52j_s}\right).$$

Equation (1) is the mathematical statement of Anderson's method. This becomes evident when it is demonstrated that the expression $S_t^U -$

$(1 + 0.52i'_t)S^U_{t-1}$ is exactly equal to the book profit at the *end* of policy year t.

Let A_t and S^C_t denote the "self-generated" assets and contingency reserves and V_t the statutory reserve at the end of policy year t for the block of issues. Furthermore, let $CF^{\text{in}}_{t;\{r\}}$ and $CF^{\text{out}}_{t;\{s\}}$ represent the various cash inflows (premiums) and cash outflows (benefit payments, commissions, expenses, premium taxes, and federal income taxes) during policy year t for the block of business. In these symbols, $\{r\}$ and $\{s\}$ denote sets of fractional durations during policy year t at which the explicit cash flows occur. In particular,

$$CF^{\text{in}}_{t;\{r\}} \equiv \{CF^{\text{in}}_{t;r_1}; CF^{\text{in}}_{t;r_2}; \ldots ; CF^{\text{in}}_{t;r_u}\} , \tag{2}$$

where $CF^{\text{in}}_{t;r_k}$ represents all cash inflow at duration $t + r_k - 1, 0 \le r_k \le 1$. Similarly,

$$CF^{\text{out}}_{t;\{s\}} \equiv \{CF^{\text{out}}_{t;s_1}; CF^{\text{out}}_{t;s_2}; \ldots ; CF^{\text{out}}_{t;s_v}\} . \tag{3}$$

The following shorthand notation is useful.

$$CF^{\text{in}}_{t;\{r\}}(1 + i'_t)^{1-\{r\}} \equiv \sum_{k=1}^{u} CF^{\text{in}}_{t;r_k}(1 + i'_t)^{1-r_k} ; \tag{4}$$

$$CF^{\text{out}}_{t;\{s\}}(1 + i'_t)^{1-\{s\}} \equiv \sum_{k=1}^{v} CF^{\text{out}}_{t;s_k}(1 + i'_t)^{1-s_k} . \tag{5}$$

The increase in unassigned surplus, $S^U_t - S^U_{t-1}$, is equal to the book profit for the block of issues during policy year t. An expression for $S^U_t - S^U_{t-1}$ can be derived by filling out Page 4 of the NAIC Annual Statement (Summary of Operations and Capital and Surplus Account).

$$\begin{aligned} S^U_t - S^U_{t-1} = i'_t A_{t-1} + CF^{\text{in}}_{t;\{r\}}(1 + i'_t)^{1-\{r\}} - CF^{\text{out}}_{t;\{s\}}(1 + i'_t)^{1-\{s\}} \\ - (V_t - V_{t-1}) - (S^C_t - S^C_{t-1}) . \end{aligned} \tag{6}$$

Using the fundamental accounting equation,

$$A_{t-1} = V_{t-1} + S^C_{t-1} + S^U_{t-1} , \tag{7}$$

we find that equation (6) can be expressed as

$$\begin{aligned} S^U_t - S^U_{t-1} = i'_t S^U_{t-1} + [(V_{t-1} + S^C_{t-1})(1 + i'_t) - (V_t + S^C_t)] \\ + CF^{\text{in}}_{t;\{r\}}(1 + i'_t)^{1-\{r\}} - CF^{\text{out}}_{t;\{s\}}(1 + i'_t)^{1-\{s\}} . \end{aligned} \tag{8}$$

Equation (8) looks very similar to the usual expression for the year-end book profit except for the first term on the right-hand side. The investment earnings on the shareholders' capital investment and the federal income taxes thereon have not been included in equation (8). Anderson's method assumes that the assets associated with the block of business are equal to the statutory reserve plus the contingency reserve, not just the assets generated from premiums less commissions, expenses, and taxes. To obtain the familiar expression for the book profit, we need to add the after-tax investment earnings on the shareholders' capital investment, $-0.52i'_t S^U_{t-1}$, to each side of equation (8).

$$
\begin{aligned}
BP_t &= S^U_t - (1 + 0.52i'_t)S^U_{t-1} \\
&= [(V_{t-1} + S^C_{t-1})(1 + i'_t) - (V_t + S^C_t)] \\
&\quad + CF^{\text{in}}_{t;\{r\}}(1 + i'_t)^{1-\{r\}} - CF^{\text{out}}_{t;\{s\}}(1 + i'_t)^{1-\{s\}} + 0.48i'_t S^U_{t-1} .
\end{aligned}
\tag{9}
$$

(In his formulation, Anderson used an expression for the book profit at the beginning of the year—the right-hand side of eq. [9] divided by $1 + i'_t$.)

The present value at issue (using the shareholders' yield rates) of the first n annual book profits is

$$
Z_n(j) = \sum_{t=1}^{n} F_t(j) BP_t . \tag{10}
$$

The statement that the shareholders' pre-tax return on the capital investment is equal to $j_1, j_2, \ldots, j_n$ over an investment horizon of n years is equivalent to the condition $Z_n(j) = 0$. If $j_t = i'_t$ for $1 \leq t \leq n$, the condition $Z_n(i') = 0$ is equivalent to $S^U_n = 0$, since equation (10) becomes a "telescoping" sum.

It probably has been noticed that the discount factor $F_t(j)$ includes discount at interest only. This is proper, since the variables representing the year-end values of assets, statutory reserves, contingency reserves, unassigned surplus, and book profits are computed for an entire block of $l_{[x]}$ contracts issued to annuitants aged x and thus include the effect of survivorship since issue. This approach is equivalent to Anderson's original formulation in which F_t was discounted for survivorship and interest but in which the assets, reserves, and book profits were calculated *per policy in force* at the appropriate duration. The approach used in this paper allows the simplest treatment of the statutory reserves, V_t. For an annuity with a period of guaranteed payments followed by a life-contingent payout, the reserve for the guaranteed payout must be held during the guaranteed period for *all* contracts issued, but the reserve for the life-contingent payout is held only for surviving annuitants. Therefore, the expressions are simpler, with the two different pieces of V_t

reflecting survivorship as appropriate rather than including survivorship directly in F_t.

The details of the pricing theory for single premium immediate annuities are contained in the explicit form of the annual year-end book profits, BP_t. All symbols in the following equations are defined by formula or in the Appendix.

$$BP_t = [EP\delta_{t;1} + (V_{t-1} + S^C_{t-1})](1 + i'_t) - E_t(1 + i'_t)^{1/2} - B_t - (V_t + S^C_t) - TAX_t + 0.48i'_t S^U_{t-1}\,, \tag{11}$$

where

$$EP = \left\{P[1 - C(1 + E^C)] - \frac{E^{AQ}}{SP}\right\} l_{[x]}\,, \tag{12a}$$

$$\delta_{t;1} = 1 \quad \text{if } t = 1$$
$$= 0 \quad \text{if } t \neq 1\,,$$

$P = 1{,}000$ if the pricing is for the annuity payout rate

$= PF^{(k)}$ (a trial policy fee) if the pricing is for the policy fee ;

$$V_t = mR_m(l_{[x]}\,{}_tV^G + l_{[x]+t-1}\,{}_tV^L)\,, \qquad V_0 = 0\,; \tag{12b}$$

$$S^C_t = S^C_{t-1} + CR^K_t l_{[x]} + CR^{\%}_t NII_t\,, \qquad S^C_0 = 0\,; \tag{12c}$$

$$E_t = E'_1\,, \qquad t = 1$$
$$= E'_t \prod_{s=1}^{t-1}(1 + i^{\text{INF}}_s) \qquad t > 1\,; \tag{12d}$$

$$E'_t = \frac{E^M}{SP}\, l_{[x]} \qquad \text{during the guaranteed period}$$
$$= \frac{E^M}{SP}\, l_{[x]+t-1/2} \qquad \text{for } t \text{ after the guaranteed period}\,; \tag{12e}$$

$$B_t = R_m l_{[x]} \sum_{s=1}^{m} (1 + i'_t)^{1-s/m} \qquad \text{during the guaranteed period}$$
$$= R_m \sum_{s=1}^{m} l_{[x]+t+s/m-1}(1 + i'_t)^{1-s/m} \qquad \text{for } t \text{ after the guaranteed period}\,. \tag{12f}$$

Certain assumptions underlie equations (11) and (12).

1. The single premium immediate annuity contract provides annuity benefits at the end of $m$ths of a year.

2. The annuity payout rate R_m is the amount of $m$thly benefit that can be purchased by $1,000 of single premium *after* the deduction of any policy fee and state premium tax payable.
3. Maintenance expenses are lumped at the middle of each policy year and can be inflated in later policy years above the level in the first policy year.
4. Federal income taxes are paid at the end of each policy year.
5. The number of survivors at fractional durations can be obtained by any acceptable method. For the sample calculations presented in Section III, linear interpolation was used:

$$l_{[x]+t+k} = (1-k)l_{[x]+t} + kl_{[x]+t+1} \quad (0 \leq k \leq 1)\,.$$

Year-end assets after federal income tax are given by

$$A_t = A_{t-1} + NII_t + EP\delta_{t:1} + CF_t - TAX_t\,, \qquad (13)$$

where $A_0 = 0$. The calculation of the contingency reserves S_t^C, the statutory reserves V_t, the cash flow CF_t, the net investment income NII_t, and the federal income tax TAX_t is discussed in subsequent sections.

The remainder of this section deals with the computation of the annuity rate R_m^* or the policy fee PF^* satisfying a specified profit objective Z_n^*. Suppose the pricing is focused on the annuity rate.

The present value at issue of the first n year-end book profits, Z_n, is a function of the annuity rate R_m. The object of the pricing is to invert this functional relationship so that, given a profit objective Z_n^*, the corresponding annuity rate R_m^* is determined. If the functional dependence of Z_n on R_m were linear or some other simple form, the inversion would be straightforward. Unfortunately, Z_n is a complicated nonlinear function of R_m when net investment income is allocated to the block of issues by an investment-generation method. This can be seen from the following line of reasoning.

In traditional pricing using the portfolio method of allocating net investment income, the portfolio net earned rates are supplied by the actuary as part of the pricing assumptions. In an investment-generation method of allocating net investment income, the investment assumptions consist of new-money interest rates and the rates at which investments are rolled over for reinvestment; the portfolio net earned rates then must be determined by the method described in Section II, E. The result is that the rates i_t' are nonlinear functions of R_m. In general, variables that do not depend either explicitly or implicitly on the rates i_t' are linear in R_m. In particular, S_t^U is linear in R_m unless TAX_t depends on i_t' (see Sec. II, F). Since $BP_t = S_t^U - (1+0.52i_t')S_{t-1}^U$, both BP_t and Z_n are nonlinear functions of R_m.

Newton-Raphson iteration can be used to determine the annuity payout rate R_m^* corresponding to the profit objective Z_n^*. The $(k+1)$st trial rate is determined from the kth trial rate as follows:

$$R_m^{(k+1)} = R_m^{(k)} - [Z_n(R_m^{(k)}) - Z_n^*]\left(\frac{dZ}{dR_m}\right)^{-1}_{R_m^{(k)}} . \qquad (14)$$

The derivative $(dZ/dR_m)_{R_m^{(k)}}$ is approximated by

$$\left(\frac{dZ}{dR_m}\right)_{R_m^{(k)}} \doteq \frac{1}{\epsilon}[Z_n(R_m^{(k)} + \epsilon) - Z_n(R_m^{(k)})] . \qquad (15)$$

The choice $\epsilon = 0.0001$ has proved convenient in calculations. Substituting expression (15) in equation (14) results in the final expression

$$R_m^{(k+1)} = R_m^{(k)} - \epsilon\left[\frac{Z_n(R_m^{(k)}) - Z_n^*}{Z_n(R_m^{(k)} + \epsilon) - Z_n(R_m^{(k)})}\right] . \qquad (16)$$

The procedure is simple. Start with a trial rate $R_m^{(0)} = 0$ and calculate the present value of book profits $Z_n(0)$ and $Z_n(\epsilon)$. From these values, determine $R_m^{(1)}$. Evaluate Z_n at trial rates $R_m^{(1)}$ and $R_m^{(1)} + \epsilon$, and thus determine $R_m^{(2)}$. Continue this process until $|R_m^{(k+1)} - R_m^{(k)}| < \delta$, where δ is some specified degree of accuracy. With $\delta = 10^{-6}$, it has been found that the iteration ceases at $R_m^{(3)}$ or $R_m^{(4)}$. The process is represented graphically in Figure 1.

As pointed out earlier in this section, if $j_t = i'_t$ for $1 \leq t \leq n$, the profit objective $Z_n = 0$ is equivalent to $S_n^U = 0$. Unlike Z_n, however, the unassigned surplus S_n^U is a *linear* function of the trial rate $R_m^{(k)}$ when TAX_t is a linear function of $R_m^{(k)}$. Thus, two trial values, $R_m^{(0)}$ and $R_m^{(1)}$, suffice to determine the rate R_m^* that produces zero unassigned surplus at the end of n years.

$$R_m^* = R_m^{(1)} - (R_m^{(1)} - R_m^{(0)})\left[\frac{S_n^U(R_m^{(1)}) - 0}{S_n^U(R_m^{(1)}) - S_n^U(R_m^{(0)})}\right] . \qquad (17)$$

Choosing $R_m^{(0)} = 0$ and $R_m^{(1)} = \epsilon$, we obtain

$$R_m^* = \frac{\epsilon S_n^U(0)}{S_n^U(0) - S_n^U(\epsilon)} . \qquad (18)$$

For the particular profit objective $Z_n = 0$ with $j_t = i'_t$ for $1 \leq t \leq n$, equation (18) gives the desired break-even rate R_m^* with much less effort and computer time than the iterative procedure.

If the purpose of the pricing is to determine the policy fee meeting a specified profit objective, the preceding equations are generally valid

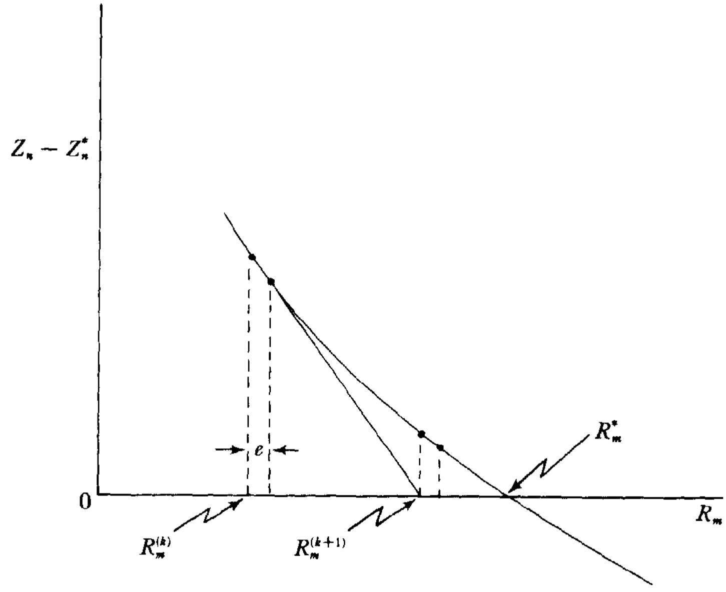

FIG. 1.—Iterative procedure to determine the annuity rate R_m^* meeting the profit objective Z_n^*.

with the substitution of $PF^{(k)}$ for $R_m^{(k)}$ and PF^* for R_m^* throughout. When pricing is for the purpose of determining the policy fee, all terms relating to statutory reserves and annuity benefits are eliminated from the equations.

B. *Contingency Reserves*

In Section II, A, it was stated that in this paper the pricing would be based on reasonable "expected" assumptions for mortality, expenses, and investment results, with specific additional provision for the accumulation of a contingency reserve. The specific provision for the mortality, expense, and investment risks is considered in this section.

The pricing theory has been presented for *nonparticipating* single premium immediate annuities, so contingency reserves are certainly not directly returnable to surviving annuitants. The contingency reserves could be considered indirectly returnable to contract holders if the pricing for the annuity rate were based on the assumption that contingency reserves would be released to general surplus at such time as there was no longer a need for them. This would mean a smaller capital investment in the block of business, resulting in a lower annuity rate for given yield rates on the shareholders' investment. The sample pricings in Section III assume that contingency reserves are fully nonreturnable.

1. MORTALITY RISK

Annuitant mortality generally is reflected in the pricing through a "static" mortality table and an appropriate projection scale. The static mortality table is representative of mortality levels at the time of pricing. It includes the effects of selection and distinguishes between males and females. The projection scale should be set at a conservative level to cover improvement in mortality from the time of issue to the ultimate death of the annuitant. In addition to the risk of future improvement in mortality, there also is a risk in using a static mortality table based on current expected mortality without margins. As will be shown in Section III, a flat percentage margin in the static mortality table can be approximated by using the static mortality table without modification and building up a contingency reserve via annual contributions of a specified flat dollar amount per $1,000 of single premium. The contributions are made in each year in which the payment of the annuity benefit is contingent on the survival of the annuitant. For the examples presented in Section III, flat annual contributions of $2.00 and $1.50 per $1,000 single premium for males and females, respectively, provide a mortality margin at all ages of approximately 5.8 and 6.1 percent, respectively. The use of this approach resulted from an investigation of the level and incidence of annual contingency reserve contribution required to approximate the inclusion of a flat percentage margin in the static mortality table. I have been unable to justify the method on purely theoretical grounds.

2. EXPENSE RISK

The unit expenses for annuity pricing can be based on results of the most recent company expense analysis. The only expense risk considered in this paper is that the per contract expenses are subject to inflation. Consistency between inflation rates and new-money interest rates is maintained by requiring that the difference between these rates equal a fixed "real" interest rate of 2–3 percent. Hence, if new-money interest rates are graded down from an initial level to an ultimate level, the same should apply to the inflation rates for per contract expenses. No specific contingency reserve against expense fluctuations is considered in this paper.

3. INVESTMENT RISK

The investment risk consists of two parts: the risk that actual investment earnings will fall short of those assumed in the pricing and the risk that the borrower will default in repaying either interest or principal or both.

The essence of the first risk is that the level of new-money rates in the future is unknown. The companion paper in these *Transactions* is devoted to an analysis of an investment strategy known as "matching." Because of the preponderance of short-term prepayment clauses in mortgage agreements and call provisions in bond indentures, it is unrealistic to attempt to extend an initial-investment matching strategy beyond fifteen years from issue. (In practice, asset-liability matching beyond the fifteenth year can be achieved by proper *reinvestment* of funds.) If the single premiums less commissions and acquisition expenses can be invested so that the resulting investment cash flow equals or exceeds the cash-flow needs of the block of annuity business for the first fifteen years, and if new-money interest rates beyond this period are assumed at a conservative level, there is practically no risk that actual investment earnings will fall short of the pricing assumptions.

If a matching investment strategy is adopted, only the risk of default need be considered. This is precisely the risk covered by the NAIC Annual Statement liability mandatory securities valuation reserve (MSVR) component for fixed-income securities. The annual formula contributions to Component I of the MSVR depend on the risk characteristics of the security (credit rating, earnings coverage ratios, and so on) and are intended to cover the expected loss of asset value upon default. These formula contributions are expressed as percentages of the statement value of the assets, which is cost or amortized cost for all fixed-income securities except those not meeting required interest or principal payments. Since the source of these contributions is part of the risk premium in the yield rate, it is reasonable to build up the contingency reserve against default as a percentage of investment income after investment expenses. As a guide to the size of this contribution, one can use the MSVR percentage for the appropriate risk class divided by the effective annual net yield. This makes sense since the net yield is approximately equal to the ratio of net investment income to the par value of the asset.

C. *Reserve Factors*

As explained in Section II, A, the total reserve held for a particular annuity contract can be split into two parts: the reserve for the benefit payments during the guaranteed period, V^G, and the reserve for the payments contingent on the survival of the annuitant, V^L. It is assumed that the reserve basis depends on a single interest rate, i_V. The reserve mortality basis is any one of the accepted tables for statutory valuation. Let g denote the length of the period of guaranteed annuity payments, measured in years. The reserve factors based on a *unit* annual benefit are

$$\begin{aligned} V^G &= a^{(m)}_{\overline{g-t}|iV}, \quad && 1 \le t \le g-1 \\ &= 0, && g \le t < \infty, \end{aligned} \tag{19a}$$

$$\begin{aligned} {}_tV^L &= {}_{g-t|}a^{(m)}_{x+t}, \quad && 1 \le t \le g-1 \\ &= a^{(m)}_{x+t} && g \le t < \infty. \end{aligned} \tag{19b}$$

D. *Cash Flow*

The variable CF_t includes cash outflow during policy year t from maintenance expenses and benefit payments.

$$\begin{aligned} CF_t &= -E_t - mR_m l_{[x]}, && 1 \le t \le g \\ &= -E_t - R_m \sum_{s=1}^{m} l_{[x]+t+s/m-1}, \quad && g < t < \infty. \end{aligned} \tag{20}$$

For purposes of determining the net investment income for year t, the partial year's interest on these components of the cash flow must be computed. For reasons that will become apparent in later sections, the calculation is done for a general interest rate i. The following equations will be used later.

$$\begin{aligned} I_t^{\mathrm{CF}}(i) &= -E_t[(1+i)^{1/2} - 1] - R_m l_{[x]} \sum_{s=1}^{m} [(1+i)^{1-s/m} - 1], \\ \frac{dI_t^{\mathrm{CF}}}{di} &= -\frac{1}{2}\frac{E_t}{(1+i)^{1/2}} - R_m l_{[x]} \sum_{s=1}^{m} \left(1 - \frac{s}{m}\right)(1+i)^{-s/m}, \end{aligned} \tag{21a}$$

for $1 \le t \le g$, and

$$\begin{aligned} I_t^{\mathrm{CF}}(i) &= -E_t[(1+i)^{1/2} - 1] - R_m \sum_{s=1}^{m} l_{[x]+t+s/m-1}[(1+i)^{1-s/m} - 1], \\ \frac{dI_t^{\mathrm{CF}}}{di} &= -\frac{1}{2}\frac{E_t}{(1+i)^{1/2}} - R_m \sum_{s=1}^{m} l_{[x]+t+s/m-1}\left(1 - \frac{s}{m}\right)(1+i)^{-s/m}, \end{aligned} \tag{21b}$$

for $g < t < \infty$.

E. *Net Investment Income*

Earnings from investments are allocated to the block of annuity business using an investment-generation method. The key variables used in the equations below are the following: i_t, the new-money interest rate that an investment made in year t can be assumed to yield until its maturity (or prior call or sale), and $r^{(0)}$, $r^{(1)}$, vectors of rollover rates. The component $r_j^{(0)}$ of the vector $r^{(0)}$ specifies the fraction of assets purchased at the beginning of the first policy year that will mature (or be called or sold) at the beginning of the $(j+1)$st policy year. Thus $\Sigma_{j=1}^{\infty} r_j^{(0)} = 1$. The components of $r^{(1)}$ have a similar interpretation, except that $r^{(1)}$ is applied to investments made after the first policy year. Two vectors of rollover rates are used in recognition of the possibility that the initial investment strategy may differ from the reinvestment strategy.

An "investment cell" is identified with each policy year, and the declining index method is used. Let $\mathcal{A}_t$ represent the assets that are invested during year t in cell t at the new-money rate i_t. $\mathcal{A}_t$ consists of the following items:

1. Cash flow from insurance operations during year t, $CF_t + EP\delta_{t;1}$; plus
2. Net investment income from year $t-1$, NII_{t-1}, assumed to be available for investment at the beginning of year t; plus
3. Funds invested in years prior to year t that are made available for reinvestment (through maturity, call, or sale) at the beginning of year t, $\mathcal{A}_t''$; less
4. Federal income taxes for year $t-1$, TAX_{t-1}, assumed to be paid at the end of that year.

Since the declining index method is being used, the amount of assets in a given cell will decrease as funds mature or are called or sold. The rate at which assets are rolled over from cell s, the cell for assets acquired in policy year s, to the various cells $t > s$ is given by the components of the vector $r^{(0)}$ if $s = 1$ and by the components of $r^{(1)}$ if $s > 1$. Let $\mathcal{A}_{t;s}'$ represent the assets remaining in cell s at the beginning of year t $(t > s)$. Thus, for $t > s$, $\mathcal{A}_{t;s}'$ equals $\mathcal{A}_s$, the funds initially invested in cell s, less that portion of funds $\mathcal{A}_s$ that has rolled over in years $s+1, s+2, \ldots, t$. The variable $\mathcal{A}_{t;t}'$ is defined as the funds invested in cell t during year t, excluding any cash flow from insurance operations during year t. In other words, $\mathcal{A}_{t;t}'$ is equal to $\mathcal{A}_t''$ plus NII_{t-1} less TAX_{t-1}.

The net investment income for policy year t is comprised of the following:

1. Partial year's interest at the new-money rate i_t on the cash flow from insurance operations during year t, $I_t^{\mathrm{CF}}(i_t)$; plus
2. Full year's interest at the new-money rates $i_1, \ldots, i_t$ on the funds $\mathcal{A}_{t;1}', \ldots, \mathcal{A}_{t;t}'$, respectively; plus
3. In the first policy year, a full year's interest at the new-money rate i_1 on EP.

The following equations embody the theory of the investment-generation method that has been described.

$$\begin{aligned} \mathcal{A}_t'' &= 0\,, & t &= 1 \\ &= r_1^{(0)}\mathcal{A}_1\,, & t &= 2 \\ &= r_{t-1}^{(0)}\mathcal{A}_1 + \sum_{j=2}^{t-1} r_{t-j}^{(1)}\mathcal{A}_j\,, & t &> 2\,, \end{aligned} \tag{22a}$$

$$\begin{aligned} \mathcal{A}_{t;t}' &= 0\,, & t &= 1 \\ &= \mathcal{A}_t'' + NII_{t-1} - TAX_{t-1}\,, & t &> 1\,, \end{aligned} \tag{22b}$$

$$\mathcal{A}_t = CF_t + EP\delta_{t;1} + \mathcal{A}_{t;t}'\,; \tag{22c}$$

$$NII_t = I_t^{\mathrm{CF}}(i_t) + i_1 EP\delta_{t;1} + \sum_{s=1}^{t} i_s\mathcal{A}_{t;s}'\,; \tag{23}$$

$$\mathcal{A}_{t;1}' = \mathcal{A}_1\Big(1 - \sum_{j=1}^{t-1} r_j^{(0)}\Big), \quad t > 1\,, \tag{24a}$$

$$\mathcal{A}_{t;s}' = \mathcal{A}_s\Big(1 - \sum_{j=1}^{t-s} r_j^{(1)}\Big), \quad t > s > 1\,. \tag{24b}$$

The total assets of the block of business at year-end $t-1$ are $A_{t-1} = \Sigma_{s=1}^{t}\,\mathcal{A}_{t;s}'$. This equation gives the composition at the beginning of year t of A_{t-1} by investment generation. In year s, funds of amount $\mathcal{A}_s$ were invested at a net new-money rate i_s. At the beginning of year t ($t > s$), the assets still associated with investment-year s amount to $\mathcal{A}_{t;s}'$ and continue to earn interest at the rate i_s. The fund $\mathcal{A}_{t;t}'$ represents that portion of the total assets A_{t-1} that will be reinvested at the beginning of year t at the net new-money rate i_t.

The calculation of the book profit BP_t requires the value of the portfolio net earned rate i_t'. The value of i_t' is determined so as to produce the same value of NII_t by the portfolio method of allocating net investment income as is produced by using the investment-generation method. Thus, NII_t is determined from equation (23) and i_t' is the solution of the following equation:

$$NII_t = I_t^{\mathrm{CF}}(i_t') + i_t'(A_{t-1} + EP\delta_{t;1})\,. \tag{25}$$

The solution would be straightforward except for the nonlinear dependence of $I_t^{\mathrm{CF}}(i_t')$ on i_t' (see eqs. [21]). The value of i_t' can be found by Newton-Raphson iteration. Let $i_{t;k}'$ denote the result of the kth iteration.

Then

$$i'_{t;k+1} = i'_{t;k} - [I_t^{\mathrm{CF}}(i'_{t;k}) + i'_{t;k}(A_{t-1} + EP\delta_{t;1}) - NII_t] \times \left[\left(\frac{dI_t^{\mathrm{CF}}}{di}\right)_{i'_{t;k}} + A_{t-1} + EP\delta_{t;1}\right]^{-1}. \tag{26}$$

The iteration is stopped when $|(i'_{t;k+1} - i'_{t;k})/i'_{t;k}| < \delta$ for some appropriately selected δ. The starting value is chosen as the portfolio net earned rate based on simple interest and maintenance expense and annuity benefit cash flows concentrated at midyear.

$$i'_{t;0} = \frac{NII_t}{A_{t-1} + EP\delta_{t;1} + \frac{1}{2}CF_t}. \tag{27}$$

With this starting value and $\delta = 10^{-6}$, two iterations generally are required to determine i'_t.

F. *Federal Income Tax*

In the tax formula based on the Income Tax Act of 1959, it is necessary to distinguish between tax-qualified and nonqualified annuity business. The following variables are relevant to the discussion: mean reserve, MV_t; taxable investment income, TII_t; operating gain (loss) for tax purposes, OG_t; valuation interest rate, i_V; and portfolio earned rates for tax purposes, i_t^T. The following equations describe the tax calculation. Certain elements of the 1959 income tax formula, such as the small-business deduction, have been ignored. It has been assumed that the block of business has no fully or partially tax-exempt investment income.

$$i_t^{T'} = \min\left\{i_t^T; \frac{1}{5}\sum_{s=0}^{4} i_{t-s}^T\right\}; \tag{28}$$

$$MV_t = \tfrac{1}{2}mR_m l_{[x]}(a_{\overline{g|}i_V}^{(m)} + {}_{g|}a_x^{(m)})\delta_{t;1} + \tfrac{1}{2}(V_{t-1} + V_t) \tag{29a}$$

$$\begin{aligned} TII_t &= NII_t - i_t^T MV_t && \text{(tax-qualified)} \\ &= NII_t - i_t^{T'} MV_t[1 - 10(i_t^{T'} - i_V)] && \text{(nonqualified)}; \end{aligned} \tag{29b}$$

$$OG_t = NII_t + EP\delta_{t;1} + CF_t - (V_t - V_{t-1}). \tag{30}$$

The allocation of federal income tax to the various lines of business and within lines to various blocks of business is very much a question of management philosophy. The particular approach adopted often is based on an attempt to maintain equity among major lines of business and among blocks of business with similar characteristics. A company using marginal tax rates to allocate income tax likely would use portfolio rates i_t^T based on the income tax return for the entire company. Others believe that a more equitable approach is to set $i_t^T = i_t'$, the portfolio net earned rate for the block of business being priced. (In the latter case, it would be more technically precise to use $i_t^T = 2NII_t/(A_{t-1} + A_t)$ in lieu of $i_t^T = i_t'$.) If the portfolio earned rates of the entire company are used for allocating tax, it is reasonable to require that these rates reflect the assumptions about the level of new-money rates i_t. The following discussion provides an approximate method for doing this.

Let A_t^C represent the assets of the entire company at the beginning of year t, and let i_{t-1}^C denote the average net annual yield rate at which these assets are invested. Let r_t^C denote the fraction of the assets A_t^C that will be rolled over during year t and made available for reinvestment at the new-money rate i_t. Finally, let g_t^C represent the growth rate of the entire company's assets before federal income tax during the tth year after issue of the new block of annuity business. Thus, $(1 + g_t^C)A_t^C$ represents the company's assets at the end of year t before federal income tax, with an amount $(1 - r_t^C)A_t^C$ invested at rate i_{t-1}^C and the remaining amount $(r_t^C + g_t^C)A_t^C$ invested at rate i_t. Assuming that federal income taxes are paid on a pro rata basis from assets invested at rate i_{t-1}^C and assets invested at rate i_t, the average rate i_t^C at which assets A_{t+1}^C at the beginning of year $t + 1$ are invested is given by

$$i_t^C = \left(\frac{1 - r_t^C}{1 + g_t^C}\right) i_{t-1}^C + \left(\frac{r_t^C + g_t^C}{1 + g_t^C}\right) i_t \,. \tag{31a}$$

The portfolio rate i_t^T for purposes of the 1959 income tax formula is derived by dividing the total net interest earnings during year t by the mean of the assets at the beginning and end of year t. It is assumed that funds of amount $(r_t^C + g_t^C)A_t^C$ are invested at the middle of year t. The interest earnings during year t from the various classes of assets are as follows:

From non-rolled-over assets :	$i_{t-1}^C(1 - r_t^C)A_t^C$
From rolled-over assets :	$\frac{1}{2}(i_{t-1}^C + i_t)r_t^C A_t^C$
From growth in company assets :	$\frac{1}{2}i_t g_t^C A_t^C$

The mean assets are $(1 + \frac{1}{2}g_t^C)A_t^C$. The portfolio rate i_t^T is given by

$$i_t^T = \left(\frac{1 - \frac{1}{2}r_t^C}{1 + \frac{1}{2}g_t^C}\right) i_{t-1}^C + \frac{1}{2}\left(\frac{r_t^C + g_t^C}{1 + \frac{1}{2}g_t^C}\right) i_t \,. \tag{31b}$$

The pair of equations (31a) and (31b) can be solved recursively to generate the portfolio rates i_t^T. In practice, one would assume a rollover rate r^C and a growth rate g^C that are independent of t.

The crude approach for projecting the portfolio earned rates of the entire company is analogous to allocating federal income tax on the basis of marginal tax rates calculated for the entire company and then using assumed rollover rates for reinvestment of company assets, assumed growth rates of company assets, and assumed future new-money interest rates to compute new marginal tax rates for future years. It is necessary to project the marginal tax rates because they are calculated as partial derivatives (according to formulas given by Fraser [2]) at a particular point in time, and vary as the company grows in size and the composition of its asset portfolio changes. The analogy with marginal tax rates is not altogether correct, since true marginal tax rates depend on additional factors such as the size of mean qualified and mean nonqualified reserves of the entire company, and these, too, would have to be projected. However, the analogy is useful because it illustrates the nature of a tax allocation that uses the portfolio earned rate of the entire company in lieu of the particular rate for the block of business being priced.

The calculation of the federal income tax TAX_t from TII_t and OG_t depends on the tax situation. A question arises as to whether the tax situation of the entire company should be used to allocate the taxes or whether the position of each block of business should determine its own tax situation for purposes of the allocation. The former approach is used in this paper. Three distinct tax situations are recognized in the formula for TAX_t.

$$TAX_t = \left.\begin{cases} 0.48TII \\ 0.48OG_t \\ 0.24(TII_t + OG_t) \end{cases}\right\} \begin{array}{l}\text{depending on the tax}\\ \text{situation in year } t\,.\end{array} \tag{32}$$

A negative value of TAX_t in any year is treated as a tax credit to the block of business. For a stock company, another term would have to be added to equation (32) for taxable transfers from the policyholders' surplus account, and appropriate assumptions would have to be made about the amount and incidence of such transfers.

G. *Fitting a Rate Basis*

After the actuary has determined annuity rates at certain pivotal issue ages, his next task is to adopt a rate basis for calculating rates at all issue ages and for all variations in the refund feature. This basis should reproduce the results of the pricing at the pivotal ages. The rate basis for single premium annuity rates generally uses annuity factors calculated according to specified mortality and interest, then loaded to cover expenses, contingencies, and profit. The mortality basis should be the same as the experience mortality used in the pricing. The interest rates used in calculating the annuity factors should reflect the investment strategy underlying the pricing and should be on an after-tax basis. A two- or three-tiered interest rate structure is often used. Specifically, the basis might assume a rate $\bar{i}_1$ for the first p_1 years, $\bar{i}_2$ for the next p_2 years, and $\bar{i}_3$ thereafter. As a guide to setting the interest rate structure of the annuity basis, it is helpful to determine theoretical rates $\bar{i}_1$, $\bar{i}_2$, and $\bar{i}_3$ based on the pricing.

The after-tax interest rates that accumulate the same assets, and hence the same surplus, at the end of each year using all of the same pricing assumptions (except for zero tax each year) can be derived as outlined in Section II, E. Equations (26) and (27) are used, but NII_t is replaced in each of these equations by $NII_t - TAX_t$. Let $\bar{i}'_t$ denote the after-tax rates so determined. Let A_s^{lower} and A_s^{upper} represent the after-tax assets at the beginning and end, respectively, of period s, $s = 1, 2, 3$.

$$\begin{aligned} A_1^{\text{lower}} &= EP\,, & A_2^{\text{lower}} &= A_{p_1}\,, & A_3^{\text{lower}} &= A_{p_1+p_2}\,; \\ A_1^{\text{upper}} &= A_{p_1}\,, & A_2^{\text{upper}} &= A_{p_1+p_2}\,, & A_3^{\text{upper}} &= A_n\,. \end{aligned} \tag{33}$$

The level rate $\bar{i}_s$ for period s is the interest rate that will accumulate assets A_s^{upper}, starting with assets A_s^{lower}, and recognizing the maintenance expense and annuity benefit cash flows, CF_t, for $p_{s-1} + 1 \leq t \leq p_s$ (taking $p_0 = 0$ and $p_3 = n - p_1 - p_2$). It is useful to define a function $g(\bar{i})$ as follows:

$$\begin{aligned} g(\bar{i}) = A_s^{\text{upper}} &- A_s^{\text{lower}}(1+\bar{i})^{p_s-p_{s-1}} \\ &- \sum_{t=p_{s-1}+1}^{p_s} [CF_t + I_t^{\text{CF}}(\bar{i})](1+\bar{i})^{p_s-t}\,. \end{aligned} \tag{34}$$

The desired value, $\bar{i}_s$, satisfies $g(\bar{i}_s) = 0$. The function $g(\bar{i})$ is not even as simple as a polynomial in $(1+\bar{i})$ because of the dependence of $I_t^{\text{CF}}(\bar{i})$ on $\bar{i}$. It is straightforward to solve for $\bar{i}_s$, however, by using Newton-Raphson iteration once again.

$$\begin{aligned} \frac{dg(\bar{i})}{d\bar{i}} = &- A_s^{\text{lower}}(p_s - p_{s-1})(1+\bar{i})^{p_s-p_{s-1}-1} \\ &- \sum_{t=p_{s-1}+1}^{p_s} [CF_t + I_t^{\text{CF}}(\bar{i})](p_s - t)(1+\bar{i})^{p_s-t-1} \\ &- \sum_{t=p_{s-1}+1}^{p_s} \left[\frac{dI_t^{\text{CF}}(\bar{i})}{d\bar{i}}\right](1+\bar{i})^{p_s-t}\,; \end{aligned} \tag{35}$$

$$\bar{i}_{s;k+1} = \bar{i}_{s;k} - g(\bar{i}_{s;k}) \left[\frac{dg(\bar{i})}{d\bar{i}}\right]^{-1}_{\bar{i}_{s;k}} . \tag{36}$$

An appropriate starting value for the iteration is the arithmetic mean of $\bar{i}'_t$ for $p_{s-1} + 1 \leq t \leq p_s$.

$$\bar{i}_{s;0} = \left(\frac{1}{p_s - p_{s-1}}\right) \sum_{t=p_{s-1}+1}^{p_s} \bar{i}'_t . \tag{37}$$

The iterations are continued until a convergence criterion is met, for example, $|g(\bar{i}_{s;k+1})| < \delta$.

One obvious difficulty with this theoretical method is that it is impractical to use a different interest rate basis for males and females, for different issue ages, and for annuities with differing refund features. Yet the theoretical rates $\bar{i}_1$, $\bar{i}_2$, and $\bar{i}_3$ derived from the procedure of this section will differ somewhat among the various issue-age/sex/refund-feature groups. If the differences are not large, an average of $\bar{i}_s$ over all the pricing groups should serve as an adequate first approximation to the interest rate basis.

III. SAMPLE PRICINGS

The theory presented in Section II is applied in this section to the pricing of straight-life and guaranteed-ten nonqualified single premium immediate annuities for male and female lives aged 65 at issue. Also, some results of an investigation of the relationship between margins in the static mortality table and the level of contributions to a contingency reserve against mortality fluctuations are presented.

A. *Pricing Assumptions*

In the following sections, the listed pricing assumptions are appropriate for calendar year 1976.

1. MORTALITY

Mortality assumptions include both a static mortality table and a projection scale. The projection scale used in this paper is a 1 percent improvement in mortality per year at all ages. The mortality tables were constructed by using the ratios of actual to expected mortality by amount of annual income based on the unprojected Annuity Table for 1949 Ultimate rates for expected and the 1967–71 Society of Actuaries study [4] for actual.

Since the 1967–71 study presented mortality ratios separately for the first five contract years and for contract years 6 and over, two tables could be constructed for each sex: one representing average mortality at each attained age during the first five contract years and one representing average mortality at each attained age for contract years 6 and over. These mortality tables, assumed to represent mortality levels in 1969, were adjusted to expected 1976 levels by projecting mortality improvement by an amount depending on age. For this purpose, the rate of mortality improvement was taken as 1 percent per year, and the number of years of projection used for each age is shown below.

Age (x)	Years Projected
0–69	0
70–76	$x-69$
77 and over	7

The four mortality tables were constructed to fit the mortality ratios of the 1967–71 study for age groups 60–69, 70–79, and 80 and over. These ratios are listed in Table 3 of the 1967–71 study reported in *TSA, 1973 Reports*. Separate ratios are given for age groups 80–89 and 90 and over. However, Table C in the Appendix of the 1967–71 study, which lists the exposure by age group, shows only the exposure for the combined group 80 and over. Moreover, the figures listed in Table C suggest that the experience for the age group 90 and over may be sparse, and certainly is so relative to the experience for the age group 80–89. Rather than placing too much emphasis on the stated ratio for the age group 90 and over, it was decided to use the actual-to-expected ratio for the age group 80 and over and to attempt a fit in which this ratio is attained at pivotal age 90. This method produces higher mortality rates near the end of the table than does a fit that recognizes the stated 90 and over ratio explicitly, and thus produces less conservative mortality rates near the end of the table. However, the projection scale of 1 percent per year is continued to the end of the mortality table (age 109), so the mortality rates at the higher ages are sufficiently conservative.

The a-1949 Table is inherently smooth, since its final graduation utilized a Makeham formula. To ensure smoothness of the mortality basis used in the sample pricings, the mortality rates in both the select tables and the ultimate tables were calculated from a mathematical function.

$$
\begin{aligned}
q_x &= f(x)q_x^{a\text{-}1949}\,; \\
f(x) &= 1\,, && x \le x_0 && (38)\\
&= (1-a) + [a + b(x-x_0)^2]e^{-c(x-x_0)^2}\,, && x_0 < x \le 109\,.
\end{aligned}
$$

The form of $f(x)$ for $x > x_0$ guarantees that both $f(x)$ and $f'(x)$ are continuous at $x = x_0$. With $b = 0$, the general shape of $f(x)$ is as shown in Figure 2. This form was used to fit the male and female select tables and the female ultimate table. Since the 1967–71 study showed a departure from the slope of male ultimate mortality by age from that of the a-1949 Table, it was necessary to use $f(x)$ with $b > 0$ to produce mortality ratios in excess of 100 percent. Figure 3 illustrates the general shape of $f(x)$ for the situation where b is sufficiently large to produce a local maximum.

The parameters x_0, a, b, and c were chosen to yield a good fit to the 1967–71 intercompany experience. The values of these parameters for

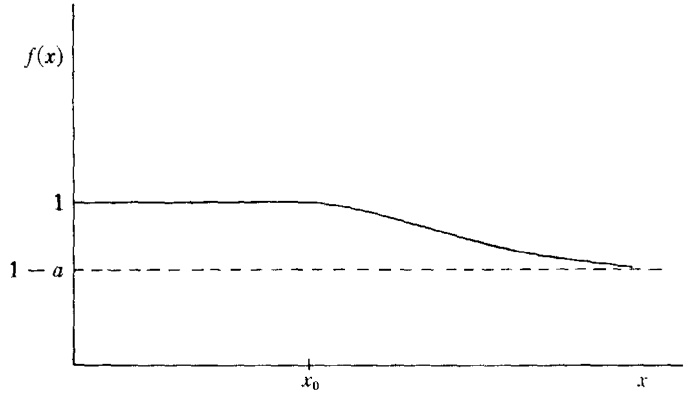

FIG. 2.—Shape of $f(x)$ defined in eq. (38) with $b = 0$

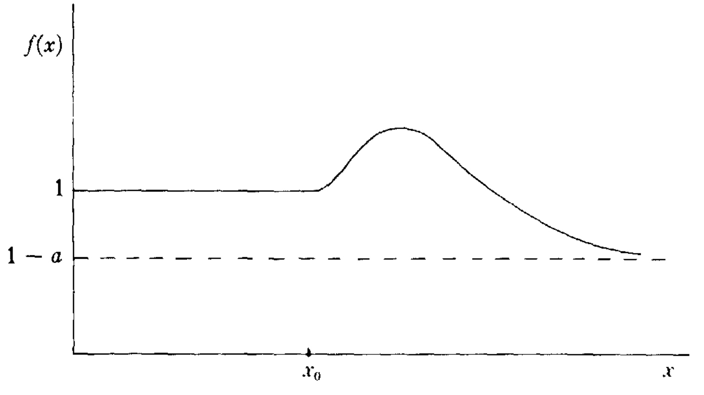

FIG. 3.—Shape of $f(x)$ defined in eq. (38) with b positive and sufficiently large to produce a local maximum for $x > x_0$.

each of the four annuitant mortality tables used in the sample pricings are listed below.

PARAMETER	MORTALITY TABLE			
	Male Select	Male Ultimate	Female Select	Female Ultimate
x_0	50	60	50	50
a	0.4	0.2	0.36	0.17
b	0	0.00173	0	0
c	0.000460	0.00333	0.00284	0.00166

After the fit was accomplished, it was observed that the ratio of female ultimate mortality to male ultimate mortality at ages 101–109, inclusive, was greater than 100 percent. Since this does not seem to be characteristic of any other annuitant mortality table, it was decided to use for ultimate female mortality at these ages the corresponding ultimate male mortality rates derived from the fit mentioned above.

The most obvious way to incorporate the effects of selection is to use both select mortality rates and ultimate mortality rates and to grade the former into the latter over an appropriate select period. For both males and females, the unprojected mortality rates used in the sample pricing of a single premium immediate annuity issued to a life aged x are given by

$$\begin{aligned} q_{x+t} &= \left(1-\frac{t}{s}\right) q_{x+t}^{\text{select}} + \left(\frac{t}{s}\right) q_{x+t}^{\text{ultimate}}\,, && 0 \le t \le s \\ &= q_{x+t}^{\text{ultimate}}\,, && s < t\,, \end{aligned} \tag{39}$$

where s is the select period measured in years. Even though the mortality table, q^{select}, was fitted to the 1967–71 intercompany experience of the first five contract years, it seemed that $s = 10$ was more consistent with statements made in the *1973 Reports* about the degree of selection. Such a choice is certainly more conservative than $s = 5$.

The mortality basis described above was fitted to the 1967–71 experience for nonrefund and refund annuities combined and was used in the sample pricing of both straight-life and guaranteed-ten annuities. It generally is recognized that mortality is higher under contracts with a refund feature than under those without any such feature. Some companies may choose to construct separate mortality bases for nonrefund and refund annuities, although it is unlikely that there will be sufficient experience to do this in other than an arbitrary fashion.

2. INTEREST

A method for developing appropriate investment assumptions is discussed in the subsequent paper in these *Transactions*. It is assumed that net investment income is allocated according to the investment-generation method described in Section II, E. The new-money interest assumption used in the sample pricings was 9.0 percent at issue, grading downward by 0.25 percent per year to an ultimate level of 5.0 percent in years 17 and later. The components of the vectors of rollover rates for the initial investment and subsequent reinvestments are listed in Table 1. The actual rollover rates used in the pricings were determined by the method derived in the following paper.

TABLE 1

INVESTMENT ROLLOVER RATES

Duration	Initial Investment	Reinvestment	Duration	Initial Investment	Reinvestment
1	2.26%	0.32%	11	4.50%	5.63%
2	2.28	0.35	12	4.55	5.68
3	2.31	0.39	13	4.60	5.74
4	2.34	0.43	14	4.65	5.81
5	2.37	0.47	15	15.90	19.91
6	6.91	6.14	16	1.04	1.29
7	6.95	6.19	17	1.05	1.31
8	6.99	6.24	18	1.06	1.33
9	7.04	6.30	19	1.08	1.35
10	17.63	19.53	20	4.49	5.59

3. COMMISSIONS AND EXPENSES

The unit expenses used in the sample pricings are listed below.

Commission	2.5 percent of the single premium
Acquisition expenses	115 percent of commission; $55.00 per contract
Maintenance expenses	$15.00 per contract
Average size of policy	$25,000 single premium

The per contract maintenance expenses were inflated at a rate equal to 2.5 percent less than the new-money interest rate for the prior year, that is, maintenance expenses for the second policy year are 6.5 percent higher than those for the first policy year, maintenance expenses for the third policy year are 6.25 percent higher than those for the second policy year, and so on.

4. STATUTORY RESERVES AND CONTINGENCY RESERVES

Statutory reserves were calculated on the basis of 6 percent interest and mortality according to the 1971 Individual Annuity Mortality

Table for male lives without projection. A six-year age setback of the male table was used for the valuation of annuities on female lives.

Annual contributions to the contingency reserve were set at \$2.00 for males and \$1.50 for females per \$1,000 single premium (excluding the period of guaranteed payments) and 1 percent of net investment income. At a net earned rate of 9.0 percent, the latter corresponds roughly to an annual contribution of 0.09 percent of assets, just about equal to 0.1 percent, the MSVR formula rate for the highest quality class of fixed-income securities. Unlike the MSVR, however, the pricing calls for contingency reserves to be held against default on mortgages as well as bonds.

5. FEDERAL INCOME TAXES

It was assumed that federal income tax is allocated to the annuity business according to the portfolio earned rate and the tax situation of the entire company. Taxes were based on taxable investment income, a situation typical of a large mutual company. The portfolio earned rates i_t^T of the entire company were projected on the basis of equations (31a) and (31b) with $g_t^C = 10$ percent and $r_t^C = 5$ percent for all t. The initial portfolio rates were assumed to be $i_{-3}^T = 5.5$ percent, $i_{-2}^T = 5.75$ percent, $i_{-1}^T = 6$ percent, and $i_0^T = i_0^C = 6.25$ percent.

6. PROFIT OBJECTIVE

Since separate provision was made in the pricing to cover adverse fluctuations in both mortality experience and default experience on assets backing the block of business, it was assumed that a fair pre-tax rate of return to shareholders is the pre-tax portfolio net earned rate for the block of business being priced. The object of the pricing was to determine the break-even monthly annuity payout rate per \$1,000 of single premium using these shareholders' yield rates. In terms of the notation of Section II, A, the profit objective can be stated as $Z_n = 0$ with $j_t = i_t'$ for $1 \leq t \leq n$. The investment horizon n was chosen as the number of years to the end of the mortality table. As shown in Section II, A, this profit objective is equivalent to "zero unassigned surplus for the block of issues at the end of the mortality table."

B. *Results*

Four sample pricings were carried out. These have been labeled A–D, as follows:

A: Male age 65, straight life
B: Male age 65, guaranteed ten
C: Female age 65, straight life
D: Female age 65, guaranteed ten

The results are presented in Tables 2–5. Table 6 gives the portfolio net earned rates before and after tax for each policy year under each of the pricings, and Table 7 shows the equivalent level after-tax earned rates as defined in Section II, G, for an initial period of twelve years, a middle period from year 13 through year 20, and a final period from year 21 to the end of the mortality table. As mentioned in that section, the theoretical level rates for any given period differ among the various pricing cells.

C. *Contingency Reserves and Mortality Margins*

In Section II, B, it was stated that for annuities on male lives an annual contribution of $2.00 per $1,000 single premium, excluding the guaranteed period, is approximately equivalent to a flat margin of about 5.8 percent in the static male mortality table. Support for this statement is given in this section.

The result of a pricing is an annuity payout rate meeting the stipulated profit objective. Using this payout rate, decreasing both the select and ultimate mortality rates by p percent, eliminating the annual $2.00 per $1,000 single premium contingency reserve contribution, and keeping all other assumptions exactly the same as in the pricing, the amount of unassigned surplus at the end of the mortality table can be determined. By varying p, the percentage reduction in the mortality rates, a value can be found that produces zero unassigned surplus at the end of the mortality table. For the male age 65 straight-life annuity pricing, the required value of p was 5.8. Table 8 shows that the flat contingency reserve contribution results in roughly the same incidence of unassigned surplus as a flat 5.8 percent mortality margin would produce.

For annuities on female lives, an annual contribution of $1.50 per $1,000 single premium (excluding the guaranteed period) is equivalent to a mortality margin of 6.1 percent. The second part of Table 8 gives results for the female age 65 straight-life annuity pricing analogous to those in the first part for male lives. Similar results were obtained for the ten-year guaranteed annuities and are not shown here.

IV. CONCLUSION

At first sight, it may seem that the theory presented in this paper is extremely complicated and thus that the methods outlined are impractical. It is certainly true that the theory can be applied only with the help of a computer. If the pricing is targeted on a nonzero present value at issue of book profits, an iterative technique is needed to solve for the annuity rate that meets the profit objective. This would be in addition

TABLE 2

MALE AGE 65, STRAIGHT-LIFE ANNUITY

(Monthly Income per $1,000 Single Premium: $9.069)

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
1	1.000	1.000	$80.55	$835.94	$8.10	$908.39	$1,036.65	$ 2.81	−$131.07
2	0.978	0.978	77.33	− 105.65	7.56	872.51	985.44	5.58	− 118.51
3	0.955	0.955	74.14	− 103.06	7.15	836.45	933.69	8.32	− 105.56
4	0.930	0.930	70.98	− 100.31	6.88	800.24	881.47	11.03	− 92.26
5	0.904	0.904	67.85	− 97.41	6.74	763.94	828.89	13.71	− 78.66
6	0.876	0.876	64.77	− 94.34	6.71	727.66	776.34	16.36	− 65.03
7	0.847	0.847	61.16	− 91.11	6.50	691.21	723.97	18.97	− 51.73
8	0.817	0.817	57.50	− 87.73	6.29	654.68	671.96	21.54	− 38.82
9	0.785	0.785	53.80	− 84.19	6.09	618.20	620.52	24.08	− 26.40
10	0.752	0.752	50.05	− 80.49	5.87	581.89	569.90	26.58	− 14.59
11	0.717	0.717	44.07	− 76.64	4.57	544.75	520.38	29.02	− 4.64
12	0.681	0.681	40.86	− 72.70	4.55	508.37	472.69	31.43	4.25
13	0.645	0.645	37.66	− 68.68	4.49	472.86	426.81	33.81	12.24
14	0.608	0.608	34.48	− 64.60	4.38	438.35	382.98	36.15	19.22
15	0.570	0.570	31.31	− 60.48	4.22	404.97	341.43	38.47	25.07
16	0.532	0.532	24.64	− 56.34	2.30	370.98	302.33	40.71	27.93
17	0.495	0.495	22.50	− 52.21	2.47	338.80	265.79	42.94	30.07
18	0.457	0.457	20.51	− 48.12	2.63	308.57	231.86	45.14	31.57
19	0.420	0.420	18.60	− 44.10	2.72	280.36	200.57	47.33	32.46
20	0.384	0.384	16.76	− 40.17	2.74	254.21	171.93	49.50	32.79

TABLE 2—*Continued*

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
21	0.348	0.348	$13.69	−$ 36.35	$2.09	$229.47	$ 145.94	$ 51.63	$ 31.90
22	0.314	0.314	12.27	− 32.67	2.13	206.94	122.61	53.76	30.58
23	0.281	0.281	10.99	− 29.15	2.15	186.62	101.89	55.86	28.87
24	0.250	0.250	9.83	− 25.81	2.15	168.49	83.73	57.96	26.79
25	0.220	0.220	8.80	− 22.66	2.14	152.49	68.02	60.05	24.42
26	0.193	0.193	7.67	− 19.72	2.01	138.44	54.61	62.13	21.71
27	0.167	0.167	6.94	− 16.99	2.00	126.39	43.31	64.20	18.88
28	0.143	0.143	6.31	− 14.49	1.98	116.23	33.94	66.26	16.03
29	0.121	0.121	5.78	− 12.22	1.96	107.82	26.27	68.32	13.23
30	0.102	0.102	5.33	− 10.18	1.94	101.04	20.09	70.37	10.58
31	0.084	0.084	4.89	− 8.38	1.87	95.68	15.16	72.42	8.09
32	0.069	0.069	4.65	− 6.80	1.88	91.66	11.29	74.47	5.90
33	0.055	0.055	4.48	− 5.43	1.89	88.81	8.28	76.51	4.02
34	0.044	0.044	4.36	− 4.27	1.90	87.00	5.98	78.56	2.46
35	0.034	0.034	4.29	− 3.31	1.92	86.06	4.25	80.60	1.21
36	0.026	0.026	4.24	− 2.51	1.94	85.84	2.97	82.64	0.24
37	0.020	0.020	4.25	− 1.88	1.97	86.24	2.03	84.68	− 0.47
38	0.015	0.015	4.28	− 1.37	2.01	87.14	1.37	86.73	− 0.95
39	0.011	0.011	4.33	− 0.98	2.05	88.44	0.90	88.77	− 1.23
40	0.007	0.007	4.41	− 0.69	2.09	90.06	0.58	90.81	− 1.34
41	0.005	0.005	4.49	− 0.48	2.14	91.93	0.37	92.86	− 1.30
42	0.004	0.004	4.59	− 0.32	2.19	94.00	0.23	94.90	− 1.13
43	0.002	0.002	4.69	− 0.21	2.25	96.24	0.14	96.95	− 0.85
44	0.002	0.002	4.81	− 0.14	2.31	98.61	0.08	99.00	− 0.48
45	0.001	0.001	4.93	− 0.08	2.36	101.09	0.04	101.05	0.00

TABLE 3

MALE AGE 65, GUARANTEED-10 ANNUITY

(Monthly Income per $1,000 Single Premium: $8.313)

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
1........	1.000	1.000	$80.89	$843.70	$8.69	$915.89	$1,027.09	$ 0.81	−$112.01
2........	1.000	0.978	78.24	− 100.39	8.14	885.59	984.43	1.59	− 100.43
3........	1.000	0.955	75.42	− 100.43	7.70	852.89	938.95	2.35	− 88.41
4........	1.000	0.930	72.44	− 100.47	7.37	817.49	890.42	3.07	− 76.00
5........	1.000	0.904	69.27	− 100.51	7.14	779.10	838.59	3.76	− 63.25
6........	1.000	0.876	65.90	− 100.56	7.00	737.45	783.32	4.42	− 50.29
7........	1.000	0.847	61.77	− 100.60	6.68	691.94	724.35	5.04	− 37.45
8........	1.000	0.817	57.31	− 100.64	6.36	642.25	661.40	5.61	− 24.76
9........	1.000	0.785	52.53	− 100.68	6.04	588.06	594.18	6.14	− 12.26
10.......	1.000	0.752	47.44	− 100.72	5.74	529.03	522.40	6.61	0.02
11.......	0.717	0.717	40.67	− 70.31	4.32	495.07	477.00	9.02	9.05
12.......	0.681	0.681	37.62	− 66.69	4.25	461.75	433.29	11.39	17.07
13.......	0.645	0.645	34.59	− 63.01	4.15	429.17	391.23	13.74	24.21
14.......	0.608	0.608	31.58	− 59.27	4.00	397.48	351.06	16.06	30.37
15.......	0.570	0.570	28.59	− 55.49	3.81	366.77	312.97	18.34	35.45
16.......	0.532	0.532	22.08	− 51.69	1.86	335.29	277.13	20.56	37.60
17.......	0.495	0.495	20.12	− 47.91	2.02	305.49	243.64	22.76	39.09
18.......	0.457	0.457	18.32	− 44.15	2.18	277.48	212.53	24.95	40.00
19.......	0.420	0.420	16.61	− 40.46	2.28	251.35	183.85	27.11	40.39
20.......	0.384	0.384	14.99	− 36.86	2.33	227.15	157.59	29.26	40.29

TABLE 3—*Continued*

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
21.......	0.348	0.348	$12.06	−$ 33.36	$1.68	$204.17	$ 133.77	$31.38	$ 39.01
22.......	0.314	0.314	10.79	− 29.98	1.73	183.25	112.39	33.49	37.37
23.......	0.281	0.281	9.65	− 26.75	1.77	164.37	93.40	35.59	35.39
24.......	0.250	0.250	8.64	− 23.69	1.80	147.53	76.75	37.68	33.10
25.......	0.220	0.220	7.74	− 20.80	1.81	132.67	62.35	39.75	30.57
26.......	0.193	0.193	6.67	− 18.10	1.66	119.58	50.05	41.82	27.71
27.......	0.167	0.167	5.99	− 15.59	1.65	108.32	39.70	43.88	24.74
28.......	0.143	0.143	5.41	− 13.30	1.64	98.79	31.11	45.93	21.75
29.......	0.121	0.121	4.92	− 11.22	1.62	90.88	24.08	47.98	18.81
30.......	0.102	0.102	4.52	− 9.35	1.60	84.45	18.41	50.03	16.01
31.......	0.084	0.084	4.08	− 7.69	1.53	79.32	13.90	52.07	13.35
32.......	0.069	0.069	3.85	− 6.24	1.52	75.41	10.35	54.11	10.95
33.......	0.055	0.055	3.68	− 4.99	1.53	72.58	7.59	56.14	8.84
34.......	0.044	0.044	3.56	− 3.92	1.53	70.68	5.48	58.18	7.01
35.......	0.034	0.034	3.48	− 3.04	1.54	69.57	3.89	60.21	5.46
36.......	0.026	0.026	3.42	− 2.31	1.56	69.13	2.72	62.25	4.17
37.......	0.020	0.020	3.42	− 1.72	1.58	69.25	1.86	64.28	3.10
38.......	0.015	0.015	3.43	− 1.26	1.61	69.81	1.25	66.32	2.24
39.......	0.011	0.011	3.47	− 0.90	1.64	70.74	0.83	68.35	1.56
40.......	0.007	0.007	3.52	− 0.64	1.67	71.95	0.53	70.39	1.03
41.......	0.005	0.005	3.59	− 0.44	1.71	73.39	0.34	72.42	0.63
42.......	0.004	0.004	3.66	− 0.29	1.75	75.01	0.21	74.46	0.34
43.......	0.002	0.002	3.75	− 0.19	1.79	76.77	0.13	76.50	0.14
44.......	0.002	0.002	3.84	− 0.13	1.84	78.64	0.08	78.54	0.03
45.......	0.001	0.001	3.93	− 0.07	1.88	80.61	0.04	80.57	0.00

TABLE 4

FEMALE AGE 65, STRAIGHT-LIFE ANNUITY

(Monthly Income per $1,000 Single Premium: $7.915)

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
1........	1.000	1.000	$81.10	$849.01	$8.33	$921.77	$1,047.50	$ 2.31	−$128.04
2........	0.990	0.990	79.00	− 94.05	7.76	898.96	1,015.41	4.60	− 121.05
3........	0.979	0.979	76.88	− 92.98	7.30	875.57	982.26	6.87	− 113.57
4........	0.966	0.966	74.72	− 91.80	6.95	851.53	948.03	9.12	− 105.61
5........	0.953	0.953	72.51	− 90.52	6.71	826.81	912.65	11.34	− 97.18
6........	0.939	0.939	70.27	− 89.13	6.56	801.39	876.27	13.54	− 88.42
7........	0.923	0.923	67.41	− 87.62	6.22	774.96	838.90	15.72	− 79.66
8........	0.907	0.907	64.40	− 85.98	5.89	747.49	800.59	17.86	− 70.96
9........	0.889	0.889	61.23	− 84.19	5.54	718.99	761.41	19.98	− 62.39
10.......	0.869	0.869	57.91	− 82.26	5.19	689.46	721.43	22.05	− 54.02
11.......	0.848	0.848	52.20	− 80.16	3.72	657.77	680.76	24.08	− 47.06
12.......	0.825	0.825	49.14	− 77.92	3.56	625.44	639.94	26.07	− 40.57
13.......	0.800	0.800	45.99	− 75.53	3.36	592.54	598.94	28.03	− 34.42
14.......	0.774	0.774	42.74	− 72.99	3.13	559.16	557.90	29.95	− 28.70
15.......	0.747	0.747	39.40	− 70.29	2.87	525.40	517.03	31.85	− 23.48
16.......	0.718	0.718	32.40	− 67.44	0.84	489.52	476.50	33.67	− 20.65
17.......	0.687	0.687	29.86	− 64.43	0.95	454.00	436.52	35.47	− 17.99
18.......	0.655	0.655	27.43	− 61.27	1.07	419.10	397.33	37.25	− 15.48
19.......	0.621	0.621	25.04	− 57.97	1.15	385.01	359.14	39.00	− 13.12
20.......	0.586	0.586	22.69	− 54.54	1.19	351.97	322.18	40.72	− 10.93

TABLE 4—*Continued*

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
21.......	0.550	0.550	$19.07	−$ 51.01	$0.56	$319.47	$ 286.68	$42.41	−$ 9.62
22.......	0.512	0.512	17.09	− 47.37	0.65	288.53	252.84	44.08	− 8.39
23.......	0.474	0.474	15.24	− 43.68	0.73	259.37	220.84	45.74	− 7.21
24.......	0.435	0.435	13.52	− 39.94	0.80	232.15	190.84	47.37	− 6.06
25.......	0.396	0.396	11.95	− 36.19	0.88	207.03	162.95	48.99	− 4.91
26.......	0.357	0.357	10.31	− 32.48	0.84	184.02	137.32	50.59	− 3.89
27.......	0.319	0.319	9.09	− 28.83	0.94	163.34	114.06	52.19	− 2.90
28.......	0.281	0.281	8.00	− 25.29	1.03	145.02	93.26	53.77	− 2.01
29.......	0.245	0.245	7.05	− 21.90	1.11	129.06	74.99	55.34	− 1.27
30.......	0.211	0.211	6.24	− 18.70	1.19	115.41	59.24	56.90	− 0.72
31.......	0.179	0.179	5.46	− 15.73	1.21	103.93	45.92	58.45	− 0.44
32.......	0.149	0.149	4.94	− 13.01	1.29	94.57	34.90	60.00	− 0.33
33.......	0.122	0.122	4.51	− 10.57	1.35	87.16	25.98	61.55	− 0.37
34.......	0.098	0.098	4.18	− 8.43	1.41	81.51	18.93	63.09	− 0.52
35.......	0.077	0.077	3.93	− 6.58	1.46	77.40	13.50	64.63	− 0.73
36.......	0.060	0.060	3.75	− 5.03	1.49	74.63	9.41	66.17	− 0.95
37.......	0.045	0.045	3.64	− 3.76	1.54	72.97	6.42	67.70	− 1.15
38.......	0.033	0.033	3.58	− 2.75	1.58	72.22	4.30	69.24	− 1.31
39.......	0.024	0.024	3.56	− 1.97	1.62	72.20	2.82	70.77	− 1.40
40.......	0.017	0.017	3.58	− 1.39	1.66	72.73	1.81	72.31	− 1.39
41.......	0.012	0.012	3.61	− 0.95	1.70	73.70	1.14	73.85	− 1.29
42.......	0.008	0.008	3.67	− 0.64	1.74	74.99	0.70	75.38	− 1.10
43.......	0.005	0.005	3.74	− 0.42	1.78	76.52	0.43	76.92	− 0.82
44.......	0.003	0.003	3.82	− 0.27	1.82	78.24	0.25	78.46	− 0.47
45.......	0.002	0.002	3.91	− 0.16	1.87	80.12	0.12	80.00	0.00

TABLE 5

FEMALE AGE 65, GUARANTEED-10 ANNUITY

(Monthly Income per $1,000 Single Premium: $7.627)

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
1.	1.000	1.000	$81.22	$851.92	$8.27	$924.87	$1,052.88	$ 0.81	−$128.82
2.	1.000	0.990	79.37	− 92.17	7.75	904.32	1,022.47	1.61	− 119.76
3.	1.000	0.979	77.39	− 92.21	7.32	882.18	990.17	2.38	− 110.37
4.	1.000	0.966	75.29	− 92.25	6.99	858.24	955.79	3.13	− 100.69
5.	1.000	0.953	73.04	− 92.29	6.75	832.24	919.14	3.86	− 90.76
6.	1.000	0.939	70.65	− 92.33	6.59	803.97	880.08	4.57	− 80.68
7.	1.000	0.923	67.52	− 92.37	6.22	772.89	838.41	5.24	− 70.77
8.	1.000	0.907	64.10	− 92.42	5.85	738.72	793.90	5.89	− 61.06
9.	1.000	0.889	60.39	− 92.46	5.47	701.18	746.27	6.49	− 51.57
10.	1.000	0.869	56.38	− 92.50	5.08	659.98	695.23	7.05	− 42.31
11.	0.848	0.848	50.26	− 77.28	3.57	629.39	656.04	9.06	− 35.71
12.	0.825	0.825	47.26	− 75.12	3.38	598.15	616.71	11.03	− 29.59
13.	0.800	0.800	44.16	− 72.82	3.16	566.33	577.19	12.97	− 23.83
14.	0.774	0.774	40.99	− 70.37	2.92	534.02	537.65	14.88	− 18.51
15.	0.747	0.747	37.73	− 67.77	2.65	501.33	498.25	16.76	− 13.68
16.	0.718	0.718	30.81	− 65.02	0.61	466.50	459.20	18.57	− 11.26
17.	0.687	0.687	28.35	− 62.12	0.71	432.03	420.67	20.35	− 9.00
18.	0.655	0.655	26.02	− 59.07	0.83	398.14	382.90	22.11	− 6.87
19.	0.621	0.621	23.73	− 55.89	0.92	365.06	346.10	23.85	− 4.88
20.	0.586	0.586	21.49	− 52.59	0.97	332.99	310.48	25.56	− 3.05

TABLE 5—*Continued*

Year	Number of Contracts	Number of Lives	Net Investment Income	Cash Flow excluding Investments	Federal Income Tax	Assets	Statutory Reserve	Contingency Reserve	Unassigned Surplus
21	0.550	0.550	$17.95	−$ 49.18	$0.34	$301.43	$ 276.27	$27.24	−$ 2.08
22	0.512	0.512	16.05	− 45.68	0.43	271.38	243.66	28.90	− 1.18
23	0.474	0.474	14.29	− 42.11	0.51	243.04	212.83	30.54	− 0.33
24	0.435	0.435	12.66	− 38.51	0.59	216.59	183.91	32.17	0.52
25	0.396	0.396	11.17	− 34.90	0.68	192.19	157.03	33.78	1.37
26	0.357	0.357	9.56	− 31.32	0.63	169.80	132.33	35.38	2.09
27	0.319	0.319	8.37	− 27.80	0.72	149.65	109.92	36.96	2.77
28	0.281	0.281	7.32	− 24.39	0.81	131.78	89.88	38.53	3.36
29	0.245	0.245	6.40	− 21.12	0.88	116.18	72.27	40.10	3.81
30	0.211	0.211	5.61	− 18.04	0.95	102.80	57.09	41.66	4.06
31	0.179	0.179	4.84	− 15.17	0.96	91.51	44.25	43.20	4.06
32	0.149	0.149	4.33	− 12.55	1.03	82.26	33.63	44.75	3.88
33	0.122	0.122	3.91	− 10.20	1.09	74.88	25.04	46.29	3.55
34	0.098	0.098	3.57	− 8.13	1.14	69.19	18.25	47.82	3.12
35	0.077	0.077	3.32	− 6.35	1.18	64.98	13.01	49.35	2.62
36	0.060	0.060	3.13	− 4.85	1.21	62.05	9.07	50.89	2.10
37	0.045	0.045	3.02	− 3.63	1.24	60.20	6.19	52.42	1.59
38	0.033	0.033	2.95	− 2.65	1.28	59.21	4.14	53.95	1.13
39	0.024	0.024	2.91	− 1.90	1.31	58.92	2.72	55.47	0.73
40	0.017	0.017	2.91	− 1.34	1.34	59.15	1.75	57.00	0.40
41	0.012	0.012	2.94	− 0.92	1.37	59.80	1.10	58.53	0.17
42	0.008	0.008	2.97	− 0.62	1.40	60.75	0.68	60.06	0.01
43	0.005	0.005	3.03	− 0.41	1.44	61.93	0.41	61.59	− 0.07
44	0.003	0.003	3.09	− 0.26	1.47	63.28	0.24	63.12	− 0.09
45	0.002	0.002	3.16	− 0.16	1.51	64.77	0.12	64.66	0.00

TABLE 6

PORTFOLIO NET EARNED RATES BEFORE AND AFTER FEDERAL INCOME TAX

Year	Male Age 65, Straight Life		Male Age 65, Guaranteed 10		Female Age 65, Straight Life		Female Age 65, Guaranteed 10	
	Before Tax	After Tax	Before Tax	After Tax	Before Tax	After Tax	Before Tax	After Tax
1	9.00%	8.10%	9.00%	8.03%	9.00%	8.08%	9.00%	8.08%
2	8.99	8.11	8.99	8.05	8.99	8.10	8.99	8.11
3	8.98	8.11	8.98	8.06	8.97	8.12	8.97	8.12
4	8.97	8.11	8.97	8.06	8.96	8.13	8.96	8.13
5	8.98	8.08	8.97	8.05	8.95	8.12	8.95	8.12
6	8.98	8.05	8.98	8.03	8.94	8.10	8.94	8.10
7	8.91	7.97	8.93	7.96	8.85	8.03	8.86	8.04
8	8.83	7.86	8.86	7.88	8.75	7.95	8.77	7.97
9	8.73	7.74	8.80	7.79	8.63	7.85	8.66	7.88
10	8.61	7.60	8.74	7.69	8.50	7.73	8.55	7.78
11	8.06	7.22	8.18	7.32	7.99	7.42	8.04	7.47
12	7.99	7.10	8.10	7.18	7.90	7.33	7.94	7.37
13	7.90	6.96	7.99	7.03	7.78	7.21	7.82	7.26
14	7.78	6.79	7.85	6.86	7.64	7.08	7.67	7.12
15	7.62	6.60	7.68	6.66	7.48	6.93	7.50	6.97
16	6.50	5.89	6.44	5.89	6.55	6.38	6.53	6.40
17	6.49	5.77	6.42	5.78	6.49	6.29	6.47	6.31
18	6.48	5.65	6.43	5.66	6.44	6.19	6.42	6.22
19	6.45	5.51	6.42	5.54	6.38	6.09	6.37	6.12
20	6.40	5.35	6.40	5.40	6.30	5.97	6.30	6.02
21	5.77	4.89	5.70	4.90	5.80	5.63	5.78	5.68
22	5.73	4.73	5.67	4.76	5.74	5.52	5.73	5.57
23	5.68	4.57	5.65	4.61	5.68	5.41	5.67	5.47
24	5.63	4.40	5.63	4.46	5.61	5.28	5.62	5.36
25	5.57	4.22	5.62	4.31	5.55	5.14	5.57	5.23
26	5.35	3.95	5.37	4.03	5.37	4.93	5.38	5.02
27	5.32	3.79	5.33	3.86	5.33	4.78	5.34	4.88
28	5.27	3.62	5.30	3.70	5.28	4.60	5.30	4.71
29	5.23	3.45	5.26	3.53	5.23	4.41	5.25	4.53
30	5.18	3.30	5.22	3.38	5.18	4.20	5.21	4.33
31	5.04	3.11	5.05	3.16	5.05	3.94	5.06	4.06
32	5.03	3.00	5.04	3.05	5.05	3.73	5.05	3.85
33	5.03	2.91	5.04	2.95	5.04	3.53	5.04	3.64
34	5.02	2.83	5.03	2.86	5.03	3.34	5.03	3.43
35	5.02	2.77	5.02	2.79	5.02	3.16	5.02	3.24
36	5.00	2.71	5.00	2.73	5.00	3.01	5.00	3.07
37	5.00	2.68	5.00	2.69	5.00	2.89	5.00	2.94
38	5.00	2.65	5.00	2.66	5.00	2.80	5.00	2.83
39	5.00	2.63	5.00	2.64	5.00	2.73	5.00	2.76
40	5.00	2.62	5.00	2.63	5.00	2.69	5.00	2.70
41	5.00	2.61	5.00	2.62	5.00	2.65	5.00	2.66
42	5.00	2.61	5.00	2.61	5.00	2.63	5.00	2.64
43	5.00	2.61	5.00	2.61	5.00	2.62	5.00	2.62
44	5.00	2.60	5.00	2.60	5.00	2.61	5.00	2.61
45	5.00	2.60	5.00	2.60	5.00	2.61	5.00	2.61

TABLE 7

LEVEL AFTER-TAX NET EARNED RATES

	Period 1 (Years 1-12)	Period 2 (Years 13-20)	Period 3 (Years 21-45)
Male age 65, straight life	7.95%	6.25%	3.76%
Male age 65, guaranteed 10	7.94	6.30	3.84
Female age 65, straight life	7.99	6.65	4.70
Female age 65, guaranteed 10	8.01	6.69	4.82

to the iterations required every policy year for each trial run to determine the portfolio net earned rate appearing in the expression for the year-end book profit. Despite the lengthy calculations, however, all the programming was accomplished conveniently in APL, and less than 30 seconds of central processing unit time on an IBM 370/168 was needed for even the most complicated pricing situation with nonzero profit objective.

The theory presented in this paper has applications beyond those already discussed.

1. Since federal income taxes are included directly in the book profits, it is possible to analyze the size of the discount from nonqualified rates that should be given to account for the favorable tax treatment of qualified annuities.
2. The expression for the year-end book profits can be extended to the case of single premium deferred annuities.
3. A general iterative technique for determining the premium rate has been developed for the case where the interest assumption is based on new-money rates and an investment-generation method.
4. In contrast to many other methods, this approach includes contingency reserves explicitly in the pricing equations.

TABLE 8

MORTALITY CONTINGENCY RESERVE VERSUS MORTALITY TABLE MARGIN

YEAR	UNASSIGNED SURPLUS PER $1,000 SINGLE PREMIUM					
	Male Age 65 Straight-Life Annuity R_{12} = $9.069			Female Age 65 Straight-Life Annuity R_{12} = $7.915		
	Basis 1*	Basis 2*	Difference	Basis 1*	Basis 2*	Difference
1	−$131.07	−$129.68	$1.39	−$128.04	−$126.42	$ 1.62
2	− 118.51	− 115.84	2.67	− 121.05	− 117.88	3.17
3	− 105.56	− 101.74	3.82	− 113.57	− 108.96	4.61
4	− 92.26	− 87.40	4.86	− 105.61	− 99.65	5.96
5	− 78.66	− 72.88	5.78	− 97.18	− 89.98	7.20
6	− 65.03	− 58.45	6.58	− 88.42	− 80.08	8.34
7	− 51.73	− 44.45	7.28	− 79.66	− 70.30	9.36
8	− 38.82	− 30.97	7.85	− 70.96	− 60.69	10.27
9	− 26.40	− 18.09	8.31	− 62.39	− 51.35	11.04
10	− 14.59	− 5.93	8.66	− 54.02	− 42.34	11.68
11	− 4.64	4.25	8.89	− 47.06	− 34.88	12.18
12	4.25	13.30	9.05	− 40.57	− 28.01	12.56
13	12.24	21.38	9.14	− 34.42	− 21.61	12.81
14	19.22	28.38	9.16	− 28.70	− 15.75	12.95
15	25.07	34.21	9.14	− 23.48	− 10.49	12.99
16	27.93	36.95	9.02	− 20.65	− 7.77	12.88
17	30.07	38.94	8.87	− 17.99	− 5.32	12.67
18	31.57	40.27	8.70	− 15.48	− 3.09	12.39
19	32.46	40.98	8.52	− 13.12	− 1.09	12.03
20	32.79	41.12	8.33	− 10.93	0.67	11.60
21	31.90	40.01	8.11	− 9.62	1.49	11.11
22	30.58	38.47	7.89	− 8.39	2.19	10.58
23	28.87	36.53	7.66	− 7.21	2.81	10.02
24	26.79	34.21	7.42	− 6.06	3.39	9.45
25	24.42	31.58	7.16	− 4.91	3.96	8.87
26	21.71	28.59	6.88	− 3.89	4.42	8.31
27	18.88	25.46	6.58	− 2.90	4.86	7.76
28	16.03	22.30	6.27	− 2.01	5.24	7.25
29	13.23	19.17	5.94	− 1.27	5.49	6.76
30	10.58	16.18	5.60	− 0.72	5.57	6.29
31	8.09	13.34	5.25	− 0.44	5.42	5.86
32	5.90	10.81	4.91	− 0.33	5.11	5.44
33	4.02	8.59	4.57	− 0.37	4.66	5.03
34	2.46	6.70	4.24	− 0.52	4.12	4.64
35	1.21	5.12	3.91	− 0.73	3.53	4.26
36	0.24	3.83	3.59	− 0.95	2.92	3.87
37	− 0.47	2.80	3.27	− 1.15	2.34	3.49
38	− 0.95	2.00	2.95	− 1.31	1.80	3.11
39	− 1.23	1.39	2.62	− 1.40	1.33	2.73
40	− 1.34	0.93	2.27	− 1.39	0.94	2.33
41	− 1.30	0.60	1.90	− 1.29	0.62	1.91
42	− 1.13	0.35	1.48	− 1.10	0.37	1.47
43	− 0.85	0.19	1.04	− 0.82	0.19	1.01
44	− 0.48	0.07	0.55	− 0.47	0.05	0.52
45	0.00	0.00	0.00	0.00	0.00	0.00

* Basis 1—contingency reserve, no mortality margin; Basis 2—mortality margin, no contingency reserve.

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APPENDIX

DEFINITION OF SYMBOLS

In the following definitions, a subscript t refers to policy year t unless otherwise indicated.

BP_t = Year-end book profit, including the effect of survivorship since issue;

F_t = Interest discount factor for BP_t;

n = Shareholders' investment horizon (often taken as the period to the end of the mortality table);

Z_n = Present value at issue of the first n annual book profits;

j_t = Shareholders' pre-tax rate of return for use of surplus funds;

i_t = Net (after investment expenses) new-money interest rate;

i'_t = Portfolio net earned rate;

$i'_{t:k}$ = Approximation to i'_t after the kth iteration;

$\bar{i}'_t$ = Portfolio net after-tax earned rate;

$\bar{i}'_{t:k}$ = Approximation to $\bar{i}'_t$ after the kth iteration;

i^C_t = Actual portfolio yield rate at the beginning of year $t+1$ for A^C_{t+1};

i^T_t = Portfolio net earned rate for purposes of federal income tax;

$i^{T'}_t$ = Adjusted reserves rate for purposes of federal income tax;

i_V = Interest rate for calculation of statutory reserves;

i^{INF}_t = Inflation rate for per contract maintenance expenses in year $t+1$;

g = Period of guaranteed annuity payments, in years;

m = Frequency of periodic annuity benefit payments;

R_m = Annuity payout rate: $m$thly income per \$1,000 single premium after policy fee and state premium tax;

$R_m^{(k)}$ = kth trial annuity payout rate;

R_m^* = Annuity payout rate meeting profit objective Z_n^*;

PF = Policy fee;

$PF^{(k)}$ = kth trial policy fee;

PF^* = Policy fee meeting profit objective Z_n^*;

$Z_n(R_m^{(k)})$ = Present value of book profits when annuity payout rate equals $R_m^{(k)}$;

Z_n^* = Profit objective;

$CF_{t;r_k}^{\text{in}}$ = A general component of cash inflow at fractional duration r_k during policy year t (premiums), $0 \leq r_k \leq 1$;

$CF_{t;s_k}^{\text{out}}$ = A general component of cash outflow at fractional duration s_k during policy year t (commissions, expenses, benefits, and federal income taxes), $0 \leq s_k \leq 1$;

CF_t = Cash outflow from benefit payments and maintenance expenses;

$I_t^{\text{CF}}(i)$ = Interest earnings during policy year t on CF_t, assuming an effective annual net earned rate i;

NII_t = Total net investment income;

$r_t^{(0)}$ = Fraction of principal invested at issue which is rolled over t years later;

$r_t^{(1)}$ = Same interpretation as $r_t^{(0)}$ except that $r_t^{(1)}$ applies only to rollover from investments made after the first policy year;

$\mathcal{A}_t''$ = Funds invested in years prior to year t that are made available for reinvestment at the beginning of year t;

$\mathcal{A}_{t;t}'$ = Funds invested in cell t during year t, excluding any cash flow from insurance operations during year t;

$\mathcal{A}_t$ = Total funds invested during policy year t at the net new-money rate i_t;

$\mathcal{A}_{t;s}'$ = Total assets in cell s at the beginning of policy year t ($t > s$);

A_t = Year-end total assets for the block of issues;

V_t = Year-end statutory reserve for the block of issues;

${}_tV^G$ = Terminal reserve factor for guaranteed annuity benefits;

${}_tV^L$ = Terminal reserve factor for life-contingent annuity benefits;

S_t^C = Year-end contingency reserve for the block of issues;

CR_t^K = Contribution per \$1,000 single premium to the contingency reserve;

$CR_t^{\%}$ = Fraction of NII_t contribution to the contingency reserve;

S_t^U = Year-end unassigned surplus for the block of issues;

C = Commission rate;

E^C = Commission-related acquisition expense;
E^{AQ} = Per contract acquisition expense;
E^M = Uninflated per contract annual maintenance expense;
SP = Average size of contract, in thousands of single premium;
E'_t = Total uninflated maintenance expenses for the block of issues;
E_t = Total maintenance expenses for the block of issues;
B_t = Total annuity benefits for the block of issues accumulated to year-end at rate i'_t;
MV_t = Mean reserve for purposes of federal income tax;
A^C_t = Assets of the entire company at the beginning of policy year t;
r^C_t = Fraction of assets A^C_t made available for reinvestment during year t;
g^C_t = Growth rate (before payment of federal income tax) of the entire company's assets;
TII_t = Taxable investment income for purposes of federal income tax;
OG_t = Taxable operating gain for purposes of federal income tax;
TAX_t = Federal income tax;
$\bar{i}_s$ = Theoretical level net after-tax earned rate for period s, used in the annuity rate basis;
$\bar{i}_{s;k}$ = Approximation to $\bar{i}_s$ after the kth iteration;
A^{lower}_s = Total assets for the block of issues at the beginning of period s;
A^{upper}_s = Total assets for the block of issues at the end of period s;
x_0, a, b, c = Constants used in fitting the male and female select and ultimate mortality bases used in the sample pricings;
q^{select}_{x+t} = Average select mortality rate at attained age $x + t$;
$q^{\text{ultimate}}_{x+t}$ = Average ultimate mortality rate at attained age $x + t$;
$l_{[x]+t-1}$ = Number of annuitants surviving at the beginning of policy year t.

DISCUSSION OF PRECEDING PAPER

LARRY R. PETERSON:

In Section I of Mr. Tilley's fine paper on the pricing of nonparticipating single premium immediate annuities (SPIAs), we find a discussion of the nonparticipating nature of these annuities. To quote: "There is seldom any consideration of equitable distribution of surplus to policyholders." Later, Mr. Tilley discusses some advantages and disadvantages of this approach.

I wish to draw attention to reasons why an actuary should consider the development of participating SPIAs. These comments apply equally to life income settlement agreements issued under current (net SPIA) rates. I offer the following thoughts.

1. *Nonparticipating Contracts Are Counter to Mutual Philosophy*

Bypassing the consideration of equitable distribution of surplus is counter to the actuarial philosophy espoused by the profession, especially by practicing actuaries in mutual companies and fraternal benefit societies. From a philosophical viewpoint, a case can be made that annuitants should benefit from the favorable experience (or share in the loss) of their equity class, just as those insureds who hold single premium life insurance or deferred annuity policies do.

2. *Several Experience Factors May Require Adjustment*

Some conservatism is in order in the pricing of the SPIA. Mr. Tilley recognizes this in the important mortality factor. Since experience is different from that assumed in pricing, the actuary should review and adjust, as necessary, the dividend scale to account for the actual cost of the annuity. Experience factors under the actuary's review include the following:

1. Mortality.
2. Interest, based on the chosen interest rate philosophy and experience as a result of operating under that philosophy.
3. Expenses, especially those of ongoing policy maintenance.
4. Contingencies, to the extent that adjustments must be made to the rate at which contingency reserves are released over the life of the policy.

There are, of course, some practical and theoretical problems involved in paying more (or less) dividends to continuing annuitants when those who die early deserve the reward (or burden) of participation. Some problems are mitigated by the actuary's annual review of the dividend scale.

I suppose one reason why most SPIAs are nonparticipating is that it is difficult to develop a method for adjusting dividends in line with experience factors. Two available methods for adjusting SPIA dividends are described below:

1. *Accumulated asset share method.*—The asset shares may be accumulated for surviving annuitants, on the basis of actual experience. Each asset share is then applied to the present value of future income payments and expenses, based on current assumptions and adjusted for duration from issue, to determine the new total income payment. From this, the new dividend can be established.
2. *Present value of future income method.*—The present values of future income payments and expenses may be calculated on the basis of assumptions used at issue (or as of the last dividend adjustment, if later). Each present value is then applied to the present value of future income payments and expenses, based on current assumptions and adjusted for duration from issue, to determine the new total income payment.

The accumulated asset share method is theoretically more correct, since, under this method, all experience prior to the point of dividend determination can be reflected. However, the method may be difficult to apply in practice for lack of reliable experience data. In this case, the actuary may rely on the present value of future income method, especially when dividends have been kept up to date periodically.

A modification of each method is to use the same mortality assumptions employed in the original "new business" pricing and to adjust only for the other experience factors. This is practical where actual current experience is too small to use, and future experience is not expected to depart markedly from that originally assumed. This modification, especially for the accumulated asset share method, avoids the possibility of a tontine effect at later durations.

A further practical modification is the curtailment of dividend scale changes after some advanced age, such as 90. This avoids problems of assumption and dividend instabilities that involve relatively few surviving annuitants.

3. *Participating SPIAs Are Competitive*

Competition is significant in the SPIA market. An advantage of the nonparticipating SPIA contract is that decisions can be made on the basis of a single number, the guaranteed amount of periodic income. However, nonparticipating contracts should not be an axiom of SPIA competition.

Companies can offer attractively priced participating SPIAs with lower income guarantees but higher current incomes. This should appeal both to annuitants, who receive favorable incomes, and to the actuary, whose job is made easier. Less conservative assumptions may be used in setting the current income levels, and an attractive product can be provided to the intensely competitive marketplace.

4. *Participating SPIAs Produce Less Statutory Surplus Strain*

Nonparticipating SPIAs create initial statutory surplus strain because the guaranteed periodic income must be valued by the use of more conservative assumptions than those used in pricing.

This problem is reduced or eliminated with participating SPIAs, since reserves are based on the guaranteed periodic income at the legal valuation rates of interest and mortality. The current dividend is held in the annual statement under the provision for policyholders' dividends payable in the following calendar year.

Since the guaranteed rates of interest and mortality used in pricing are closer to those used in reserve valuation, less surplus strain is created. This gives the actuary more flexibility in pricing and the company more flexibility in sales. As always, it is very important for the actuary to exercise judgment as to the adequacy of reserves, and, if necessary, to establish reserves stronger than those provided under the minimums of the Standard Valuation Law.

J. ALAN LAUER:

This is a very interesting and worthwhile paper for actuaries interested in single premium immediate annuities. Many of the concepts and considerations discussed in the paper are applicable to other products as well. A particularly laudable aspect of the paper is that it records in the literature the manner in which Anderson's method can be applied to a product such as the single premium immediate annuity.

The author describes two approaches to determining an appropriate rate of return to shareholders. He then indicates a preference for the second approach, and goes on to develop the rest of the paper on the basis of that approach. I share Mr. Tilley's preference for the second approach, under which the risk premium can be related to the nature of the particular risk, although it is true that a contingency reserve held for a particular risk would be available in case of adverse experience with reference to some other risk. However, even if one prefers the second approach, the shareholders' rate of return under the first approach is still of interest. It should not be difficult to estimate that rate of return if the pricing has been done by the second approach using the

Anderson technique described in the paper. One would simply set the contingency reserve to zero and determine by trial and error (Newton-Raphson method or otherwise) an effective rate of return under the first approach. This might be expressed as a level rate or as a constant percentage of the effective rate of return under the second approach. Of course, the answer would vary with plan, age, and sex.

Since the contingency reserve remains after the last policy in the block of business has gone off the books, Mr. Tilley's annual contributions to the contingency reserve would seem to be no different from what sometimes are referred to as "permanent contributions to surplus." While Mr. Tilley's methods for determining these contributions are not the only correct ones, they are very reasonable and illustrate quite well a rational approach to the question.

The author describes the manner in which he has developed mortality assumptions for his pricing and points out correctly the importance of exercising care in selecting these assumptions. Mr. Tilley indicates that he has relied heavily on Table 3 of the 1967-71 study reported in *TSA, 1973 Reports*. Let us first agree that the *1973 Reports* are the latest available at the time the pricing is done, and are therefore appropriate. Also, while some actuaries might prefer to use Table 1 (experience on nonrefund annuities) and Table 2 (experience on refund annuities), Table 3 (experience on nonrefund and refund annuities combined) has the advantage of being based on the combined experience of Tables 1 and 2 and therefore is less subject to random fluctuation. However, a review of Table 12 of the *1973 Reports* shows that even Table 3 should not be used directly without analysis and exercise of some actuarial judgment.

Table 12 presents the mortality ratios in the six most recent studies for individual immediate nonrefund and refund annuities combined. The ratios from Table 12 that are shown below are based on expected deaths according to the Annuity Table for 1949 Ultimate, without projection. Consider first the following figures applicable to females at attained ages 60-69 in contract years 6 and over:

Years	Experience between Anniversaries	Years	Experience between Anniversaries
1941-48	119%	1958-63	120%
1948-53	116	1963-67	102
1953-58	108	1967-71	89

Consider the actuary to whom the latest data available were from the 1958-63 experience. Now that we can see the experience for 1963-67 and 1967-71, it is quite evident that it would have been unwise for our actuary to assume a continuation of the 1958-63 mortality ratio of 120 percent. That is, while the mortality ratio for that period was, in fact, 120 percent, the wise (or lucky) actuary would have recognized this as a random fluctuation and would have assumed some lower ratio.

Now let us see how this principle would apply to the most recent experience. Let us consider the following data for males at attained ages 70-79 in contract years 1-5:

Years	Experience between Anniversaries	Years	Experience between Anniversaries
1941-48........	107%	1958-63.........	85%
1948-53........	97	1963-67.........	81
1953-58........	94	1967-71.........	93

In my opinion, the prudent actuary would assume that the 93 percent ratio for 1967-71 represents a fluctuation, and would assume some lower ratio (perhaps 80 percent) in his pricing. By the time this discussion is published, it is possible that the results of the 1971-76 annuity mortality study will have been published and will have proved me right or wrong.

In setting mortality assumptions, it is helpful to calculate the ratio of the select mortality ratio to the ultimate mortality ratio for each of the age groups in each of the studies. The ratio of assumed select mortality to assumed ultimate mortality should be consistent with the trend.

The author's approach to graduation of his mortality assumptions is quite elegant. It is also in keeping with the admirably complete approach he has taken to the total problem of pricing the immediate annuity. Many actuaries probably would feel that the degree of elegance achieved by Mr. Tilley in his graduation is far greater than that necessary in practice. It must be remembered that the product has a single premium, is nonparticipating, and has no cash values. In particular, I suspect that little was gained by departing from the original four mortality tables and constructing a pair of true select and ultimate tables using formula (39). This may have had some impact on the book profits of individual years (although I suspect that the impact was not large), but I doubt very much whether it had any real effect on the final answer, which is monthly income per $1,000 single premium.

There is a consideration that is worth discussing that Mr. Tilley may have felt to be outside the scope of his paper. This is the necessity of avoiding anomalous results when highly complicated assumptions are used in developing rates for immediate annuities. The most likely anomaly, if one is not careful, is that the periodic income purchased by a given amount of premium is greater for a life annuity with a short certain period than for a straight life annuity, or is greater for a life annuity with a given certain period than for a life annuity with a shorter certain period. This is particularly likely to happen at the younger ages at issue (below ages 50 or 55 for single life annuities, and below age 60 for joint life annuities). At the younger ages the near-term mortality rates are quite low, and the difference in annuity rates for life annuities with only slightly different certain periods is very small when both rates are based on identical pricing assumptions (other than the length of the certain period). When the assumptions for the two annuities are different, this delicate balance can be upset. For instance, Mr. Tilley has used an annual contribution to the contingency reserve for the mortality risk that is not made during the period of guaranteed payments. This approach is not unreasonable, but it does favor the annuity with the longer certain period and thus possibly could result in anomalies if care were not taken. Similarly, Mr. Tilley has used interest assumptions that vary slightly by plan. Although the differences would appear to be small, the resulting annuity rates still should be inspected carefully. While Mr. Tilley has used the same mortality assumptions for refund and nonrefund annuities, some actuaries might choose to use different assumptions. The use of different assumptions would favor the refund annuities and might produce anomalous results at the young ages.

Finally, I would like to say a word about the mathematical formulas in the paper. There are many of them, and it does take some time to go through them all. The author has done a clear, thorough job of describing his technique, which does require many calculations. However, I detected no mathematics in the paper that was not covered on the examinations when I began to take them twenty-five years ago. Actuaries interested in the subject of single premium immediate annuities should not allow the many formulas to deter them from reading this very fine paper.

(AUTHOR'S REVIEW OF DISCUSSION)

JAMES A. TILLEY:

I would like to thank Messrs. Lauer and Peterson for their discussion of my paper.

Mr. Peterson states the case for participating single premium immediate annuities very well, and his discussion is a welcome addition to the paper. I suspect that the pressure to quote a competitive guaranteed annuity payment as well as the difficulties, both theoretical and practical, of developing credible dividend practices have prevented more companies from marketing participating single premium immediate annuities.

Mr. Lauer is correct in emphasizing that considerable judgment is required in determination of the credibility of actual-to-expected mortality ratios. It is useful to estimate 95 percent confidence intervals for the ratios, and these should be included in the reports of the Society's mortality studies. In examining the mortality ratio for males at attained ages 70–79 in contract years 1–5, Mr. Lauer concludes that the 93 percent ratio for 1967–71 was an upside statistical fluctuation from a more reasonable value of 80 percent. Another (perhaps less plausible) interpretation is that the 1963–67 value was a downside fluctuation, and that over the period 1958–71 the mortality ratio was fairly stable between 85 and 90 percent. A mortality ratio of 90 percent at age 75 is obtained when the parameters for the male select curve are substituted in equation (38).

Mr. Lauer has stressed the importance of scrutinizing the annuity rate basis for anomalies, especially if complicated pricing assumptions that differ by plan/age/sex cell have been used. The accompanying table indicates that anomalies probably do not arise (or are insignificant) for straight life and guaranteed annuities issued to males at ages 40, 50, and 60, despite my particular treatment of the mortality contingency reserve. The pricings utilized the assumptions described in Section III of the paper.

Mr. Lauer has remarked that slightly different interest assumptions were used in pricing different plans. This is not true. The investment

ANNUITY PAYOUT RATES PER $1,000
SINGLE PREMIUM (MALES)

Issue Age	Guaranteed Period (Years)			
	0	5	10	20
40	$6.030	$6.028	$6.006	$5.915
50	6.910	6.877	6.778	6.478
60	8.287	8.163	7.854	7.051

assumptions consist of new-money interest rates and rollover rates for the initial investment and subsequent reinvestments. The assumptions used in the sample pricings are described in Section III, A, 2, and do *not* vary by plan/age/sex cell. The values of i'_t and $\bar{i}'_t$ shown in Table 6 do vary by pricing cell, but these interest rates are *not* the a priori assumptions. They are calculated from equation (27) *after* net investment income has been determined from equations (22)–(24), which do not depend on i'_t and $\bar{i}'_t$ at all. Before- and after-tax portfolio rates that differ slightly by plan/age/sex cell occur because of differences in the amount and incidence of cash flow among the pricing cells. This must be dealt with when fitting the annuity rate basis, as discussed in Section II, G, of the paper.

Many of the calculations (and associated computer time) can be eliminated if the pricing objective is a stipulated amount of unassigned surplus at the end of a specified period. In this case, the portfolio rates i'_t and $\bar{i}'_t$ need be determined only for the fitting of the interest rate part of the annuity rate basis—that is, only on the last trial run.

Several actuaries who have discussed this paper with me orally have raised interesting questions about the "shareholders' rate of return." This phrase is somewhat ambiguous and is used very loosely in the literature. I have used it in the paper to mean the rate of return payable to the general surplus account of a mutual company or to the shareholders' surplus account of a stock company. Arguments were given in the paper to suggest that such a rate of return is thought of most fruitfully as an after-tax rate. However, this means "after-tax" only as far as the surplus account of the insurance company is concerned, not with respect to individual shareholders to whom actual dividends will be paid. Except for a dividend exclusion, individual shareholders must pay personal income tax on after-tax earnings of corporations that are distributed as dividends—the so-called double taxation of dividends. Thus, after-corporate-tax earnings from the block of business to the shareholders' surplus account form the basis for pre-personal-income-tax dividends to shareholders.

In equation (1) of the paper, the denominator of the discount factor contains factors $(1 + 0.52j_s)$, where the j_s's are pre-tax rates of return to the shareholders' surplus account. The use of the factor 0.52 assumes that surplus funds of the insurer are taxed at a marginal rate of 48 percent, which is valid only if the underlying investments produce fully taxable investment income. The discount factor $F_t(j)$ was written in this form primarily for the later convenience of identifying $Z_n = 0$ with

$j_t = i'_t$ as break-even pricing. What is relevant, of course, is the after-corporate-tax return to the shareholders' surplus account, because this is the source of actual shareholder dividends and of retained earnings that permit the company to grow and its stock to appreciate.

On the basis of the preceding discussion, it can be seen that a break-even profit objective probably will not result in an attractive overall yield to individual shareholders. For example, with a net new-money rate of 10 percent on fully taxable fixed-income investments, the after-tax yield to the insurer's surplus account is 5.2 percent. If this is paid out fully to shareholders, their pre-tax yield is also 5.2 percent; this is not unattractive by itself, but, without any retained earnings, there are no prospects for long-term capital appreciation. Individual shareholders might expect to get a combined (dividends plus appreciation) pre-tax return of about 10 percent in this example, and the pre-corporate-tax rate of return to the shareholders' surplus account would have to be about twice the prevailing new-money rate on fully taxable fixed-income investments.

I am grateful to Messrs. Lauer and Peterson for taking time to work through my lengthy paper and for their written comments. I would also like to thank those who offered me their opinions orally.