Guaranteed benefits in incomplete markets and risk analysis

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Abstract
This paper presents a methodology of pricing the guaranteed minimum death benefit of a variable annuity in a market model with jumps. Recent developments in the stock market make variable annuities very attractive products from the insured point of view, but less attractive for insurers. The insured still has the possibility of investment benefits, while avoiding the risk of a stock market collapse. The insurer wants to minimize its risk and yet sell a competitive product.
The financial market model consists of one riskless asset and one risky asset whose price jumps in proportions J at some random times τ which correspond to the jump times of a Poisson process. The model describes incomplete markets and there is no perfect hedging.
In the second part of the paper, we describe a possible method of risk analysis for binomial tree models.

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0. Introduction

In the Black-Scholes model, the share price is a continuous function of time. Some rare events (which are rather frequent lately), can accompany "jumps" in the share price. In this case the market model is incomplete, hence there is no perfect hedging of options.
We consider a market model with one riskless asset and one risky asset whose price jumps in proportions $J_1, J_2, \ldots, J_n, \ldots$ at some random times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ which correspond to the jump times of a Poisson process. Between the jumps the risky asset follows the Black-Scholes model.
The mathematical model consists of a probability space $(\Omega, \mathcal{F}, P)$, a Brownian motion (W_t) and a Poisson process $(\mathrm{N}_t)_{t\geq 0}$ with parameter λ. The jumps J_n are independent and identically distributed on $(-1, \infty)$ and $(F_t)_t$ is the filtration which incorporates all information available at time t. The price process (S_t) of the risky asset is described as follows:
On $[\tau_j, \tau_{j+1})$, $dS_t = S_t(\mu dt + \sigma dW_t)$ i.e. Black-Scholes model;
At time τ_j, the jump of (S_t) is given by $\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j^-} = S_{\tau_j} - J_j$;
In other words, $S_{\tau_j} = S_{\tau_j^-}(1 + J_j)$; As defined, (S_t) is a right-continuous process.
It is straightforward to see that we have the following formula for the price process:

$$S_t = S_0\left(\prod_{j=1}^{N_t}(1 + J_j)\right)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{1}$$

1 Guaranteed minimum death benefits

A variable annuity is an investment wrapped with a life insurance contract. The convenient tax deferral characteristic of the variable annuities makes them a very interesting and popular investment and retirement instrument. The average age at which people buy their first variable annuity is 50. There are a few different types of GMDB options associated with variable annuities. The most popular are:
1. **Return of premium** - the death benefit is the larger of the account value on the date of death or the sum of premiums less partial withdrawals;
2. **Reset** - the death benefit is automatically reset to the current account value every x years;
3. **Roll-up** - the death benefit is the larger of the account value on the day of death or the accumulation of premiums less partial withdrawls accumulated at a specified interest rate (e.g. 1.5% in many 2003 contracts);
4. **Ratchet** (look back) - same as reset, except that the death benefit is not allowed to decrease, except for withdrawals.
Let ω be the expiry date for a variable annuity with a return of premium GMDB option associated with (S_t). Let T be the random variable that models the future lifetime of the insured (buyer of the contract).Then the payoff of the product is:

$$P(T) = \begin{cases} H(t) & \text{if } T \leq \omega \\ S(\omega) & \text{if } T > \omega \end{cases}$$

where $H(t) = \max(S(0), S(t)) = S(t) + \max(S(0) - S(t), 0) = S(t) + (S(0) - S(t))_+$
Basically, the value of the guarantee at time 0 is given by the price of a put option with stochastic expiration date. It can be shown that in discrete settings and when the benefit is paid at the end of the year of death,

$$PV(GMDB) = \sum_{m=1}^{\omega-x} {}_{m-1|}q_x P(m, S_0) \tag{2}$$

where $P(m, S_0)$ is the price of the put option with expiry m and strike S_0, in the Black-Scholes model.
If the benefit is paid at the moment of death, then

$$PV(GMDB) = \int_0^\infty f_T(t) P(t, S_0) dt \tag{3}$$

where $f_T(t)$ is the pdf of the future lifetime random variable. Closed form expressions can be obtained for appropriate assumptions on T (constant force, UDD, Balducci etc).
Next we want to determine the price of the put option associated with GMDB in the market model described in the introduction, which minimizes the risk at maturity.
Suppose $E(J_1) < \infty$ and let $\widetilde{S}_t = e^{-rt} S_t$ for $s \leq t$. Then

$$E(\widetilde{S}_t | F_s) = \widetilde{S}_s E\left(e^{(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)} \prod_{j=N_s+1}^{N_t} ((1+J_j)|F_s)\right)$$

$$= \widetilde{S}_s E\left(e^{(\mu - r - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)} \prod_{j=1}^{N_t - N_s} (1 + J_{N_s+j})\right)$$

because $W_t - W_s$ and $N_t - N_s$ are independent of F_s.
Hence

$$E(\widetilde{S}_t | F_s) = \widetilde{S}_s e^{(\mu - r)(t-s)} E(\prod_{j=N_s+1}^{N_t} (1 + J_j))$$

But

$$E(\prod_{j=N_s+1}^{N_t} (1 + J_j)) = E(\prod_{j=1}^{N_t} (1 + J_j)) - E(\prod_{j=1}^{N_t} (1 + J_j))$$

and

$$E(\prod_{j=1}^{N_t}(1+J_j)) = \sum_{n=1}^{\infty} E(\prod_{j=1}^{n}(1+J_j))P(N_t = n)$$

$$= \sum_{n=1}^{\infty}(1+E(J_j))^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t(1+E(J)))^n}{n!}$$

$$= e^{-\lambda t}e^{\lambda t(1+E(J))} = e^{\lambda t E(J)}$$

So

$$E(\widetilde{S}_t|F_s) = \widetilde{S}_s e^{(\mu-r)(t-s)}e^{\lambda(t-s)E(J)}$$

Hence $(\widetilde{S}_t)$ is a martingale iff $\mu = r - \lambda E(J)$. In our case, we want to price a put option with strike S_0 and expiry T.
Let $f(x) = (S_0 - x)_x$. The price of the put option which minimizes the risk at time t is given by:

$$E(e^{-r(T-t)}f(S_t)|F_t) = E\left(e^{-r(T-t)}f\left(S_t e^{(\mu-r-\frac{\sigma^2}{2})(t-s)+\sigma(W_t-W_s)}\prod_{j=N_t+1}^{N_T}(1+J_j)\right)\Big|F_t\right)$$

$$= E\left(e^{-r(T-t)}f\left(S_t e^{(\mu-r-\frac{\sigma^2}{2})(t-s)+\sigma(W_t-W_s)}\prod_{j=1}^{N_{T-t}}(1+J_j)\right)\right)$$

$$= E\left(P\left(t, S_t e^{-\lambda(T-t)E(J)}\prod_{j=1}^{N_{T-t}}(1+J_j)\right)\right)$$

where $P(t,x)$ is the function that gives the price of the option for the Black-Scholes model. As N_{T-t} is Poisson with parameter $\lambda(T-t)$,

$$E(e^{-r(T-t)}f(S_t)|F_t) = \sum_{n=0}^{\infty} E\left(P\left(t, S_t e^{-\lambda(T-t)E(J)}\prod_{j=1}^{n}(1+J_j)\right)\right)\frac{e^{-\lambda(T-t)}\lambda^n(T-t)^n}{n!}$$

Let us now assume that J takes values in $\{u,d\}$ and $P(J=u)=p$, $P(J=d)=1-p$. We will use the following:

Lemma 1: Let N be Poisson with parameter λ.
Let $S = \sum_{n=1}^{N} V_n$ with $P(V_n = u) = p$, and $P(V_n = d) = 1-p$. Then law(S) =law$(uN_1 + dN_2)$, where N_1 is Poisson λp and N_2 is Poisson $(\lambda(1-p))$.

Proof: One method would be to show that the two random variables have the same moment generating function.
Another method would be to re-write $S = \sum_{n=1}^{N}(u + (d-u)I_n)$, where $I_n = 0$ with probability p and $I_n = 1$ with probability $1-p$. So,

$$S = uN + (d-u)\sum_{n=1}^{N} I_n = uN_1 + dN_2,$$

because $\sum_{n=1}^{N} I_n$ is Poisson $(\lambda(1-p))$. This completes the proof of the lemma.
Now,

$$\prod_{j=1}^{N_{T-t}}(1+J_j) = \prod_{j=1}^{N_{T-t}} e^{ln(1+J_j)} = e^{\sum_{j=1}^{N_{T-t}} ln(1+J_j)},$$

and using the lemma we have $\sum_{j=1}^{N_{T-t}} ln(1+J_j)$ has the same law as
$ln(1+u)N_1 + ln(1+d)N_2$ where N_1 and N_2 are iid with parameters λp and $\lambda(1-p)$ respectively.

So, the price of the option at time t is given by:

$$\underbrace{\sum_{n_1,n_2}\sum_{k_1,k_2}}_{\alpha k_1+\beta k_2=\alpha n_1+\beta n_2} P\Big(t, S_0 e^{-\lambda(T-t)[pu+(1-p)d]} e^{ln(1+u)n_1+ln(1+d)n_2}\Big) e^{-\lambda}\frac{\lambda^{k_1+k_2}p^{k_1}(1-p)^{k_2}}{(k_1)!(k_2)!}$$

where $\alpha = ln(1+u)$ and $\beta = ln(1+d)$.
Replacing now the price of the put option in formula (2) we get the price for GMDB paid at the end of the year of death or in formula (3) we get the price of the GMDB for continuous time model, with benefit paid at the moment of death.
Most of the time, α and β are linearly independent over Z, so in this case the decomposition $\alpha n_1 + \beta n_2$ is unique, and the price of the option is given by:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P\Big(t, S_0 e^{-\lambda(T-t)[pu+(1-p)d]} e^{ln(1+u)n_1+ln(1+d)n_2}\Big) e^{-\lambda}\frac{\lambda^{n_1+n_2}p^{n_1}(1-p)^{n_2}}{(n_1)!(n_2)!}$$

2. Other market model with jumps

The problem of the price jumps can be analyzed in other models too. Another model could be described as follows: only one jump whose time occurance is uniformly distributed on the contract length. Let ω be the expiration date of the contract and T_j the random variable modeling the time of occurance of the jump. Let also T_d be the random variable that models the lifetime of the insurer. Let's assume for simplicity that T_d is exponential, i.e. $f_{T_d} = \lambda e^{-\lambda t}$.
The probability that the jump occurs before the death is

$$P(T_j < T_d) = \int_0^{\omega}\int_{t_j}^{\infty}\frac{1}{\omega}\lambda e^{-\lambda t_d}dt_d dt_j = \int_0^{\omega}\frac{1}{\omega}e^{-\lambda t_j}dt_j =$$

$$= \frac{1}{\omega}\frac{e^{-\lambda t_j}}{-\lambda}\Big|_0^{\omega} = \frac{1-e^{-\lambda\omega}}{\lambda\omega}$$

Let τ be the random time of the jump. Then,

$$\begin{cases} S_t = S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} & \text{for } t<\tau \text{ and} \\ S_t = S_0(1+J)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} & \text{for } t\geq\tau \end{cases}$$

As in the first model, the discounted price process is a martingale for specific jump processes and the GMDB price can be found similarly.

3. Risk analysis

We focus our attention now on a binomial tree model, and for simplicity we will assume that the price process (S_t) of a risky asset follows a simple random walk, going up one unit with probability 1/2 and down one unit with probability 1/2. For simplicity we will assume that $S_0 = 0$, using a translation of the random variable that models the stock price. Let $\tau_N = inf\{k \geq 0 : |S_k| = N\}$ be the first time the random walk is at distance N from the origin. If we think about the stock price, τ_N is the random time when S_n goes up or down N units, for the first time. Hence, τ_N can be interpreted as a measure of risk.
First, it is quite easy to show that the distribution of τ_N has an exponential tail and hence has moments of all orders. Let $T_N = inf\{k \geq 0 : S_k = N\}$. If $\omega \in \{\tau_N = n\}$, then $\omega \in \{T_N = n\} \cup \{T_{-N} = n\}$.
So $P(\tau_N = n) \leq P(T_N = n) + P(T_{-N} = n) \underbrace{=}_{symmetry} 2P(T_N = n)$.
So, $P(\tau_N = n) \leq 2P(T_N = n)$. Now, $P(T_N = n) = \sqrt{\frac{2}{\pi}}\frac{N}{\sqrt{n^3}}e^{-\frac{N^2}{2n}}$ by [1], hence the conclusion.
Next we want to look directly at $P(\tau_N = n)$. Let $k \in N$ be fixed, and $l \in N$ such that $-k \leq -l \leq l \leq k$.
Let y_l^n be the probability of a path from $(0,0)$ to $(n, \pm l)$ without passing through $\pm k$.

Note: (a, b) means getting to the value b at time a.
We have the following recurrence relations:

$$\begin{cases} y_0^n = \frac{1}{2} y_1^{n-1} \\ y_1^n = y_0^{n-1} + \frac{1}{2} y_2^{n-1} \\ \dots \\ y_l^n = \frac{1}{2} y_{l-1}^{n-1} + \frac{1}{2} y_{l+1}^{n-1}, \quad \text{for } 2 \leq l \leq k-2 \\ \dots \\ y_{k-1}^n = \frac{1}{2} y_{k-2}^{n-1} \\ y_k^n = \frac{1}{2} y_{k-1}^{n-1} \end{cases}$$

We want to find $p_n := y_k^n = \frac{1}{2} y_{k-1}^{n-1}$. Then we will take k=N and get the distribution of τ_N.

Let $\begin{pmatrix} y_0^n \\ y_1^n \\ . \\ . \\ . \\ y_{k-1}^n \end{pmatrix}$ be a column vector.

The recurrence relations can be written as:

$$y^n = A y^{n-1}$$

where A is a (k)x(k) matrix:

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{pmatrix}$$

Let $P_A(t) = det(tI - A)$. Let $B_k := tI - A$ and let $P_k := det(B_k)$. Then, using the last row of matrix B_k,

$$det B_k = det \begin{pmatrix} t & -\frac{1}{2} & 0 & \dots & 0 \\ -1 & t & -\frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & t & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -\frac{1}{2} & t \end{pmatrix} = -(-\frac{1}{2}) det(C) + t det(B_{k-1})$$

Then again, using the last row of matrix C,

$$det(C) = -(-\frac{1}{2}) det(D) + (-\frac{1}{2}) det(B_{k-2}) = \frac{1}{2} det(D) - \frac{1}{2} det(P_{k-2}).$$

But D has the last column 0, so $det(D) = 0$. Hence $det(C) = -\frac{1}{2} P_{k-2}$ and so

$$P_k = -\frac{1}{4} P_{k-2} + t P_{k-1} \tag{4}$$

The recurrence relation $P_k - tP_{k-1} + \frac{1}{4} P_{k-2} = 0$ has characteristic polynomial $x^2 - tx + \frac{1}{4}$. The roots for this polynomial are $x_{1,2} = \frac{t \pm \sqrt{t^2-1}}{2}$ and so,

$$P_k = \alpha_1 x_1^k + \alpha_2 x_2^k.$$

But $P_1 = t$, because $A = 0$ when $k = 1$ and $P_2 = t^2 - \frac{1}{2}$.
So we can identify $\alpha_1 = \alpha_2 = 1$. Hence:

$$P_k = x_1^k + x_2^k = \frac{1}{2^k} \left((t + \sqrt{t^2 - 1})^k + (t - \sqrt{t^2 - 1})^k \right).$$

Let $P(t) = (t+\sqrt{t^2-1})^k + (t-\sqrt{t^2-1})^k = 2^k P_k = 2^k P_A(t)$.
Then $P(A) = 2^k P_A(A) = 0$, by Cayley's theorem. If $P(t) = a_0 t^k + \cdots + a_k$, then

$$a_0 A^k + \cdots + a_k I = 0. \tag{5}$$

Next, let's multiply (5) to the right by y^{n-k}, which is a column vector, for $n \geq k$. We get

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_k y^{n-k} = 0$$

In particular, if we read only the last line we get:

$$a_0 y^n_{k-1} + a_1 y^{n-1}_{k-1} + \cdots + a_k y^{n-k}_{k-1} = 0, \ \forall\, n \geq k.$$

But $p_n = \frac{1}{2} y^{n-1}_{k-1}$, so we get the recurrence:

$$a_0 p_n + a_1 p_{n-1} + \cdots + a_k p_{n-k} = 0, \text{ for } n \geq k+1$$

Let now $Q(t) = a_0 + \cdots + a_k t^k$. Consider also the power series $S(t) = p_0 + p_1 t + p_2 t^2 + \ldots$
Let $Q(t)S(t) = c_0 + c_1 t + c_2 t^2 + \ldots$
For $n \leq k$, $c_n = a_0 p_n + a_1 p_{n-1} + \cdots + a_n p_0$.
For $n \geq k+1$, $c_n = a_0 p_n + a_1 p_{n-1} + \cdots + a_k p_{n-k} = 0$.
But as $p_0 = p_1 = \cdots = p_{k-1} = 0$, we get that $c_0 = c_1 = \cdots = c_{k-1} = 0$ and $c_k = a_0 p_k$.
Hence $Q(t)S(t) = c_k t^k = a_0 p_k t^k$ and so

$$S(t) = \frac{a_0 p_k t^k}{Q(t)} \tag{6}$$

We have:
$Q(t) = a_0 + a_1 t + \cdots + a_k t^k$, and
$P(t) = a_0 t^k + \cdots + a_k$.
These two polynomials are reciprocal and

$$Q(t) = t^k P(\frac{1}{t}) = t^k \left(\left(\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1} \right)^k + \left(\frac{1}{t} - \sqrt{\frac{1}{t^2} - 1} \right)^k \right)$$

$$(1+\sqrt{1-t^2})^k + (1-\sqrt{1-t^2})^k.$$

In particular, $a_0 = Q(0) = 2^k$. Also, $p_k = \frac{1}{2^{k-1}}$ (one gets to $\pm k$ after k steps iff there are $k\ +1's$ or $k\ -1's$, and any of these two events happen with probability $\frac{1}{2^k}$, hence $p_k = 2\frac{1}{2^k}$).
We then get $a_0 p_k = 2$, so

$$S(t) = \frac{a_0 p_k t^k}{Q(t)} = \frac{2t^k}{(1+\sqrt{1-t^2})^k + (1-\sqrt{1-t^2})^k}.$$

As a conclusion, $P(\tau_N = n)$ is the coefficient of t^n in the Taylor series of

$$S(t) = \frac{2t^N}{(1+\sqrt{1-t^2})^N + (1-\sqrt{1-t^2})^N} \tag{7}$$

4. Conclusions:
This paper has two parts. In the first part, a closed-form equation is deduced for the price of the GMDB option that minimizes the risk at issue. In the second part, we make a risk analysis in a binomial market model, by looking at the distribution of the first time a stock price goes up or down some fixed units of price.

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