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ACHIEVING CONSISTENCY BETWEEN INVESTMENT PRACTICE AND INVESTMENT ASSUMPTIONS FOR SINGLE PREMIUM NEW-MONEY PRODUCTS

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ABSTRACT

This paper investigates a technique for developing investment assumptions for pricing that are consistent with the investment practices of the company. An investment strategy is defined to be a specific allocation of investable funds among given representative fixed-income instruments. The investment strategy problem consists of two parts: how to invest funds initially received on a block of new issues and how to reinvest any excess of investment cash flow over the cash-flow needs of the block of business at later durations.

Since capital market conditions are known at the time of issue but are largely uncertain for the future, the initial-investment problem is considerably easier to analyze than is the reinvestment problem. Also, for single premium business, the amount of investable funds at issue is significantly larger than the amount of investable funds at later times. Thus, the initial-investment strategy is more important than the reinvestment strategy for single premium business.

This paper analyzes initial-investment strategies that produce asset-liability matching for specified durations. It shows how to express a subset of the entire region of feasible initial-investment strategies in a way that can be visualized easily and communicated to investment department officers. Practical applications of the theory to the pricing of single premium immediate annuities are given.

I. INTRODUCTION

SEVERAL papers have been written on the matching of assets and liabilities. One criticism often aimed at theoretical papers on this subject is that they ignore the realities of the actual operation of the investment department of an insurance company. Another criticism is that the investment strategies that result from the application of these theoretical approaches are so restrictive that they lead to uncompetitive rate structures. Furthermore, it is often claimed that investment vehicles with an appropriate pattern of contractual interest and principal repayment to effect a proper matching of assets and liabilities do not exist.

This paper attempts to examine the practical aspects of the relationship between actual investment strategies and investment assumptions for the pricing of single premium immediate annuities. The ideas presented also should find application in the pricing of other single premium products and of "accumulation" products to which investment income is allocated by an investment-generation method.

It is important to understand some of the difficulties in specifying investment assumptions for the pricing of new-money products. The following discussion centers on a new block of single premium immediate annuity business.

At the time business is written, the capital market conditions are known—in particular, the yield curves for various investment vehicles and the yield spreads between different types of investment. The investment department can formulate appropriate policy for the investment of funds arising from the single premiums less the state premium taxes, commissions, and acquisition expenses. The major difficulty in setting investment strategy and assumptions for pricing is the uncertainty of future money market conditions. A model of the block of new issues could be used to project the expected cash-flow requirements in each future year—annuity benefit payments, maintenance expenses, and federal income taxes. Ignoring bond calls, mortgage prepayments and refinancings, and the sale of assets prior to maturity, one could project the scheduled interest and principal payments from the portfolio of assets acquired through the initial investment of the single premiums less commissions, expenses, and taxes. Upon comparison of the cash-flow requirements of the block of annuity business with the investment cash flow from the portfolio of assets, it generally will be found that there is either a surplus or a deficiency of investment cash flow in each future year. Any surplus of investment cash flow over cash-flow requirements must be reinvested. However, the capital market conditions at the time of reinvestment are not known at issue, and neither economists nor actuaries currently possess the ability to predict interest rates accurately beyond a very short period.

What to do with a deficiency of investment cash flow compared with cash-flow requirements constitutes an even more difficult problem. Should the investment department liquidate assets to meet the deficiency? If so, which investments should be liquidated? If capital gains are realized as a result of liquidation in a period of declining interest rates, are they to be offset by liquidating other investments at a loss to minimize the federal income taxes on capital gains? How are capital gains and losses to be allocated equitably among blocks of business? In actuality, the flow of investable funds from the entire company's operations or from the entire line of single premium annuity business may be positive, so that it is possible to superimpose the deficiency arising in one block of business on the overall positive flow of investable funds. This is known as "constructive liquidation"—there is no liquidation of any actual asset, merely a decrease in the amount of investable funds.

Constructive liquidation can produce inequities in the allocation of investment income among blocks of business unless care is taken. Suppose that interest rates have fallen since the time of issue of a block of single premium immediate annuities and that this block is currently experiencing a negative flow of investable funds. If a constructive liquidation approach is used in lieu of actual liquidation, this block will be permitted to retain its high-yielding assets while effectively taking a low interest rate loan from the blocks of new business that are providing the net positive flow of investable funds. It might be argued that the new block of business should "lend" its funds to the old block at an interest rate in excess of the current new-money rate in recognition of the fact that the old block (and the entire company) is spared from having to incur a taxable capital gain and transaction costs for liquidation as well as the loss of high-yielding assets. Alternatively, it could be argued that new business should not expect to earn more by investing in an old block of business than through conventional fixed-income instruments. If the latter philosophy is adopted, it makes no difference to the new block of business whether constructive liquidation or actual liquidation is utilized, since only the old block is affected.

The previous considerations indicate some of the difficulties encountered in deciding upon investment assumptions for single premium immediate annuities. Since most business is written on a nonparticipating basis, there is no opportunity to adjust payments to annuitants for differences in actual investment performance from that assumed in setting the annuity rates. Nevertheless, it is possible to reduce the importance of the *reinvestment* strategy by adopting for the *initial-investment* strategy an approach known as "matching." Matching essentially means choosing an initial asset mix and maturity distribution so that in any year the investment cash flow (after investment expenses) from the resulting portfolio of assets equals or exceeds the cash-flow needs of the block of annuity issues. The theoretical idealization in which these two cash flows are exactly equal is known as "absolute matching." The following sections of this paper investigate a practical approach for matching asset and liability cash flows for blocks of single premium annuity business.

II. DETERMINING INVESTMENT STRATEGY

In determining how funds are to be invested, the insurance company's investment department will make commitments to various borrowers to lend specified amounts of money at certain times in the future, based on projections of investable funds arising from insurance operations and of investment cash flow. One of the functions of the investment department is to determine the allocation of investable funds among the various classes of assets, and within each class among credit-risk/maturity-date categories. The primary objectives of investment policy are to satisfy the liquidity needs of the insurance operation and to optimize overall yield at an acceptable level of risk. A further objective is to achieve consistency between the maturity structure of the asset portfolio and the terms of the company's obligations under the contracts it issues. Absolute matching of the investment cash flow with the cash-flow requirements of the insurance operation fulfills the third objective completely; however, absolute matching is unattainable for practical investment operations and, even if attainable, would probably be undesirable, since it is so restrictive that uncompetitive rates would result.

Since fixed-income investments provide contractual payments of principal and interest at specified intervals, they are used to back the obligations under conventional fixed-income annuities. Only mortgages and bonds will be considered in this paper. The pattern of principal and interest payments is fundamental to achieving a matching of assets and liabilities, so this is discussed first.

Mortgages typically require periodic level payments (comprised of principal and interest), with the size of the level payment determined by the contractual interest rate, the amount of the initial loan, and the period of amortization of the loan. There is often a maturity date before the expiration of the amortization period, at which point a "balloon" payment of the remaining outstanding principal is made. In current markets, ten-year protection against prepayment of the mortgage is common, with the penalty for prepayment assessed as a percentage of the outstanding principal. The penalty actually forms a decreasing scale of percentages, highest at the date of earliest prepayment and zero at the maturity date.

Public bonds commonly provide semiannual coupons. Full principal repayment occurs at the maturity date. The call-protection period is usually five or ten years, the latter associated with the longer-maturity public bonds. Call premiums are assessed as a percentage of the par value of the bond, according to a scale that decreases from its highest value at the date of earliest call to zero at maturity.

The provisions of a private placement bond are somewhat flexible and are usually negotiated by the lender and the borrower. A particular issue may be so large that several insurance companies participate. The contractual provisions currently found in typical private placement bond indentures are as follows:

1. There is often a moratorium on repayment of principal for a specified number of years, followed by provision for equal principal repayments at par (regardless of the level of new-money rates at time of payment) each year thereafter until the final maturity date.
2. As with public bonds, coupons usually are paid semiannually at a stated rate applied to the outstanding principal.
3. Ten-year call protection is common.
4. Call premiums are assessed as a percentage of outstanding principal according to a decreasing scale similar to that described for public bonds.

Practicality is built into the theoretical treatment that follows. Investment department officers are allowed to decide upon an appropriate number of investment "cells" that represent typical investments that exist in sufficient quantity to be acquired readily. For example, there may be n_1 mortgage cells and n_2 bond cells for a total of $n_1 + n_2 = n$ cells. Within each broad class, the different cells would have variations in the parameters that define the pattern of principal and interest payments: contractual interest rate, amortization period, maturity date, and so on. The analysis presented in this section assumes that there are no calls, sales, or prepayments prior to the maturity date; in real life, of course, these will occur. Because of the preponderance of call provisions in bond indentures and prepayment clauses in mortgage agreements, it is unrealistic to attempt to extend an initial-investment matching strategy beyond fifteen years. (Matching can be achieved after the fifteenth year by taking account of the investment cash flow arising from the *reinvestment* of funds during the first fifteen years.) If a matching strategy can be achieved for the first fifteen years by using only assets that have at least a ten-year call protection, it can be assumed that any calls or prepayments occurring from the eleventh to the fifteenth year can be reinvested in such a way that matching or near-matching for this five-year period still exists. Practical applications of the theory presented in this paper have borne out this assumption.

An investment strategy is defined to be an n-component vector $(p_1, p_2, \ldots, p_n)$ that specifies the partition of one dollar of investable funds among the n cells into amounts of $p_1, p_2, \ldots, p_n$, respectively, where $\Sigma_{j=1}^{n} p_j = 1$. For each investment cell j, let a_{ij} represent the total amount of scheduled principal and interest payments in year i after acquisition. Any payments to be received during year i are assumed to be reinvested to the end of the year at a specified short-term interest rate. The investment cash flow in year i arising from the initial investment of one dollar is thus

$$A_i = \sum_{j=1}^{n} a_{ij} p_j . \tag{1}$$

A model office can be used to project annual cash outflow from a block of annuity contracts on the basis of assumptions about the volume (amount of single premium) and the distribution of new business by issue-age/sex/contract-type cell. The initial amount of investable funds is assumed to be the total single premiums received, less all state premium taxes, commissions, and acquisition expenses. If the model office assumes that the block of issues is written over a one-year period, the premiums, premium taxes, commissions, and acquisition expenses would be discounted at the short-term interest rate to the beginning of the first year. Similarly, in the determination of the annual cash-flow demands of the block of annuity business, any benefits or maintenance expenses incurred during the year are accumulated to the year-end at the short-term interest rate. The division of the annual cash-flow requirements by the initial amount of investable funds results in a vector of annual cash-flow requirements per initial dollar invested, $\{B_i\}$.

The mathematical statement of a "matching" strategy is

$$A_i \geq B_i \tag{2}$$

for some specified set of years i. In the following discussion it is assumed that the matching condition is applicable to m years, and these are labeled 1, . . . , m for convenience, even though it may not be intended that matching apply to the first m consecutive years. The solution of the investment problem is the $(n-1)$-dimensional region R of feasible investment strategies that consists of all points $P = (p_1, \ldots, p_n)$ satisfying the constraints

$$p_j \geq 0 , \qquad 1 \leq j \leq n ; \tag{3a}$$

$$\sum_{j=1}^{n} p_j = 1 ; \tag{3b}$$

$$\sum_{j=1}^{n} a_{ij} p_j \geq b_i , \quad 1 \leq i \leq m . \tag{3c}$$

Expression (3c) is derived by substituting equation (1) in expression (2) and making the change of notation $b_i \equiv B_i$ for all i.

All constraints defining the region R are linear in the variables p_j. Constraints (3a) and (3b) ensure that R is bounded. Constraints (3*c*) will be referred to hereafter as the matching constraints. Let H_0 denote the $(n-1)$-dimensional plane defined by constraint (3b). Region R is bounded by a subset of all the $(n-2)$-dimensional planes formed by the intersections of H_0 with the boundary planes defined by the nonnegativity and the matching constraints. Region R is known as a polytope. As a specific example of these statements, a possible region R is shown in Figure 1 for a three-dimensional problem. For this example, H_0 is an

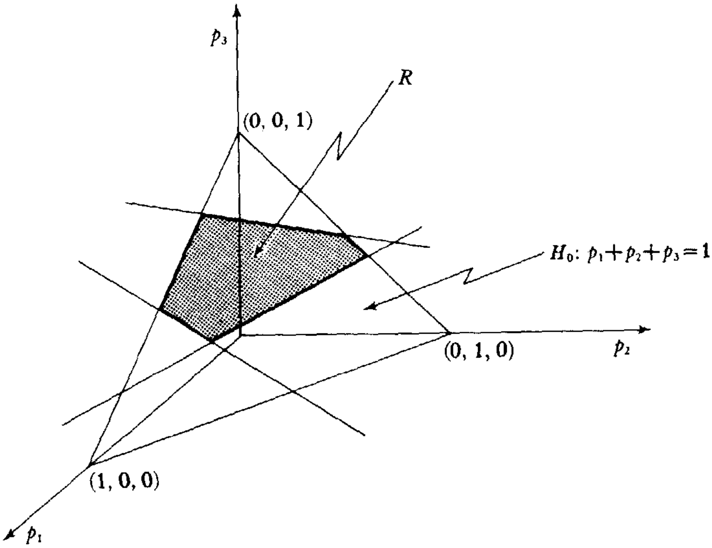

FIG. 1.—Investment-strategy geometry for three investment cells. The five-sided polygon R lying in the plane $p_1 + p_2 + p_3 = 1$ is bounded by three matching constraints and two of the three nonnegativity constraints.

ordinary two-dimensional plane, and R is a polygon bounded by five straight lines (one-dimensional planes).

Unless the investment strategy is restricted further by demanding that yield be optimized, the region R, if it is nonempty, is the solution of the investment problem. This is the essence of the approach adopted in this paper: to allow the investment department to define representative investment cells, to allow the actuary to specify the years for which a matching constraint is to be applied, and then to determine a region of feasible investment strategies that meets these constraints, if one exists. It is important to present the solution as a region of feasible strategies rather than as a single strategy or a small number of point strategies because the investment department may not actually be able to meet a particular strategy but may be able to acquire a portfolio of assets falling within a specified region. The trouble with claiming success as soon as the polytope R is defined is that a polytope is difficult to visualize. It would be impossible to communicate the solution to the investment department or perhaps even to other actuaries. Moreover, it is desirable to perform a sensitivity analysis in which the mortality or expense assumptions underlying the cash flows b_i are changed. The resulting region R also would change, but since R is difficult to visualize it would be hard to characterize the change as a distortion, compression, expansion, translation, or rotation (or some combination of these) of the original region. Even though the polytope defined by the constraints (3a), (3b), and (3c) represents the solution of the investment problem, it is not a useful solution.

Suppose that, in lieu of the polytope R, the boundary and interior of the largest $(n - 1)$-dimensional sphere S that can be inscribed in R is used as the solution of the investment problem. Some of the investment strategies contained in R are not contained in S, so S represents only a part of the full solution. However, S is much easier to visualize. A sphere is characterized completely by a center point and a radius. The results of the sensitivity analysis mentioned previously can be understood in terms of the translation of the center of the sphere and an expanding or shrinking of its radius.

It is probably true that the officers of the investment department would understand a sphere no more easily than a polytope. Instead of examining region R and its maximal in-sphere S, it may be useful to look at the region R' obtained by projecting R onto the coordinate plane $p_n = 0$. Let S' be the maximal in-sphere of R'. By inscribing the largest $(n - 1)$-dimensional hypercube in S', it is possible to present the investment strategy in a very understandable fashion. Let $r'_{\max}$ and $P'_0 \equiv (p'_{01}, \ldots, p'_{0,n-1}, 0)$ denote the radius and center, respectively, of the sphere S'. The investment department should attempt to invest annuity funds as follows:

$$p'_{0j} - \frac{r'_{\max}}{\sqrt{(n-1)}} \leq p_j \leq p'_{0j} + \frac{r'_{\max}}{\sqrt{(n-1)}}, \quad j = 1, 2, \ldots, n-1;$$
$$p_n = 1 - \sum_{j=1}^{n-1} p_j . \tag{4}$$

The fraction of annuity funds to be invested in cell j ($j = 1, 2, \ldots, n - 1$) must lie within the range shown in formula (4), but can be chosen *independently* of the fractions invested in all other cells. Once it has been decided how much is to be invested in cells 1 through $n - 1$, the balance of investable funds goes into cell n. Of course, the hypercube of feasible investment strategies defined by (4) contains even fewer strategies than does the sphere S', but it is simple to communicate and easy to understand.

One step that remains to be done in this section is to exhibit the method of finding the largest sphere S that can be inscribed in the region R defined by the constraints (3). Since the sphere is a quadratic surface, this may appear to be a difficult problem. As will be shown, however, the radius and center of the maximal in-sphere can be obtained as the solution of a standard linear programming problem.

For a given center point $P_0 = (p_{01}, \ldots, p_{0n})$ lying on the plane H_0, the largest $(n-1)$-dimensional sphere $S(P_0)$ lying within R can be found by geometrical construction. Let H_i denote the $(n-1)$-dimensional planes $p_i = 0$ for $1 \leq i \leq n$, and let H_{n+i} denote the $(n-1)$-dimensional planes $\Sigma_{j=1}^{n} a_{ij}p_j = b_i$ for $1 \leq i \leq m$. Let T_i represent the $(n-2)$-dimensional plane formed by the intersection of H_0 with H_i for $1 \leq i \leq n + m$. Construct lines from P_0 perpendicular to each of the planes T_i, and let these lines be of length $l_i(P_0)$. As shown in Appendix I, $l_i(P_0)$ is a *linear* function of P_0, that is,

$$l_i(P_0) = \sum_{j=1}^{n} \alpha_{ij} p_{0j} - \beta_i \, , \quad 1 \leq i \leq n + m \, , \tag{5}$$

for α_{ij} and β_i independent of P_0. The largest $(n-1)$-dimensional sphere S lying within the region R and having given center P_0 has radius $r(P_0)$ given by

$$r(P_0) = \min l_i(P_0) \, , \quad 1 \leq i \leq n + m \, . \tag{6}$$

The largest $(n-1)$-dimensional sphere S inscribed in the region R has the center P_0 that maximizes the function $r(P_0)$, subject to the constraint $P_0 \in R$. This is a max-min problem, and it is straightforward to convert it into a standard linear programming problem. Let r_{max} denote the radius of the largest sphere S. Then

$$r_{max} = \max_{P_0 \in R} \{ \min_{1 \leq i \leq n+m} l_i(P_0) \} \, . \tag{7}$$

If we let

$$y = \min_{1 \leq i \leq n+m} l_i(P_0) \, ,$$

it is clear that

$$y \leq l_i(P_0) = \sum_{j=1}^{n} \alpha_{ij} p_{0j} - \beta_i \, , \quad 1 \leq i \leq n + m \, ,$$

which can be rewritten as

$$\sum_{j=1}^{n} \alpha_{ij} p_{0j} - y \geq \beta_i \, , \quad 1 \leq i \leq n + m \, . \tag{8}$$

The linear programming problem equivalent to the original problem is the following.

Maximize y, subject to

$$y \geq 0\,, \quad p_{0j} \geq 0\,, \qquad 1 \leq j \leq n\,; \tag{9a}$$

$$\sum_{j=1}^{n} p_{0j} = 1\,; \tag{9b}$$

$$\sum_{j=1}^{n} a_{ij} p_{0j} \geq b_i\,, \qquad 1 \leq i \leq m\,; \tag{9c}$$

$$\sum_{j=1}^{n} \alpha_{ij} p_{0j} - y \geq \beta_i\,, \qquad 1 \leq i \leq n + m\,. \tag{9d}$$

The number of constraints can be reduced even further when the explicit form of $l_i(P_0)$ is considered. Let C_i represent the 2×2 matrix, defined in equation (A5) of Appendix I, that results from the problem of finding $l_i(P_0)$ for $1 \leq i \leq n + m$.

$$\left.\begin{aligned} C_i &= \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} \\ C_i^{-1} &= \begin{bmatrix} \dfrac{1}{n-1} & \dfrac{-1}{n-1} \\ \dfrac{-1}{n-1} & \dfrac{n}{n-1} \end{bmatrix} \end{aligned}\right\}, \qquad 1 \leq i \leq n\,; \tag{10a}$$

$$\left.\begin{aligned} C_{n+i} &= \begin{bmatrix} n & \sum_{j=1}^{n} a_{ij} \\ \sum_{j=1}^{n} a_{ij} & \sum_{j=1}^{n} a_{ij}^2 \end{bmatrix} \\ C_{n+i}^{-1} &= \begin{bmatrix} \dfrac{\sum_{j=1}^{n} a_{ij}^2}{D_i} & \dfrac{-\sum_{j=1}^{n} a_{ij}}{D_i} \\ \dfrac{-\sum_{j=1}^{n} a_{ij}}{D_i} & \dfrac{n}{D_i} \end{bmatrix} \\ D_i &= n \sum_{j=1}^{n} a_{ij}^2 - \left(\sum_{j=1}^{n} a_{ij} \right)^2 \end{aligned}\right\}, \qquad 1 \leq i \leq m\,. \tag{10b}$$

Using equations (10) and equation (A12) of Appendix I, the expressions for $l_i(P_0)$ can be derived.

$$\begin{aligned} l_i(P_0) &= \left(\frac{n}{n-1}\right)^{1/2} |p_{0i}| \\ &= \left(\frac{n}{n-1}\right)^{1/2} p_{0i}, \quad 1 \le i \le n, \quad P_0 \in R; \end{aligned} \tag{11a}$$

$$\begin{aligned} l_{n+i}(P_0) &= \left(\frac{n}{D_i}\right)^{1/2} \left| \sum_{j=1}^{n} a_{ij} p_{0j} - b_i \right| \\ &= \left(\frac{n}{D_i}\right)^{1/2} \left(\sum_{j=1}^{n} a_{ij} p_{0j} - b_i \right), \quad 1 \le i \le m, \quad P_0 \in R. \end{aligned} \tag{11b}$$

It is useful to define new constants a'_{ij} and b'_i in terms of a_{ij} and b_i as follows:

$$a'_{ij} \equiv \left(\frac{n}{D_i}\right)^{1/2} a_{ij}, \quad 1 \le i \le m, \quad 1 \le j \le n; \tag{12a}$$

$$b'_i \equiv \left(\frac{n}{D_i}\right)^{1/2} b_i, \quad 1 \le i \le m. \tag{12b}$$

The linear programming problem stated in (9) now can be restated in terms of the new coefficients a'_{ij} and b'_i as follows:

Maximize y, subject to

$$y \ge 0, \quad p_{0j} \ge 0, \qquad 1 \le j \le n; \tag{13a}$$

$$\sum_{j=1}^{n} p_{0j} = 1; \tag{13b}$$

$$\sum_{j=1}^{n} a'_{ij} p_{0j} \ge b'_i, \qquad 1 \le i \le m; \tag{13c}$$

$$p_{0i} - \left(\frac{n-1}{n}\right)^{1/2} y \ge 0, \quad 1 \le i \le n; \tag{13d}$$

$$\sum_{j=1}^{n} a'_{ij} p_{0j} - y \ge b'_i, \qquad 1 \le i \le m. \tag{13e}$$

If constraints (13e) are satisfied, then constraints (13c) also are satisfied and are thus redundant. By eliminating the redundant constraints, the problem finally can be stated as follows:

Maximize y, subject to

$$y \ge 0, \quad p_{0j} \ge 0, \qquad 1 \le j \le n; \tag{14a}$$

$$\sum_{j=1}^{n} p_{0j} = 1; \tag{14b}$$

$$p_{0j} - \left(\frac{n-1}{n}\right)^{1/2} y \geq 0 , \quad 1 \leq j \leq n ; \tag{14c}$$

$$\sum_{j=1}^{n} a'_{ij} p_{0j} - y \geq b'_i , \quad 1 \leq i \leq m . \tag{14d}$$

Actually, if constraints (14c) are satisfied and $y \geq 0$, it is necessarily true that $p_{0j} \geq 0$ for $1 \leq j \leq n$. The full form of (14a) is retained, however, since it is customary to list the nonnegativity constraints for a linear programming problem. The point $P_0 = (p_{01}, \ldots , p_{0n})$ that maximizes y is the center of the sphere, and the value of the objective function, y, at this extreme point is the radius of the maximal sphere, $r_{\max}$.

III. INVESTMENT ASSUMPTIONS FOR PRICING

The purpose of this section is to show how the results of Section II can be applied to the pricing of single premium immediate annuities.

The pricing theory presented in the preceding paper in this volume of the *Transactions* requires investment assumptions in the form of new-money interest rates by policy year and vectors of investment rollover rates for both the initial investment and subsequent reinvestments. The pricing described in the preceding paper can be carried out in conjunction with the determination of investment strategy described in this paper. The first step is to have the investment department decide upon representative investment cells and the approximate fraction of investable funds currently being allocated to each of these investment cells. Let these fractions be denoted by $(p_1^{(0)}, \ldots , p_n^{(0)})$ for the n investment cells. The pricing proceeds as follows:

1. Using $p_j^{(0)}$ as the weight for the net annual yield $i_{1;j}$ associated with cell j, compute a composite net new-money rate i_1 as

$$i_1 = \sum_{j=1}^{n} p_j^{(0)} i_{1;j} . \tag{15}$$

The composite net new-money rates i_t for $t \geq 2$ can be obtained by grading down i_1 to some chosen ultimate level after a specified number of years (see Sec. III, A, 2, of the preceding paper). Let v_{ij} be the fraction of the principal repaid in year i for investment cell j. The rollover rates for the initial investment are equal to

$$r_i^{(0)} = \sum_{j=1}^{n} p_j^{(0)} v_{ij} , \tag{16}$$

where $r_i^{(0)}$ represents the fraction of the assets purchased at the beginning of the first policy year that will mature or be called or sold at the beginning of the $(i + 1)$st policy year. This is the same notation as used in Section II, E, of the preceding paper in this volume. Even though the theory derived in Section II of this paper assumes no call or sale of bonds, and no prepayment of mortgages, v_{ij} can reflect assumptions with respect to calls, sales, and prepayments. For the initial pricing, it is assumed that the rollover rates appropriate to the reinvestment of funds are the same as those appropriate to the initial investment, namely, $r^{(1)} = r^{(0)}$, using the notation of the preceding paper.

2. The method of the preceding paper is used to determine the annuity payout rate per \$1,000 of single premium, R_m, that meets the profit objectives of the company. This rate is then used in a model-office projection of a block of annuity business to determine the normalized cash-flow requirement vector $\{b_i\}$ as outlined in Section II of this paper.
3. The technique described in Section II is used to find the maximal in-sphere S of investment strategies meeting the asset-liability matching constraints for specified years, say, years

The center point of the sphere obtained in step 3, $(p_1^{(1)}, \ldots, p_n^{(1)})$, is used in place of $(p_1^{(0)}, \ldots, p_n^{(0)})$ in step 1 to compute revised new-money interest rate assumptions, i_t, and vectors of rollover rates, $r^{(0)}$ and $r^{(1)}$. Step 2 is performed to determine a new annuity payout rate from the pricing and a new cash-flow requirement vector from the model office. Step 3 then produces a new maximal sphere, with center $(p_1^{(2)}, \ldots, p_n^{(2)})$. This iterative process is terminated when a reasonable degree of convergence is attained with respect to the annuity payout rate, and the location of the center and the length of the radius of the maximal in-sphere of investment strategies. Figure 2 is a flowchart of the complete process of pricing and determination of investment strategy.

The description of the procedure for pricing single premium immediate annuities consistently with the practices of the investment department (steps 1–3) has glossed over a few points. First, as will be seen in Section IV, it generally is not possible to match asset and liability cash flows for years 1–15, even though it is possible to match them for years 1–10 or years 6–15. Second, no mention has been made of how to determine the reinvestment strategy embodied in the vector of rollover rates $r^{(1)}$. Third, step 2 suggests running a separate model-office projection for each issue-age/sex/contract-type pricing cell. Perhaps a better approach would be to price annuity rates R_m for all pricing cells and then to run a single model-office projection based on an assumed distribution of new business by issue-age/sex/contract-type cell. The first and second points will be discussed in the next section. The balance of this section is devoted to a consideration of the third point.

It is theoretically interesting to determine a separate maximal in-sphere of investment strategies for each issue-age/sex/contract-type pricing cell. If there is a nonempty intersection of all the separate maximal in-spheres, the single investment strategy adopted for single premium immediate annuity funds might be required to lie in this intersection. Then, regardless of the distribution of new business by pricing cell, the overall investment strategy would result in asset-liability matching. However, it might very well happen that the various spheres do not all intersect, or that, even if they do, the intersection is a small region. If the former situation occurs, it is not clear how to establish the investment policy for annuity funds. If the latter situation occurs, the intersection

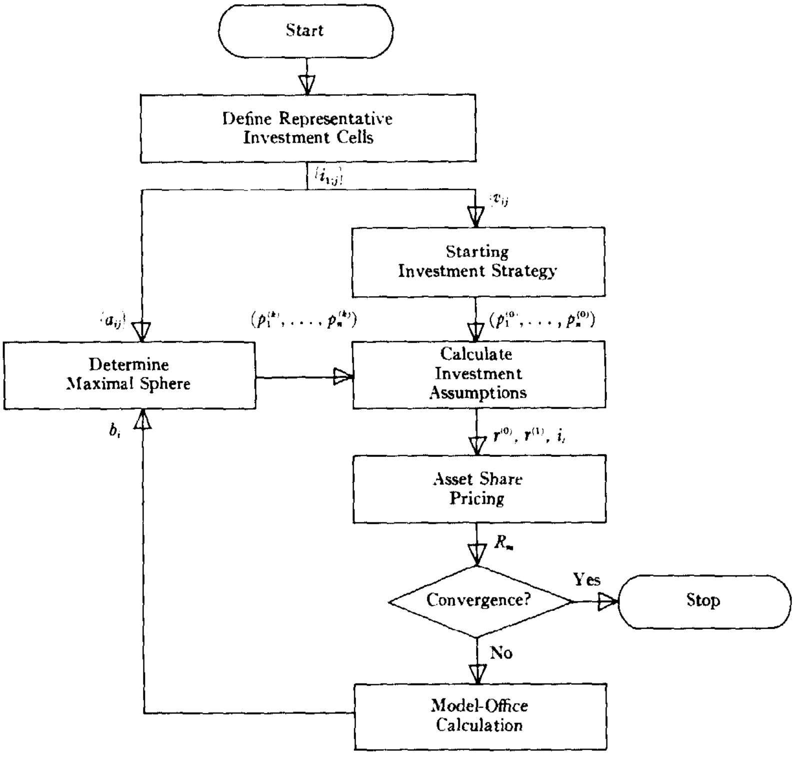

FIG. 2.—Flowchart for the complete process of pricing and determination of investment strategy. The variables shown in the flowchart represent output from the immediately preceding step and input for the succeeding step(s) in the process. The flowchart assumes that a nonempty set of investment strategies exists.

may be so small that there is not sufficient latitude in the investment strategy to be practical. Moreover, the region defined by the intersection itself is not a sphere and is as difficult to visualize as is a polytope. Thus, the most practical approach may be to run only a single model office for the combined business in all pricing cells. A single maximal in-sphere results from this procedure. At those points within this single sphere that lie outside the maximal in-sphere for a particular pricing cell, matching may not exist for that pricing cell. It cannot be certain that matching does not exist, because the full region of feasible matching strategies for the particular pricing cell is a polytope that includes points lying outside the maximal in-sphere. This situation is depicted in two dimensions in Figure 3.

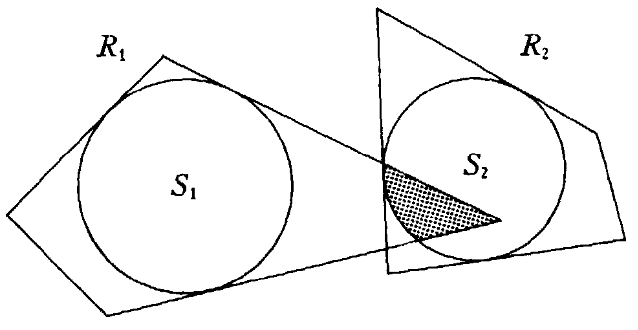

FIG. 3.—R_1 is the polytope of feasible investment strategies for a particular issue-age/sex/contract-type pricing cell, S_1 is the maximal in-sphere of R_1, R_2 is the polytope of feasible investment strategies for a combined model office of all pricing cells, and S_2 is the maximal in-sphere of R_2. The polytopes R_1 and R_2 intersect, but the spheres S_1 and S_2 do not. The shaded region represents points within S_2 that are feasible matching strategies for the separate pricing cell even though they do not lie within S_1.

IV. SAMPLE CALCULATIONS

In this section the method described in Section III is applied to the calculation of annuity rates and investment strategies. The mortality and expense assumptions used in both the pricing and the model-office projection are those described in Section III of the preceding paper in this volume of the *Transactions*. The model office is used to make separate projections of one year's issues of single premium immediate annuity business for each pricing cell. It is assumed that new business is written at the middle of each month of the first calendar year, with an equal amount of single premium written each month. Federal income tax for each year is based solely on the taxable investment income for the block of business. Only non-tax-qualified business is studied. Profit objectives for the pricing and the basis used for statutory reserves are described in Section III of the preceding paper. Annuity rates are determined from the pricing model described in the preceding paper, except for certain differences in the calculation of the federal income tax and contingency reserves.

Six representative investment cells are used, three for private placement bonds and three for mortgages. The parameters specifying the contractual provisions of each type of investment are listed in Tables 1 and 2. For the reasons stated in Section II, when determining feasible in-

TABLE 1

INVESTMENT CELLS 1, 2, AND 3: PRIVATE PLACEMENT BONDS

	CELL INDEX		
	1	2	3
1. Period to final maturity	10 years	15 years	20 years
2. Moratorium on principal repayment	0 year	5 years	5 years
3. Call protection	10 years	10 years	10 years
4. Percentage called after call protection expires	0%	10%	10%
5. Call premium as a percentage of outstanding principal	0%	3.0%	4.5%
6. Contractual coupon rate, convertible semiannually	8.55%	8.80%	8.80%
7. Reduction in effective annual yield for investment expenses	0.20%	0.20%	0.20%

TABLE 2

INVESTMENT CELLS 4, 5, AND 6: MORTGAGES

	CELL INDEX		
	4	5	6
1. Amortization period	25 years	30 years	30 years
2. Period to maturity	15 years	20 years	15 years
3. Prepayment protection	10 years	10 years	10 years
4. Percentage prepaid after prepayment protection expires	30%	30%	30%
5. Prepayment penalty as a percentage of outstanding principal	5%	5%	5%
6. Contractual interest rate, convertible monthly	9.25%	9.25%	9.25%
7. Reduction in effective annual yield for investment expenses	0.40%	0.40%	0.40%

vestment strategies one assumes that there are no bond calls or mortgage prepayments. However, as mentioned in Section III, it is proper to make assumptions about calls and prepayments when determining the vector of rollover rates to be used in the pricing. The assumptions concerning calls and prepayments are shown in Tables 1 and 2, respectively. It has been assumed that the bond calls and mortgage prepayments take place as soon as the respective protection periods expire and that the listed call premiums and prepayment penalties are appropriate to call or prepayment at that time.

Mortgage cells 5 and 6 produce identical principal and interest payments through the fifteenth year, except for the "balloon" payment at the end of the fifteenth year for cell 6. Investment strategies are determined by using only cells 1–5, but rollover rates are based on all six cells. Before determination of the rollover rates, the assumption is made that, of all investments in mortgages with thirty-year amortization periods, 50 percent are in mortgages having a balloon payment after fifteen years and 50 percent are in mortgages having a balloon payment after twenty years. This is in accord with the actual investment practices observed by the author.

The matrix $\{v_{ij}\}$ of principal repayment in years $1 \leq i \leq 20$ for investment cells $1 \leq j \leq 6$, *including* bond calls and mortgage prepayments, appears in Table 3. The matrix $\{a_{ij}\}$ of interest and principal payments in years $1 \leq i \leq 15$ for investment cells $1 \leq j \leq 5$, *excluding* bond calls and mortgage prepayments, is shown in Table 4. The actual values of $\{v_{ij}\}$ and $\{a_{ij}\}$ used in the computer programs are carried to more decimal places than are shown in Tables 3 and 4. It will be noticed in Table 4 for the columns labeled "Cell 4" and "Cell 5" that the sum of the principal and interest payments is not quite level by year. This arises because the level mortgage payment is based on a contractual rate before accounting for investment expenses. A fixed proportion of each interest payment is then deducted for investment expenses. When the net interest payment is added to the principal repayment, this "reconstituted" amount increases slightly each year. For each investment cell, principal and interest payments made during the year have been accumulated to the end of the year at a short-term interest rate of 4 percent. This applies to $\{a_{ij}\}$, which is used for determining investment strategies, but not to $\{v_{ij}\}$, which is used only for determining rollover rates corresponding to a given investment strategy.

The net annual yield for each cell is computed by converting the contractual interest rates shown in line 6 of Tables 1 and 2 to effective annual rates and then subtracting the amount for investment expenses

TABLE 3

PRINCIPAL REPAYMENT PATTERN
MATRIX $\{v_{ij}\}$

Year i	Investment Cell j					
	1	2	3	4	5	6
1	0.1	0	0	0.0107	0.0065	0.0065
2	0.1	0	0	0.0117	0.0071	0.0071
3	0.1	0	0	0.0129	0.0078	0.0078
4	0.1	0	0	0.0141	0.0086	0.0086
5	0.1	0	0	0.0155	0.0094	0.0094
6	0.1	0.1	0.06*	0.0170	0.0103	0.0103
7	0.1	0.1	0.06	0.0186	0.0113	0.0113
8	0.1	0.1	0.06	0.0204	0.0124	0.0124
9	0.1	0.1	0.06	0.0224	0.0136	0.0136
10	0.1	0.15	0.13	0.2742	0.2844	0.2844
11	0	0.09	0.06	0.0188	0.0114	0.0114
12	0	0.09	0.06	0.0207	0.0125	0.0125
13	0	0.09	0.06	0.0227	0.0137	0.0137
14	0	0.09	0.06	0.0248	0.0151	0.0151
15	0	0.09	0.06	0.4955	0.0165	0.5760
16	0	0	0.06	0	0.0181	0
17	0	0	0.06	0	0.0198	0
18	0	0	0.06	0	0.0218	0
19	0	0	0.06	0	0.0239	0
20	0	0	0.06	0	0.4760	0

* 0.06 ≅ 0.0666 . . . ; 0.13 ≡ 0.1333 . . . .

TABLE 4

INVESTMENT CASH-FLOW PATTERN
MATRIX $\{a_{ij}\}$

Year i	Investment Cell j				
	1	2	3	4	5
1	0.1843	0.0869	0.0869	0.1006	0.0965
2	0.1759	0.0869	0.0869	0.1006	0.0965
3	0.1675	0.0869	0.0869	0.1007	0.0965
4	0.1590	0.0869	0.0869	0.1007	0.0965
5	0.1506	0.0869	0.0869	0.1008	0.0966
6	0.1422	0.1869	0.1535	0.1009	0.0966
7	0.1337	0.1782	0.1477	0.1009	0.0967
8	0.1253	0.1695	0.1419	0.1010	0.0967
9	0.1169	0.1608	0.1361	0.1011	0.0968
10	0.1084	0.1521	0.1304	0.1012	0.0968
11	0	0.1434	0.1246	0.1013	0.0969
12	0	0.1347	0.1188	0.1014	0.0970
13	0	0.1261	0.1130	0.1015	0.0970
14	0	0.1174	0.1072	0.1017	0.0971
15	0	0.1087	0.1014	0.1018	0.0972

shown in line 7. This results in rates of 8.53 percent for cell 1, 8.79 percent for cells 2 and 3, and 9.25 percent for cells 4, 5, and 6. The investment strategy assumed initially is shown in column 2 of Table 5. These are the fractions ($p_1^{(0)}, \ldots, p_n^{(0)}$) defined in Section III. Using equation (15), an initial net new-money rate of 8.96 percent results. This is rounded to 9 percent for convenience. The new-money rates are graded by policy year, as discussed in Section III, A, 2, of the preceding paper. Rollover rates for the first-iteration pricing are obtained by using equation (16) and the data from Table 3 and column 2 of Table 5.

The results of the first-iteration pricing appear in Table 6 for each of

TABLE 5

INVESTMENT STRATEGIES FOR FIRST- AND SECOND-ITERATION PRICINGS

Cell (1)	Investment Strategy: First Iteration (2)	Investment Strategy: Second Iteration (3)
1	0.06	0.20
2	0.33	0.35
3	0.21	0.15
4	0.20	0.15
5	0.10	0.075
6	0.10	0.075
Total	1.00	1.000

TABLE 6

MONTHLY ANNUITY PAYOUT RATES FOR FIRST- AND SECOND-ITERATION PRICINGS

Pricing Cell*	Monthly Income per $1,000 of Single Premium: First Iteration	Monthly Income per $1,000 of Single Premium: Second Iteration
A	$8.911	$8.887
B	8.172	8.150
C	7.734	7.712
D	7.463	7.442

* Pricing cell A: male age 65, straight-life annuity; Pricing Cell B: male age 65, guaranteed-10 annuity; Pricing Cell C: female age 65, straight-life annuity; Pricing Cell D: female age 65, guaranteed-10 annuity.

the four pricing cells. These monthly payout rates per $1,000 of single premium are used in the model-office projection. The model office produces normalized cash-flow vectors $\{b_i\}$, $1 \leq i \leq 15$, for each of the four pricing cells. Annuity benefits and maintenance expenses incurred during the calendar year are accumulated to the end of the year at a short-term interest rate of 4 percent. The first-iteration cash-flow vectors are shown in Table 7.

TABLE 7

MODEL-OFFICE NORMALIZED CASH FLOW

(Annuity Benefits, Maintenance Expenses, and Federal Income Taxes)

YEAR	PRICING CELL A*		PRICING CELL B*		PRICING CELL C*		PRICING CELL D*	
	Iteration 1	Iteration 2	Iteration 1	Iteration 2	Iteration 1	Iteration 2	Iteration 1	Iteration 2
1......	0.1228	0.1225	0.1154	0.1151	0.1081	0.1078	0.1053	0.1051
2......	0.1247	0.1245	0.1204	0.1202	0.1118	0.1116	0.1102	0.1100
3......	0.1214	0.1211	0.1200	0.1198	0.1102	0.1100	0.1099	0.1096
4......	0.1180	0.1177	0.1197	0.1194	0.1085	0.1082	0.1096	0.1093
5......	0.1146	0.1143	0.1196	0.1192	0.1069	0.1065	0.1095	0.1091
6......	0.1112	0.1108	0.1196	0.1191	0.1052	0.1048	0.1094	0.1090
7......	0.1073	0.1069	0.1193	0.1188	0.1032	0.1027	0.1091	0.1086
8......	0.1034	0.1028	0.1191	0.1184	0.1010	0.1004	0.1088	0.1082
9......	0.0992	0.0985	0.1188	0.1181	0.0987	0.0980	0.1085	0.1077
10.....	0.0949	0.0941	0.1038	0.1029	0.0962	0.0953	0.1008	0.1000
11.....	0.0888	0.0882	0.0815	0.0808	0.0918	0.0912	0.0885	0.0879
12.....	0.0845	0.0838	0.0775	0.0768	0.0892	0.0885	0.0859	0.0852
13.....	0.0800	0.0793	0.0733	0.0726	0.0863	0.0856	0.0831	0.0824
14.....	0.0753	0.0746	0.0690	0.0683	0.0832	0.0824	0.0801	0.0794
15.....	0.0706	0.0699	0.0647	0.0639	0.0798	0.0791	0.0768	0.0761

* Pricing Cell A: male age 65, straight-life annuity; Pricing Cell B: male age 65, guaranteed-10 annuity; Pricing Cell C: female age 65, straight-life annuity; Pricing Cell D: female age 65, guaranteed-10 annuity.

A quick inspection of Tables 4 and 7 indicates that it is impossible for the asset flows to match the liability flows unless investment cell 1 is used. Cell 1 represents a ten-year private placement bond with full call protection and with 10 percent of the initial par value repaid each year. It is this early principal repayment that makes cell 1 useful for matching asset flows with liability flows in years 1–5. However, if too large a fraction of the investable funds is invested in cell 1, it will be difficult to match in years 11–15 because the ten-year private placement produces no investment cash flow in these years.

There were no feasible investment solutions that produced matching in years 1–15, inclusive. Two other runs were performed for each of the pricing cells; these involved matching from years 1–10 and from years 6–15. The center and radius of the maximal in-sphere for each pricing cell are shown in Table 8. Since there were no solutions that produced matching in years 1–15, a decision had to be made whether to base the investment strategy on the matching solution for years 1–10, the matching solution for years 6–15, or perhaps some hybrid of these two solutions. The solution for years 1–10 emphasizes strongly cell 1, the ten-year private placement bond. As a practical point, it should be noted that the supply of these shorter-term private placements is not as abundant

TABLE 8

MAXIMAL IN-SPHERES OF INVESTMENT STRATEGIES: FIRST ITERATION

Pricing Cell*	Matching Years	Center of Maximal In-Sphere					Radius
		Cell 1	Cell 2	Cell 3	Cell 4	Cell 5	
A.......	1–10	0.509	0.123	0.123	0.123	0.123	0.137
	6–15	0.146	0.417	0.146	0.146	0.146	0.163
B......	1–10	0.561	0.189	0.084	0.084	0.084	0.093
	6–15	0.165	0.339	0.165	0.165	0.165	0.185
C.......	1–10	0.398	0.150	0.150	0.150	0.150	0.168
	6–15	0.122	0.513	0.122	0.122	0.122	0.136
D.......	1–10	0.434	0.142	0.142	0.142	0.142	0.158
	6–15	0.134	0.463	0.134	0.134	0.134	0.150

* Pricing Cell A: male age 65, straight-life annuity; Pricing Cell B: male age 65, guaranteed-10 annuity; Pricing Cell C: female age 65, straight-life annuity; Pricing Cell D: female age 65, guaranteed-10 annuity.

as the supply of longer-term private placements with maturities from fifteen to twenty years. If there were a significant increase in the demand for the shorter-term securities, the coupon rates would fall, thus decreasing their desirability. It might be reasonable to add a further constraint to the investment strategy problem by limiting the fraction of funds that can be invested in cell 1—for example, $p_1 \leq 0.2$.

Suppose there had been a matching solution for years 1–15. It is quite probable that the midpoint of the line joining the centers of the maximal in-spheres for the 1–10 and 6–15 problems would lie closer to the maximal in-sphere for the 1–15 problem (and perhaps *within* this sphere) than would either of the center points of the 1–10 and 6–15 maximal in-spheres. Let us adopt this "hybridization" of the 1–10 and 6–15 solutions even though there was no feasible solution of the 1–15 matching problem. The hybridized points for the pricing cells A, B, C, and D will be denoted by

P_A, P_B, P_C, and P_D, respectively. To arrive at a single investment strategy for the second iteration, this crude averaging process can be continued. Let P_M represent the midpoint of the line joining P_A and P_B, and P_F the midpoint of the line joining P_C and P_D. Finally, let $\bar{P}$ be the midpoint of the line joining P_M and P_F. This geometry is shown schematically in Figure 4. The jth coordinate of the point $\bar{P}$ can be obtained by averaging all the entries in the column under cell j in Table 8. The result is $\bar{P} = (0.309, 0.292, 0.133, 0.133, 0.133)$. If we implement the limitation $p_1 \leq 0.2$ (as discussed in the previous paragraph) by setting

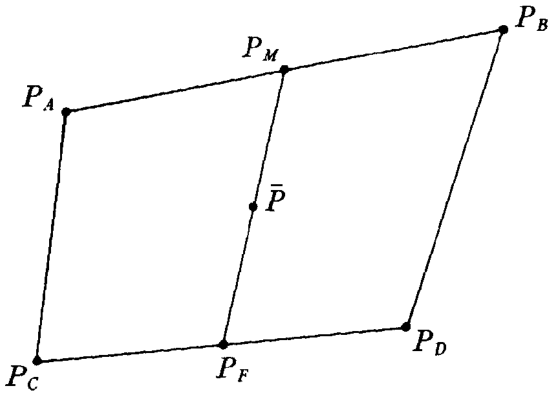

FIG. 4.—Geometry of a crude averaging technique to arrive at a single investment strategy $\bar{P}$ for use in the second-iteration pricing. Points P_A, P_B, P_C, and P_D are representative matching strategies after the first-iteration pricing for the four pricing cells labeled A, B, C, and D.

$p_1 = 0.2$ and rounding $p_3 = p_4 = p_5$ to 0.15, we obtain $p_2 = 0.35$. Thus, as the investment strategy for the pricing in the second iteration, we choose $P^{(1)} = (0.20, 0.35, 0.15, 0.15, 0.075, 0.075)$—see Table 5. Note that the 15 percent of the investable funds channeled into mortgages with thirty-year amortization periods has been split equally between cell 5 (twenty-year maturity) and cell 6 (fifteen-year maturity), as discussed earlier in this section.

It may seem that all these manipulations are arbitrary, but it must be recognized that a crude approach can be used during the first iteration of the solution to the pricing/investment-policy problem, since it is required only that the second and higher-order iterations converge. The purpose of the manipulations is only to produce a better starting investment strategy for the pricing than was used at the outset (col. 2 of Table 5).

What can be said about the reinvestment problem? Since all the bond cells used in this sample calculation are paying back principal during years 6–10, it can be expected that any investment strategy that places a significant fraction of investable funds into bonds will produce in years 6–10 significant excesses of investment cash flow over the cash-flow needs of the block of annuity business. It was mentioned earlier that in the current capital market ten-year private placements are less easily acquired than fifteen-year or twenty-year private placements. Moreover, Table 8 shows clearly that the ten-year private placement bonds are needed for matching only in years 1–5. Therefore, it makes sense to invest none of the excesses (of investment cash flow over annuity cash-flow requirements for years 6–10) in investment cell 1. By taking the 20 percent of investable funds that otherwise would have been invested in cell 1 according to the revised investment strategy and prorating it among cells 2–6 in proportion to the percentages to be invested in these cells under the revised strategy, the reinvestment strategy for the second-iteration pricing becomes $P^{(1)\prime} = (0, 0.4375, 0.1875, 0.1875, 0.09375, 0.09375)$. The reinvestment strategy is used to determine the vector of rollover rates for reinvested funds.

The whole cycle begins again. The coordinates of $P^{(1)}$ are used to weight the net annual yield for each cell. By using equation (15) and column 3 of Table 5, an initial net new-money rate of 8.88 percent is obtained. In practice this would be rounded to 8.9 percent and the net new-money rates for subsequent years graded down from there. For the purposes of these sample calculations, however, it was easier to round 8.88 percent to 9 percent so that only the rollover rates would have to be changed in the pricings. The annuity payout rates arising from the second pricing are shown in Table 6. For each of the pricing cells, the monthly income per \$1,000 of single premium is about \$0.020–\$0.025 less than the corresponding figure for the first-iteration pricing. The second-iteration model-office normalized cash-flow vectors $\{b_i\}$ are shown in Table 7. Finally, the solutions of the matching problems are presented in Table 9. As in the first iteration, there were no feasible solutions for matching in years 1–15, inclusive. The similar results in Tables 8 and 9 indicate that there is little to be gained by performing another iteration.

For each pricing cell, it is worthwhile to examine how closely the investment cash flow arising from the initial-investment strategy alone (that is, without reinvestment) matches the projected cash-flow needs of the annuity business. This is shown in Table 10, using the cash-flow vectors $\{b_i\}$ from Table 7, iteration 2, and the investment cash-flow matrix $\{a_{ij}\}$ from Table 4. The extent of nonmatching for any year is

TABLE 9

MAXIMAL IN-SPHERES OF INVESTMENT STRATEGIES: SECOND ITERATION

PRICING CELL*	MATCHING YEARS	CENTER OF MAXIMAL IN-SPHERE					RADIUS
		Cell 1	Cell 2	Cell 3	Cell 4	Cell 5	
A.......	1–10	0.507	0.123	0.123	0.123	0.123	0.138
	6–15	0.147	0.410	0.147	0.147	0.147	0.165
B.......	1–10	0.557	0.179	0.088	0.088	0.088	0.098
	6–15	0.167	0.332	0.167	0.167	0.167	0.187
C.......	1–10	0.394	0.152	0.152	0.152	0.152	0.169
	6–15	0.125	0.500	0.125	0.125	0.125	0.140
D.......	1–10	0.429	0.143	0.143	0.143	0.143	0.160
	6–15	0.138	0.450	0.138	0.138	0.138	0.154

* Pricing Cell A: male age 65, straight-life annuity; Pricing Cell B: male age 65, guaranteed-10 annuity; Pricing Cell C: female age 65, straight-life annuity; Pricing Cell D: female age 65, guaranteed-10 annuity.

TABLE 10

EXTENT OF NONMATCHING AFTER SECOND ITERATION*

$$\left(\frac{1}{b_i}\sum_{j=1}^{5} a_{ij}p_j\right) - 1$$

YEAR i	PRICING CELL†			
	A	B	C	D
1...........	−10.3%	− 4.5%	+ 1.9%	+ 4.6%
2...........	−13.1	−10.0	− 3.1	− 1.6
3...........	−12.0	−11.1	− 3.2	− 2.8
4...........	−10.9	−12.2	− 3.1	− 4.1
5...........	− 9.7	−13.4	− 3.1	− 5.4
6...........	32.2	23.0	39.8	34.4
7...........	31.8	18.6	37.2	29.7
8...........	31.6	14.3	34.8	25.1
9...........	31.7	9.9	32.4	20.5
10..........	32.0	20.7	30.3	24.2
11..........	11.8	22.0	8.1	12.2
12..........	13.0	23.3	7.0	11.2
13..........	14.6	25.2	6.1	10.3
14..........	16.6	27.4	5.6	9.6
15..........	18.9	30.1	5.1	9.2

* Results are shown as percentages of cash-flow requirements for the year.

† Pricing Cell A: male age 65, straight-life annuity; Pricing Cell B: male age 65, guaranteed-10 annuity; Pricing Cell C: female age 65, straight-life annuity; Pricing Cell D: female age 65, guaranteed-10 annuity.

given as a percentage of the cash-flow requirement for that year. Table 10 suggests that the initial-investment strategy $P^{(1)} = (0.20, 0.35, 0.15, 0.15, 0.15)$ would produce only a very small deficiency in years 1–5 and excesses thereafter for a reasonable distribution of business by sex and type of contract.

V. CONCLUSION

It is necessary, from a competitive standpoint, to use new-money interest rate assumptions in the pricing of single premium products. Considerations of equity between classes of policyholders may require that net investment income be allocated to such products via an investment-generation method. Under these circumstances, it is important to use investment assumptions for pricing that are consistent with the actual practices of the company's investment department. Primarily, it is the actuary's responsibility to ensure this consistency. By using the method described in this paper, the actuary can develop strategies for the investment of the initial single premiums less expenses and taxes, so that investment cash flow from the resulting portfolio of assets meets or exceeds the expected cash-flow needs of the block of business at later durations. The sensitivity of the investment strategy to the experience assumptions for mortality and expenses can be analyzed conveniently. Moreover, the results can be communicated easily to investment officers.

The ideas discussed in this paper are simple. The mathematics of the maximal in-sphere problem is superficially complicated, but the final result that the center and radius of the sphere are the solution of a standard linear programming problem makes the approach very practical. Software packages for solving linear optimization problems are available in several computer languages, using either the conventional simplex method or the revised simplex method.

VI. ACKNOWLEDGMENT

The author would like to express his gratitude to Amy J. Rupert for her considerable efforts in programming and testing the computer model office for single premium immediate annuities, in developing the programs to define the representative investment cells and the corresponding cash-flow and principal repayment matrices, and in modifying the input format of the APL revised simplex program so that the maximal in-sphere problem could be handled easily.

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APPENDIX I

The notation used in this appendix refers only to the geometrical problem considered here; symbols used in the appendix that are identical with symbols used in the main body of the paper do not necessarily have the same interpretation.

Consider the following geometrical problem.

Given a point $P_0 \equiv (p_{01}, \ldots, p_{0n})$ in n-space, and m planes T_i ($m \leq n$) of the form

$$T_i: \quad \sum_{j=1}^{n} a_{ij}p_j = b_i\,, \quad 1 \leq i \leq m\,, \tag{A1}$$

find the length l of the line from P_0 perpendicular to the "surface" formed by the intersection of the m planes T_i, $1 \leq i \leq m$.

This problem can be solved by *minimizing* the function

$$f(p_1, \ldots, p_n) = \sum_{j=1}^{n} (p_j - p_{0j})^2 \tag{A2}$$

subject to the constraint that the point $P \equiv (p_1, \ldots, p_n)$ lies on each of the planes T_i, $1 \leq i \leq m$. Introduce m Lagrange multipliers λ_i, $1 \leq i \leq m$, and define functions g_i and G as follows:

$$g_i(p_1, \ldots, p_n) = \sum_{j=1}^{n} a_{ij}p_j - b_i\,, \quad 1 \leq i \leq m\,; \tag{A3a}$$

$$G(p_1, \ldots, p_n) = f(p_1, \ldots, p_n) - 2\sum_{i=1}^{m} \lambda_i g_i(p_1, \ldots, p_n)\,. \tag{A3b}$$

The factor of 2 in equation (A3b) is included only for later convenience. The point P minimizing $f(p_1, \ldots, p_n)$ can be found by solving the equations $\partial G/\partial p_j = 0$, $1 \leq j \leq n$, together with the constraints $g_i(p_1, \ldots, p_n) = 0$, $1 \leq i \leq m$. Specifically,

$$\frac{\partial G}{\partial p_j} = 2(p_j - p_{0j}) - 2\sum_{i=1}^{m} \lambda_i a_{ij} = 0\,, \quad 1 \leq j \leq n\,,$$

or

$$p_j = p_{0j} + \sum_{i=1}^{m} \lambda_i a_{ij}\,, \quad 1 \leq j \leq n\,. \tag{A4}$$

Define a symmetric $m \times m$ matrix C as follows:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk}\,, \quad 1 \leq i \leq m\,, \quad 1 \leq j \leq m\,. \tag{A5}$$

In the remainder of this appendix it is assumed that C is nonsingular and possesses an inverse $C^{-1} \equiv \{c_{ij}^{-1}\}$. In Appendix II the conditions under which C^{-1} exists will be shown. Since C is symmetric, so is C^{-1}. Upon substituting equation (A4) in equation (A2), the following expression for l^2 can be derived.

$$l^2 = \sum_{j=1}^{n} (p_j - p_{0j})^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} c_{rs}\lambda_r\lambda_s \,. \qquad \text{(A6)}$$

From equation (A4), we also derive

$$\sum_{j=1}^{n} a_{ij}p_j = \sum_{j=1}^{n} a_{ij}p_{0j} + \sum_{r=1}^{m} \lambda_r c_{ri} \,, \quad 1 \leq i \leq m \,. \qquad \text{(A7)}$$

The constraint $g_i(p_1, \ldots, p_n) = 0$ is equivalent to $\Sigma_{j=1}^{n} a_{ij}p_j = b_i$. The latter relationship allows equation (A7) to be rewritten as

$$\sum_{r=1}^{m} \lambda_r c_{ri} = \sum_{j=1}^{n} a_{ij}p_{0j} - b_i \,, \quad 1 \leq i \leq m \,. \qquad \text{(A8)}$$

Multiplying equation (A8) by c_{is}^{-1}, summing over the index i from 1 to m, and using the result $\Sigma_{i=1}^{m} c_{ri}c_{is}^{-1} = \delta_{rs}$, we derive

$$\lambda_s = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}p_{0j} - b_i \right) c_{is}^{-1}. \qquad \text{(A9)}$$

The definition of the Kronecker function δ_{rs} is

$$\begin{aligned} \delta_{rs} &= 1 \,, \quad r = s \\ &= 0 \,, \quad r \neq s \,. \end{aligned}$$

Substituting equation (A9) in equation (A6), we obtain

$$\begin{aligned} l^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} &\left[\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}p_{0j} - b_i \right) c_{ir}^{-1} \right] \\ &\times \left[\sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{kj}p_{0j} - b_k \right) c_{ks}^{-1} \right] c_{rs} \,. \end{aligned} \qquad \text{(A10)}$$

Using the fact that

$$\sum_{s=1}^{m} c_{rs}c_{ks}^{-1} = \sum_{s=1}^{m} c_{rs}c_{sk}^{-1} = \delta_{rk}$$

and carrying the sum out over r, we finally arrive at

$$l^2 = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{ik}^{-1} \left(\sum_{j=1}^{n} a_{ij}p_{0j} - b_i \right) \left(\sum_{j=1}^{n} a_{kj}p_{0j} - b_k \right). \qquad \text{(A11)}$$

Equation (A11) is more general than is needed for the discussion in Section III of the paper. In terms of the notation of Section III, the specific problem is to find the length of the shortest line from a point P_0 lying on a plane H_0 to the surface formed by the intersection of H_0 with another plane H_i. Thus, the point P_0, from which the perpendicular extends, itself lies on one of the planes. In terms of the notation of this appendix, $m = 2$, and P_0 lies on T_1 but not necessarily on T_2. Hence, in equation (A11), all terms in the outer two summations vanish except for the single term with $i = 2$ and $k = 2$. The expression for l^2 is a perfect square, and the square root can be extracted easily.

$$l = (c_{22}^{-1})^{1/2} \left| \sum_{j=1}^{n} a_{2j} p_{0j} - b_2 \right| , \qquad \text{(A12)}$$

where c_{22}^{-1} is the (2, 2)th element of the inverse of the matrix C defined in equation (A5).

APPENDIX II

Before stating and proving the theorem relating to the nonsingularity of the matrix C defined by equation (A5), it is necessary to review briefly a few definitions and theorems pertaining to vector spaces and matrices.

Let $\mathcal{R}^n$ denote the vector space of n-component vectors, each component of which is a real number. Let $\mathbf{0}^{(n)}$ denote the n-vector having all components equal to zero. Let $\boldsymbol{v}_i$, $1 \leq i \leq m \leq n$, represent m nonzero vectors in $\mathcal{R}^n$, and let ξ_i, $1 \leq i \leq m$, represent m scalars in $\mathcal{R}^1$. The vectors $\boldsymbol{v}_i$, $1 \leq i \leq m \leq n$, are said to be linearly independent if and only if $\Sigma_{i=1}^{m} \xi_i \boldsymbol{v}_i = \mathbf{0}^{(n)}$ implies $\xi_i = 0$, $1 \leq i \leq m$.

A square $m \times m$ matrix, all the components of which are real (that is, belonging to $\mathcal{R}^m \times \mathcal{R}^m$), is nonsingular if and only if its m row vectors are linearly independent vectors in $\mathcal{R}^m$. Every nonsingular matrix possesses an inverse.

The scalar product of two vectors $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ belonging to $\mathcal{R}^n$ is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{k=1}^{n} u_k v_k .$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be orthogonal if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

The vector subspace of $\mathcal{R}^n$ spanned by m linearly independent vectors $\boldsymbol{v}_i$, $1 \leq i \leq m$, consists of all vectors of the form $\Sigma_{i=1}^{m} \xi_i \boldsymbol{v}_i$, where the ξ_i's are arbitrary scalars in $\mathcal{R}^1$. This subspace is known as the linear span of the set of vectors $\boldsymbol{v}_i$, $1 \leq i \leq m$, and is m-dimensional. Let it be represented by V_r. The vector subspace of $\mathcal{R}^n$ containing all the vectors in $\mathcal{R}^n$

orthogonal to every vector in V_v is called the orthogonal complement of V_v and is denoted by $V_v^{\perp}$. $V_v^{\perp}$ is $(n - m)$-dimensional. The only vector belonging to both V_v and $V_v^{\perp}$ is $\mathbf{0}^{(n)}$, the zero vector.

THEOREM. *Let $\boldsymbol{a}_i$, $1 \leq i \leq m \leq n$, represent m nonzero vectors in $\mathcal{R}^n$, and define the matrix C in $\mathcal{R}^m \times \mathcal{R}^m$ so that the element in the ith row and the jth column is equal to the scalar product of $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. Then C is nonsingular if and only if the vectors $\boldsymbol{a}_i$, $1 \leq i \leq m$, are linearly independent.*

Proof: Assume that the vectors $\boldsymbol{a}_i$, $1 \leq i \leq m \leq n$, are linearly independent. Let $\boldsymbol{r}_i$ represent the ith row vector of C. Let V_a denote the linear span of the vectors $\boldsymbol{a}_i$, $1 \leq i \leq m$, and $V_a^{\perp}$ its orthogonal complement. The statement $\Sigma_{i=1}^{m} \xi_i \boldsymbol{r}_i = \mathbf{0}^{(m)}$ implies $\Sigma_{i=1}^{m} \xi_i \boldsymbol{a}_i \cdot \boldsymbol{a}_j = 0$ for $1 \leq j \leq m$. Thus, $\Sigma_{i=1}^{m} \xi_i \boldsymbol{a}_i$ belongs to $V_a^{\perp}$. By definition, however, it also belongs to V_a. Hence, $\Sigma_{i=1}^{m} \xi_i \boldsymbol{a}_i = \mathbf{0}^{(n)}$, implying that $\xi_i = 0$, $1 \leq i \leq m$, since the vectors $\boldsymbol{a}_i$, $1 \leq i \leq m$, are linearly independent. This proves that the row vectors of C are linearly independent and that C is nonsingular.

Next, assume that C is nonsingular. Thus, the row vectors of C are linearly independent vectors in $\mathcal{R}^m$. Suppose that $\Sigma_{i=1}^{m} \xi_i \boldsymbol{a}_i = \mathbf{0}^{(n)}$. It follows that $(\Sigma_{i=1}^{m} \xi_i \boldsymbol{a}_i) \cdot \boldsymbol{a}_j = 0$, $1 \leq j \leq m$, which further implies that $\Sigma_{i=1}^{m} \xi_i \boldsymbol{r}_i = \mathbf{0}^{(m)}$. Hence, $\xi_i = 0$, $1 \leq i \leq m$, and the vectors $\boldsymbol{a}_i$, $1 \leq i \leq m$, are linearly independent. This completes the proof of the theorem.

The matrix C defined in equation (A5) is precisely the matrix defined in the statement of the theorem. The 2×2 matrices C_i, $1 \leq i \leq n + m$, defined in equations (10a) and (10b) are clearly nonsingular for $n > 1$.