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THE DYNAMICS OF PENSION FUNDING: CONTRIBUTION THEORY

NEWTON L. BOWERS, JR., JAMES C. HICKMAN, AND CECIL J. NESBITT

ABSTRACT

A general model for a pension plan involving growth with respect to the population, salaries, and retirement benefits is used to study contribution patterns that may arise under different actuarial cost methods. Detailed results are presented for the case where the growth of population and the growth of salaries are described by exponential functions. Economic implications are presented and discussed.

I. INTRODUCTION

THE paper "Introduction to the Dynamics of Pension Funding" [2] presented a mathematical model for a pure pension plan (no benefits other than for retirement) under conditions of growth in regard to the covered population and salaries. This model may be used to provide answers to a wide variety of pension funding questions.

In the earlier paper, the theory was developed for all members of the pension plan, both active and retired. In this paper the emphasis will shift to contribution theory, which simplifies the formulas because only the subgroup of active members is considered (see Richard K. Kischuk's discussion of [2]). The resulting formulas are somewhat simpler and provide answers to questions about contributions more readily than do those involving the whole group. In the final section of the paper, corresponding formulas for the whole group will be outlined.

In several developments in this paper exponential growth functions are used. This does not mean that it is anticipated that any real pension plan will have a covered population and corresponding salaries that will proceed smoothly along exponential paths. Instead, these special cases are developed because of their pedagogical value; they are easy to derive and interpret. In addition, the authors do not have the foresight to fix the jagged paths that the experience of real pension plans probably will follow. The long-range cost estimates of the social security OASDI systems have been based on exponential economic assumptions for similar reasons.

The Model

A summary of the model along the lines of that given in Section II of [2] is repeated here. All new entrants join the group at age a, and all retirements occur at age r. Only retirement benefits are considered. For both active and retired participants, survivorship is deterministic and in accordance with the function l_x, which is independent of the time variable t. At time zero, the density of new retirants at age r is l_r, and thereafter this density increases by a factor $g_1(t)$. This establishes a generation pattern of growth for participants. It is assumed that $g_1(t) > 0$. This implies a positive density $g_1(t + r - x)l_x$ of members aged x, $a \leq x$, at time t. Salary rates at time zero are represented by the function $s(x)$ for a member aged x, $a \leq x < r$. The function $s(x)$ captures the merit component of salary changes. Thereafter, salary rates increase by a factor $g_2(t)$. This establishes a year-of-experience pattern of salary growth. The function $g_2(t)$ is designed to capture the influences of productivity and inflation on salaries. The rate of initial annual pension payment, commencing at age r, is a fixed positive fraction b of the final salary rate. Pension payment rates increase during retirement by a factor $\beta(x)$.

We now will combine several of these functions. For $a \leq x < r$, the density of new pensions to be incurred at time $t + r - x$ with respect to the survivors of members aged x at time t is given by the function

$$h(t + r - x) = g_1(t + r - x)g_2(t + r - x)l_r s(r)b \,.$$

For $x \geq r$, $h(t + r - x)$ is the density of new pensions incurred at time[1] $t - (x - r)$ for those who were then aged r. Therefore, $h(t + r - x)(l_x/l_r)\beta(x)$ is the density of existing pensions for retirees aged x at time t. In this expression, $\beta(x)$ is the aforementioned pension adjustment factor, and $\beta(r) = 1$.

Outline

In Section II a number of pension funding functions in regard to active members are considered. These are the following:

- $(aA)(t)$ Present value at time t of future benefits for the then active members.[2]
- $P(t)$ Annual rate of normal cost for the plan at time t.
- $(aV)(t)$ Supplemental present value (accrued liability) of the plan at time t for the then active members.

[1] In [2], p. 186, this time was incorrectly called t.

[2] In R. K. Kischuk's discussion of [2], the notation $(Aa)(t)$ was used for this quantity. To avoid inconsistency with the notation $(Pa)(t)$ and with annuity notation, the revised symbol $(aA)(t)$ will be used.

$(Pa)(t)$ Present value at time t of future normal costs for the then active members.

${}^{T}P(t)$ Annual rate of terminal funding normal cost for the plan at time t. This is a special case of $P(t)$ and implies that ${}^{T}P(t)dt$ is the present value at time t of future benefits payable to those retiring at that time.

The basic income allocation equation relative to active members is discussed.

In Section III ratios of each of the first four of the functions listed above to the fifth function are exhibited. These ratios take on particularly simple and easily interpreted forms in the exponential growth case, that is, when the growth functions $g_1(t)$, $g_2(t)$ are exponential.

In Section IV various contribution rates determined by combinations of normal cost and rates of amortization of unfunded supplemental present value or by aggregate funding are explored. In the exponential growth case, aggregate funding is found to achieve the same level of funding as some individual cost methods with selected patterns of amortizing the unfunded supplemental present value.

Section V outlines some of the corresponding theory that emerges if the functions relate to the whole group of members, both active and retired, instead of to only the active group. In many ways this section provides a bridge to the developments in the original paper [2]. In this section it is natural to consider $B(t)$, the annual rate of pension outgo at time t, in place of ${}^{T}P(t)$, the annual rate of terminal funding normal cost, as a key function.

As in [2], the presentation will be mathematical. However, the mathematics is elementary and leads to natural verbal interpretations. It is hoped that other actuaries will become interested in developing these ideas further by numerical examples.

II. FUNDING FUNCTIONS FOR ACTIVE MEMBERS

Basic Functions

Central to the study of contribution theory is the annual rate of terminal funding cost for the plan at time t. This rate will be denoted by ${}^{T}P(t)$. It will serve as a building block and as a standard of comparison for other contribution patterns. For the model plan, the density of pensions expected to arise $r - x$ years later with respect to members aged x at time t, $a \leq x \leq r$, is $h(t + r - x)$. Therefore, the annual rate of terminal funding cost for the plan at time $t + r - x$ will be

$$^{T}\boldsymbol{P}(t + r - x) = h(t + r - x)\bar{a}_r^{\beta}, \tag{1}$$

where the annuity symbol $\bar{a}_r^{\beta}$ is given by

$$\bar{a}_r^{\beta} = \int_r^{\infty} e^{-\delta(x-r)}\, {}_{x-r}p_r \beta(x)dx\,; \tag{2}$$

δ is the annual force of interest. Note that the annuity function incorporates the pension adjustment function $\beta(x)$. With the terminal funding cost rate function, it is now easy to define the present value of future benefits for the active group as

$$(\boldsymbol{aA})(t) = \int_a^r e^{-\delta(r-x)}\ {}^{T}\boldsymbol{P}(t + r - x)dx\,. \tag{3}$$

By using the pension purchase density function $m(x)$ and the accrual function $M(x)$ (see [2], p. 182, and [3]), we may write immediately the annual normal cost rate at time t for the actuarial cost method associated with $m(x)$ as

$$\boldsymbol{P}(t) = \int_a^r e^{-\delta(r-x)}\ {}^{T}\boldsymbol{P}(t + r - x)m(x)dx\,. \tag{4}$$

Then the supplemental present value (accrued liability) at time t for active members, based on the actuarial cost method described by $m(x)$, is

$$(\boldsymbol{aV})(t) = \int_a^r e^{-\delta(r-x)}\ {}^{T}\boldsymbol{P}(t + r - x)M(x)dx\,. \tag{5}$$

The present value at time t of future normal costs of the plan for active members is

$$(\boldsymbol{Pa})(t) = \int_a^r e^{-\delta(r-x)}\ {}^{T}\boldsymbol{P}(t + r - x)[1 - M(x)]dx\,. \tag{6}$$

Income Allocation Equation

In [2] a liability growth equation in regard to all members, both active and retired, was expressed in formula (40) and rearranged in formula (43). Corresponding to formula (43), we have the equation, relevant to active members,

$$\boldsymbol{P}(t) + \delta(\boldsymbol{aV})(t) = {}^{T}\boldsymbol{P}(t) + \frac{d(\boldsymbol{aV})(t)}{dt}\,. \tag{7}$$

This equation may be described as an *income allocation equation;* the normal cost and assumed interest are allocated to the terminal funding cost and the change in the supplemental present value for active members. Formula (7) is very general. Using the notation developed to describe our basic model, we may verify formula (7) by using integration by parts on formula (4) as follows:

$$\begin{aligned} P(t) &= \int_a^r e^{-\delta(r-x)}\ {}^TP(t+r-x)m(x)dx \\ &= e^{-\delta(r-x)}\ {}^TP(t+r-x)M(x)\Big|_a^r \\ &\quad - \int_a^r M(x)e^{-\delta(r-x)}d[{}^TP(t+r-x)] \\ &\quad - \int_a^r M(x)\delta e^{-\delta(r-x)}\ {}^TP(t+r-x)dx \\ &= {}^TP(t) + \frac{d}{dt}\int_a^r M(x)e^{-\delta(r-x)}\ {}^TP(t+r-x)dx \\ &\quad - \delta\int_a^r M(x)e^{-\delta(r-x)}\ {}^TP(t+r-x)dx \\ &= {}^TP(t) + \frac{d}{dt}(aV)(t) - \delta(aV)(t)\,. \end{aligned}$$

III. FUNCTION RATIOS

In this section we shall consider the ratios of each of the functions $(aA)(t)$, $P(t)$, $(aV)(t)$, and $(Pa)(t)$ to ${}^TP(t)$, the annual rate of terminal funding. These ratios lead to insights because, according to the income allocation equation, formula (7), ${}^TP(t)$ may be thought of as the outgo function in regard to funding for active members. First, we consider the ratio of the value of future benefits for the active group to the annual rate of terminal funding cost,

$$(aA)(t)/{}^TP(t) = \int_a^r [h(t+r-x)/h(t)]e^{-\delta(r-x)}dx\,, \tag{8}$$

which may be viewed as the present value of a varying annuity certain. In the exponential growth case where $g_1(t) = e^{\alpha t}$ (population growth), $g_2(t) = e^{\gamma t}$ (salary growth), and $h(t) = g_1(t)g_2(t)l_rs(r)b = e^{\tau t}l_rs(r)b$ for $\tau = \alpha + \gamma$, $[h(t+r-x)/h(t)]e^{-\delta(r-x)} = e^{-(\delta-\tau)(r-x)}$.

At this stage it is necessary to distinguish the three cases $\delta > \tau$, $\delta = \tau$, and $\delta < \tau$. In the next three subsections these three cases will be explored and the analyses extended to the other ratios of interest.

Case 1: $\delta > \tau$

In this case the annual force of interest is greater than the combined rates of salary and population growth. We have

$$[h(t + r - x)/h(t)]e^{-\delta(r-x)} = v^{r-x} \tag{9}$$

and formula (8) simplifies to

$$(aA)(t)/{}^{T}P(t) = \bar{a}_{\overline{r-a}|} \tag{10}$$

evaluated at annual force $\theta = \delta - \tau$. *Here, and in the sequel, if the force to be used in the evaluation of a compound interest function is not stated, it is to be taken as* $\theta = \delta - \tau$.

It is clear that in this case $(aA)(t)/{}^{T}P(t)$ is a decreasing function of $\theta = \delta - \tau$. If $\tau < 0$, that is, if the combination of population and salaries is on a decreasing rather than an increasing exponential path, the ratio of the present value of future benefits to the current terminal funding normal cost rate may be fairly small.

From formula (10) one sees that the present value of future benefits for active members is equal to the discounted value at the force θ of an annuity of ${}^{T}P(t)$ for $r - a$ years. More completely, in the exponential growth case $(aA)(t)$ equals the discounted value at annual force δ of an increasing annuity with a payment rate from formula (1) of ${}^{T}P(t + u) = {}^{T}P(t)e^{\tau u}$ at time $t + u$, $0 \leq u < r - a$. That is,

$$(aA)(t) = \int_0^{r-a} {}^{T}P(t + u)e^{-\delta u}du = \int_0^{r-a} [{}^{T}P(t)e^{\tau u}]e^{-\delta u}du \,.$$

The key point is that, in the exponential growth case, the terminal funding normal cost increases at the total growth rate $\tau = \alpha + \gamma$, the sum of the population and salary growth rates. Note also that the ratio in formula (10) is independent of t as a result of the exponential growth functions.

The function $(aA)(t)$, the present value of future benefits for active members at time t, is independent of the actuarial cost method. However, the remaining funding functions defined by formulas (4)–(6) depend on the actuarial cost method through the accrual function $M(x)$.

For example, the ratio of the annual normal cost rate at time t to the annual rate of terminal cost funding is

$$P(t)/{}^{T}P(t) = \int_a^r [h(t + r - x)/h(t)]e^{-\delta(r-x)}m(x)dx \,. \tag{11}$$

In the exponential growth case,

$$
\begin{aligned}
P(t)/{}^{T}P(t) &= \int_a^r v^{r-x} m(x)dx \\
&= v^r \int_a^r e^{x(\delta-\tau)} m(x)dx \\
&= v^{r-x(\theta)} .
\end{aligned}
\tag{12}
$$

Here $x(\theta)$ is calculated from the equation

$$
e^{\theta x(\theta)} = \int_a^r e^{\theta x} m(x)dx . \tag{13}
$$

The existence of a value of $x(\theta)$ on the interval from a to r is assured by the mean-value theorem for integrals.

We will call $x(\theta)$ the *average age of normal cost payment* associated with the actuarial cost method defined by $m(x)$ and the combination of interest, population, and salary forces $\theta = \delta - \tau = \delta - \alpha - \gamma$. Two extreme cases need special attention. For terminal funding, $M(x) = 0$ for $a \leq x < r$, $M(x) = 1$ for $r \leq x$, and $x(\theta) = r$. For *initial funding*, the whole pension cost is funded for each entrant by a lump-sum payment at entry, so $M(x) = 1$ for $a \leq x$, and $x(\theta) = a$. In the model plan with exponential growth, the number $x(\theta)$ can tell us some of the characteristics of the actuarial cost method with which it is associated.[3]

Formula (13) has an interesting interpretation. Note that the right-hand side, to be denoted by $\psi(\theta)$, may be interpreted as the moment generating function associated with the actuarial cost method defined by $m(x)$. This leads to the formal conclusion that if two actuarial cost methods yield the same value of $x(\theta)$ for all values of θ on an interval containing zero, then their associated moment generating functions are the same on the interval, and the two actuarial cost methods are identical.

It is more practical to examine the relationship between $x(\theta)$ and the characteristics of the associated actuarial cost method. For two con-

[3] For a given value of θ, the number $x(\theta)$, which depends on θ, does not characterize an actuarial cost method. For fixed θ and $x(\theta)$, $a < x(\theta) < r$, it is easy to construct a density function $m(x)$ that will yield

$$
\int_a^r e^{\theta x} m(x)dx = e^{\theta x(\theta)}
$$

by using step functions or exponential density functions as described in [3].

tinuous pension purchase density functions, $m(x)$ and $m_1(x)$, with $M(a) = M_1(a) = 0$ we have

$$e^{\theta x(\theta)} - e^{\theta x_1(\theta)} = \int_a^r e^{\theta x} d[M(x) - M_1(x)] .$$

Using integration by parts, we have

$$e^{\theta x(\theta)} - e^{\theta x_1(\theta)} = -\theta \int_a^r [M(x) - M_1(x)] e^{\theta x} dx .$$

If $m(x)$ is associated with a decelerating cost method ($m'(x) < 0$), and $m_1(x)$ is associated with an accelerating cost method ($m_1'(x) > 0$), then $M(x) - M_1(x) > 0$ for $a < x < r$, and for θ positive or negative we have $x(\theta) < x_1(\theta)$. We are led to the following obvious result: in comparing decelerating ($m(x)$) and accelerating ($m_1(x)$) cost methods, we have from formula (12)

$$P(t)/{}^TP(t) = v^{r-x(\theta)} < v^{r-x_1(\theta)} = P_1(t)/{}^TP(t) .$$

This states that the ratio of annual normal cost rate to the terminal cost funding rate is less for decelerating cost methods than for accelerating cost methods. A closely related argument will be used in the next subsection in comparing the pro rata accrued benefit and entry age normal cost methods.

Using our new symbol, formula (12) may be rearranged as

$$P(t) = v^{r-x(\theta)}\ {}^TP(t) , \tag{14}$$

or, by using formula (1) for the exponential growth case, as

$$\begin{aligned} P(t) &= e^{-\delta[r-x(\theta)]} e^{r[r-x(\theta)]}\ {}^TP(t) \\ &= e^{-\delta[r-x(\theta)]}\ {}^TP(t + r - x(\theta)) . \end{aligned} \tag{15}$$

This shows that the annual normal cost rate at time t may be thought of as remaining in the fund for $r - x(\theta)$ years and then utilized to provide the terminal funding cost ${}^TP(t + r - x(\theta))$. The *total funding term* for current active members is $r - a$ years. We shall see that this may be thought of as consisting of a *past funding term* of $r - x(\theta)$ years and a *future funding term* of $N = x(\theta) - a$ years. Of course, $x(\theta)$ depends on the actuarial cost method. In the special case of terminal funding $x(\theta) = r$, that is, the past funding term for the active group is zero years and all funding is in the future. For initial funding

$${}^IP(t) = e^{-\delta(r-a)}\ {}^TP(t + r - a) \tag{16}$$

and $x(\theta) = a$. Thus the future funding term is zero years, and the past funding term is $r - a$ years.

The next ratio we will consider is the ratio of the supplemental present value at time t for active members to the terminal funding cost rate at time t. We remain in the situation where $\delta > \tau$. We have, for a cost method characterized by $M(x)$,

$$({}^{a}V)(t)/{}^{T}P(t) = \int_a^r [h(t + r - x)/h(t)]e^{-\delta(r-x)}M(x)dx \, .$$

This ratio may be calculated directly in the exponential growth case from the equation

$$({}^{a}V)(t)/{}^{T}P(t) = \int_a^r v^{r-x}M(x)dx \, , \tag{17}$$

but it is simpler to note that

$$\frac{d}{dt}({}^{a}V)(t) = \tau({}^{a}V)(t) \, . \tag{18}$$

When this result is substituted in formula (7), we have

$$P(t) + \delta({}^{a}V)(t) = {}^{T}P(t) + \tau({}^{a}V)(t) \tag{19}$$

or

$$({}^{a}V)(t) = [{}^{T}P(t) - P(t)]/(\delta - \tau) \, . \tag{20}$$

By use of formula (14) this becomes

$$({}^{a}V)(t) = {}^{T}P(t)\bar{a}_{\overline{r-x(\theta)}|} \tag{21}$$

or

$$({}^{a}V)(t) = P(t)\bar{s}_{\overline{r-x(\theta)}|} \, . \tag{22}$$

Formula (21) exhibits the supplemental present value (accrued liability) for active members as the present value of the current terminal funding cost over the next $r - x(\theta)$ years. Formula (22) looks at the same quantity as the accumulated value of current normal costs for the past $r - x(\theta)$ years. These interpretations of formulas (21) and (22) are somewhat incomplete. To state it another way, $({}^{a}V)(t)$ is the present value at force of interest δ of the terminal funding cost ${}^{T}P(t + u) = {}^{T}P(t)e^{\tau u}$, where $0 \leq u \leq r - x(\theta)$;

$$({}^{a}V)(t) = \int_0^{r-x(\theta)} {}^{T}P(t + u)e^{-\delta u}du = \int_0^{r-x(\theta)} {}^{T}P(t)e^{\tau u}e^{-\delta u}du$$
$$= {}^{T}P(t)\bar{a}_{\overline{r-x(\theta)}|} \, .$$

We also know that $({}^{a}V)(t)$ is the accumulated value, at force of interest δ, of the annual normal costs $P(t - u) = P(t)e^{-\tau u}$, where $0 \leq u \leq r - x(\theta)$.

Thus,

$$
\begin{aligned}
(\boldsymbol{aV})(t) &= \int_0^{r-x(\theta)} \boldsymbol{P}(t-u)e^{\delta u}du \\
&= \int_0^{r-x(\theta)} \boldsymbol{P}(t)e^{-\tau u}e^{\delta u}du \\
&= \boldsymbol{P}(t)\bar{s}_{\overline{r-x(\theta)}|}\,.
\end{aligned}
$$

The present value of benefits for active members must equal the supplemental present value plus the present value of future normal costs. That is, adding formulas (5) and (6), we obtain

$$(\boldsymbol{aA})(t) = (\boldsymbol{aV})(t) + (\boldsymbol{Pa})(t)\,.$$

One now may apply formulas (10) and (21) to show that

$$(\boldsymbol{aA})(t) = {}^T\boldsymbol{P}(t)\bar{a}_{\overline{r-a}|} = {}^T\boldsymbol{P}(t)\bar{a}_{\overline{r-x(\theta)}|} + (\boldsymbol{Pa})(t)\,,$$

and by rearranging we have

$$
\begin{aligned}
(\boldsymbol{Pa})(t) &= {}^T\boldsymbol{P}(t)\bar{a}_{\overline{r-a}|} - {}^T\boldsymbol{P}(t)\bar{a}_{\overline{r-x(\theta)}|} \\
&= {}^T\boldsymbol{P}(t)v^{r-x(\theta)}\bar{a}_{\overline{x(\theta)-a}|}\,.
\end{aligned}
\tag{23}
$$

An application of formula (14) yields

$$(\boldsymbol{Pa})(t) = \boldsymbol{P}(t)\bar{a}_{\overline{x(\theta)-a}|}\,. \tag{24}$$

Formula (23) exhibits the present value of future normal costs for active lives at time t as the present value of terminal funding costs that will arise during the future funding term of $N = x(\theta) - a$ years. Formula (24) expresses the same quantity as the present value of normal costs at rate $\boldsymbol{P}(t)$ for the next $N = x(\theta) - a$ years. This value will provide the terminal funding costs $r - x(\theta)$ years later. In reviewing these interpretations, recall that the compound interest functions are evaluated at $\delta - \tau = \delta - \alpha - \gamma$, that is, each interest function depends on the annual forces of interest, population change, and salary change.

These concepts may be clarified by the Lexis-type diagram in Figure 1. Cohorts of entrants may be viewed as moving along diagonal lines in the figure. Vertical lines depict cross-sectional views of the funding for active lives at a fixed point of time. The dashed diagonal line may be viewed as the mean path followed by the group active at time t. The paths to be followed in tracing the relationships are indicated by arrows indexed by relationship numbers. Along diagonal lines the final salary and population growth functions are fixed—only interest contributes to changed values. Payments distributed along horizontal lines are of different amounts because of τ and have different present values because of δ. Normal cost payments made at the same time (same vertical line) differ because they are made on behalf of different cohorts and because of the different times until they are used to purchase retirement benefits ($\theta = \delta - \tau$). The relationship numbers (1)–(6) denote the following:

(1) Formula (15): $P(t) = e^{-\delta[r-x(\theta)]}\ {}^{T}P(t + r - x(\theta))$

(2) Formula (16): ${}^{I}P(t) = e^{-\delta(r-a)}\ {}^{T}P(t + r - a)$

(3) Formula (21): $(aV)(t) = {}^{T}P(t)\bar{a}_{\overline{r-x(\theta)}|}$

(4) Formula (22): $(aV)(t) = P(t)\bar{s}_{\overline{r-x(\theta)}|}$

(5) Formula (24): $(Pa)(t) = P(t)\bar{a}_{\overline{x(\theta)-a}|}$

(6) Formula (12): $P(t) = e^{-\theta[r-x(\theta)]}\ {}^{T}P(t)$

Because of the interpretations provided by relationships (4) and (5), $r - x(\theta)$ was called the *past funding term* and $N = x(\theta) - a$ was called

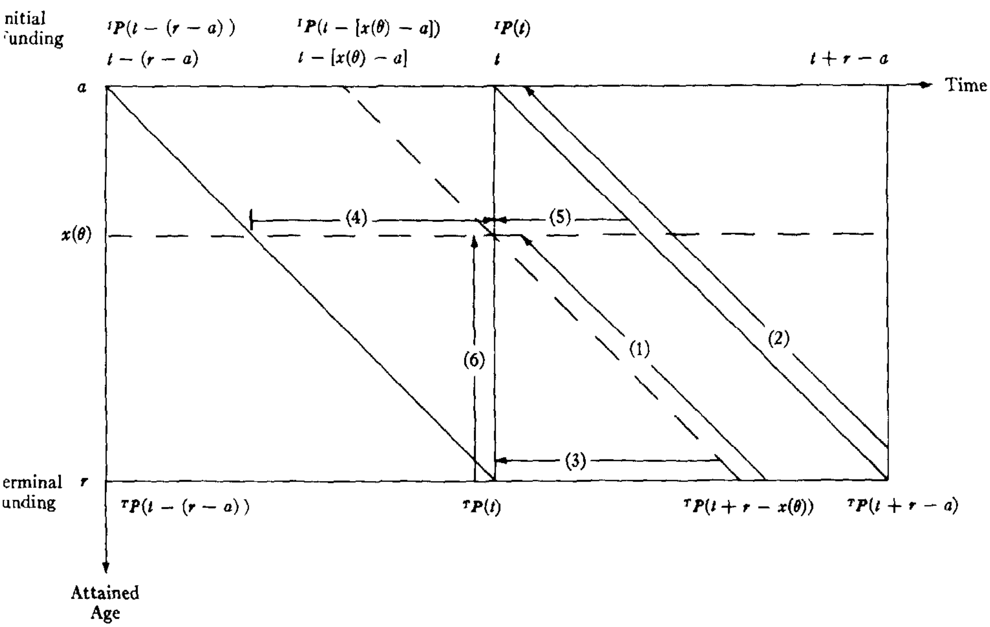

FIG. 1.—Illustration of relations among funding functions: active lives; exponential growth.

the *future funding term*. Since $(aA)(t) = (aV)(t) + (Pa)(t)$, we have a natural division of the present value of future benefits for active lives into a past and future component and of the funding term $r - a$ into associated terms of length $r - x(\theta)$ and $N = x(\theta) - a$.

Application of Case 1: $\delta > \tau$

a) Terminal funding.—For terminal funding, it has been noted already that the future funding term is $N = r - a$ and the past funding term is zero. Then $(aV)(t) = 0$ and $(Pa)(t) = {}^TP(t)\bar{a}_{\overline{r-a}|} = (aA)(t)$.

b) Pro rata accrued benefit cost method.—Here $m(x) = 1/(r - a)$ where $a < x < r$, and $m(x) = 0$ elsewhere. By Jensen's inequality [4], for a random variable X and a function such that $\varphi''(x) > 0$,

$$E[\varphi(X)] > \varphi(E[X]) ,$$

where E denotes mathematical expectation. If $\varphi(x) = e^{(\delta-\tau)x}$, then $\varphi''(x) > 0$. Now let $m(x)$ play the role of a uniform probability density function. Then from formula (13) and Jensen's inequality,

$$e^{(\delta-\tau)x(\theta)} = E[e^{(\delta-\tau)X}] > e^{(\delta-\tau)E[X]} \tag{25}$$

and

$$x(\theta) > E[X] . \tag{26}$$

With the uniform density $m(x) = 1/(r - a)$, we know that $E[X] = (a + r)/2$. We then may rewrite inequality (26) as

$$x(\theta) > (a + r)/2 , \tag{27}$$

where $x(\theta)$ pertains to a pro rata accrued benefit cost method. In other words, for this cost method the future funding term N is more than one-half of the total funding term of $r - a$ years.[4]

c) Entry age normal cost method.—A choice of $m(x)$ is

$$\begin{aligned} m(x) &= e^{-(\delta-\gamma)x} l_x s(x)\Big[\int_a^r e^{-(\delta-\gamma)y} l_y s(y)dy\Big]^{-1} , && a \le x < r \\ &= 0 , && \text{elsewhere .}^5 \end{aligned}$$

Then from formula (4), after substituting for ${}^TP(t + r - x)$ and $m(x)$ in the exponential case and rearranging, we have

$$P(t) = e^{-(\delta-\gamma)r} s(r) l_r b \bar{a}_r^\beta \Big[\int_a^r e^{-(\delta-\gamma)y} s(y) l_y dy\Big]^{-1} \times \int_a^r e^{\tau t + \alpha(r-x)} l_x s(x) dx , \tag{28}$$

[4] This proof was suggested by Dr. Hans U. Gerber.

[5] See [2], p. 183, for a discussion of this choice for $m(x)$.

or

$$P(t) = {}^{e}\pi_a W(t) . \tag{29}$$

In formula (29) ${}^{e}\pi_a$ is the entry age normal cost rate—as a level fraction of salary—for an employee entering at age a, retiring at age r, and having an annual salary rate $e^{\gamma(x-a)}s(x)$ at age x, where $a \leq x < r$. The symbol $W(t)$ is the annual payroll at time t (see [2], formula [77]). Note that ${}^{e}\pi_a$ is independent of t and of the population growth rate α.

Here a direct application of Jensen's inequality does not lead to a statement about $x(\theta)$, the average age of normal cost payment. However, an indirect application (see Appendix) shows that, if $e^{-\alpha x}s(x)l_x$ is a decreasing function of x and if, as throughout this subsection, $\delta > \tau$, then

$$x(\theta) < (r + a)/2 , \tag{30}$$

where $x(\theta)$ pertains to the entry age normal method. That is, under the stated condition, for entry age normal funding the future funding period $N = x(\theta) - a$ is less than one-half of the entire funding term $r - a$. It follows from formulas (14), (27), and (30) that the normal cost for the entry age normal method is less than for the accrued benefit method, and from formula (21) the reverse relation holds for the supplemental present values.

Case 2: $\delta = \tau$

In this case the annual force of interest equals the combined rate of salary and population growth. We have $[h(t + r - x)/h(t)]e^{-\delta(r-x)} = 1$, and formulas (10), (21), (12), and (23) may be replaced respectively by

$$(aA)(t) = {}^{T}P(t)(r - a) , \tag{31}$$

$$(aV)(t) = {}^{T}P(t)(r - \bar{x}) , \tag{32}$$

$$P(t) = {}^{T}P(t) , \tag{33}$$

$$(Pa)(t) = {}^{T}P(t)(\bar{x} - a) , \tag{34}$$

where

$$r - \bar{x} = r - \int_a^r xm(x)dx = \int_a^r M(x)dx .$$

These formulas relate to certain of the formulas in [2], Section VI, "The Exponential Growth Case," in the following fashion: formulas (31), (32), and (33) are the active members' analogues of formulas (89), (90), and (88) in [2], and formula (34) is identical to (92) of [2].

When $\theta = \delta - \tau = 0$, formula (13) does not lead to a definition of

$x(\theta)$. There is a natural way out of this difficulty that leads to insights about $x(\theta)$. From formula (13) we have

$$\begin{aligned} e^{\theta x(\theta)} &= \int_a^r e^{\theta x} m(x)\,dx \,, \\ x(\theta) &= \frac{1}{\theta} \ln \left[\int_a^r e^{\theta x} m(x)\,dx \right] \\ &= \frac{C(\theta)}{\theta} \,, \end{aligned} \tag{35}$$

where $C(\theta) = \ln \psi(\theta)$ is the cumulant generating function associated with the density $m(x)$ ([1], p. 307). Now we define $x(\theta)$ in the case $\theta = 0$ as

$$\begin{aligned} x(\theta) &= \lim_{\theta \to 0} C(\theta)/\theta \\ &= \bar{x} \,. \end{aligned}$$

The evaluation of the limit requires one application of L'Hospital's rule.

Case 3: $\delta < \tau$

a) Basic ratios.—This is the case where τ, the sum of the generation growth rate α and the salary growth rate γ, is such that $\delta < \tau$. In this case the basic mathematical formulas remain. The number $x(\theta)$ is still defined by formula (13) for each actuarial cost method. However, the formulas involving compound interest functions take on a new significance for $v = e^{-(\delta-\tau)} = e^{(\tau-\delta)} > 1$, and where we had $\bar{a}_{\overline{n}|}$ evaluated at force $\delta - \tau > 0$ we now have $\bar{s}_{\overline{n}|}$ evaluated at $\tau - \delta > 0$. In general, where a discount effect was observed in case 1 an accumulation effect is observed in case 3.

Corresponding to formulas (10), (14), (21), (22), and (24), respectively, we now have

$$(aA)(t) = {}^TP(t)\bar{s}_{\overline{r-a}|} \,, \tag{36}$$

$$P(t) = {}^TP(t)e^{(\tau-\delta)[r-x(\theta)]} \,, \tag{37}$$

$$(aV)(t) = {}^TP(t)\bar{s}_{\overline{r-x(\theta)}|} \,, \tag{38}$$

$$(aV)(t) = P(t)\bar{a}_{\overline{r-x(\theta)}|} \,, \tag{39}$$

$$(Pa)(t) = P(t)\bar{s}_{\overline{N}|} \,, \tag{40}$$

where each of the compound interest functions is evaluated at annual force of interest $\tau - \delta$.

The application of Jensen's inequality yields

$$x(\theta) < (a + r)/2 \, ,$$

where $x(\theta)$ pertains to a pro rata accrued benefit cost method. This is the opposite of the relation for $\delta > \tau$ found in inequality (27). In this case it seems difficult to obtain a useful general statement about $x(\theta)$ for the entry age normal cost method.

b) *Income allocation.*—The income allocation equation (19) takes on the form

$$P(t) = {}^{T}P(t) + (\tau - \delta)(\boldsymbol{a}V)(t) \, .$$

That is, the normal cost must provide not only for the terminal funding cost but also for the growth required in the supplemental present value, $(\boldsymbol{a}V)(t)$, in excess of interest income. In case 3, terminal funding has the lowest cost among the methods that complete funding by age r. Further, if one cost method defines a higher supplemental value than a second cost method, the normal cost for the first method will have a *higher* normal cost than the second. Thus in case 3, among the cost methods completing funding during the working lifetime of members, initial funding would result in not only the highest supplemental present value $(\boldsymbol{a}V)(t)$ but also the highest normal cost.

c) *Discussion.*—The practical implications of interest rates below the sum of the growth rates of salaries and population are enormous. Traditional economic arguments in favor of funded pensions begin to lose their validity. The change to current cost funding of pensions in nations with a high rate of wage inflation and relatively low interest rates would seem to be a realization of this theoretical result (see [7] for a discussion of this important point).

In this paper the stress has been on contribution theory. Current cost funding, or pay-as-you-go funding, has not been one of the cost methods under consideration. We have not made any assumptions about the benefit adjustment function $\beta(x)$. However, it is clear that there are many patterns of postretirement benefit adjustments that would result in a current cost rate below the terminal funding cost rate.

IV. CONTRIBUTION THEORY

In this section several patterns of contribution rates are developed. These patterns are selected to build up funds to meet the cost of the plan in regard to active members. With slight changes, the theory could be developed for the whole group, active or retired. The contribution patterns will be related to individual actuarial cost methods. While the patterns could be developed in a more general context, the exponential growth case will be assumed throughout. Amortization factors $(1/\bar{a}_{\overline{n}|})$ will be evaluated at force $\theta = \delta - \tau$ to provide for amortization of the unfunded supplemental present value as a level percentage of payroll that is increasing at an annual rate of $\tau = \alpha + \gamma$, the sum of the population and salary growth rates. This will be done in lieu of considering amortization by level amounts.

Normal Cost plus Amortization over Fixed Term

The objective is to reach fully funded status for some actuarial cost method at the end of n years measured from an arbitrary initial time. For convenience the initial time will be denoted by zero. The annual contribution rate $(\boldsymbol{aC})(t)$ at time t, $0 \leq t < n$, in regard to active members is denoted by the formula

$$(\boldsymbol{aC})(t) = \boldsymbol{P}(t) + 0.01f\,\boldsymbol{W}(t)\,.$$

In this equation $\boldsymbol{W}(t)$ is the rate of payroll payment at time t (see formula [77] in [2]), and f is a level percentage of payroll determined so as to amortize the initial unfunded supplemental present value over n years. If $(\boldsymbol{aF})(t)$ denotes the fund available for the active members at time t, and $(\boldsymbol{aU})(t) = (\boldsymbol{aV})(t) - (\boldsymbol{aF})(t)$ is the *unfunded supplemental present value* for active members at time t, then

$$\begin{aligned}(\boldsymbol{aU})(0) = (\boldsymbol{aV})(0) - (\boldsymbol{aF})(0) &= 0.01f\int_0^n e^{-\delta t}\boldsymbol{W}(t)dt \\ &= 0.01f\,\boldsymbol{W}(0)\int_0^n e^{-\delta t}e^{\tau t}dt \\ &= 0.01f\,\boldsymbol{W}(0)\bar{a}_{\overline{n}|}\,,\end{aligned}$$

where $\bar{a}_{\overline{n}|}$ is calculated at force $\delta - \tau$ as previously noted. In this result we have used the fact that $\boldsymbol{W}(t) = e^{\tau t}\boldsymbol{W}(0)$, and in developing formula (41) we also shall use the fact that $\boldsymbol{P}(t) = e^{\tau t}\boldsymbol{P}(0)$. These results are achieved in [2], p. 199. Now substituting in the expression for the annual contribution rate, we have

$$\begin{aligned}(\boldsymbol{aC})(t) &= \boldsymbol{P}(t) + \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}\frac{\boldsymbol{W}(t)}{\boldsymbol{W}(0)} \\ &= \boldsymbol{P}(t) + \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}e^{\tau t} \qquad (41) \\ &= \left[\boldsymbol{P}(0) + \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}\right]e^{\tau t}\,.\end{aligned}$$

From these formulas it is clear that $(\boldsymbol{aC})(t)/\boldsymbol{W}(t)$ is a constant, namely,

$$\left[\boldsymbol{P}(0) + \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}\right]\boldsymbol{W}(0)^{-1} .$$

We now consider how convergence to a fully funded status occurs. With contributions determined by $(\boldsymbol{aC})(t)$ as in formula (41), the active member fund $(\boldsymbol{aF})(t)$ grows according to the differential equation

$$\begin{aligned}\frac{d(\boldsymbol{aF})(t)}{dt} &= (\boldsymbol{aC})(t) + \delta(\boldsymbol{aF})(t) - {}^{T}\boldsymbol{P}(t) \\ &= \boldsymbol{P}(t) + \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}e^{\tau t} + \delta(\boldsymbol{aF})(t) - {}^{T}\boldsymbol{P}(t) .\end{aligned} \tag{42}$$

The income allocation formula (7) may be rearranged as

$$\frac{d(\boldsymbol{aV})(t)}{dt} = \boldsymbol{P}(t) + \delta(\boldsymbol{aV})(t) - {}^{T}\boldsymbol{P}(t) .$$

When formula (42) is subtracted from this expression, the result is

$$\frac{d(\boldsymbol{aU})(t)}{at} = \delta(\boldsymbol{aU})(t) - \frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}e^{\tau t} . \tag{43}$$

Changing t to h, multiplying through by the integrating factor $e^{-\delta h}$, and rearranging yield

$$d[e^{-\delta h}(\boldsymbol{aU})(h)] = -\frac{(\boldsymbol{aU})(0)}{\bar{a}_{\overline{n}|}}e^{-(\delta-\tau)h}dh .$$

Integrating from 0 to t and solving for $(\boldsymbol{aU})(t)$, we have

$$\begin{aligned}(\boldsymbol{aU})(t) &= (\boldsymbol{aU})(0)e^{\delta t} - (\boldsymbol{aU})(0)e^{\tau t}\bar{s}_{\overline{t}|}\bar{a}_{\overline{n}|}^{-1} \\ &= (\boldsymbol{aU})(0)e^{\tau t}[e^{(\delta-\tau)t} - \bar{s}_{\overline{t}|}\bar{a}_{\overline{n}|}^{-1}] \\ &= (\boldsymbol{aU})(0)e^{\tau t}\bar{a}_{\overline{n-t}|}/\bar{a}_{\overline{n}|} .\end{aligned} \tag{44}$$

It may be verified that formula (44) holds whether δ is greater than, equal to, or less than τ. It is now clear from formula (44) that $(\boldsymbol{aU})(n) = 0$, which implies that $(\boldsymbol{aF})(n) = (\boldsymbol{aV})(n)$. Also, from formulas (43) and (44) we note that

$$\frac{d(\boldsymbol{aU})(t)}{dt} = (\boldsymbol{aU})(t)(\delta - \bar{a}_{\overline{n-t}|}^{-1}) , \tag{45}$$

and if $\bar{a}_{\overline{n}|}^{-1} < \delta$ then $d(\boldsymbol{aU})(t)/dt > 0$ until time t_0, when $\bar{a}_{\overline{n-t_0}|}^{-1} = \delta$. That is, $(\boldsymbol{aU})(t)$ increases to a maximum at time t_0 and then decreases.

In practice one might select what appears to be a more flexible contribution system, to be denoted by superimposing tildes on the functions, such that

$$(\boldsymbol{a\tilde{C}})(t) = \boldsymbol{P}(t) + (\boldsymbol{a\tilde{U}})(t)/\bar{a}_{\overline{n-t}|}\,, \tag{46}$$

with $(\boldsymbol{a\tilde{U}})(0) = (\boldsymbol{aV})(0) - (\boldsymbol{aF})(0) = (\boldsymbol{aU})(0)$. Again the objective is to amortize the unfunded supplemental present value over a fixed n years. In application, formula (46) allows for spreading experience gains and losses. However, in the deterministic model we are studying, gains and losses do not appear, and we might guess that identical results will be produced by contributions defined by formulas (41) and (46). To confirm this guess we can write the equation analogous to formula (42), subtract it from the income allocation equation, and obtain

$$\frac{d(\boldsymbol{a\tilde{U}})(t)}{dt} = (\boldsymbol{a\tilde{U}})(t)(\delta - \bar{a}_{\overline{n-t}|}^{-1})\,, \tag{47}$$

which is the same differential equation as (45). Since the two differential equations involve the same initial condition, we can conclude that $(\boldsymbol{a\tilde{U}})(t) = (\boldsymbol{aU})(t)$, $0 \leq t \leq n$. This in turn implies that $(\boldsymbol{a\tilde{C}})(t) = (\boldsymbol{aC})(t)$, $0 \leq t \leq n$, which may be confirmed by substituting from formula (44) in formula (46) and comparing with formula (41).

Normal Cost plus Amortization over a Moving Term

In this case, the objective is to attain fully funded status for some actuarial cost method asymptotically. For $t \geq 0$, the annual contribution rate at time t for active members, $(\boldsymbol{aC})(t)$, is defined by the formula

$$(\boldsymbol{aC})(t) = \boldsymbol{P}(t) + (\boldsymbol{aU})(t)/\bar{a}_{\overline{n}|}\,. \tag{48}$$

That is, amortization is over a term of n years from the current time t rather than from time zero. The amortization term continually moves forward.

The formula analogous to formula (43), derived by a similar chain of steps using $(\boldsymbol{aC})(t)$ as in formula (48), is

$$\frac{d}{dt}(\boldsymbol{aU})(t) = -(\bar{a}_{\overline{n}|}^{-1} - \delta)(\boldsymbol{aU})(t)\,. \tag{49}$$

Solving this differential equation yields

$$(\boldsymbol{aU})(t) = (\boldsymbol{aU})(0)\exp\left[-(\bar{a}_{\overline{n}|}^{-1} - \delta)t\right]. \tag{50}$$

Now if $a_{\overline{n}|}^{-1} > \delta$, then

$$\lim_{t\to\infty} [(aV)(t) - (aF)(t)] = \lim_{t\to\infty} (aU)(t) = 0 \tag{51}$$

and

$$(aF)(t) \to (aV)(t) \quad \text{as} \quad t \to \infty\ . \tag{52}$$

It is clear that convergence as described in formula (52) will occur for certain sets of (n, δ, τ). For example, the conditions for convergence may be written as

$$n < \frac{\ln\delta - \ln\tau}{\delta - \tau}, \qquad \delta \neq \tau$$
$$< \frac{1}{\delta}, \qquad \delta = \tau\ .$$

If $\delta > 0$ and $\tau = 0$, any n, $0 < n < \infty$, will assure convergence. If $\delta = \tau$ and $n < 1/\delta$, then as $\tau \to \infty$ the upper bound on n approaches zero. The point is that if pension obligations are growing very rapidly as a result of a large total growth rate relative to the interest rate, the rolling amortization period n must be small to assure convergence.

If $\bar{a}_{\overline{n}|}^{-1} = \delta$ the functions $(aF)(t)$ and $(aV)(t$ [illegible] bounded, but

$$(aU)(t) = (aU)([illegible] , \tag{53}$$

and no progress is made toward reducing the unfunded present supplemental value in an absolute sense. However, in a relative sense progress is made. This may be seen by rearranging formula (53) as

$$(aV)(t) - (aF)(t) = (aV)(0) - (aF)(0)\ .$$

In the exponential growth case, we may observe that formula (21) tells us that $(aV)(t)$ grows at the same rate as terminal funding cost ${}^TP(t)$. From formula (6) we learn that ${}^TP(t)$ grows as $h(t)$ grows, and in the exponential growth case $h'(t) = \tau h(t)$. Therefore, $(aV)(t) = (aV)(0)e^{\tau t}$, and we may divide each side of our re-formed formula (53) by this expression to obtain

$$1 - \frac{(aF)(t)}{(aV)(t)} = \left[1 - \frac{(aF)(0)}{(aV)(0)}\right]e^{-\tau t}\ .$$

This implies that if $\tau > 0$, then

$$\lim_{t\to\infty}\left[1 - \frac{(aF)(t)}{(aV)(t)}\right] = 0\ ,$$

or

$$\lim_{t\to\infty} [(aF)(t)][(aV)(t)]^{-1} = 1\ . \tag{54}$$

If $\bar{a}_{\overline{n}|}^{-1} < \delta$, then $(aU)(t)$ increases indefinitely as $t \to \infty$. Again there is a relative kind of convergence. Rearrange formula (50) as follows:

$$(aV)(t) - (aF)(t) = [(aV)(0) - (aF)(0)] \exp[-(\bar{a}_{\overline{n}|}^{-1} - \delta)t]\ .$$

We then divide by $(aV)(t) = (aV)(0)e^{\tau t}$ to obtain

$$1 - \frac{(aF)(t)}{(aV)(t)} = \left[1 - \frac{(aF)(0)}{(aV)(0)}\right] \exp\{-[\bar{a}_{\overline{n}|}^{-1} - (\delta - \tau)]t\}$$

$$= \left[1 - \frac{(aF)(0)}{(aV)(0)}\right] \exp(-\bar{s}_{\overline{n}|}^{-1}t)$$

where $\bar{s}_{\overline{n}|}$ is valued at $\delta - \tau$ and n is a finite number. In this case formula (54) holds once more and the same type of relative convergence takes place. The unfunded supplemental present value becomes indefinitely large, but the funding ratio $(aF)(t)/(aV)(t)$ approaches 1.

The ratio of the annual contribution rate defined by formula (48) to the rate of payroll payment, $W(t)$, each evaluated at time t, is a decreasing function of t. To establish this fact we use the same ideas as for formula (41). We start with formula (48) and substitute formula (50) to obtain

$$\frac{(aC)(t)}{(W)(t)} = \frac{P(t)}{W(t)} + \frac{(aU)(t)}{\bar{a}_{\overline{n}|}W(t)}$$

$$= \frac{P(0)}{W(0)} + \left[\frac{(aU)(0)}{\bar{a}_{\overline{n}|}W(0)}\right] \exp(-\bar{s}_{\overline{n}|}^{-1}t) .$$

Since $0 < \bar{s}_{\overline{n}|}^{-1}$, $(aC)(t)/W(t)$ decreases as t increases, and

$$\lim_{t\to\infty} \frac{(aC)(t)}{W(t)} = \frac{P(0)}{W(0)}, \tag{55}$$

that is, the ratio of the contribution rate to the payroll rate approaches the ratio at time zero of the normal cost rate to the payroll rate.

Aggregate Cost Method

It is natural to ask the following: what is the result if, for the normal cost plus amortization over a moving term method, discussed in the preceding subsection, the term n is taken equal to $N = x(\theta) - a$, the future funding term for the actuarial cost method defined by the accrual function $M(x)$? We then would have

$$(aC)(t) = P(t) + (aU)(t)/\bar{a}_{\overline{N}|}, \tag{56}$$

which may be rearranged as

$$\begin{aligned}(aC)(t) &= [P(t)\bar{a}_{\overline{N}|} + (aV)(t) - (aF)(t)]/\bar{a}_{\overline{N}|} \\ &= [(aA)(t) - (aF)(t)]/\bar{a}_{\overline{N}|},\end{aligned} \tag{57}$$

since $P(t)\bar{a}_{\overline{N}|} = (\boldsymbol{Pa})(t)$ by formula (24) and $(\boldsymbol{aV})(t) = (\boldsymbol{aA})(t) - (\boldsymbol{Pa})(t)$. In the notation of formula (56) of [2], $(\boldsymbol{Pa})(t)/P(t)$ equals $\bar{a}(t)$, a mean temporary annuity. However, because we are discussing the exponential growth case, $\bar{a}(t) = \bar{a}_{\overline{N}|}$ is independent of t ([2], formula [85]). We now recognize that

$$(\boldsymbol{aC})(t) = [(\boldsymbol{aA})(t) - (\boldsymbol{aF})(t)]/\bar{a}(t) \tag{58}$$

is the aggregate cost contribution rate defined by formula (58) of [2]. This demonstrates that, for the model plan and exponential growth case, contributions determined by normal cost plus amortization over a moving term N (the future funding term of the given actuarial cost method) are the same as the contributions under aggregate cost funding with mean temporary annuity value $\bar{a}(t)$. Further, since the contribution rates are the same in the two cases, the same fund $(\boldsymbol{aF})(t)$ develops, and the funds' convergence in relation to $(\boldsymbol{aV})(t)$ is the same as indicated in the preceding subsection. Finally, formula (55) shows that $(\boldsymbol{aC})(t)/\boldsymbol{W}(t) \rightarrow \boldsymbol{P}(0)/\boldsymbol{W}(0)$ as $t \rightarrow \infty$, which is to be expected under the aggregate cost method.

In a more general case where exponential growth is not assumed, one can show that the contribution rate defined by

$$(\boldsymbol{aC})(t) = \boldsymbol{P}(t) + (\boldsymbol{aU})(t)/\bar{a}(t) \tag{59}$$

is equivalent to the aggregate cost contribution rate in formula (58). In this case, the amortization annuity value $\bar{a}(t)$ may vary with t.

The discussion here is related to the generalized aggregate cost method first described by Trowbridge [5] and further described in Trowbridge and Farr ([6], p. 62), where the role of $\bar{a}^{-1}_{\overline{N}|}$ is played by the constant k.

V. ANALOGOUS THEORY, ALL MEMBERS

In the preceding two sections a body of theory about pension contributions for the subgroup of active members was developed. A parallel theory exists for the whole group, both actives and retireds. The development of this parallel theory is very similar to that for the active subgroup, and therefore it does not seem necessary to present full details. Instead, some of the modifications of key formulas will be given and an outline of the theory indicated.

The concepts and corresponding symbols to be used in this section also appear in early sections of this paper or in [2]. For example, $\boldsymbol{P}(t)$, the normal cost rate at time t, and $(\boldsymbol{Pa})(t)$, the present value of future normal costs for active members, remain as defined in Section II. Table 1 provides

TABLE 1

BASIC SYMBOLS

	Whole Group		Active Group	
	Symbol	Definition	Symbol	Definition
Value of benefits	$A(t)$	[2], formula (28)	$(aA)(t)$	Formula (3)
Contribution rate	$C(t)$	[2], page 192	$(aC)(t)$	Formulas (41) and (48)
Fund	$F(t)$	[2], formula (54)	$(aF)(t)$	Formula (42)
Unfunded supplemental present value	$U(t)$	[2], formula (49)	$(aU)(t)$	Explanation of formula (41)
Supplemental present value	$V(t)$	[2], formula (37)	$(aV)(t)$	Formula (5)

a glossary of other needed symbols. Some reference to [2] is required to complete the developments in this section.

Within the larger framework of this section, we work with $B(t)$ ([2], formula [26]), the annual rate of pension outgo at time t, instead of with ${}^{T}P(t)$, the terminal funding annual cost rate. For simplicity, the discussion will be limited to the exponential growth case.

As a replacement of the income allocation equation, formula (7), one now has

$$P(t) + \delta V(t) = B(t) + \frac{dV(t)}{dt}, \tag{60}$$

which is formula (43) of [2]. For the exponential growth case, this becomes

$$P(t) + \delta V(t) = B(t) + \tau V(t), \tag{61}$$

which, for $\delta \neq \tau$, may be rearranged as

$$V(t) = [B(t) - P(t)]/(\delta - \tau). \tag{62}$$

Also, in the exponential growth case, there are the following relations:

$$\begin{aligned} B(t) &= \int_r^\infty e^{\tau(t+r-x)} s(r) b l_x \beta(x) dx \\ &= e^{\tau t} l_r s(r) b \bar{a}_r'^{\beta}, \end{aligned} \tag{63}$$

where $\bar{a}_r'^{\beta}$ is valued at force of interest τ. Then

$$\begin{aligned} B(t) &= e^{\tau t} l_r s(r) b \bar{a}_r^{\beta} (\bar{a}_r'^{\beta}/\bar{a}_r^{\beta}) \\ &= {}^{T}P(t) (\bar{a}_r'^{\beta}/\bar{a}_r^{\beta}). \end{aligned} \tag{64}$$

Thus, if δ (the force of interest) is greater than τ (the force of total growth), then $B(t) > {}^T P(t)$; if $\delta < \tau$, then $B(t) < {}^T P(t)$; and if $\delta = \tau$, then $B(t) = {}^T P(t)$.

Further, according to the discussion following formula (12), the annual normal cost rate at time t under initial funding is given by

$$^I P(t) = v^{r-a} \; {}^T P(t) , \tag{65}$$

where, in accordance with the earlier convention, the discount factor is valued at force $\theta = \delta - \tau$. At first we assume $\delta \neq \tau$. Then, upon substitution from formula (64),

$$^I P(t) = B(t)(\bar{a}^{\beta}_r / \bar{a}'^{\beta}_r) v^{r-a} . \tag{66}$$

Before proceeding with the development, we will examine the function $\bar{a}^{\beta}_r v^r / \bar{a}'^{\beta}_r$. We have

$$\bar{a}^{\beta}_r v^r / \bar{a}'^{\beta}_r = \int_r^{\infty} e^{-(\delta-\tau)y} \left\{ e^{-\tau(y-r)} {}_{y-r}p_r \beta(y) \Big[\int_r^{\infty} e^{-\tau(y-r)} {}_{y-r}p_r \beta(y) dy \Big]^{-1} \right\} dy$$

$$= E[e^{-(\delta-\tau)Y}] = e^{-(\delta-\tau)y(\theta)} ,$$

where the expectation is taken with respect to the density function within the braces. The number $y(\theta)$ exceeds r as a result of the mean-value theorem for integrals. We find that

$$\lim_{\tau \to \delta} y(\theta) = r + E[Y - r] = \bar{y} ,$$

where $\bar{y}$ is the expected value associated with the density function within braces when $\delta = \tau$. Thus $\bar{y}$ may be interpreted as the *average age of pension payment.*

With this result, we may write formula (66) as

$$^I P(t) = B(t) e^{-\theta[y(\theta)-a]} . \tag{67}$$

If $\delta > \tau$, then $^I P(t) < B(t)$; if $\delta = \tau$, then $^I P(t) = B(t)$; and if $\delta < \tau$, then $^I P(t) > B(t)$.

Next, since $A(t)$ equals $^I V(t)$, the supplemental present value under initial funding, and since the expanded income allocation equation and formula (62) hold for initial, terminal, and continuous cost methods, one finds that

$$A(t) = {}^I V(t) = [B(t) - {}^I P(t)]/(\delta - \tau) .$$

By substituting from formula (67), we have

$$\begin{aligned} A(t) &= B(t)(1 - v^{y(\theta)-a})/(\delta - \tau) \\ &= B(t) \bar{a}_{\overline{y(\theta)-a|}} . \end{aligned} \tag{68}$$

Here $y(\theta)$ is determined as in formula (67). From formulas (65) and (67) we obtain

$$^{T}P(t) = B(t)e^{-\theta[y(\theta)-r]} . \tag{69}$$

Formula (69) then permits the transformation of formula (37) or (14) into

$$P(t) = B(t)e^{-\theta[y(\theta)-x(\theta)]} , \tag{70}$$

and use of formula (62) leads to

$$V(t) = B(t)\bar{a}_{\overline{y(\theta)-x(\theta)}|} . \tag{71}$$

Finally, from formulas (71) and (68) we obtain

$$(Pa)(t) = A(t) - V(t) = B(t)v^{y(\theta)-x(\theta)}\bar{a}_{\overline{x(\theta)-a}|} = P(t)\bar{a}_{\overline{x(\theta)-a}|} , \tag{72}$$

which is a restatement of formula (24).

Table 2 shows the correspondence between the formulas of this section and Section III.

TABLE 2

CORRESPONDENCE AMONG FORMULAS

Entire Group (Sec. V)	Active Group ($\delta > \tau$) (Sec. III)
(68)	(10)
(70)	(15)
(71)	(21)
(72)	(24)

The concepts of this section may be clarified by the Lexis-type diagram in Figure 2. The general format is similar to that of Figure 1. The dashed diagonal line again may be viewed as the mean path followed by the group active at time t. The paths to be followed in tracing the relationships are indicated by relationship numbers. Payments distributed along horizontal lines are of different amounts because of τ and have different present values because of δ. Normal cost payments made at the same time (same vertical line) differ because they are made on behalf of different cohorts and because of the different periods of time until they are used to pay retirement benefits ($\theta = \delta - \tau$). The relationship numbers here denote the following:

(1) Formula (67): $^{I}P(t) = B(t)e^{-\theta[y(\theta)-a]}$

(2) Formula (68): $A(t) = B(t)\bar{a}_{\overline{y(\theta)-a}|}$

(3) Formula (69): $^{T}P(t) = B(t)e^{-\theta[y(\theta)-r]}$

(4) Formula (70): $P(t) = B(t)e^{-\theta[y(\theta)-x(\theta)]}$

(5) Formula (71): $V(t) = B(t)\bar{a}_{\overline{y(\theta)-x(\theta)}|}$

(6) Formula (72): $(Pa)(t) = P(t)\bar{a}_{\overline{x(\theta)-a}|}$

Additional relationships can be seen by dividing $A(t)$ and $V(t)$ into components for active and retired lives. For example, $(rV)(t)$, the present value of benefits for retired lives, is represented by $B(t)\bar{a}_{\overline{y(\theta)-r}|}$, relationship (7), or by ${}^{T}P(t)\bar{s}_{\overline{y(\theta)-r}|}$, relationship (8).

Of course, one may examine the formulas that correspond to formulas (31)–(34) for $\delta = \tau$ and to formulas (36)–(39) for $\delta < \tau$. The theory of Section IV concerning contributions for active lives may be recast in terms of the whole group. The changes are primarily notational, except that the total rate of pension outgo $B(t)$ plays the role of the terminal funding cost ${}^{T}P(t)$ and the global income allocation formula (61) is used.

VI. CONCLUSION

In this paper a theory of contributions to fund pensions during workers' periods of employment, under dynamic economic and demographic as-

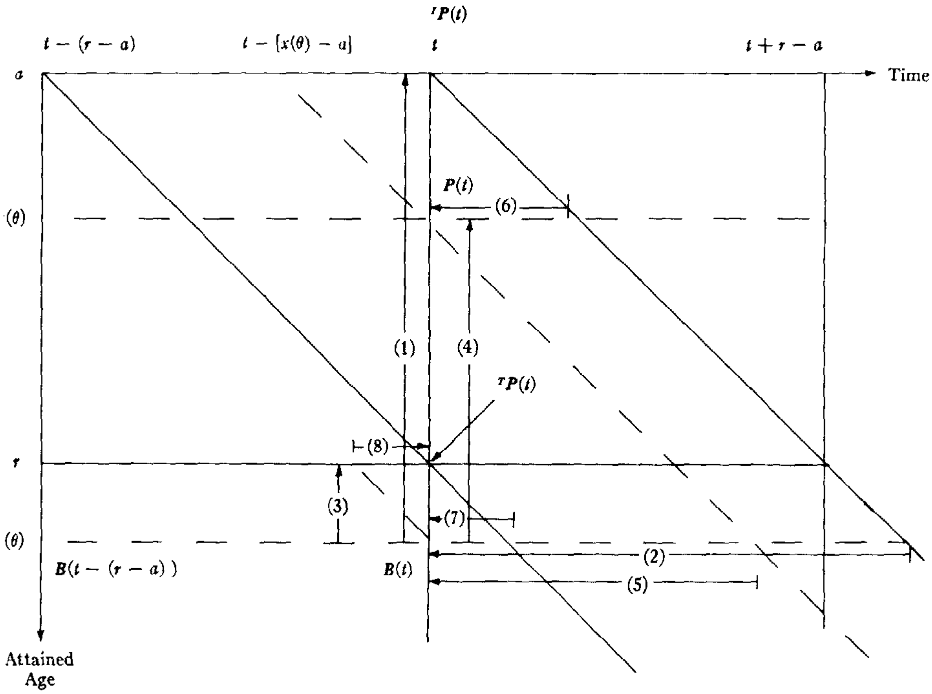

FIG. 2.—Illustration of relations among funding functions: whole population; exponential growth.

sumptions, has been developed. Relationships among the contribution patterns that may arise under different cost methods also have been developed. The theory has economic implications. For example, if the sum of the rates of population increase and salary increase exceeds the interest rate, the terminal funding normal cost is below that of any cost method funding pensions during the working lifetimes of members.

REFERENCES

1. Bowers, Newton L., Jr. "An Approximation of the Distribution of Annuity Costs," *TSA*, XIX (1967), 295.
2. Bowers, Newton L., Jr., Hickman, James C.; and Nesbitt, Cecil J. "Introduction to the Dynamics of Pension Funding," *TSA*, XXVIII (1976), 177.
3. Cooper, Steven L., and Hickman, James C. "A Family of Accrued Benefit Actuarial Cost Methods," *TSA*, XIX (1967), 53.
4. Olson, Gary E. "On Some Actuarial Inequalities," *TSA*, XXVII (1975), 11; discussion by Hans U. Gerber and Donald A. Jones, p. 24.
5. Trowbridge, Charles L. "The Unfunded Present Value of Pension Funding Methods," *TSA*, XXV (1963), 15.
6. Trowbridge, Charles L., and Farr, Charles E. *The Theory and Practice of Pension Funding*. Homewood, Ill.: Richard D. Irwin, 1976.
7. Trowbridge, J. R. "Assessmentism—an Alternative to Pension Funding," *JIA*, CIV (1977), 173.

APPENDIX

PROOF OF FORMULA (30), $\delta > \tau$

The pension purchase density function associated with entry age normal funding is given by

$$m(x) = e^{-(\delta-\gamma)x}s(x)l_x \Big/ \int_a^r e^{-(\delta-\gamma)y}s(y)l_y dy\,, \qquad 0 \le x < r\,.$$

Upon substituting in formula (13) and using $\theta = \delta - \tau = \delta - \alpha - \gamma > 0$, we have

$$\begin{aligned} e^{\theta \bar{x}(\theta)} &= \int_a^r e^{\theta x}\left[e^{-(\delta-\gamma)x}s(x)l_x \Big/ \int_a^r e^{-(\delta-\gamma)y}s(y)l_y dy\right]dx \\ &= \int_a^r e^{-\alpha x}s(x)l_x dx \Big/ \int_a^r e^{-(\delta-\tau)y-\alpha y}s(y)l_y dy\,, \end{aligned}$$

so that

$$e^{-\theta \bar{x}(\theta)} = \int_a^r e^{-(\delta-\tau)y}\left[e^{-\alpha y}s(y)l_y \Big/ \int_a^r e^{-\alpha x}s(x)l_x dx\right]dy\,.$$

The function within the brackets may be interpreted as a density function. Using Jensen's inequality, we have

$$e^{-\theta x(\theta)} = E[e^{-\theta Y}] > e^{-\theta E[Y]} .$$

Now if $e^{-\alpha y}s(y)l_y$ is a decreasing function, $E[Y] < (a + r)/2$. Using this result, we may strengthen the previous inequality as follows:

$$e^{-\theta x(\theta)} > e^{-\theta E[Y]} > e^{-\theta[(a+r)/2]}$$

or

$$x(\theta) < \frac{a + r}{2} ,$$

$$x(\theta) - a = N < \frac{r - a}{2} .$$

C. L. TROWBRIDGE:

Mr. Harry Sarason, in his discussion of the earlier (1976) paper by these same three authors, predicted that the second paper in the series might bear the title "The Dynamics of Pension Funding." Have the authors added the final two words to the title of this most recent work simply to prove that Mr. Sarason isn't 100 percent clairvoyant? Perhaps we shall find out when still another of the series appears, as I suspect it will. There seems to be no end to the development of this elusive subject.

Certainly Drs. Bowers, Hickman, and Nesbitt have come up with a worthwhile extension of their previous work. The mathematical model is the same, but the analyses are carried further. Some simplification is obtained by splitting apart the active and retired portions of the mathematical analysis, in accordance with a suggestion from Mr. Kischuk, an approach which leads into what the authors call contribution theory. This discussant has not encountered this terminology before, but it strikes him as appropriate.

One new function that the careful reader will note is $x(\theta)$, calculated from equation (13). This is the attained age at which the normal contribution $\boldsymbol{P}(t)$ can be viewed as "centered." In the special cases of initial and terminal funding, $x(\theta)$ becomes a and r, respectively. Otherwise $a < x(\theta) < r$, the exact place of $x(\theta)$ within this range depending on both the actuarial cost method and the "excess" interest function θ.

As in the earlier paper, it is important to look at the three cases $\delta > \tau$, $\delta = \tau$, and $\delta < \tau$ separately. For many applications α can be treated as zero, and, since $\tau = \alpha + \gamma$, the three cases become $\delta > \gamma$, $\delta = \gamma$, and $\delta < \gamma$. Advance funding loses much of its theoretical appeal in the last of these. This discussant's British namesake, whose *JIA* paper is No. 7 in the reference list, clearly shows why "pay-as-you-go" is predominant in France. The authors warn us when they note that "the practical implications of interest rates below . . . the growth rates of salaries . . . are enormous." We will do well to heed this warning.

JOHN W. PENNISTEN:

The authors' paper is an outstanding addition to the literature on the funding of pension plans.

I. *Relative Levels of Contributions under Different Funding Methods*

When alternate funding methods for a particular plan are considered, it is natural to consider the patterns of annual contributions generated by the methods. For example, it is known that the normal cost of a unit credit funding method will increase as the average age of the covered active population increases; but to what ultimate level will it rise?

Applying the "equation of maturity" for stationary populations with mature pension funds from [2], we have

$$C^1 + dF^1 = B^1 , \qquad C^2 + dF^2 = B^2 .$$

Since plan benefits are independent of the funding method, $B^1 = B^2$, and

$$C^2 - C^1 = d(F^2 - F^1) . \tag{1}$$

That is, when the population covered by a pension plan (actives, vested terminations, pensioners, and survivor beneficiaries) has matured and become stationary, and the funding under alternate methods would have progressed to the point where there was no change in the fund accumulated under either method from one year to the next, the contribution required under one funding method would differ from that under the other method by the amount of the discounted interest earnings on the difference between the accumulated funds.

A more general result, using continuous functions, may be obtained by applying the "liability growth equation" from the authors' previous paper [1]:

$$P^1(t) + \delta V^1(t) = B^1(t) + \frac{dV^1(t)}{dt} ;$$

$$P^2(t) + \delta V^2(t) = B^2(t) + \frac{dV^2(t)}{dt} .$$

Again, $B^1(t) = B^2(t)$ regardless of the funding method, so that

$$P^2(t) - P^1(t) = \delta[V^1(t) - V^2(t)] + \left[\frac{dV^2(t)}{dt} - \frac{dV^1(t)}{dt}\right] . \tag{2}$$

For the exponential growth case examined by the authors,

$$\frac{dV(t)}{dt} = \tau V(t)$$

and

$$P^2(t) - P^1(t) = \theta[V^1(t) - V^2(t)] . \tag{3}$$

It should be noted (*a*) that both equations (2) and (3) measure differences in normal cost, rather than total contributions, and (*b*) that they are independent of any funding that may have occurred.

The results in this section of the discussion are illustrated in Tables 1–7 in the next section.

II. Numerical Examples

At the end of Section I of their paper, the authors request numerical examples. Although the examples given in this section of the discussion support the concepts of the paper, they also supplement examples given elsewhere (see [2], [3], and [4]) and develop the authors' previous comments in [1] regarding directions for further work (see Sec. III of this discussion).

The most significant difference between the projections in this discussion and projections given elsewhere is that one hundred and twenty-five years, instead of fifty years, are shown. This helps to illustrate (*a*) the progress of funding after the original group of active employees and pensioners has died and been completely replaced, (*b*) the large amounts that emerge ultimately even with low rates of exponential growth, and (*c*) the magnitude of the asymptoticity of the amortization of experience gains and losses under frozen initial liability funding methods.

The projections in this section are based on curtate, rather than continuous, functions. "Plan termination liabilities" are based on 100 percent vesting of all benefits based on salary and service accrued at date of termination. Annual plan valuations, including those in Table 5, are on a "closed group" basis, that is, excluding future new entrants. Under the terminal funding method, any initial accrued liabilities are amortized over ten years to prevent negative assets in the early years.

The pension plan in Table 1 requires no employee contributions and provides 1 percent of final salary for each year of service upon normal retirement at age 65. There are no preretirement death, disability, or withdrawal benefits and no postretirement death or survivor benefits (straight life annuity form only). New employees are assumed to enter at age 30 and there are no withdrawals from active employment other than by death or normal retirement. Pre- and postretirement mortality is assumed to follow the 1971 Group Annuity Mortality Table (1971 GAM) for men with radix of 100 at age 65, so that there are 122.70 new entrants each year at age 30, $(L_{30} - L_{65}) = 4{,}091.57$ active employees, and $L_{65} = 1{,}561.20$ pensioners. Annual interest and salary scale under noninflationary conditions are assumed to be $4\frac{1}{2}$ and $2\frac{1}{2}$ percent, respectively. Each of the new employees at age 30 is assumed to start with a \$10,000 annual salary, so that each new retiree at age 65 has an annual benefit of $(0.01)(35)(\$10{,}000)(1.025)^{35}$, or \$8,306. In Table 1 there are no automatic postretirement benefit increases, no population growth or decline, and no experience gains or losses.

In Table 1, and all other tables of this discussion, normal cost under

TABLE 1

INITIALLY MATURE POPULATION; 4½ PERCENT INTEREST, 2½ PERCENT SALARY SCALE (NONINFLATIONARY); NO EXPERIENCE GAINS OR LOSSES; FINAL SALARY PLAN; NO AUTOMATIC POSTRETIREMENT BENEFIT INCREASES; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$63.4	$178.8	$13.0	$20.2	$ 7.6	$4.5	$11.6	$ 3.2	6.33%	$12.2	$ 3.4
2....	63.4	178.8	13.0	20.2	15.5	4.5	11.6	6.6	6.33	12.2	7.0
3....	63.4	178.8	13.0	20.2	23.8	4.5	11.6	10.2	6.33	12.2	10.7
4....	63.4	178.8	13.0	20.2	32.5	4.5	11.6	13.8	6.33	12.2	14.6
5....	63.4	178.8	13.0	20.2	41.5	4.5	11.6	17.7	6.33	12.2	18.6
10...	63.4	178.8	13.0	20.2	93.3	4.5	11.6	39.8	6.33	12.2	41.9
11...	63.4	178.8	13.0	8.9	93.3	4.5	11.6	44.8	6.32	12.2	47.2
15...	63.4	178.8	13.0	8.9	93.3	4.5	11.6	67.3	6.33	12.2	70.8
20...	63.4	178.8	13.0	8.9	93.3	4.5	11.6	101.5	6.33	12.2	106.9
25...	63.4	178.8	13.0	8.9	93.3	4.5	11.6	144.2	6.33	12.2	151.8
30...	63.4	178.8	13.0	8.9	93.3	4.5	11.6	197.5	6.33	12.2	207.9
31...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
40...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
41...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
50...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
75...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
100..	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
125..	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

the frozen initial liability method is shown as a percentage of payroll. Thus the pension fund (in millions of dollars) at the end of the fifth year may be derived from the pension fund at the end of the fourth year as follows: [(\$14.6 + (0.0633 × \$63.4) + \$12.2 − \$13.0)] × 1.045 = \$18.6 million.

It can also be seen from Table 1 (using amounts in *thousands* of dollars) that, after the thirtieth year, the difference between the normal cost under the unit credit method and that under the frozen initial liability method is equal to the discounted interest earnings on the difference between the accumulated funds:

$$\$4{,}464.7 - (0.063310 \times \$63{,}436.7) = \$448.5$$

$$= (0.045/1.045)(\$207{,}871.7 - \$197{,}456.5)\ .$$

Table 2 is similar to Table 1 except that the covered group is assumed to have started thirty-five years ago with no pensioners and a 5 percent (linear) increase in the number of new entrants at age 30 for each of the first twenty years of its existence. Thus the number of active employees at the commencement of funding is

$$(L_{30} - L_{46}) + \sum_{k=46}^{64} (0.05)(65 - k)l_k = 3{,}046.55$$

and there are no pensioners. Under these conditions, the dollar amount of normal cost increases during the first twenty years and remains constant thereafter, whereas, since ω for the 1971 GAM is 110, the pension payrolls, pension funds, and plan termination liability do not mature until (110 − 45) = 65 years (at the beginning of the first year, the active employee group below age 46 is already mature). Table 2 thus illustrates the general rule that pensioner populations require much more time to attain full maturity than do active employee populations (see also Table 5).

Table 3 is similar to Table 1, except that annual interest and salary scale under inflationary conditions are assumed to be 6 and 4 percent respectively. Thus there is an annual inflation element of (1.06/1.045) − 1, or 1.44 percent, in the interest rate and (1.04/1.025) − 1, or 1.46 percent, in the salary scale. Inflation at these rates is assumed to have been occurring indefinitely into the past, so that a retired employee aged x at the beginning of the first year has an annual benefit of $\$8{,}306(1.025/1.04)^{x-65}$ and the initial pension payroll is less than in Table 1 (see also Table 4). In accordance with this general inflation, each year's new employees at age 30 have starting salaries 1.46 percent higher than the previous year's new employees.

TABLE 2

INITIALLY IMMATURE POPULATION; 4½ PERCENT INTEREST, 2½ PERCENT SALARY SCALE (NONINFLATIONARY); NO EXPERIENCE GAINS OR LOSSES; FINAL SALARY PLAN; NO AUTOMATIC POSTRETIREMENT BENEFIT INCREASES; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-Go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$42.5	$ 34.5	$ 0	$0	$ 0	$2.6	$2.5	$ 5.4	6.33%	$2.9	$ 5.9
2....	44.5	38.8	0.04	0.5	0.4	2.8	2.5	11.1	6.33	2.9	12.1
3....	46.4	43.5	0.1	0.9	1.2	3.0	2.5	17.2	6.33	2.9	18.6
4....	48.3	48.5	0.2	1.3	2.5	3.1	2.5	23.6	6.33	2.9	25.5
5....	50.0	53.8	0.4	1.8	4.0	3.3	2.5	30.3	6.33	2.9	32.5
10...	57.1	83.0	1.7	4.0	16.3	3.9	2.5	66.4	6.33	2.9	70.6
11...	58.3	89.2	2.1	4.5	19.5	4.0	2.5	73.9	6.33	2.9	78.4
15...	61.7	113.7	3.8	6.3	34.1	4.3	2.5	104.0	6.33	2.9	109.7
20...	63.4	140.7	6.5	8.5	55.1	4.5	2.5	138.6	6.33	2.9	145.5
25...	63.4	159.2	9.1	8.9	73.7	4.5	2.5	166.3	6.33	2.9	174.8
30...	63.4	170.1	10.9	8.9	84.5	4.5	2.5	188.7	6.33	2.9	199.1
31...	63.4	171.5	11.2	8.9	86.0	4.5	0	190.1	6.33	0	200.5
40...	63.4	177.8	12.6	8.9	92.3	4.5	0	196.5	6.33	0	206.9
41...	63.4	178.1	12.7	8.9	92.6	4.5	0	196.7	6.33	0	207.1
50...	63.4	178.8	12.9	8.9	93.3	4.5	0	197.4	6.33	0	207.8
75...	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
100..	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9
125..	63.4	178.8	13.0	8.9	93.3	4.5	0	197.5	6.33	0	207.9

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

TABLE 3

INITIALLY MATURE POPULATION; 6 PERCENT INTEREST, 4 PERCENT SALARY SCALE (INFLATIONARY); NO EXPERIENCE GAINS OR LOSSES; FINAL SALARY PLAN; NO AUTOMATIC POSTRETIREMENT BENEFIT INCREASES; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-Go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$ 63.4	$148.6	$11.4	$17.9	$ 6.9	$ 4.0	$11.7	$ 4.7	5.75%	$12.4	$ 4.9
2....	64.6	150.7	11.5	18.0	14.2	4.1	11.7	9.5	5.75	12.4	10.0
3....	65.3	153.0	11.7	18.1	21.9	4.2	11.7	14.5	5.75	12.4	15.3
4....	66.3	155.2	11.9	18.3	30.0	4.3	11.7	19.7	5.75	12.4	20.8
5....	67.2	157.5	12.0	18.4	38.5	4.3	11.7	25.1	5.75	12.4	26.4
10...	72.3	169.3	13.0	19.0	88.6	4.6	11.7	55.2	5.75	12.4	58.2
11...	73.4	171.8	13.1	9.3	89.9	4.7	11.7	62.0	5.75	12.4	65.4
15...	77.7	182.1	13.9	9.9	95.3	5.0	11.7	91.9	5.75	12.4	96.9
20...	83.6	195.8	15.0	10.6	102.5	5.3	11.7	137.1	5.75	12.4	144.6
25...	89.9	210.6	16.1	11.4	110.2	5.7	11.7	193.4	5.75	12.4	204.0
30...	96.7	226.4	17.3	12.3	118.5	6.2	11.7	264.2	5.75	12.4	278.7
31...	98.1	229.7	17.6	12.5	120.2	6.3	0	268.1	5.75	0	282.7
40...	111.8	261.8	20.0	14.2	137.0	7.1	0	305.5	5.75	0	322.2
41...	113.4	265.7	20.3	14.4	139.0	7.2	0	310.0	5.75	0	326.9
50...	129.3	302.8	23.1	16.5	158.5	8.2	0	353.3	5.75	0	372.6
75...	185.9	435.4	33.3	23.7	227.8	11.9	0	508.0	5.75	0	535.8
100..	267.3	626.0	47.9	34.0	327.7	17.0	0	730.5	5.75	0	770.4
125..	384.3	900.2	68.8	48.9	471.1	24.5	0	1,050.4	5.75	0	1,107.8

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

By applying equation (1) from [2] (which is a curtate equivalent of the liability growth equation of [1] but is based on pension funds and total contributions rather than on normal costs and accrued liabilities), one may still obtain results from Table 3 similar to those obtained in the discussion of Table 1:

$$C^1 + dF^1 = B^1 + v\Delta F^1 ,$$

$$C^2 + dF^2 = B^2 + v\Delta F^2 ;$$

$$(C^2 - C^1) = d(F^1 - F^2) + v(\Delta F^2 - \Delta F^1) . \tag{4}$$

For a mature population, with supplemental liabilities fully funded and no experience gains or losses, and subject to constant inflationary forces such that both F^1 and F^2 are only growing each year at the same rate of salary inflation (which is 1.46 percent in Table 3),

$$(C^2 - C^1) = d'(F^1 - F^2) , \tag{5}$$

where "interest" is net of salary inflation; that is, in Table 3, $[1.06/(1.04/1.025)] - 1$, or 4.4712 percent (in the authors' notation, $\delta - \gamma = \ln 1.044712$ and $\alpha = 0$ in Table 3). Then, comparing the unit credit and frozen initial liability methods (in *thousands* of dollars) in the forty-first year in Table 3,

$$\$7{,}234.1 - (0.057476 \times \$113{,}428.0) = \$714.7$$

$$= (0.044712/1.044712)(\$322{,}229.6 - \$305{,}531.1) .$$

Table 4 is similar to Table 3, except that there are now automatic postretirement benefit increases of 1 percent per year ($\beta = \ln 1.01$), in the authors' example of an exponential $\beta(x)$ in [1]; this is equivalent to a postretirement interest rate of $(1.06/1.01) - 1$, or 4.95 percent, not 5 percent). Since in Table 3 it is assumed that inflation of approximately $1\frac{1}{2}$ percent per year has been occurring in all past years, such postretirement benefit increases are made retroactive (based on years since retirement) for the initial pensioner group to compensate partially for such inflation. Thus the initial pension payroll in Table 4 is greater than that in Table 3 but still not so large as that in Table 1. Further, for the calculation of plan termination liabilities, it is assumed that such automatic benefit increases will occur from the normal retirement age of 65 only and that there will be no pension adjustments during the deferred period from date of termination of employment to age 65. Comparison of Table 4 with Table 3 demonstrates the dramatic impact on costs and liabilities that automatic pension adjustments as small as 1 percent per year can have.

TABLE 4

INITIALLY MATURE POPULATION; 6 PERCENT INTEREST, 4 PERCENT SALARY SCALE (INFLATIONARY); NO EXPERIENCE GAINS OR LOSSES; FINAL SALARY PLAN; 1 PERCENT PER YEAR AUTOMATIC POSTRETIREMENT BENEFIT INCREASES, ALSO RETROACTIVE; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-Go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$ 63.4	$165.0	$12.4	$19.9	$ 7.9	$ 4.3	$12.9	$ 5.2	6.17%	$13.6	$ 5.4
2....	64.4	167.4	12.6	20.0	16.3	4.4	12.9	10.5	6.17	13.6	11.0
3....	65.3	169.8	12.8	20.2	25.1	4.5	12.9	16.0	6.17	13.6	16.9
4....	66.3	172.3	13.0	20.3	34.4	4.5	12.9	21.7	6.17	13.6	22.9
5....	67.2	174.8	13.2	20.4	44.1	4.6	12.9	27.7	6.17	13.6	29.2
10...	72.3	188.0	14.2	21.1	101.4	4.9	12.9	61.0	6.17	13.6	64.2
11...	73.4	190.8	14.4	10.0	102.9	5.0	12.9	68.5	6.17	13.6	72.1
15...	77.7	202.2	15.2	10.6	109.0	5.3	12.9	101.6	6.17	13.6	107.0
20...	83.6	217.4	16.4	11.4	117.3	5.7	12.9	151.5	6.17	13.6	159.6
25...	89.9	233.8	17.6	12.3	126.1	6.2	12.9	213.7	6.17	13.6	225.1
30...	96.7	251.4	18.9	13.2	135.6	6.6	12.9	292.0	6.17	13.6	307.5
31...	98.1	255.1	19.2	13.4	137.6	6.7	0	296.2	6.17	0	312.0
40...	111.8	290.7	21.9	15.3	156.8	7.7	0	337.6	6.17	0	355.5
41...	113.4	295.0	22.2	15.5	159.1	7.8	0	342.6	6.17	0	360.8
50...	129.3	336.2	25.3	17.7	181.3	8.8	0	390.4	6.17	0	411.1
75...	185.9	483.4	36.4	25.4	260.7	12.7	0	561.4	6.17	0	591.2
100..	267.3	695.1	52.3	36.5	374.9	18.3	0	807.3	6.17	0	850.1
125..	384.3	999.5	75.3	52.5	539.1	26.3	0	1,160.8	6.17	0	1,222.4

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

Table 5 is similar to Table 4, except that the number of new employees at age 30 each year is 2 percent greater than the number for the previous year. This differs from the authors' assumption as to population growth, which is based on a mature population. Table 5 thus illustrates the build up to such mature state, which requires $(65 - 30) = 35$ years for the active employee group and $(110 - 30) = 80$ years for all employees and pensioners, assuming no net eligible "immigration" (e.g., transfers) at the older ages. After thirty-five years the total active employee payroll is growing by $(1.02)(1.04/1.025) - 1$, or 3.49 percent per year (in the authors' notation, $e^{\tau} - 1$), and after $(110 - 30) = 80$ years the pension payroll is growing by the same amount (automatic postretirement benefit increases, which were made retroactive for the initial pensioner group, affect only existing pensions, not new pensions). At the beginning of the one hundred and twenty-fifth year, the initial group of 4,091.57 active employees has grown to 35,045.66 and the initial group of 1,561.20 pensioners has grown to 7,613.37.

Table 6 is based on the immature population of Table 2 and the inflationary conditions of Table 3. However, during years 16–25, when supplemental liabilities have been only partially funded under the unit credit and frozen initial liability funding methods, a period of severe inflation is assumed to occur, with seven percent interest and seven and one-half percent total salary increases, and with experience reverting to the actuarial assumptions both before and after this period. Under these conditions, during each of the ten years 16–25 there are net experience losses under both the unit credit and the frozen initial liability funding methods but net experience *gains* under the terminal funding method, since the liabilities under that method are independent of the salary experience of active employees. (Later on, of course, higher contributions are required under the terminal funding method when the employees with the higher earnings ultimately retire.) Table 6 demonstrates the extreme asymptoticity of the amortization of experience gains and losses under a frozen initial liability method; even one hundred years after the occurrence of the experience, the normal cost percentage is still significantly greater than its original entry-age level. Further, the more rapid fifteen-year amortization of the net experience losses under the unit credit method produces, during years 38–39, a *greater* pension fund under the unit credit method than under the frozen initial liability method.

Table 7 is similar to Table 6, except that the pension plan now has a career average salary benefit formula of 1 percent of each year's earnings (salary increases are assumed to occur at the beginning of each year). As in Table 6, there are net experience gains under the terminal funding

TABLE 5

INITIALLY MATURE POPULATION; 6 PERCENT INTEREST, 4 PERCENT SALARY SCALE (INFLATIONARY); NO EXPERIENCE GAINS OR LOSSES; FINAL SALARY PLAN; 1 PERCENT PER YEAR AUTOMATIC POSTRETIREMENT BENEFIT INCREASES, ALSO RETROACTIVE; 2 PERCENT PER YEAR INCREASE IN NEW ENTRANTS AT AGE 30

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-Go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$ 63.4	$ 165.0	$ 12.4	$ 19.9	$ 7.9	$ 4.3	$12.9	$ 5.2	6.17%	$13.6	$ 5.4
2....	64.4	167.4	12.6	20.0	16.3	4.4	12.9	10.5	6.17	13.6	11.0
3....	65.4	169.8	12.8	20.2	25.1	4.5	12.9	16.0	6.17	13.6	16.9
4....	66.4	172.3	13.0	20.3	34.4	4.5	12.9	21.8	6.17	13.6	22.9
5....	67.5	174.9	13.2	20.4	44.1	4.6	12.9	27.7	6.17	13.6	29.2
10...	73.7	188.1	14.2	21.1	101.4	5.0	12.9	61.3	6.17	13.6	64.6
11...	75.1	190.9	14.4	10.0	102.9	5.1	12.9	68.8	6.17	13.6	72.6
15...	81.6	202.6	15.2	10.6	109.0	5.5	12.9	102.7	6.17	13.6	108.4
20...	91.6	218.7	16.4	11.4	117.3	6.1	12.9	154.8	6.17	13.6	163.8
25...	104.5	237.3	17.6	12.3	126.1	6.9	12.9	221.7	6.17	13.6	235.0
30...	121.2	259.9	18.9	13.2	135.6	7.9	12.9	309.0	6.17	13.6	328.1
31...	125.1	265.1	19.2	13.4	137.6	8.2	0	315.8	6.17	0	335.6
40...	169.4	330.1	22.2	16.5	159.7	11.1	0	399.1	6.17	0	426.0
41...	175.3	339.8	22.7	17.1	163.3	11.5	0	411.1	6.17	0	439.0
50...	238.8	451.3	28.7	23.3	210.9	15.6	0	548.4	6.17	0	586.4
75...	563.4	1,059.4	66.2	55.0	492.3	36.8	0	1,288.6	6.17	0	1,378.1
100..	1,329.1	2,499.2	156.3	129.7	1,161.5	86.8	0	3,039.8	6.17	0	3,251.0
125..	3,135.4	5,895.8	368.7	306.1	2,739.9	204.8	0	7,171.1	6.17	0	7,669.4

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

TABLE 6

INITIALLY IMMATURE POPULATION; 6 PERCENT INTEREST, 4 PERCENT SALARY SCALE (INFLATIONARY); 7 PERCENT INTEREST, 7½ PERCENT TOTAL SALARY INCREASES DURING YEARS 16–25; FINAL SALARY PLAN; NO AUTOMATIC POSTRETIREMENT BENEFIT INCREASES; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

YEAR	PAYROLL	PLAN TERMINATION LIABILITY	PAY-AS-YOU-GO BENEFIT PAYMENTS	TERMINAL FUNDING		UNIT CREDIT*			FROZEN INITIAL LIABILITY†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$ 42.5	$ 26.7	$ 0	$ 0	$ 0	$ 2.4	$2.6	$ 5.4	5.75%	$3.1	$ 5.9
2....	45.2	30.8	0.04	0.4	0.4	2.6	2.6	11.2	5.75	3.1	12.2
3....	47.8	35.4	0.1	0.8	1.2	2.8	2.6	17.4	5.75	3.1	19.0
4....	50.4	40.4	0.3	1.3	2.3	2.9	2.6	24.2	5.75	3.1	26.2
5....	53.0	45.8	0.4	1.7	3.8	3.1	2.6	31.3	5.75	3.1	33.8
10...	65.1	78.1	1.9	4.1	16.4	4.0	2.6	72.3	5.75	3.1	77.1
11...	67.4	85.5	2.3	4.7	19.8	4.2	2.6	81.3	5.75	3.1	86.7
15...	75.6	116.6	4.5	6.9	36.4	4.8	2.6	119.4	5.75	3.1	126.5
20...	95.4	175.7	8.3	11.4	67.9	6.1	3.7	180.4	6.77	3.1	189.4
25...	121.1	243.6	13.5	14.9	109.5	7.7	5.4	255.5	7.65	3.1	264.7
30...	134.6	289.5	18.7	16.5	142.7	8.6	5.8	326.8	7.50	3.1	333.2
31...	136.6	297.6	19.7	16.7	148.2	8.7	3.1	338.1	7.43	0	343.0
40...	155.7	360.2	26.6	19.7	186.4	9.9	0.4	421.0	6.93	0	421.5
41...	157.9	366.4	27.2	20.1	190.1	10.1	0	428.1	6.89	0	429.5
50...	180.0	421.3	32.1	22.9	220.3	11.5	0	491.6	6.55	0	500.6
75...	258.8	606.2	46.3	33.0	317.2	16.5	0	707.4	6.05	0	736.2
100..	372.2	871.6	66.6	47.4	456.2	23.7	0	1,017.1	5.86	0	1,067.3
125..	535.1	1,253.3	95.8	68.2	655.9	34.1	0	1,462.6	5.79	0	1,539.5

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

TABLE 7

INITIALLY IMMATURE POPULATION; 6 PERCENT INTEREST, 4 PERCENT SALARY SCALE (INFLATIONARY); 7 PERCENT INTEREST, 7½ PERCENT TOTAL SALARY INCREASES DURING YEARS 16–25; CAREER AVERAGE SALARY PLAN; NO AUTOMATIC POSTRETIREMENT BENEFIT INCREASES; NO POPULATION GROWTH OR DECLINE

(In $ Millions)

Year	Payroll	Plan Termination Liability	Pay-as-You-Go Benefit Payments	Terminal Funding		Unit Credit*			Frozen Initial Liability†		
				Contribution	Pension Fund	Normal Cost	Past Service	Pension Fund	Normal Cost	Past Service	Pension Fund
1....	$ 42.5	$ 17.3	$ 0	$ 0	$ 0	$ 1.3	$1.41	$ 2.9	3.065%	$1.6	$ 3.1
2....	45.2	19.7	0.02	0.2	0.2	1.4	1.41	6.0	3.065	1.6	6.5
3....	47.8	22.3	0.07	0.4	0.6	1.5	1.41	9.3	3.065	1.6	10.1
4....	50.4	25.2	0.1	0.7	1.2	1.6	1.41	12.9	3.065	1.6	14.0
5....	53.0	28.3	0.2	0.9	2.0	1.7	1.41	16.7	3.065	1.6	18.0
10...	65.1	46.5	1.0	2.2	8.7	2.1	1.41	38.5	3.065	1.6	41.1
11...	67.4	50.6	1.3	2.5	10.6	2.2	1.41	43.4	3.065	1.6	46.2
15...	75.6	67.7	2.4	3.7	19.4	2.5	1.41	63.7	3.065	1.6	67.4
20...	95.4	90.4	4.3	5.4	34.7	3.1	1.44	94.5	3.01	1.6	99.8
25...	121.1	114.2	6.4	6.2	51.3	3.7	1.43	128.5	2.93	1.6	136.2
30...	134.6	138.4	8.4	7.2	63.5	4.2	1.42	159.2	2.94	1.6	169.6
31...	136.6	143.1	8.8	7.4	65.8	4.3	0.01	164.0	2.94	0	174.7
40...	155.7	183.9	12.2	9.6	85.7	5.1	(0.01)	205.8	2.98	0	219.2
41...	157.9	188.3	12.5	9.8	88.0	5.2	0	210.5	2.98	0	224.0
50...	180.0	227.9	15.7	11.9	108.9	6.1	0	252.5	3.01	0	268.1
75...	258.8	341.7	24.7	17.6	169.0	8.8	0	377.0	3.04	0	398.3
100..	372.2	491.7	35.5	25.3	243.3	12.7	0	542.4	3.057	0	572.4
125..	535.1	707.0	51.1	36.3	349.8	18.2	0	779.9	3.062	0	822.8

NOTE.—Payroll, contributions, and benefits payable as of the beginning of each year; plan termination liability and pension funds calculated as of the end of each year, prior to contributions and benefit payments then due for next year. Initial accrued liability under terminal funding, if any, is amortized by constant dollar amount over ten years and under unit credit and frozen initial liability is amortized by constant dollar amount over thirty years.

* Benefits projected to normal retirement age by salary scale and prorated by years of service; net experience gains or losses are amortized by constant dollar amount over fifteen-year period following year of occurrence.

† Entry age normal form with normal cost expressed as a constant percentage of salary.

method. However, since benefits that accrued prior to plan years 16–25 are now not affected by the higher salary increases during those years, whereas *all* of the plan's assets benefit from the increased investment earnings, there are now net experience *gains* during all years under the frozen initial liability method. Further, since the unit credit method has smaller asset accumulations and its costs are prorated into constant dollar amounts rather than being spread over future salary increases as under the frozen initial liability method, net *losses* are experienced during years 16–21, followed by net *gains* during years 22–26, when more assets have accumulated to take advantage of the higher interest rates. Again, as in Table 6, there is the extreme asymptoticity of the amortization of experience under the frozen initial liability method, with the normal cost percentage still 0.1 percent less ($1 - 3.062/3.065$) than its original entry-age level even one hundred years after the experience period.

III. *Summary; Future Projects*

The tables in the preceding section, and the comments by the authors in their previous paper [1], including their review of the discussions, indicate the need for further work on the effects of (1) immature populations; (2) alternate benefit formulas, including those integrated with various social insurance benefits or contributions; and, especially, (3) experience gains and losses.

REFERENCES

1. BOWERS, NEWTON L., JR.; HICKMAN, JAMES C.; and NESBITT, CECIL J. "Introduction to the Dynamics of Pension Funding," *TSA*, XXVIII (1976), 177.
2. TROWBRIDGE, CHARLES L. "Fundamentals of Pension Funding," *TSA*, IV (1952), 17.
3. TROWBRIDGE, CHARLES L., and FARR, CHARLES E. *The Theory and Practice of Pension Funding*. Homewood, Ill.: Richard D. Irwin, 1976.
4. WINKLEVOSS, HOWARD E. *Pension Mathematics, with Numerical Illustrations*. Homewood, Ill.: Richard D. Irwin, 1977.

(AUTHORS' REVIEW OF DISCUSSION)

NEWTON L. BOWERS, JR., JAMES C. HICKMAN, AND CECIL J. NESBITT

Mr. Trowbridge perceives correctly that there are several lines of development that may be followed, starting with the model for a dynamic pension environment outlined in our first joint paper on this topic. The model was the basis for a third paper by Cecil J. Nesbitt, one of our team of authors, "Exploration of Pension Funding in Case of Exact Vesting," which appeared in the 1978.2 issue of *ARCH*.

Mr. Trowbridge has also caught some of our fascination with relating concepts from pension funding to ideas in other branches of actuarial mathematics. Considered individually, each of these connections is of little importance. However, together they help to build a needed unity in actuarial mathematics. The analogy between pension purchase density functions and accrual functions on the one hand and probability density functions and cumulative distribution functions on the other is clear. The average age of normal cost payment turns out to be related closely to the cumulant generating function of probability and statistics. However, the average age of normal cost payment provides a bridge that permits pension funding ideas to be stated in terms of compound interest functions.

In addition to thanking Mr. Trowbridge for his discussion, we want to reinforce his statements about the implications when $\delta - \tau$ becomes negative, particularly if inflation is the cause. However, if $\theta = \delta - \tau$ is negative because of the population growth rate, advance funding still may be in order. In any case, the causes for the negative value of θ should be distinguished and taken into account appropriately in the funding arrangements.

We are most grateful to Mr. Pennisten for his seven well-worked-out, ingenious, and illuminating illustrations. They are a very useful contribution to the development and understanding of the theory. Since our papers are on a completely continuous basis, the illustrator either must calculate the continuous functions by approximate integration or must visualize the corresponding discrete theory and then compute the discrete functions in the usual manner. Mr. Pennisten has made the second choice, and his discussion gives various insights into the necessary modifications of the continuous theory. At some stage it may be useful to have an explicit development of the main relations for the discrete theory.

In our paper, we generally have looked at the ratios of functions, while Mr. Pennisten has provided additional insight by looking at differences between functions. From his tables it can be observed that the ratio of unit credit normal cost to terminal funding contribution tends to 0.51 for the first two tables and to 0.50 in the remaining tables, except in Table 5, where the limit is 0.67. The ultimate ratios of the frozen initial liability normal cost to terminal funding contribution are 0.45 in all except Table 5, where the ratio is 0.63. The ultimate ratios of terminal funding contribution to pay-as-you-go benefit payments vary from 0.68 to 0.71, except for 0.83 in the case of Table 5.

These variations in the ultimate ratios can be explained in terms of the net excess of the interest rate over the total growth rate (corresponding to $\theta = \delta - \tau$ of the paper). For Tables 1 and 2 this excess is 4.5 percent; for all other tables except Table 5 it is 4.4712 percent (see Mr. Pennisten's remarks on Table 3); and for Table 5 it is $(0.06 - 0.0349)/1.0349$, or 2.43 percent.

Many of these ratios are attained within the active service period of thirty-five years, but those involving the pension roll may take twice as long, and for the frozen initial liability method the period may be well over one hundred years. This illustrates again Mr. Pennisten's observations that pensioner populations require much more time to attain full maturity than do active employee populations, and that the frozen initial liability method (which is of the moving amortization type) can be very gradual.