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FUND DEVELOPMENT OF AN EARNINGS-RELATED SOCIAL INSURANCE PLAN UNDER STABILIZED CONDITIONS

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ABSTRACT

It has been well established that, under stabilized conditions, the full fund for an earnings-related social insurance plan increases at the rate of growth in total earnings and that, as a consequence, the current cost (pay-as-you-go) contribution rate will be greater than, equal to, or less than, the full cost rate, depending upon whether the rate of interest is greater than, equal to, or less than the rate of growth in total earnings, and vice versa. This is briefly summarized in Section IV. The OASDI in the United States, and the Canada and Quebec Pension Plans in Canada, are examples of plans that, with minor modifications, would be strictly subject to this theory.

The purpose of this paper is to extend this theory further and to examine how an actual fund, for any given contribution rate, would develop as a proportion of the full fund. This leads to the key equation (10) developed in Section VI, some of the implications of which are discussed in the remainder of the paper.

I. INTRODUCTION

DETAILED cost studies of earnings-related social insurance plans often consider periods of up to seventy-five years; however, few, if any, serious studies consider very long-term prospects, such as those after one hundred years. The reasons are fairly obvious: studies are expensive and time-consuming; conditions that may prevail at such a distant time are extremely uncertain, not only as to appropriate actuarial assumptions but even as to the nature of the plan let alone the form of social organization that may then exist; and there is less interest in such dim horizons than in the nearer-term future. Nevertheless, this does not mean that such remote periods should be ignored entirely. What is needed is a set of analytical methods that can be quickly and inexpensively applied and that, despite all the uncertainties involved, do provide some real insight into the possibilities that may arise. This paper was designed to develop some such methods.

Section II defines all the key variables used in the paper, except those used in a purely local context.Section III explains what is meant by "stabilized conditions," the basic assumption underlying the subsequent analyses. Section IV develops a simple formula for the growth of the full fund of an earnings-related social insurance plan, and also develops a fundamental relationship between the size of the full cost rate and the current cost (pay-as-you-go) rate of the plan. Sections V–IX are concerned with how the actual fund will evolve as a proportion of the full fund—that is, with how the funding ratio will change. Section V proves that the funding ratio may be kept constant by the charging of a specific contribution rate, which is defined as the equilibrium contribution rate and for which a suitable formula is developed. Section VI derives the key formula of the paper, which shows how the funding ratio will evolve for any contribution rate. Sections VII and VIII analyze some of the qualitative implications of this formula, and Section IX illustrates some of its quantitative implications. Section X develops some additional relationships that may have useful analytical value, such as (*a*) how to convert the estimated funding ratios into ratios of the actual fund to contributory earnings and (*b*) a formula to calculate the number of years' expenditures that will be covered by any estimated actual fund assuming no further contributions. Section XI briefly recapitulates the purpose for which the paper was written.

II. DEFINITIONS OF VARIABLES

FF = Full fund = Total liability, at any moment in time;

AF = Actual fund at any moment in time;

FR = Funding ratio = AF/FF;

C = Contributory earnings during a one-year period accumulated with interest to the end of that period;

CE = Instantaneous contributory earnings on a per annum basis;

ER = Equilibrium contribution rate = Contribution rate required to maintain FR constant;

t = Time t; may be used as a subscript attached to any of the above variables to indicate the value of the variable at time t (e.g., FF_t = Full fund at time t, and C_t = Contributory earnings between t and $t + 1$, accumulated with interest to $t + 1$);

FC = Full cost contribution rate = Contribution rate that must be charged in connection with a full fund to maintain the fund at the FF level;

PG = Current cost contribution rate = Pay-as-you-go (referred to hereinafter as pay-go) contribution rate;

CR = Actual contribution rate;
n = $(CR - PG)/(FC - PG)$;
i = Rate of interest;
p = Rate of growth in population;
s = Economic (nonpromotional) rate of growth in individual earnings;

$$r = \frac{1 + i}{(1 + p)(1 + s)}.$$

III. STABILIZED CONDITIONS

Generally speaking, the term "stabilized conditions" means not only that plan benefits are assumed to have matured but also that economic, demographic, and other similar variables are assumed to have become constant.

In particular, it is assumed that the rate of interest i, the economic rate of growth in individual earnings s, and other similar economic variables, such as the rate of growth in the CPI, will be constant at current levels indefinitely into the future.

It is also assumed that the rates of fertility, mortality, and immigration will be constant indefinitely into the future. It is not assumed that the population will be constant (stationary), but it is assumed that the percentage distribution of the population by age and sex will be constant. Thus, the rate of growth in the population, p, which is assumed to be constant in the future, is applicable not only to the population as a whole but also to each of its components, such as the aged population, the working-age population, and so on.

All other factors that affect costs, such as participation rates, earnings distributions, disability rates, proportions married, and so on, are also assumed to remain constant at current levels indefinitely into the future.

It follows from these assumptions that both the full cost contribution rate FC under many full-funding methods, and the pay-go contribution rate PG, are assumed to remain constant at current levels indefinitely into the future (as are i, p, s, and therefore r, as indicated above).

It is obvious that such assumptions represent a highly idealized and simplified set of circumstances, which will never be realized, and probably will never be even closely approximated, at any time in the future. It might be argued that the analysis that follows, which is based on such assumptions, is of more theoretical than practical interest. On the other hand, a thorough grasp of this simplified theoretical model may yield a number of valuable insights that will facilitate the understanding of more complicated real-world situations, and, at least in that sense, the analysis may have a definite practical value.

It should be noted that the full-funding method on which *FF* and *FC* are based is not defined. The reasoning in this paper is applicable in connection with any full-funding method, provided that, under stabilized conditions, the method produces a positive full fund *FF*, and the contribution rate *FC* associated with that method is, or can be expressed as, a level proportion of contributory earnings.

IV. EVOLUTION OF *FF*, AND RELATIONSHIP BETWEEN *FC* AND *PG*

Under the assumptions outlined in Section III, any basic monetary element of a plan for any year (such as contributory earnings of 43-year-old males, or widows' pensions payable to 81-year-old females), is equal to the corresponding element for the preceding year multiplied by the factor $(1 + p)(1 + s)$, which is equal to 1 plus the rate of growth in total earnings. Under such circumstances, it can be shown that the full fund, *FF*, increases each year by the same factor; that is,

$$FF_{t+1} = (1 + p)(1 + s)FF_t \,. \tag{1}$$

Another method of calculating FF_{t+1}, which would yield identical results, is given by the formula

$$FF_{t+1} = (1 + i)FF_t + (FC - PG)C_t \,. \tag{2}$$

In this formula $PG\ C_t$ represents the value of benefits accumulated to the end of the year. *FC* is the contribution rate underlying both formulas. An examination of the two formulas indicates that $PG = FC$ implies $(1 + i) = (1 + p)(1 + s)$, that is, $r = 1$. Similarly, $PG > FC$ implies $r > 1$, and $PG < FC$ implies $r < 1$. Since the logic also operates in the opposite direction, we have

$$PG \gtreqless FC \Leftrightarrow r \gtreqless 1 \,. \tag{3}$$

Throughout the balance of this paper it is assumed that $r \neq 1$. Virtually all of the following reasoning would be incorrect if $r = 1$, usually because, in one way or another, it would involve division by zero.

V. CIRCUMSTANCES UNDER WHICH THE FUNDING RATIO, *FR*, WILL REMAIN CONSTANT—THE EQUILIBRIUM CONTRIBUTION RATE *ER*

In Section IV, we analyzed the progress of the full fund, *FF*, and relationships between the full cost contribution rate, *FC*, and the pay-go rate, *PG*. As a practical matter, when a plan is mature, the actual fund, *AF*, probably will be only a fraction of the full fund, *FF*, although theoretically it might exceed it, and, unless the plan is being financed on a fairly strict pay-go basis with no real concern for fund levels, legislators will be considering a range of possible contribution rates, probably lying between the full cost rate *FC* and the pay-go rate *PG*, but quite possibly extending beyond this range. Not only at the time of plan maturity, but well in advance of it, planners will be interested in developing an appreciation of how the actual fund *AF* might progress under various circumstances. They may be concerned with such questions as (1) how the funding ratio will change over various specified periods of time for any given contribution rate *CR*; (2) whether a particular *CR* will be sufficient to maintain the solvency of the fund indefinitely, and, if not indefinitely, for how long; (3) what *CR* would be required to meet a specific funding objective over a specified period of time; and so on. Relatively simple procedures for evaluating answers to most such questions, assuming stabilized conditions, are developed later on in this paper, but in this section we shall confine ourselves to the problem of developing the conditions that must be imposed on the actual contribution rate *CR*, given an actual fund *AF*, to ensure that the fund increases at the same rate as the full fund *FF*, that is, to ensure that the funding ratio *FR* remains constant.

We may write

$$AF_{t+1} = AF_t(1 + i) + (CR - PG)C_t ,$$

or

$$\begin{aligned}(CR - PG)C_t &= AF_{t+1} - AF_t(1 + i)\\ &= FR_{t+1}FF_{t+1} - FR_tFF_t(1 + i) .\end{aligned}$$

Formula (2) in Section IV may be rewritten as

$$(FC - PG)C_t = FF_{t+1} - FF_t(1 + i) ,$$

whence

$$\frac{(CR - PG)C_t}{(FC - PG)C_t} = \frac{CR - PG}{FC - PG} = \frac{FR_{t+1}FF_{t+1} - FR_tFF_t(1 + i)}{FF_{t+1} - FF_t(1 + i)} .$$

If the funding ratio remains constant, that is, if $FR_{t+1} = FR_t = FR$, the preceding formula reduces to

$$\frac{CR - PG}{FC - PG} = FR ;$$

therefore, the contribution rate required to keep the funding ratio constant and equal to FR is

$$CR = FR\,FC + (1 - FR)PG = ER\,. \tag{4}$$

Relationship (4) states that for the actual fund AF to develop at the same rate as the full fund FF, that is, for the funding ratio FR to remain constant, the necessary and sufficient condition is that the contribution rate CR for the actual fund AF be set equal to the equilibrium contribution rate ER. This equilibrium contribution rate is a weighted average of the full cost contribution rate FC and the pay-go contribution rate PG, where the weight assigned to FC is the funding ratio FR, and the weight assigned to PG is $(1 - FR)$.

Relationship (4) may also be developed from general reasoning. Let us assume that we have a partial fund equal to, say, 30 percent of the full fund. We could consider the plan as consisting of two portions, a 30 percent portion that is fully funded and a 70 percent portion that is completely unfunded. If we charged the full cost rate FC for the first 30 percent, that portion would remain fully funded, while if we charged the pay-go rate PG for the remaining 70 percent, that portion would remain unfunded. Recombining the two portions would give a composite contribution rate equal to $0.3FC + (1 - 0.3)PG$, and the combined fund as it developed would clearly remain at 30 percent of the fully funded level. If, throughout the above discussion, we replace the 0.3 and 30 percent terms with FR, we obtain relationship (4).

VI. EVOLUTION OF THE FUNDING RATIO FR, IF THE CONTRIBUTION RATE CR IS NOT SET EQUAL TO THE EQUILIBRIUM CONTRIBUTION RATE ER

It should be obvious that, if CR is set higher than ER, FR should increase, while if CR is set lower, FR should decrease. The purpose of this section is to determine the extent of the increase or decrease.

From the definition of n in Section II, it follows that

$$CR = n\,FC + (1 - n)PG\,. \tag{5}$$

It is clear from a comparison of this formula with relationship (4) that $CR = ER$ implies $n = FR$, and vice versa. In this section we shall examine what happens when $CR \neq ER$, that is, $n \neq FR$.

Let us assume that the current time (at some future date, under stabilized conditions) is zero, so that the value of the current actual fund is AF_0, the current full fund is FF_0, the current funding ratio is $FR_0 = AF_0/FF_0$, and so forth. Since $n \neq FR_0$, the contribution rate CR is not the equilibrium contribution rate for the actual fund AF_0. It should be clear from equations (4) and (5), however, that CR is the equilibrium contribution rate for a hypothetical actual fund HF_0 set equal to $n\ FF_0$. Let D represent the difference between HF and AF, for example, $D_0 = HF_0 - AF_0$. Since AF_0 and CR are known quantities, and, presumably, n and FF_0 can be estimated, it follows that FR_0, HF_0, and D_0 can be calculated. The contribution rate for the full fund FF is FC, while the contribution rate for the HF and AF funds is CR. After t years, these funds will become FF_t, HF_t, and AF_t, respectively.

Extension of formula (1) in Section IV indicates that

$$FF_t = (1 + p)^t(1 + s)^t FF_0 . \tag{6}$$

Since the HF fund is subject to its equilibrium contribution rate CR, its funding ratio n does not change with time; therefore,

$$HF_t = n\ FF_t . \tag{7}$$

We may also write, as a matter of definition,

$$D_t = HF_t - AF_t . \tag{8}$$

Any fund at time t can be considered as equal to the initial fund at time 0 accumulated with interest to time t, plus contributions less expenditures accumulated with interest. When considering the two funds HF and AF, it should be noted that, since they have the same contributions, expenditures, and interest rate, the difference between the two at time t, namely, D_t, is simply equal to the difference between their initial amounts, accumulated with interest. Hence, we may write

$$D_t = (1 + i)^t D_0 . \tag{9}$$

From these relationships it follows that

$$\begin{aligned} FR_t = \frac{AF_t}{FF_t} = \frac{HF_t - D_t}{FF_t} &= n - \frac{D_t}{FF_t} \\ &= n - \frac{(1 + i)^t D_0}{(1 + p)^t(1 + s)^t FF_0} = n - r^t \frac{D_0}{FF_0} \\ &= n - r^t \frac{HF_0 - AF_0}{FF_0} = n - r^t(n - FR_0) . \end{aligned}$$

Hence,

$$FR_t = n - r^t(n - FR_0) . \tag{10}$$

VII. IMPLICATIONS IF $PG < FC$

The possibility that $PG < FC$ is traditionally regarded as unlikely, presumably on the grounds that viable economic conditions more or less require that the rate of interest exceed the rate of growth in total earnings in the long run. However, since most of us are strongly antipathetic to the notion that the pay-go rate might be less than the full cost rate indefinitely, this possibility may be more likely than we are inclined to admit.

Relationship (3) states that, if $PG < FC$, then $r < 1$. It follows that $r^t \longrightarrow 0$ as $t \longrightarrow \infty$, and, consequently, from equation (10),

$$\text{If } PG < FC , \quad \text{then } FR_t \longrightarrow n \text{ as } t \longrightarrow \infty . \tag{11}$$

It can be seen from equation (5) that the actual contribution rate, CR, can be considered as a weighted average of the full cost rate, FC, and the pay-go rate, PG, with n being the weight assigned to FC and $(1 - n)$ the weight assigned to PG.

Relationship (11) states that, provided that PG is less than FC, the funding ratio will tend to n, the weight assigned to FC in calculating CR, and this is true regardless of the initial level of the actual fund, AF_0, or the initial level of the funding ratio, FR_0.

For example, if CR is set at 25 percent of the full cost rate FC and 75 percent of the pay-go rate PG, the fund will tend toward 25 percent of the full fund, regardless of whether it started as 5, 30, or 70 percent, or any other percentage of the full fund.

As long as CR is set somewhere between PG and FC, n will have a value somewhere between 0 and 1, and the fund will tend to some positive fraction of the full fund. This will not be the case, however, if CR is set outside these limits. If CR is set below the pay-go rate, PG, then n is negative and the fund eventually will become negative no matter how large its initial amount. If CR is set above the full cost rate, FC, then $n > 1$ and the fund eventually will exceed the full fund, no matter how small its initial amount. In the special case where CR is set equal to PG, the funding ratio will tend toward zero (but never reach it), no matter how large the initial fund, and in the special case where CR is set equal to FC, the funding ratio will tend toward 1 (but never reach it), no matter how small the initial fund.

VIII. IMPLICATIONS IF $PG > FC$

The situation $PG > FC$ may be of more practical interest than that analyzed in Section VII, because of the prevailing belief that, in the long run, it is more likely that PG will exceed FC than the reverse.

Relationship (3) states that $PG > FC$ implies $r > 1$, and that therefore $r^t \longrightarrow +\infty$ *as* $t \longrightarrow \infty$. It follows from equation (10) that

$$\begin{aligned} &\text{If } PG > FC \text{ and } n < FR_0 \text{ (i.e., } CR > ER_0) , \\ &\text{then } FR_t \longrightarrow +\infty \text{ as } t \longrightarrow \infty ; \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\text{If } PG > FC \text{ and } n > FR_0 \text{ (i.e., } CR < ER_0) , \\ &\text{then } FR_t \longrightarrow -\infty \text{ as } t \longrightarrow \infty . \end{aligned} \tag{13}$$

In contrast to the situation where $PG < FC$, analyzed in Section VII (which revealed that the funding ratio, whatever its initial value FR_0, tended to converge to n), in the situation where $PG > FC$ the funding ratio diverges from n. If the initial funding ratio, FR_0, is greater than n, the funding ratio moves away from n in the direction of $+\infty$, while if FR_0 is less than n, the funding ratio moves away from n in the direction of $-\infty$. Only if the contribution ratio CR is set equal to the equilibrium contribution rate ER_0, in which case the funding ratio will remain constant and equal to n, do we have an inherently stable situation for a constant CR. Otherwise, the fund will be set on a course such that eventually it will exceed the full fund and, in fact, become indefinitely large relative to the full fund, or it will be set on a course that eventually will lead to its disappearance. Furthermore, while in the case $PG < FC$ the absolute rate of change in FR decreases with the passage of time, and eventually becomes insignificant as FR approaches n, in the case where $PG > FC$ the absolute rate of change in FR increases with the passage of time and eventually becomes indefinitely large.

Two special cases may be worth noting when $PG > FC$. If CR is set equal to the full cost rate FC, the fund eventually will become negative, no matter how large its initial value, provided that the initial value is less than the full fund. If CR is set equal to the pay-go rate PG, the fund eventually will exceed the full fund and, in fact, will become indefinitely large relative to the full fund, no matter how small its initial value, provided that it is positive.

IX. SOME NUMERICAL EXAMPLES

Sections VII and VIII discussed trends that tend to occur under certain circumstances, some favorable, some unfavorable, but no attempt was made to measure the time horizons involved in these trends. Obviously, such time horizons can be important in assessing the timing and magnitude of any critical decisions. Equation (10), together with some of the other equations, permits us to make a number of helpful quantitative estimates in this regard, depending on the nature of the problem.

For illustrative purposes, let us suppose that $PG = 0.09$, $FC = 0.08$, and $r = 1.004$. Incidentally, these assumptions are very roughly consistent with the long-range hypotheses and conclusions of *Statutory Actuarial Report No. 6* on the Canada Pension Plan. Let us further assume that $FR_0 = 0.2$. From relationship (4), we know that the equilibrium contribution rate $ER_0 = 0.088$.

1. For one reason or another, there may be pressure to establish CR at a somewhat lower level than the equilibrium contribution rate, perhaps equal to the full cost rate, 0.08. In that event, n would equal 1. There may be interest in finding out how much this would affect the funding ratio over a twenty-year period. Substituting $n = 1$, $r = 1.004$, $t = 20$, and $FR_0 = 0.2$ in equation (10), we would estimate $FR_{20} = 0.134$ as the funding ratio after twenty years.
2. Although the fund is still positive after twenty years, relationship (13) indicates that, under the circumstances postulated, it will eventually disappear. To determine when this will happen, we first transform equation (10) as follows:

$$t = \frac{\ln(|n - FR_t|) - \ln(|n - FR_0|)}{\ln r} \,. \tag{14}$$

   The fund will disappear when FR_t equals zero. Substituting $n = 1$, $FR_t = 0$, $FR_0 = 0.2$, and $r = 1.004$ in equation (14) yields $t = 55.9$ as the estimated number of years for which the fund will survive.
3. The problem may, of course, be quite different. For example, it may be desired to fully fund the plan over a period such as fifty years, and the problem may consist of determining the contribution rate that would be required. Another simple transformation of equation (10) gives

$$n = \frac{r^t FR_0 - FR_t}{r^t - 1} \,. \tag{15}$$

   Substituting $r = 1.004$, $t = 50$, $FR_0 = 0.2$, and $FR_t = 1$ in this equation yields $n = -3.421$. Then substituting $n = -3.421$, $FC = 0.08$, and $PG = 0.09$ in equation (5) yields $CR = 0.12421$.
4. Heretofore, we have assumed that CR remains constant throughout the period covered by our calculations. It is possible, however, to determine the effect of varying CR from time to time by determining the status at the end of the first stage for which CR is held constant, and using this as a starting position for determining the effect of a new CR for the second stage, and so on. For example, as a result

full fund and contributory earnings increase at the rate of increase in total earnings.

B. *Actual Fund Expressed as a Proportion of Contributory Earnings*

Some may feel that ratios of the actual fund to the full fund, as developed in prior sections of this paper, are of limited interest on the grounds that the hypothetical full fund of a social insurance plan is not particularly significant, and would prefer to relate the actual fund to some more meaningful economic variable. They may wish to convert these funding ratios into ratios of the actual fund to contributory earnings, by multiplication by the ratio given in formula (16), that is,

$$\frac{AF_t}{CE_t} = FR_t \frac{PG - FC}{\ln r} . \tag{17}$$

Actually, a neat, compact expression for the ratio of the actual fund to contributory earnings that does not explicitly involve FR_t may be developed as follows:

$$\begin{aligned} \frac{AF_t}{CE_t} &= \frac{FR_t(PG - FC)}{\ln r} \\ &= \frac{PG - PG + FR_tPG - FR_tFC}{\ln r} \\ &= \frac{PG - [FR_tFC + (1 - FR_t)PG]}{\ln r} , \end{aligned}$$

or

$$\frac{AF_t}{CE_t} = \frac{PG - ER_t}{\ln r} . \tag{18}$$

This formula, while slightly more compact than formula (17), normally may not be as easy to apply in practice, since it requires calculating ER_t.

If desired, the ratio of any fund, full or actual, to contributory earnings may be converted into a ratio of the fund to (*a*) annualized expenditures—by dividing by *PG*—or (*b*) annualized contributions—by dividing by the contribution rate (*FC* or *CR*, as the case may be).

C. *Number of Years' Expenditures Covered by a Fund, Assuming No Further Contributions*

Another criterion sometimes used in social insurance to evaluate the strength of a fund, AF_t, is to estimate the number of years' expenditures,

Y_t, that would be covered by the fund if no further contributions were collected. Under stabilized conditions, the present value of future expenditures over a period of y years is given by

$$\int_0^y \frac{PG\,CE_t(1+p)^x(1+s)^x}{(1+i)^x}\,dx = PG\,CE_t \int_0^y r^{-x}dx$$

$$= \frac{PG\,CE_t(r^y - 1)}{r^y \ln r}\,.$$

Setting this expression equal to AF_t, dividing both sides by CE_t, and using expression (18), we obtain

$$\frac{PG - ER_t}{\ln r} = \frac{PG(r^y - 1)}{r^y \ln r}\,,$$

or

$$1 - \frac{ER_t}{PG} = 1 - r^{-y}\,.$$

Substituting Y_t for y, we obtain

$$Y_t = \frac{\ln PG - \ln ER_t}{\ln r}\,. \tag{19}$$

Formula (19) means that the actual fund is exactly sufficient, without further contributions, to cover expenditures for a period of years, Y_t, equal to the excess of the logarithm of the pay-go rate over the logarithm of the equilibrium contribution rate, divided by the logarithm of r. In the special case of a full fund, formula (19) becomes

$$Y_t = \frac{\ln PG - \ln FC}{\ln r}\,.$$

D. *A Relationship between Full Funds and Full Cost Rates*

For a given plan structure and given actuarial assumptions, CE, PG, and r are independent of the full-funding method, but FF and FC are not. If we let ${}_1FF_t$ and ${}_1FC$ refer to the full fund and full cost rate under full-funding method 1, and ${}_2FF_t$ and ${}_2FC$ refer to the same functions under full-funding method 2, then, from formula (16),

$$\frac{{}_1FF_t}{{}_2FF_t} = \frac{PG - {}_1FC}{PG - {}_2FC}\,. \tag{20}$$

It is well known that those full-funding methods involving the lowest mature full cost rates normally produce the highest full funds. Formula (20) not only supports this proposition provided that $r > 1$ (the reverse would be true if $r < 1$), but also provides a mathematical relationship under stabilized conditions between the factors involved—a relationship that may be of more interest in private pension plan analysis than in social insurance analysis. For example, it enables one to determine the relative sizes of the ultimate funds involved under different full-funding methods knowing only the associated full cost rates and the pay-go rate, without the necessity of making any fund projections.

E. *Independence of the Equilibrium Contribution Rate from the Full-funding Method*

The equilibrium contribution rate, ER_t, is the contribution rate that will cause the actual fund, AF_t, to increase at the rate of increase in total earnings if all actuarial assumptions are met. Viewed as such, ER_t is clearly independent of the full-funding method used as a comparison basis. At the same time, relationship (4), which expresses ER_t as a function of FR_t and FC, both of which depend upon the full-funding method used as a comparison basis, seems to imply the opposite. That this is an illusion may be demonstrated by means of formula (20) in the following manner (where prefixed subscripts 1 and 2 pertain to full-funding methods 1 and 2, respectively):

$$\begin{aligned} {}_1ER_t &= {}_1FR_t\,{}_1FC + (1 - {}_1FR_t)PG \\ &= AF_t\left(\frac{{}_1FC - PG}{{}_1FF_t}\right) + PG \\ &= AF_t\left(\frac{{}_2FC - PG}{{}_2FF_t}\right) + PG \\ &= {}_2ER_t\,. \end{aligned}$$

F. *A Direct Formula for Projecting the Equilibrium Contribution Rate*

Normally, ER_t would be calculated by first estimating n, using the definition in Section II, then calculating FR_t using formula (10), and, finally, calculating ER_t using relationship (4). However, considering the potential importance of ER_t, not only in its own right but as a possible working variable in formulas such as (18) and (19), it may be useful to develop a more direct expression for it. Bearing in mind that $FR_t = n - r^t(n - FR_0)$ and $n = (CR - PG)/(FC - PG)$, we have

$$
\begin{aligned}
ER_t &= FR_t\,FC + (1 - FR_t)PG \\
&= [n - r^t(n - FR_0)](FC - PG) + PG \\
&= (1 - r^t)(CR - PG) + r^t(FR_0)(FC - PG) + PG \\
&= CR(1 - r^t) + r^t[FR_0\,FC + (1 - FR_0)PG] \\
&= CR(1 - r^t) + r^t\mathrm{ER}_0\,;
\end{aligned}
$$

whence

$$ER_t = CR - r^t(CR - ER_0)\,. \tag{21}$$

The similarity to formula (10) is striking.

It should be noted that, if AF_0 and CE_0 are known, formula (18) may be used to calculate ER_0, formula (21) to project ER_t, and formula (18) to calculate AF_t/CE_t and, for that matter, AF_t itself, since $CE_t = (1 + p)^t(1 + s)^tCE_0$. This procedure has the advantage, when compared with that based on formula (10), of permitting the projection of AF_t without making any assumption as to a full-funding method to be used as a comparison basis, and without requiring the calculation of FF_0 *or* FC. It has the disadvantage, however, that it does not seem to yield such valuable insights into the mechanics of the funding process as those that can be obtained from formula (10).

XI. CONCLUSIONS

For many earnings-related social insurance plans, such as OASDI and the Canada and Quebec Pension Plans, detailed long-range cost estimates are made by official and unofficial agencies for up to seventy-five years in the future. They cover periods during which plan costs are expected to change dramatically as a result of more or less predictable economic, demographic, and other trends; and perhaps the major purpose of such studies is to assess the scope of such changes. Clearly, the techniques outlined in this paper are unsuitable for analyzing such periods.

However, toward the end of the periods covered by such studies, not only do prognostications become more and more uncertain, but presently discernible trends tend to disappear as well. In effect, a time is being approached when, for all practical present purposes, an assumption of stabilized conditions seems to be as appropriate as any other. The main purpose of this paper is to provide an analytical method to sharpen our perceptions, both of a qualitative and of a quantitative nature, of what may or may not be feasible at such a time.

A word of caution is called for. While many fascinating calculations can easily be made using the techniques outlined in this paper, the various assumptions underlying such calculations must be chosen with care in order to ensure their mutual compatibility. The long-range hypotheses and findings of the detailed cost estimates referred to in the first paragraph of this section may serve as a useful guide in this respect.

If an earnings-related social insurance plan should actually approach stabilized conditions to the extent that analyses of the relatively short-range future conducted by the methods outlined in this paper would approximate fairly closely the findings of much more detailed cost studies, there should be a quickening interest in this approach. It would not supplant the need for detailed investigation of any concrete proposals that might be implemented, but it could be used (*a*) to analyze a tremendous number of possibilities and, from them, to select those worthy of more detailed examination; (b) to explain, in readily intelligible terms, trends inherent in various proposals that might be difficult to detect or isolate from more elaborate inquiries; and (*c*) to permit actuaries, and perhaps others, lacking ready access to detailed cost-study systems, to satisfy themselves that all reasonable possibilities have received due consideration and fair appraisal.

DISCUSSION OF PRECEDING PAPER

CECIL J. NESBITT:

Professor Bowers, Professor Hickman, and I have written several papers ([1], [2], [3]) on the dynamics of pension funding. In these papers we have often illustrated ideas or developed the theory in relation to what we called the exponential growth case, which lends itself to mathematical exploration. We considered the exponential growth case to be rather special and unrealistic. The author, however, whose entire paper is devoted to an exponential growth case, has given us new insights into this case, and has indicated how useful it may be for studying alternatives in the farther-out portions of long-term projections for social insurance and pension systems. He has also developed the theory on a discrete basis, whereas our papers were in terms of continuous functions.

Analogues of many of the author's formulas follow readily from our materials on pension funding dynamics. To encourage the author, and possibly others, to read our papers, I will indicate some of the basic concepts of the author's paper in terms of our continuous theory, and the notations we employed. However, this discussion will be mainly self-contained and should be understandable with minimum reference to the prior papers.

We define the basic functions:

- $\boldsymbol{W}(t)$ Total annual (covered) payroll at time t, a continuous analogue of the author's C_t, and the same as his CE_t.
- $\boldsymbol{B}(t)$ Annual rate of pension outgo at time t. This is the continuous analogue of the author's $PG\ C_t$, and agrees with the author's $PG\ CE_t$.
- $\boldsymbol{P}(t)$ Annual rate of normal cost for the plan at time t. This is the continuous analogue of the author's $FC\ C_t$.
- $\boldsymbol{V}(t)$ Actuarial accrued liability of the plan at time t. Corresponds to the author's FF_t.

Our notations in regard to growth rates are

- δ Force of interest.
- α Rate of growth in population.
- γ Economic rate of growth of salaries.
- $\tau = \alpha + \gamma$ A total rate of growth corresponding to $\ln (1 + p)(1 + s)$.
- $\delta - \tau$ A net rate corresponding to the author's $\ln r$.

For the exponential growth case, all the basic functions grow at the constant rate τ. This implies $\boldsymbol{W}(t) = e^{\tau t}\boldsymbol{W}(0)$, $\boldsymbol{B}(t) = e^{\tau t}\boldsymbol{B}(0)$, and so forth, and any pair of these functions has constant ratio.

We shall also need the additional functions

$\boldsymbol{C}(t)$	Annual rate of contribution at time t, the continuous analogue of the author's $CR\ C_t$.
$\boldsymbol{F}(t)$	Fund on hand at time t, which corresponds to the author's AF_t.
$\boldsymbol{U}(t) = \boldsymbol{V}(t) - \boldsymbol{F}(t)$	Unfunded actuarial accrued liability at time t.
$(\boldsymbol{EC})(t)$	Equilibrium annual contribution rate at time t.

From compound interest theory, we have the following controlling equations, which hold in general:

$$\frac{d\boldsymbol{V}(t)}{dt} = \boldsymbol{P}(t) + \delta\boldsymbol{V}(t) - \boldsymbol{B}(t)\,, \tag{a}$$

$$\frac{d\boldsymbol{F}(t)}{dt} = \boldsymbol{C}(t) + \delta\boldsymbol{F}(t) - \boldsymbol{B}(t)\,. \tag{b}$$

These correspond to the author's equation (2) and his first equation in Section V. By subtraction, we also have

$$\frac{d\boldsymbol{U}(t)}{dt} = \boldsymbol{P}(t) - \boldsymbol{C}(t) + \delta\boldsymbol{U}(t). \tag{c}$$

In the exponential growth case, $d\boldsymbol{V}(t)/dt = \tau\boldsymbol{V}(t)$ (cf. the author's eq. [1]), and equation (a) can be rearranged as

$$\boldsymbol{B}(t) = \boldsymbol{P}(t) + (\delta - \tau)\boldsymbol{V}(t)\,, \tag{d}$$

which shows that $\boldsymbol{B}(t) \gtreqless \boldsymbol{P}(t)$ according as $\delta - \tau \gtreqless 0$. This is equivalent to the author's relation (3). We also have

$$\boldsymbol{V}(t) = [\boldsymbol{B}(t) - \boldsymbol{P}(t)]/(\delta - \tau)\,. \tag{e}$$

In the equilibrium case, $\boldsymbol{F}(t)$ will also be growing at rate τ; hence, from equation (b), with $\boldsymbol{C}(t)$ replaced by $(\boldsymbol{EC})(t)$, we have

$$\boldsymbol{F}(t) = [\boldsymbol{B}(t) - (\boldsymbol{EC})(t)]/(\delta - \tau)\,. \tag{f}$$

The constant ratio of $\boldsymbol{F}(t)$ to $\boldsymbol{V}(t)$ can then be expressed as k, where

$$k = [\boldsymbol{B}(t) - (\boldsymbol{EC})(t)]/[\boldsymbol{B}(t) - \boldsymbol{P}(t)] . \tag{g}$$

This yields

$$(\boldsymbol{EC})(t) = k\boldsymbol{P}(t) + (1 - k)\boldsymbol{B}(t) , \tag{h}$$

which is analogous to the author's equation (4). Again, from equation (f), we have

$$(\boldsymbol{EC})(t) = \boldsymbol{B}(t) - (\delta - \tau)\boldsymbol{F}(t) , \tag{i}$$

the analogue of the author's equation (18). As noted by the author, the equilibrium contribution rate is independent of the funding method.

In the equilibrium case, $\boldsymbol{U}(t)$ also grows at rate τ, and equation (c), with $\boldsymbol{C}(t)$ replaced by $(\boldsymbol{EC})(t)$, yields

$$(\boldsymbol{EC})(t) = \boldsymbol{P}(t) + (\delta - \tau)\boldsymbol{U}(t) . \tag{j}$$

In words, the equilibrium contribution rate equals the normal cost rate plus a contribution toward the unfunded actuarial accrued liability at the rate $(\delta - \tau)$ (cf. Sec. II of [3]).

It is a nice exercise to show from these relations for the exponential growth case that the analogue of the author's equation (21) holds.

The author has breathed new life and utility into the exponential growth case of pension funding. His discrete presentation is a useful complement to our continuous theory (see also John W. Pennisten's discussion of [2]). It was a pleasure to look at the ideas from his viewpoint.

REFERENCES

1. BOWERS, NEWTON L., JR.; HICKMAN, JAMES C.; and NESBITT, CECIL J. "Introduction to the Dynamics of Pension Funding," *TSA*, XXVIII (1976), 177.
2. ———. "The Dynamics of Pension Funding: Contribution Theory," *TSA*, XXXI (1979), 93.
3. ———. "Notes on the Dynamics of Pension Funding," *ARCH*, 1981.1, 21.

(AUTHOR'S REVIEW OF DISCUSSION)

PIERRE TREUIL:

Professor Nesbitt has prepared an interesting discussion in which he indicates that a number of my definitions are analogous to definitions of several functions used in papers prepared by himself in association with Professors Bowers and Hickman. He shows how, by the addition of a few more definitions to their papers, it would be possible to develop analogues of a number of my formulas.

He implies that had I been familiar with the Bowers-Hickman-Nesbitt papers I might have chosen to use them as a framework for mine. It is really impossible to say what I would have done under such circumstances, since the decision would have involved weighing the importance of maintaining uniformity in actuarial notation and terminology between related papers, as opposed to the desirability of presenting my subject in whatever manner I deemed most effective.

Incidentally, I do not quite understand his allegation that my theory was developed on a discrete basis, since all my formulas are essentially of a continuous nature. Perhaps the fact that some of them were developed for specific one-year periods has led to some confusion in this regard.