TRANSACTIONS OF SOCIETY OF ACTUARIES
1979 VOL. 31

ADJUSTABLE LIFE EXPENSE ALLOWANCES UNDER THE COMMISSIONERS RESERVE VALUATION METHOD

SOLOMON GOLDFINGER

ABSTRACT

As a result of the increasing interest in adjustable life insurance, the treatment of cash values and reserves for such policies has become quite significant. This paper will show how the Commissioners Reserve Valuation Method (CRVM) may be extended in a natural way to cover adjustable life policies. Formulas defining the additional expense allowance at the time of change of the policy are derived. This allowance is shown to be a natural generalization of the CRVM expense allowance at issue for a traditional policy. Setting cash values equal to CRVM reserves establishes a method for computing cash values for an adjustable life policy at any point. Formulas are derived that are equally applicable to all types of changes of coverage and to policies that do not change. These formulas do not always agree with those developed by Walter L. Chapin in his paper "Toward Adjustable Individual Life Policies"; the differences between the two sets of formulas are analyzed.

I. INTRODUCTION

THIS paper deals with certain theoretical problems of individual adjustable life policies. The problem of adjustments of the expense allowance at the time of change of the policy is emphasized. If the nonforfeiture value is taken to be the CRVM reserve, the derivations of nonforfeiture values, reserves, and net premiums all become different aspects of the same issue.

In the paper "Toward Adjustable Individual Life Policies" (*TSA*, XXVIII, 237), Walter L. Chapin has presented a comprehensive analysis of adjustable life policies. A different approach to CRVM reserves and expense allowances is taken in this paper. The first part discusses these questions from first principles and does not assume a knowledge of Mr. Chapin's paper. The second part presents the conceptual and practical differences between the two approaches. The sections of Chapin's paper describing gross premiums, pension coverage, and adaptation of customary procedures are not discussed.

II. BACKGROUND

The problem of expense allowances and nonforfeiture values for adjustable life policies was discussed in a general manner by the Society of Actuaries' Special Committee on Valuation and Nonforfeiture Laws. An "open policies" section of the committee's January, 1976, report (p. 32) states the following:

> The committee believes that the traditional prospective adjusted premium approach can accommodate most open policy–type plans by assuming that changes not stated numerically in the policy would not occur. The adjusted premium would then be determined at issue in the usual manner so that the present value of adjusted premiums is equal to the present value of future benefits plus the expense allowance. When a change in benefits or premiums does occur, new adjusted premiums would be calculated such that their present value equals
>
> *a*) The present value of future benefits less the current minimum cash value and
> *b*) Any new expense allowance resulting from the change, again assuming no further changes beyond the point of recalculation.

With respect to the expense allowance, there seems little question but that the amount of expense allowance at original issue should be the same as for a standard plan beginning with the same initial policy features. Furthermore, it should be amortized in the same manner. To reach a conclusion as to whether future changes in expense allowances should occur as policy changes occur, it is useful to consider the type and origin of excess first-year expenses, that is, compensation, underwriting, and issue. While there may be exceptions, some of these same types of expenses will likely be incurred when either the premium or face amount increases. First-year commissions will likely be paid on any increase in gross premiums. Increases in face amount, unless they are nominal, will likely result in some underwriting expense. On the other hand, since expenses of policy establishment at the time of change should be much less than at the time of issue, there are grounds for providing a smaller initial expense allowance on such increases than on a newly issued policy. How much smaller is a far more difficult question to answer. The committee concluded that, in the interest of simplicity and encouragement of experimentation, full formula expenses should be permitted to apply to increases in premium or amount at the time these occur.

This conclusion creates an inconsistency: Full initial expense allowances on policy increments could easily result in negative cash-value increments. This result is different from a new issue where negative values are uncollectable. It would be difficult to explain to a policyholder who has increased his premiums that his cash value has decreased. In fact, however, a negative increment in cash values is not unreasonable when it reflects the actual incidence of expenses incurred in connection with the change. The cost of underwriting which is incurred when the face amount increases without a corresponding increase in premium would be one example. Such costs could be reflected in the cash values only if negative increments were allowed. An inability to collect such costs from those terminating shortly after a change would mean increased costs for continuing policyholders. Being able to collect the full costs of change would put companies in a position to make the change on the most favorable terms. Recalling that the Society meeting discussions preceding formation of our committee are rife with criticisms of the current law's inflexibility in the "open policy" area, the committee's conclusion was to resolve the inconsistency in favor of flexibility and to recommend full allowances at time of change.

For completeness, the committee also considered the possibility that premiums or amounts would decrease. We do not believe these changes should reduce or reverse initial expense allowances, since the initial costs for underwriting, compensation, and issue will already have been incurred and cannot be reversed. There is even some ground to argue for more allowance at the time of such change to cover the cost of the change. However, the committee felt that it would be more appropriate to reflect any such costs in increased gross premiums rather than in reduced cash values. This is consistent with the proposal for multitrack policies.

The following two points in these recommendations may be emphasized:

1. Future changes that result in a lower premium would not reduce the expense allowances calculated at issue.
2. Changes in benefits or premiums would require the calculation of new modified premiums having a present value of
   - *a*) The present value of future benefits less the current minimum cash value, plus
   - *b*) Any new expense allowance resulting from the change, again assuming no further changes beyond the point of recalculation.

Although these comments relate to minimum nonforfeiture values, the same approach may be taken with regard to minimum reserves, since in either situation a modified premium is calculated with the present value at issue of all such modified premiums equal to the present value of future benefits plus an expense allowance. Reserves at the time of change of an adjustable life policy therefore may be handled in exactly the same manner as that described above for nonforfeiture values. The only difference would be in the size of the expense allowance.

III. ASSUMPTIONS AND NOTATION

It is assumed that reserves are based on the CRVM. The notation, defined below, is consistent with that used by Chapin.

m = Designation of the status of an adjustable life policy ($m = 1$ is status for an original issue, $m = 2$ is status after first change, etc.);

x_m = Age of the insured at the beginning of the mth status;
z = Age at expiry of a term policy or age at maturity of an endowment policy;
w = Age to which premiums are paid on a limited-payment whole life or endowment policy;
I_{x_m} = Amount of insurance in effect during the mth status;
$I_{x_m}\pi_{x_m}$ = CRVM net premium during the mth status;
${}_{\overline{x_m-x_1}}V_{x_1}$ = Terminal reserve at commencement of the mth status;
${}_{\overline{y-x_1}}V_{x_1}$ = Terminal reserve at attained age y;
$I_{x_m}\Delta_{x_m}$ = Allowance for statutory expense in the first year of status m;
$\beta^F_{x_m}$ = Full preliminary term renewal net premium for a new issue at age x_m of the plan in status m;
${}_{19}P_{x_m+1}$ = Net level premium for a nineteen-payment life plan issued at age $x_m + 1$;
$\beta^{\text{CRVM}}_{x_m}$ = Modified net premium for a new issue at age x_m of the plan in status m, as defined by the CRVM;
$P^A_{x_m}$ = Lesser of $\beta^F_{x_m}$ and ${}_{19}P_{x_m+1}$;
${}^r(P^A_{x_m} - c_{x_m})$ = $(P^A_{x_m} - c_{x_m})$ for the plan in status r.

IV. REVIEW OF THE CRVM

A review of the CRVM as defined in the Standard Valuation Law, and some formulas that result from it, are presented as further background. Although the resulting formulas for traditional policies are well known and readily available, one must start from scratch with the definitions given in the Standard Valuation Law to determine the correct expense allowances under the CRVM for adjustable policies.

Under the CRVM, reserves for life insurance and endowment benefits of policies providing for a uniform amount of benefits and requiring the payment of uniform premiums equal the excess, if any, of the present value at the date of valuation of the future guaranteed benefits over the present value of any future modified net premiums.

In actuarial notation, if β^{CRVM}_x is the modified net premium, then

$$ {}_tV_x = A_{x+t:\overline{z-x-t}|} - \beta^{\text{CRVM}}_x \ddot{a}_{x+t:\overline{w-x-t}|} , \tag{1} $$

where z equals the age at which coverage expires, t equals the duration, and w equals the age to which premiums are paid.

Where benefits and premiums are uniform by duration, the modified net premium is the amount for which the present value, at the date of issue of the policy, of all modified net premiums equals the sum of the then present value of such benefits provided for by the policy plus the excess of (A) over (B).

In actuarial notation,

$$\beta_x^{CRVM}\ddot{a}_{x:\overline{w-x}|} = A_{x:\overline{z-x}|} + [(A) - (B)]\,. \tag{2}$$

(A) is defined as a net level annual premium equal to the present value, at the date of issue, of such benefits provided for after the first policy year divided by the present value, at the date of issue, of an annuity of 1 per annum payable on the first and each subsequent anniversary of such policy on which a premium falls due; provided, however, that such net level annual premium may not exceed the net level annual premium for a nineteen-payment life plan of insurance for the same amount at an age one year higher than the age at issue of such policy.

In actuarial notation, (A) is defined to be

$$\frac{(D_{x+1}/D_x)A_{x+1:\overline{z-x-1}|}}{(D_{x+1}/D_x)\,\ddot{a}_{x+1:\overline{w-x-1}|}}\,,$$

which may be recognized as the net level premium for the same policy issued to a person one year older and which we will call β_x^F; provided, however, that (A) should equal ${}_{19}P_{x+1}$, the net level annual premium for nineteen-payment life issued at age $x+1$, if $\beta_x^F > {}_{19}P_{x+1}$. (B) is defined as a net one-year term premium, c_x, for the benefits provided in the first policy year.

To summarize, β_x^{CRVM}, the modified net premium, is defined by the equation

$$\beta_x^{CRVM}\ddot{a}_{x:\overline{w-x}|} = A_{x:\overline{z-x}|} + (\beta_x^F - c_x) \qquad \text{if } \beta_x^F \leq {}_{19}P_{x+1} \tag{3}$$

$$= A_{x:\overline{z-x}|} + ({}_{19}P_{x+1} - c_x) \quad \text{otherwise}\,. \tag{4}$$

In the first case above, the formula for β_x^{CRVM} may be simplified considerably. We know that

$$A_{x:\overline{z-x}|} = c_x + \frac{D_{x+1}}{D_x}A_{x+1:\overline{z-x-1}|}\,. \tag{5}$$

This equals $c_x + \beta_x^F(D_{x+1}/D_x)\ddot{a}_{x+1:\overline{w-x-1}|}$, by the definition of β_x^F. But $(D_{x+1}/D_x)\ddot{a}_{x+1:\overline{w-x-1}|}$ equals $a_{x:\overline{w-x-1}|}$. Substituting the resulting expression for $A_{x:\overline{z-x}|}$ in formula (3), we obtain

$$\beta_x^{CRVM}\ddot{a}_{x:\overline{w-x}|} = (c_x + \beta_x^F a_{x:\overline{w-x-1}|}) + (\beta_x^F - c_x) \tag{6}$$

or

$$\beta_x^{CRVM}\ddot{a}_{x:\overline{w-x}|} = \beta_x^F a_{x:\overline{w-x-1}|} + \beta_x^F \tag{7}$$

or

$$\beta_x^{\text{CRVM}}\ddot{a}_{x:\overline{w-x}|} = \beta_x^F \ddot{a}_{x:\overline{w-x}|} . \tag{8}$$

Therefore, $\beta_x^F = \beta_x^{\text{CRVM}}$.

Thus, if $\beta_x^F \leq {}_{19}P_{x+1}$, the modified net premium given by the CRVM, β_x^{CRVM}, equals β_x^F, the net level premium for the same policy issued at age $x+1$. While this is a consequence of the formula given by the CRVM, the method does not state *explicitly* that $\beta_x^{\text{CRVM}} = \beta_x^F$ if $\beta_x^F \leq {}_{19}P_{x+1}$. This point will be seen to be significant later.

V. DEFINITION OF THE ALLOWANCE FOR STATUTORY EXPENSE FOR AN ADJUSTABLE LIFE POLICY

When an adjustable life policy is issued, the allowance for statutory expense should be the same as that allowed by the CRVM for a traditional policy of the plan of the first status. Expressing this expense allowance at issue in terms of our adjustable life notation, we have

$$I_{x_1}\Delta_{x_1} = I_{x_1}(\beta_{x_1}^F - c_{x_1}) \qquad \text{if } \beta_{x_1}^F \leq {}_{19}P_{x_1+1} \tag{9}$$

$$= I_{x_1}({}_{19}P_{x_1+1} - c_{x_1}) \qquad \text{if } \beta_{x_1}^F > {}_{19}P_{x_1+1} . \tag{10}$$

To allow for a unified discussion we have defined $P_{x_1}^A$ as the lesser of $\beta_{x_1}^F$ and ${}_{19}P_{x_1+1}$. That is, $P_{x_1}^A$ is (A) in the expression "the excess of (A) over (B)" in the CRVM.

The CRVM as defined in the Standard Valuation Law does not apply to adjustable life policies that are changed, since the law defines the CRVM only for policies whose benefits and premiums are level. However, the law does require that reserves for policies whose premiums or benefits vary should be calculated by a method that is consistent with the method given for policies whose premiums and benefits are level. Thus, the additional expense allowance at the time of change of an adjustable life policy must be consistent with the expense allowance prescribed by the CRVM for traditional policies.

In particular, there are three situations where an adjustable life expense allowance should match the allowance given by the CRVM. First, when an adjustable life policy is originally issued, the expense allowance should be as defined by the CRVM for the plan in the first status. Second, if the adjustable life policy does not change, the adjustable life expense allowance formula should produce no additional expense allowance. The third situation is represented by an increase in face amount and premium but no change in plan. The impact is the same as if the adjustable life policy had not changed and a separate policy had been issued for the difference in the face amounts. For example, if an adjustable life policy is issued at age 45 as life paid-up at 65 for $10,000, and then at age 55 it is changed to life paid-up at 65 for $25,000, the resulting coverage is the same as if the first policy had not been changed and a new life paid-up at 65 policy for $15,000 had been issued at age 55. It is desirable to have the adjustable life expense allowance for this type of change identical with the expense allowance for a separate policy providing the increased coverage.

In devising an adjustable life additional expense allowance formula, one might begin by setting the additional expense allowance at the time of change of an adjustable life policy equal to the excess of (*a*) the expense allowance for a new issue of the plan in status m at age x_m for amount I_{x_m} over (*b*) that of a new issue of the plan in status $m-1$ at age x_{m-1} for amount $I_{x_{m-1}}$. This would result in the formula

$$I_{x_m}\Delta_{x_m} = I_{x_m}(P^A_{x_m} - c_{x_m}) - I_{x_{m-1}}(P^A_{x_{m-1}} - c_{x_{m-1}}) . \tag{11}$$

This formula, however, does not produce expense allowances that are consistent with those under the CRVM for traditional policies. It would permit an additional expense allowance where the only "change" from status $m-1$ to status m is the increased age of the insured (no change in plan or amount). Obviously, no additional expense allowance is warranted merely because the insured has grown older. However, if I_{x_m} equals $I_{x_{m-1}}$, and the plan in status m is the same as that in status $m-1$, formula (11) generally will give a positive value because the values of $(\beta^F_x - c_x)$ and $({}_{19}P_{x+1} - c_x)$ generally increase with age.

This effect may be offset by subtracting that portion of the allowance that results exclusively from the older age in status m. Let ${}^{m-1}(P^A_{x_m} - c_{x_m})$ represent the CRVM expense allowance for a new issue of the plan in status $m-1$ at age x_m, and ${}^{m-1}(P^A_{x_{m-1}} - c_{x_{m-1}})$ the CRVM expense allowance for a new issue of the plan in status $m-1$ at age x_{m-1}. Then the right-hand side of formula (11) should be reduced by the difference in these expense allowances,

$$I_{x_{m-1}}[{}^{m-1}(P^A_{x_m} - c_{x_m}) - {}^{m-1}(P^A_{x_{m-1}} - c_{x_{m-1}})] . \tag{12}$$

The resulting formula is

$$\begin{aligned} I_{x_m}\Delta_{x_m} = I_{x_m}\, {}^{m}(P^A_{x_m} - c_{x_m}) - I_{x_{m-1}}\, {}^{m-1}(P^A_{x_{m-1}} - c_{x_{m-1}}) \\ - I_{x_{m-1}}[{}^{m-1}(P^A_{x_m} - c_{x_m}) - {}^{m-1}(P^A_{x_{m-1}} - c_{x_{m-1}})] . \end{aligned} \tag{13}$$

If we combine terms, this reduces to

$$I_{x_m}\Delta_{x_m} = I_{x_m}\ {}^{m}(P^A_{x_m} - c_{x_m}) - I_{x_{m-1}}\ {}^{m-1}(P^A_{x_m} - c_{x_m})\,. \qquad (14)$$

Does this formula reproduce the CRVM expense allowance in the three situations mentioned previously? First, for a new issue, $m = 1$ and $I_{x_{m-1}} = 0$, so this formula yields the CRVM expense allowance for a new issue. Second, if there is no change in amount or plan, $I_{x_m} = I_{x_{m-1}}$ and ${}^{m}(P^A_{x_m} - c_{x_m}) = {}^{m-1}(P^A_{x_m} - c_{x_m})$; thus, the formula properly reduces to zero and satisfies the demonstrated deficiency in formula (11). Finally, if the premium and amount change but the plan remains the same, ${}^{m}(P^A_{x_m} - c_{x_m})$ and ${}^{m-1}(P^A_{x_m} - c_{x_m})$ are identical. We therefore may combine terms to obtain $I_{x_m}\Delta_{x_m} = (I_{x_m} - I_{x_{m-1}})\ {}^{m}(P^A_{x_m} - c_{x_m})$. This is easily recognized as the expense allowance under the CRVM for a new issue at age x_m of the plan in status m, for face amount $(I_{x_m} - I_{x_{m-1}})$. Thus, formula (14) meets the requirements of consistency with the CRVM expense allowances.

The question arises whether a more general type of change also may be handled through the separate-policy approach. When the plans in status m and $m - 1$ are not the same, it is possible to express the increased coverage as provided in a separate policy. However, the separate policy in this situation generally will involve a pure endowment at the last common premium-paying durations of the policy before change, if continued, and of the policy after change. The pure endowment is for an amount equal to the difference at this point in the reserves of the policy before change and the policy after change. Under this approach, the additional expense allowance at the time of change would equal the expense allowance for the separate policy just described. However, this approach is not always appropriate, since it may produce anomalous results in certain situations. Formula (14) gives correct results in all cases and is used throughout this paper. Note that when the plan does not change this formula reduces to that obtained by the separate-policy approach. (This issue is explored more fully in Sec. IX.)

In some cases $I_{x_m}\Delta_{x_m}$ as defined by formula (14) will be negative, as in a change from, say, life paid-up at 65 to term to 60. In that case, $I_{x_m}\Delta_{x_m}$ should be set to zero, and at the time of the next change the plan in status $m + 1$ should be compared again with that in status $m - 1$. This is consistent with the Special Committee's recommendation that negative expense allowances not be required since full expenses were incurred at the time of issue.

VI. THE MODIFIED NET PREMIUM

Once the amount of any additional expense allowance at the time of change has been determined, the modified net premium for the adjustable life policy in its mth status, $I_{x_m}\pi_{x_m}$, may be derived. The modified net premium is that amount for which the present value of all future modified net premiums, plus the reserve then being held, equals the present value of future benefits, plus any additional expense allowance. That is,

$$(I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|}) + {}_{\overline{x_m-x_1}|}V_{x_1} = I_{x_m}A_{x_m:\overline{z-x_m}|} + I_{x_m}\Delta_{x_m}\,. \tag{15}$$

Note that for a new issue (${}_{\overline{x_m-x_1}|}V_{x_1} = 0$) or any time the coverage is not changed ($I_{x_m}\Delta_{x_m} = 0$) formula (15) reduces to the usual modified reserve formula.

Once the modified net premium, $I_{x_m}\pi_{x_m}$, for an adjustable life policy in its mth status has been computed, the CRVM terminal reserves during the mth status may be computed either prospectively or retrospectively. Prospectively,

$${}_{\overline{y-x_1}|}V_{x_1} = I_{x_m}A_{y:\overline{z-y}|} - I_{x_m}\pi_{x_m}\ddot{a}_{y:\overline{w-y}|}\,, \tag{16}$$

where y is an attained age during the mth status and z and w are for the plan in status m. Retrospectively,

$${}_{\overline{y-x_1}|}V_{x_1} = ({}_{\overline{x_m-x_1}|}V_{x_1} - I_{x_m}\Delta_{x_m})\left(\frac{D_{x_m}}{D_y}\right) + I_{x_m}\pi_{x_m}\left(\frac{N_{x_m} - N_y}{D_y}\right) - I_{x_m}\left(\frac{M_{x_m} - M_y}{D_y}\right). \tag{17}$$

Once again, when the plan in status m is the same as that in status $m - 1$, the results should be consistent with the separate-policy approach. The modified net premium for a separate policy issued at age x_m for the amount $(I_{x_m} - I_{x_{m-1}})$ is $(I_{x_m} - I_{x_{m-1}})\,\beta^{\text{CRVM}}_{x_m}$, where $\beta^{\text{CRVM}}_{x_m} = {}_wP^{\text{NLP}}_{x_m} + (P^A_{x_m} - c_{x_m})/\ddot{a}_{x_m:\overline{w-x_m}|}$. The total modified net premium after change, $I_{x_m}\pi_{x_m}$, as defined by formula (15), therefore should equal the modified net premium of the policy before change, plus the modified net premium for the separate policy. That is,

$$I_{x_m}\pi_{x_m} = I_{x_{m-1}}\pi_{x_{m-1}} + (I_{x_m} - I_{x_{m-1}})\beta^{\text{CRVM}}_{x_m}\,. \tag{18}$$

Formula (18) may be derived from our basic formula for $I_{x_m}\pi_{x_m}$, formula (15), as follows:

$$I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} = I_{x_m}A_{x_m:\overline{z-x_m}|} + I_{x_m}\Delta_{x_m} - {}_{\overline{x_m-x_1}}V_{x_1} \quad (\text{formula [15]}) .$$

Since the type of plan has not changed,

$${}_{\overline{x_m-x_1}}V_{x_1} = I_{x_{m-1}}A_{x_m:\overline{z-x_m}|} - I_{x_{m-1}}\pi_{x_{m-1}}\ddot{a}_{x_m:\overline{w-x_m}|}$$

and

$$I_{x_m}\Delta_{x_m} = (I_{x_m} - I_{x_{m-1}})(P^A_{x_m} - c_{x_m}) .$$

Substituting in formula (15), we obtain

$$I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} = I_{x_m}A_{x_m:\overline{z-x_m}|} + (I_{x_m} - I_{x_{m-1}})(P^A_{x_m} - c_{x_m}) - (I_{x_{m-1}}A_{x_m:\overline{z-x_m}|} - I_{x_{m-1}}\pi_{x_{m-1}}\ddot{a}_{x_m:\overline{w-x_m}|}) , \tag{19}$$

or

$$I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} = I_{x_{m-1}}\pi_{x_{m-1}}\ddot{a}_{x_m:\overline{w-x_m}|} + (I_{x_m} - I_{x_{m-1}})[A_{x_m:\overline{z-x_m}|} + (P^A_{x_m} - c_{x_m})] , \tag{20}$$

or

$$I_{x_m}\pi_{x_m} = I_{x_{m-1}}\pi_{x_{m-1}} + (I_{x_m} - I_{x_{m-1}})\beta^{\text{CRVM}}_{x_m} \quad (\text{formula [18]}) .$$

Similarly, the reserve for the adjustable life policy in the mth status may be shown to equal the reserve for the coverage in the previous status plus the reserve for the new coverage if it were contained in a separate policy. This may be derived by substituting in formula (16) the expression for $I_{x_m}\pi_{x_m}$ given by formula (18) and applying some algebra. Details are left to the reader.

In formula (18) it is important to keep in mind the distinction between $I_{x_m}\pi_{x_m}$, which is the modified net premium for an adjustable life plan in status m, and $\beta^{\text{CRVM}}_{x_m}$, which is the modified net premium for a new issue of the plan in status m at age x_m. In general, these will not be equal except where $m = 1$. This may be demonstrated by substituting $\beta^{\text{CRVM}}_{x_m}$ for π_{x_m} in formula (18). Solving, the result is $\pi_{x_{m-1}} = \beta^{\text{CRVM}}_{x_m}$, which certainly is not true in the general case.

Early in the paper it was shown that under the CRVM for the case $\beta^F_x \leq {}_{19}P_{x+1}$, $\beta^{\text{CRVM}}_x = \beta^F$, where β^{CRVM}_x is the modified net premium and β^F_x is the full preliminary term renewal premium. This was indicated to be a consequence of the CRVM method that is not stated explicitly. Therefore there is no contradiction in not having the adjustable life modified net premium in the mth status, π_{x_m}, equal $\beta^F_{x_m}$, even in the case where $\beta^F_{x_m} \leq {}_{19}P_{x_m+1}$.

Instead, in the case where $\beta^F_{x_m} \leq {}_{19}P_{x_m+1}$ and $\beta^F_{x_{m-1}} \leq {}_{19}P_{x_{m-1}+1}$ and the only change is an increase in the amount and the premium, formula (18) gives the relationship

$$I_{x_m}\pi_{x_m} = I_{x_{m-1}}\pi_{x_{m-1}} + (I_{x_m} - I_{x_{m-1}})\beta^F_{x_m}.$$

VII. RELATION BETWEEN MODIFIED NET PREMIUM AND ADDITIONAL EXPENSE ALLOWANCE

The basic relation between the modified net premium and the additional expense allowance is that given by formula (15) and is always applicable. For a new issue of a regular policy, when $\beta^F_{x_1} \leq {}_{19}P_{x_1+1}$ there is the much simpler relation

$$I_{x_1}\Delta_{x_1} = I_{x_1}(\beta^F_{x_1} - c_{x_1}) = I_{x_1}(\beta^{\text{CRVM}}_{x_1} - c_{x_1}) = I_{x_1}(\pi_{x_1} - c_{x_1}) \,. \tag{21}$$

For an adjustable life policy where $\beta^F_{x_m} \leq {}_{19}P_{x_m+1}$ and $\beta^F_{x_{m-1}} \leq {}_{19}P_{x_{m-1}+1}$ and only the amount and premium change, s[illegible] interesting relationships may be derived. By formula (14),

$$I_{x_m}\Delta_{x_m} = (I_{x_m} - I_{x_{m-1}})(\beta^F_{x_m} - c_{x_m}) \,,$$

but by formula (18),

$$I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}} = (I_{x_m} - I_{x_{m-1}})\beta^F_{x_m} \,.$$

In this situation, the additional expense allowance may be expressed directly in terms of the modified net premium of the adjustable life policy in the mth status. Substituting, we have

$$I_{x_m}\Delta_{x_m} = (I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}}) - (I_{x_m} - I_{x_{m-1}})c_{x_m} \,. \tag{22}$$

In words, the *increase* in expense allowance equals the *increase* in the modified net premium, less the *increase* in the one-year term cost of insurance. Thus, this formula is the natural generalization of what is given by the CRVM for a new issue. Of course, if we set $m = 1$ and $I_{x_{m-1}} = 0$, formula (22) reduces to that for a traditional policy at issue (formula [9]):

$$I_{x_1}\Delta_{x_1} = I_{x_1}(\pi_{x_1} - c_{x_1}) \,.$$

(For a new issue, π_x equals β^F_x when $\beta^F_x \leq {}_{19}P_{x+1}$.)

In this situation, where $\beta^F_{x_m} \leq {}_{19}P_{x_m+1}$ and $\beta^F_{x_{m-1}} \leq {}_{19}P_{x_{m-1}+1}$ and the only change is that of amount and premium, other interesting formulas also may be derived. For example, the equation

$$I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}} = (I_{x_m} - I_{x_{m-1}})\ {}_wP^{\text{NLP}}_{x_m} + \frac{I_{x_m}\Delta_{x_m}}{\ddot{a}_{x_m:\overline{w-x_m}|}} \tag{23}$$

may be obtained from formula (20). Also, from formula (18),

$$I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}} = (I_{x_m} - I_{x_{m-1}})\beta^F_{x_m} = (I_{x_m} - I_{x_{m-1}})\ {}_wP^{\text{NLP}}_{x_m+1}\ ;$$

substituting, we have

$$(I_{x_m} - I_{x_{m-1}})({}_wP^{\text{NLP}}_{x_m+1} - {}_wP^{\text{NLP}}_{x_m}) = \frac{I_{x_m}\Delta_{x_m}}{\ddot{a}_{x_m:\overline{w-x_m}|}}\,. \tag{24}$$

In summary, the main formulas, which are applicable to all types of changes, are the following: (14), the additional expense allowance in status m; (15), the modified net premium in status m; and (16), the modified reserve in status m.

VIII. COMPARISON WITH CHAPIN'S FORMULAS

The formulas developed in this paper for the additional expense allowance at the time of change of an adjustable life policy now will be compared with those presented by Chapin in his paper.

In describing the allowance for statutory expense, Chapin considers four situations, each of which results in a different formula (the numbering of the formulas is ours):

1. When $\pi_{x_m} < {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} < {}_{19}P_{x_{m-1}+1}$,

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}})\,. \tag{25}$$

2. When $\pi_{x_m} < {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} \geq {}_{19}P_{x_{m-1}+1}$,

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - \sum_{r=1}^{m-1} I_{x_r}\Delta_{x_r}\,. \tag{26}$$

3. When $\pi_{x_m} \geq {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} \geq {}_{19}P_{x_{m-1}+1}$,

$$\begin{aligned} I_{x_m}\Delta_{x_m} = I_{x_m}({}_{19}P_{x_m+1} - c_{x_m}) - I_{x_{m-1}}({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}}) \\ - I_{x_{m-1}}({}_{19}P_{x_m+1} - {}_{19}P_{x_{m-1}+1})\,. \end{aligned} \tag{27}$$

4. When $\pi_{x_m} \geq {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} < {}_{19}P_{x_{m-1}+1}$,

$$\begin{aligned} I_{x_m}\Delta_{x_m} = I_{x_m}({}_{19}P_{x_m+1} - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) \\ - I_{x_{m-1}}({}_{19}P_{x_m+1} - {}_{19}P_{x_{m-1}+1})\,. \end{aligned} \tag{28}$$

In reference to the formula for case 1, Chapin states: "When $m > 1$, the increase in statutory expense in the mth status for a preliminary term policy consists of two parts: (*a*) the statutory expense for a new issue at age x_m for the amount of insurance and net premium in the mth status, less (*b*) the expense for a new issue at age x_{m-1} for the amount of insurance and net premium in the $(m - 1)$st status."

If this language is taken literally, the formula for case 1 should be

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\beta^F_{x_m} - c_{x_m}) - I_{x_{m-1}}(\beta^F_{x_{m-1}} - c_{x_{m-1}}) .$$

It already has been shown that this formula (11) gives an additional expense allowance that is not consistent with that allowed under the CRVM. In his subsequent development, however, Chapin uses π_{x_m} not as the modified net premium for a *new* issue of the plan in the mth status but rather as the modified net premium for an adjustable life plan in its mth status.

Similarly, Chapin distinguishes among his four cases by comparing π_{x_m} and $\pi_{x_{m-1}}$ with ${}_{19}P_{x_m+1}$ and ${}_{19}P_{x_{m-1}+1}$, respectively. In the Commissioners Method, however, it is β^F_x that is compared with ${}_{19}P_{x+1}$ and not the resulting modified net premium β^{CRVM}_x. Therefore, the test should be formulated in terms of $\beta^F_{x_m}$ and $\beta^F_{x_{m-1}}$ rather than π_{x_m} and $\pi_{x_{m-1}}$.

For the case where $\beta^F_{x_m} \leq {}_{19}P_{x_m+1}$ and $\beta^F_{x_{m-1}} \leq {}_{19}P_{x_{m-1}+1}$, formula (14) is

$$I_{x_m}\Delta_{x_m} = I_{x_m}{}^{m}(\beta^F_{x_m} - c_{x_m}) - I_{x_{m-1}}{}^{m-1}(\beta^F_{x_m} - c_{x_m}) .$$

When the type of plan in status m is the same as that in status $m - 1$ and the only change is in premium and amount, this reduces to

$$I_{x_m}\Delta_{x_m} = (I_{x_m} - I_{x_{m-1}})(\beta^F_{x_m} - c_{x_m}) .$$

Formula (22) shows that in this case

$$I_{x_m}\Delta_{x_m} = (I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}}) - (I_{x_m} - I_{x_{m-1}})c_{x_m} .$$

Later in his paper Chapin shows that his formula meets minimum requirements under the CRVM by establishing that the reserve that results from his additional allowance generally is greater than the reserve for the coverage of the previous status plus the reserve for the additional coverage if given in a separate policy. He does this by demonstrating that the modified net premium resulting from his formula is generally *less* than the sum of the modified net premium of the previous status and the modified net premium for the new coverage in a separate policy, since a lower modified net premium implies a lower expense allowance and a larger reserve.

It would seem more appropriate to define the modified net premium so that it *equals* the old modified net premium plus the modified net premium for the new coverage in a separate policy, so that the reserve in the mth status *equals* the reserve for the previous coverage plus the reserve for the new coverage if given in a separate policy. It has been shown that the formulas in this paper meet this requirement. Besides essentially aesthetic considerations and the desire to minimize statutory reserve requirements, it should be noted that if there is a dip in the mortality table (as at ages under ten or in the mid-twenties) Chapin's additional expense allowance in fact will *exceed* that defined by the CRVM for the new coverage if given in a separate policy.

In Chapin's expense allowance formula for case 2 (formula [26] in this paper), the amount to be subtracted could have been expressed exclusively in terms of functions of the $(m - 1)$st status. There seems no need to subtract the sum of the expense allowances of all previous statuses.

For case 3 the approach in this paper is similar to that taken by Chapin, except that $I_{x_{m-1}}[({}_{19}P_{x_m+1} - c_{x_m}) - ({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}})]$ would be subtracted rather than just $I_{x_{m-1}}({}_{19}P_{x_m+1} - {}_{19}P_{x_{m-1}+1})$. The difference in expense allowances is $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$, and the comments made for case 1 apply here as well.

The comments of case 3 also apply to case 4. Also, it is hard to justify subtracting the difference of the nineteen-payment life premiums in a case where $\pi_{x_{m-1}} < {}_{19}P_{x_{m-1}+1}$.

In addition to these specific observations on Chapin's formulas, it is clearer and more convenient to use a generally applicable formula such as (14) than to have separate formulas for various cases.

IX. HANDLING THE GENERAL CHANGE

The formulas in this paper have been applied occasionally to the special case where the change consisted of increasing the premium and face amount of a life plan without changing the premium-paying period. This has facilitated the understanding of the formulas by showing that they give the same results as would be obtained if the increased benefits were provided in a separate policy. Since the separate-policy approach generally is not appropriate for a more general type of change, general formulas have been employed in that situation.

Chapin, however, presents a method for extending the separate-policy approach to the more general type of change. His suggestion is that the lifetime of the separate policy be restricted to the period during which the premium-paying period of the policy under the new status overlaps the premium-paying period of the policy under the old status. The benefits of the separate policy consist of an increased death benefit during this period, plus a pure endowment at the end of this period equal to the difference in reserves at that point between the plan before change and the plan after change.

If curtate 1958 CSO reserves at 3 percent are used, the increased benefits given by a change at age 40 from $10,000 term to 65 to $20,000 life paid-up at 65 may be represented in a separate policy providing $10,000 term to 65 plus a pure endowment of the reserve at age 65 of a $20,000 paid-up life plan. Similarly, a change at age 35 from a $20,000 life paid-up at 65 issued at age 27 to a $30,026 endowment at 60 may be viewed as a separate policy that provides $10,026 of insurance and matures at age 60 for $18,652. This latter number equals the excess of the endowment at age 60, $30,026, over the reserve at age 60 of the $20,000 life paid-up at 65, if continued.

Chapin does not use this approach to derive his expense allowance formulas, but he implies that such an approach would be acceptable. Although this approach may work in many cases, it is not of general applicability. For one thing, the CRVM is not defined for pure endowment benefits. Another problem is in making the CRVM comparison of $\beta^F_{x_m}$ with ${}_{19}P_{x_m+1}$. As long as the type of coverage and the premium-paying period do not change, the comparison, done on a per thousand basis, will give the same result for a separate policy as for the new total policy after change. That is, if both the separate policy and the policy after change are, say, life paid-up at 65, both will fall into the same branch of the CRVM. However, for a more general change involving type of coverage, it is possible for the full preliminary term modified premium of the new benefits, viewed as a separate policy, to fall into one CRVM branch, while the modified premium of the total policy after change falls into the other CRVM branch. A related problem is that comparing unlike plans by treating the difference in their reserves at some age as a pure endowment benefit obscures the identity of the plan, which, after all, is the real test whether full preliminary term applies under the CRVM or not. In concrete terms, should the test be applied per thousand of pure endowment benefit or per thousand of the original coverage?

Another limitation of this approach is that, if the policy after change has a premium-paying period that is longer than that of the policy before change, the expense allowance for the separate policy will be amortized over a period that is different from the premium-paying period of the new plan.

These considerations demonstrate that it is both difficult and inaccurate to try to maintain the separate-policy approach where the plan changes. The approach given by formula (14), $I_{x_m}\Delta_{x_m} = I_{x_m}\ {}^m(P^A_{x_m} - c_{x_m}) - I_{x_{m-1}}\ {}^{m-1}(P^A_{x_m} - c_{x_m})$, is recommended. This views the additional expense allowance at the time of change as the allowance for a new issue of the plan in status m to age x_m, less that portion of the allowance of the new policy that derives from benefits already included in the previous status, for which an additional allowance is not warranted. This approach should provide proper expense allowances in all cases.

X. CONCLUSION

The purpose of this paper has been to illustrate the concepts involved in devising correct additional expense allowances for adjustable life policies. As shown by the issues raised, honest differences of opinion can and do exist. Moreover, in an area of such theoretical complexity it is both easy and tempting to use an allowance formula that is simple but that is not fully consistent with the CRVM. If adjustable life is to win broad acceptance, it is important that the expense allowances can be demonstrated to be fair, practical, and correct.

XI. ACKNOWLEDGMENTS

Although I am listed as the sole author of this paper, it was not a purely individual effort. The ideas in the paper took shape over several months, in the course of discussions with several of my colleagues, including Steve Moses, Michael Gallo, and Robin Welch. I am grateful to Walter N. Miller and Walter Shur for taking the time to comment on early drafts of my paper. Their suggestions improved the paper immeasurably. And I would like to single out Robin Welch for special thanks. Writing this paper was his idea, and without his prodding and encouragement it never would have been written.

DISCUSSION OF PRECEDING PAPER

JOHN E. ASCHENBRENNER:

The Commissioners Reserve Valuation Method (CRVM) contained in the Standard Valuation Law was not written or designed with policies of the adjustable life type in mind. The Standard Valuation Law states that "reserves according to the commissioners reserve valuation method for (1) life insurance policies providing for a varying amount of insurance or requiring the payment of varying premiums . . . shall be calculated by a method consistent with the principles of the preceding paragraph," where the preceding paragraph describes the CRVM for traditional level premium, level death benefit life insurance.

I find Mr. Goldfinger's approach to CRVM reserves on adjustable life very interesting and innovative. It is a reasonable and legitimate extension of the CRVM and is consistent with the principles of the Standard Valuation Law. I would use exactly the same words to describe Mr. Chapin's approach. From the standpoint of consistency with the CRVM principles I do not believe that either is any better than the other.

Both approaches require that the expense allowance and CRVM reserves on the initial issue of an adjustable life policy be calculated as if it were a traditional level premium, level death benefit policy. Mr. Goldfinger introduces the following additional constraints:

1. If the adjustable life policy does not change, the adjustable life expense formula should produce no additional expense allowance.
2. If an increase in face amount and premium takes place with no change in plan, the additional allowance should be equal to that which would have been generated if a separate policy had been issued for the difference in face amounts.

I believe Mr. Chapin's additional constraint is that the total expense allowance (including the current and prior allowances) should be equal to that which would be provided if the existing cash value from the prior status were used to purchase a new policy with the same death benefit, premium, and plan as the adjustable life policy following the adjustment. This, together with the original-issue requirements mentioned earlier, should be enough to ensure consistency with the principles of the CRVM. In addition Mr. Chapin has shown that, in the situation where the plan is not changing but the face amount and premium are increasing, his formulas would produce expense allowances no greater than those produced by issuing a separate policy for the increased face amount, unless mortality rates are decreasing by age. It can also be shown that Mr. Chapin's formula would produce no additional expense allowance if applied when no adjustment occurs, unless mortality rates are decreasing by age.

I think it is important to clear up a misconception that might arise from the reading of Mr. Goldfinger's paper. In Section VIII he displays the formula

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\beta^F_{x_m} - c_{x_m}) - I_{x_{m-1}}(\beta^F_{x_{m-1}} - c_{x_{m-1}}) .$$

He says that this formula results from taking Chapin's language literally and that it "gives an additional expense allowance that is not consistent with that allowed under the CRVM." This formula very well may produce too much expense allowance, but with Mr. Goldfinger's definition of β^F it is *not* the formula used by Mr. Chapin. Note that Mr. Goldfinger later criticizes the Chapin approach for just the opposite—that is, for producing an expense allowance which is too *low* in most instances.

This misconception may stem from the paragraph in Mr. Goldfinger's paper immediately preceding the above formula. This paragraph includes a quotation from Mr. Chapin's paper concerning "the statutory expense for a new issue at age x_m for the amount of insurance and net premium in the *m*th status." Note that Mr. Chapin refers to an allowance based on the *amount of insurance* and *net premium* in the *m*th status, *not* the plan in the *m*th status *or* the net premium that would be needed for a new issue of this plan, as Mr. Goldfinger seems to interpret it. Mr. Chapin refers to the net premium, π_{x_m}, that actually will be credited during the *m*th status, a premium that is dependent not only on the new plan and amount of insurance but also on the reserve existing under the prior status.

Mr. Goldfinger also criticizes the Chapin approach because Chapin compares π_x with ${}_{19}P_{x+1}$ instead of comparing β^F_x with ${}_{19}P_{x+1}$ in determining his expense allowance. Mr. Goldfinger indicates that the Standard Valuation Law does not state explicitly that $\beta^{\text{CRVM}}_x = \beta^F_x$ if $\beta^F_x \leq {}_{19}P_{x+1}$ even though that is a consequence of the method when applied to traditional plans. He may be putting too much emphasis on the words used in the law rather than on its actual intent. I believe the history leading up to the Standard Valuation Law shows that the CRVM was intended to be a full preliminary term valuation except at very high premium levels. The omission from the standard law of the statement that $\beta_x^{CRVM} = \beta_x^F$ probably came about in order to facilitate the inclusion of the ${}_{19}P_{x+1}$ limit. With this in mind, I think it is entirely appropriate for Mr. Chapin to base his comparison on π_x (which is his β_x^{CRVM}) rather than on β_x^F.

It may be worth reemphasizing Mr. Goldfinger's comment that his approach generally will provide more expense allowance on adjustments than does Mr. Chapin's. In some situations this difference can be very substantial.

Consider, for example, the simplified situation where (1) the previous adjustment produced a positive expense allowance, (2) both the new and the prior premium are small enough so that the ${}_{19}P_{x+1}$ limit does not come into play, and (3) the plans in status m and $m-1$ are identical. For this case, the difference between the Chapin and Goldfinger formulas can be reduced to an expression that is fairly easy to understand and evaluate. Under both methods,

$$\begin{bmatrix}\text{Present value}\\ \text{of future premiums}\end{bmatrix} - \begin{bmatrix}\text{New expense}\\ \text{allowance}\end{bmatrix} + \begin{bmatrix}\text{Reserve from}\\ \text{prior status}\end{bmatrix} = \begin{bmatrix}\text{Present value}\\ \text{of future benefits}\end{bmatrix}.$$

Thus

$$I_{x_m}(\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} - \Delta_{x_m}) + I_{x_{m-1}}(A_{x_m:\overline{z-x_m}|} - \pi_{x_{m-1}}\ddot{a}_{x_m:\overline{w-x_m}|}) = I_{x_m}A_{x_m:\overline{z-x_m}|},$$

and by rearranging terms we obtain

$$I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}} = \frac{I_{x_m}(A_{x_m:\overline{z-x_m}|} + \Delta_{x_m}) - I_{x_{m-1}}A_{x_m:\overline{z-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|}}.$$

To determine the expense allowance using the Chapin approach,

$$I_{x_m}\Delta_{x_m}^C = I_{x_m}(\pi_{x_m}^C - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}}^C - c_{x_{m-1}}),$$

where the superscript C refers to Chapin. Substituting the general formula from above,

$$I_{x_m}\Delta_{x_m}^C = \frac{I_{x_m}(A_{x_m:\overline{z-x_m}|} + \Delta_{x_m}^C) - I_{x_{m-1}}A_{x_m:\overline{z-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|}} - (I_{x_m}c_{x_m} - I_{x_{m-1}}c_{x_{m-1}}),$$

$$I_{x_m}\Delta^C_{x_m} = [(I_{x_m} - I_{x_{m-1}})A_{x_m:\overline{z-x_m}|} - (I_{x_m}c_{x_m} - I_{x_{m-1}}c_{x_{m-1}})\ddot{a}_{x_m:\overline{w-x_m}|}]\frac{1}{\ddot{a}_{x_m:\overline{w-x_m}|} - 1}$$

$$= [(I_{x_m} - I_{x_{m-1}})({}_{w-x_m}P_{x_m:\overline{z-x_m}|} - c_{x_m}) - I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})]\frac{\ddot{a}_{x_m:\overline{w-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|} - 1}\,.$$

To determine the expense allowance using the Goldfinger approach,

$$I_{x_m}\Delta^G_{x_m} = I_{x_m}\;{}^{m}[\beta^F_{x_m} - c_{x_m}] - I_{x_{m-1}}\;{}^{m-1}[\beta^F_{x_m} - c_{x_m}]\,,$$

where the superscript G refers to Goldfinger.

Mr. Goldfinger shows (formula [22]) that when the plan does not change between $m - 1$ and m, this becomes

$$I_{x_m}\Delta^G_{x_m} = I_{x_m}(\pi^G_{x_m} - c_{x_m}) - I_{x_{m-1}}(\pi^G_{x_{m-1}} - c_{x_m})\,.$$

Substituting the general formula from above, this becomes

$$I_{x_m}\Delta^G_{x_m} = \frac{I_{x_m}(A_{x_m:\overline{z-x_m}|} + \Delta^G_{x_m}) - I_{x_{m-1}}A_{x_m:\overline{z-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|}} - (I_{x_m} - I_{x_{m-1}})c_{x_m}$$

$$= [I_{x_m}A_{x_m:\overline{z-x_m}|} - I_{x_{m-1}}A_{x_m:\overline{z-x_m}|} - c_{x_m}(I_{x_m} - I_{x_{m-1}})\ddot{a}_{x_m:\overline{w-x_m}|}]\frac{1}{\ddot{a}_{x_m:\overline{w-x_m}|} - 1}$$

$$= (I_{x_m} - I_{x_{m-1}})({}_{w-x_m}P_{x_m:\overline{z-x_m}|} - c_{x_m})\frac{\ddot{a}_{x_m:\overline{w-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|} - 1}\,.$$

In this simplified situation, Chapin's approach gives an allowance equal to the Goldfinger allowance minus

$$(c_{x_m} - c_{x_{m-1}})I_{x_{m-1}}\frac{\ddot{a}_{x_m:\overline{w-x_m}|}}{\ddot{a}_{x_m:\overline{w-x_m}|} - 1}.$$

If, as in the normal situation, $c_{x_m} > c_{x_{m-1}}$, Goldfinger's approach will provide more allowance than Chapin's. The difference could sometimes be quite substantial. If x_m is considerably greater than x_{m-1} and/or $I_{x_{m-1}}$ is large compared to $I_{x_m} - I_{x_{m-1}}$, Chapin's method might provide no expense allowance in situations where it is reasonable to expect a significant expense allowance, and where in fact Goldfinger does provide an allowance. If there is a dip in the mortality table so that $c_{x_m} < c_{x_{m-1}}$, Goldfinger's approach will provide a smaller allowance than Chapin's.

Both methods can be used to (1) determine plan when face amount and premium are known; (2) determine premium when face amount and plan are known; and (3), with some difficulty, determine face amount when premium and plan are known. I believe the Goldfinger method is somewhat more complicated than the Chapin method when plan of insurance is the unknown.

Mr. Goldfinger touches on an interesting question in his discussion of Chapin's expense allowance formula for case 2. If a policy always remains on the same plan of insurance, but the face amount and premium fluctuate up and down, should a new expense allowance be generated each time the face amount and premium increase to their previous level, or should a new expense allowance be generated only when the face and premium rise above their previous high level? In the last paragraph in Section V, Mr. Goldfinger states that if $I_{x_m}\Delta_{x_m}$ is negative, it "should be set to zero, and at the time of the next change the plan in status $m + 1$ should be compared again with that in status $m - 1$." This eliminates duplicate expense allowances every time the policy goes back up to its previous level. Mr. Chapin's paper is not quite as clear on this question, but it seems to imply that duplicate expense allowances would not be permitted. I tend to agree with this position, but there are good arguments to support an additional allowance each time the policy increases to a previous level. The allowance is intended to cover expenses such as underwriting and commissions that may be incurred even though the increase is only bringing the policy back to a previous level.

I would like to congratulate both Mr. Chapin and Mr. Goldfinger for their differing but appropriate methods of extending the CRVM to adjustable life.

CHARLES CARROLL:

Mr. Goldfinger is to be congratulated for his clear analysis of CRVM expense allowances for adjustable life policies. Particularly impressive is the simplicity of his definitions and formulas.

In Section VI of the paper, Mr. Goldfinger emphasizes the fact that, except for the case where $m = 1$, the modified net premium for the adjustable life policy in status m, π_{x_m}, is *not* equal to the CRVM modified net premium for a new issue at age x_m of the plan in status m, $\beta_{x_m}^{\text{CRVM}}$. I found it interesting to discover a relationship between π_{x_m} and $\beta_{x_m}^{\text{CRVM}}$ in the general case. Following are the two key formulas (equations [14] and [15] in Mr. Goldfinger's paper):

$$I_{x_m}\Delta_{x_m} = I_{x_m}\,{}^{m}(P^A_{x_m} - c_{x_m}) - I_{x_{m-1}}\,{}^{m-1}(P^A_{x_m} - c_{x_m})\,; \qquad (1)$$

$$I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} + {}_{x_m-x_1}V_{x_1} = I_{x_m}A_{x_m:\overline{z-x_m}|} + I_{x_m}\Delta_{x_m}\,. \qquad (2)$$

Substituting for $I_{x_m}\Delta_{x_m}$ in equation (2) and rearranging terms, we obtain

$$I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w-x_m}|} = I_{x_m}A_{x_m:\overline{z-x_m}|} + I_{x_m}\;{}^{m}(P^A_{x_m} - c_{x_m}) - [I_{x_{m-1}}\;{}^{m-1}(P^A_{x_m} - c_{x_m}) + {}_{\overline{x_m - x_1}}V_{x_m}]\,,$$

$$I_{x_m}\pi_{x_m} = I_{x_m}\left[\frac{A_{x_m:\overline{z-x_m}|} + {}^{m}(P^A_{x_m} - c_{x_m})}{\ddot{a}_{x_m:\overline{w-x_m}|}}\right] - \left[\frac{I_{x_{m-1}}\;{}^{m-1}(P^A_{x_m} - c_{x_m}) + {}_{\overline{x_m - x_1}}V_{x_m}}{\ddot{a}_{x_m:\overline{w-x_m}|}}\right]. \tag{3}$$

The first expression in brackets can be recognized as an expression for $\beta^{\text{CRVM}}_{x_m}$. Equation (3) may then be restated as

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta^{\text{CRVM}}_{x_m} - \left[\frac{I_{x_{m-1}}\;{}^{m-1}(P^A_{x_m} - c_{x_m}) + {}_{\overline{x_m - x_1}}V_{x_m}}{\ddot{a}_{x_m:\overline{w-x_m}|}}\right]. \tag{4}$$

A similar expression can be derived from equation (6) in Mr. Chapin's paper. This equation defines $I_{x_m}\pi_{x_m}$ when $\pi_{x_m} < {}_{19}P_{x_m+1}$ and $I_{x_m}\Delta_{x_m} > 0$:

$$I_{x_m}\pi_{x_m} = I_{x_m}\left[\frac{M_{x_m+1} - M_z + kD_z}{N_{x_m+1} - N_w}\right] - \left\{\frac{[I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) + {}_{\overline{x_m - x_1}}V_{x_m}]D_{x_m}}{N_{x_m+1} - N_w}\right\}. \tag{5}$$

The expression in brackets on the left can be recognized as an expression for $\beta^F_{x_m}$ (using Mr. Goldfinger's notation). Note that even though $\pi_{x_m} < {}_{19}P_{x_m+1}$, it is possible that $\beta^F_{x_m} > {}_{19}P_{x_m+1}$. However, if $\beta^F_{x_m} < {}_{19}P_{x_m+1}$, the equation above can be restated as

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta^{\text{CRVM}}_{x_m} - \left[\frac{I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) + {}_{\overline{x_m - x_1}}V_{x_m}}{a_{x_m:\overline{w-x_m}|}}\right]. \tag{6}$$

It is interesting to note the similarities and differences between this equation and the one derived above (4) based on Mr. Goldfinger's formulas.

In the case where $\pi_{x_m} > {}_{19}P_{x_m+1}$, $I_{x_m}\Delta_{x_m} > 0$, and $\beta^F_{x_m} \geq {}_{19}P_{x_m+1}$, it is possible to derive the following equation based on equation (7) of Mr. Chapin's paper:

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta^{\text{CRVM}}_{x_m} - \left[\frac{I_{x_{m-1}}({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}}) + {}_{\overline{x_m - x_1}}V_{x_m}}{\ddot{a}_{x_m:\overline{w-x_m}|}}\right]. \tag{7}$$

It is clear from comparing equations (7) and (4) that in this case the modified net premium based on Mr. Chapin's approach would be less than that based on Mr. Goldfinger's and, therefore, the reserve under Mr. Chapin's approach would be greater.

It is not possible to make such a definite statement from a comparison of equations (6) and (4). However, except where the mortality rates are decreasing with increasing age, Mr. Chapin's approach apparently produces lower net premiums and greater reserves in most cases.

An interesting special case for which to compare the two approaches is a "renewable term" adjustable life policy. Assume that status $(m - 1)$ of the policy called for premiums and benefits to age $x_m = x_{m-1} + n$. At age x_m the policy is being "renewed" for another term period of n years at the same face amount. Mr. Chapin's formula for the modified net premium would be

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta_{x_m}^{\text{CRVM}} - \left[\frac{I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}})}{a_{x_m:\overline{n}|}}\right].$$

In other words, even though the term of coverage of the prior status has expired, Mr. Chapin's approach calls for a credit equal to the prior initial expense deficit. This might be appropriate in a case where no underwriting was done at renewal and where the agent's compensation was based on the difference between the old and new gross premiums. However, it does not seem appropriate to base statutory minimum reserves on these assumptions.

Under Mr. Goldfinger's approach, the formulas would be

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta_{x_m}^{\text{CRVM}} - \left[\frac{I_{x_{m-1}}\;{}^{m-1}(P_{x_m}^{A} - c_{x_m})}{\ddot{a}_{x_m:\overline{n}|}}\right].$$

But ${}^{m-1}(P_{x_m}^{A} - c_{x_m})$, which is the statutory expense allowance for a new issue at age x_m of a term policy to age x_m, would be zero. Therefore,

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta_{x_m}^{\text{CRVM}}.$$

In other words, Mr. Goldfinger's approach produces modified net premiums that agree with current practice regarding conventional renewable term policies. Mr. Chapin's approach does not. Of course, this is an artificial example, but it does indicate that Mr. Goldfinger's approach is more consistent, as does the case of a policy for which amount and premium change but plan stays the same.

Mr. Goldfinger demonstrates that if no change in status occurs, his formulas produce no additional expense allowance. Mr. Chapin's formulas also satisfy this requirement. In the case where $\pi_{x_m} < {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} < {}_{19}P_{x_{m-1}+1}$, Mr. Chapin's formula for the additional expense allowance is

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) . \tag{8}$$

If $\pi_{x_m} = \pi_{x_{m-1}}$ when no change in status occurs (which is not altogether obvious), equation (8) simplifies to

$$I_{x_m}\Delta_{x_m} = I_{x_{m-1}}c_{x_{m-1}} - I_{x_m}c_{x_m} .$$

It is obvious that, except where the slope of mortality rates is negative, the above expression is negative. But, since the additional expense allowance can never be less than zero, it would be set equal to zero in this case. Based on this quirk, one can construct cases where there would be an increase in amount and premium with no change in plan, and yet Mr. Chapin's formulas would produce no additional expense allowance. Setting $I_{x_m}\Delta_{x_m} = 0$ in equation (8),

$$I_{x_m}(\pi_{x_m} - c_{x_m}) = I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) . \tag{9}$$

From equation (6),

$$I_{x_m}\pi_{x_m} = I_{x_m}\beta_{x_m}^{\text{CRVM}} - \left[\frac{I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) + {}_{\overline{x_m - x_1}}V_{x_m}}{a_{x_m:\overline{w-x_m}|}}\right] .$$

Substituting for $I_{x_m}\pi_{x_m}$ in (9) and rearranging, we have

$$I_{x_m} = \left\{I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) + \left[\frac{I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) + {}_{\overline{x_m - x_1}}V_{x_m}}{a_{x_m:\overline{w-x_m}|}}\right]\right\} \times \frac{1}{\beta_{x_m}^{\text{CRVM}} - c_{x_m}} . \tag{10}$$

If the plan originally is issued as a whole life plan at $x_1 = 25$, the percentage increase in amount based on formula (10) and 3 percent 1958 CSO curtate functions at various ages would be as follows:

x_2	Percent Increase in Amount
26	0.3%
30	1.7
40	10.0
50	27.9
60	61.4

From a practical standpoint the percentages at the younger ages are insignificant. Also, it is likely that an adjustable policy issued at age 25 would have gone through a number of changes before the insured reached age 40, where the percentages begin to get significant.

On the whole, Mr. Goldfinger's approach seems to provide expense allowances more in line with an intuitive approach. Mr. Chapin's approach, although eminently practical, produces some rather surprising results in certain instances.

WALTER L. CHAPIN:

Mr. Goldfinger has presented a detailed examination of the Commissioners Reserve Valuation Method expense allowance for adjustable life (AL). It is encouraging that he felt the expense subject important enough for an entire paper. In my paper, "Toward Adjustable Individual Life Policies," statutory expense is covered by a statement of four formulas within two paragraphs of text.

I will refer to formulas in Mr. Goldfinger's paper by using the prefix G and in mine by using the prefix C. Illustrations are based on 1958 CSO 3 percent curtate tables.

Mr. Goldfinger's formula G(14) is intended for use in all calculations of statutory expense. The formulas in our papers give consistent results for the first and second statuses of an AL policy, after allowing for a small arbitrary omission in my formula, provided that the policy has at least twenty future premiums payable, or more in the case of some endowments. If the policy has fewer future premiums, or is in a third or later status, the expense calculations all differ. Expense allowances by G(14) and C(3) are shown below for illustrative data:

Status	Age	Plan	Amount	Goldfinger Expense	Chapin Expense
1.	25	Life @ 70	$10,000	$105.09	$ 105.09
2.	35	Life @ 70	20,000	161.62	155.69
3.	45	Life @ 60	40,000	670.94	810.26
3.	45	Life @ 60	50,000	964.14	1,183.20

The basis of my expense calculations is to assume that the current status of the AL policy will remain unchanged and to commute the varying amounts of insurance benefit and expense into a level amount, level premium whole life policy issued at the adjustable policy issue age with premiums payable for twenty years. If the net premium payable for twenty years on the equivalent amount of insurance is less than that for CRVM twenty-payment life, the AL net premium will be preliminary term; if not, it will be the CRVM modified net premium.

The criterion, given in my paper, of comparing the AL net premium for a unit, π_{x_m}, with ${}_{19}P_{x_m+1}$, to determine whether the net premium should be preliminary term or modified, is always accurate when π_{x_m} is greater than ${}_{19}P_{x_m+1}$. This situation occurs when a gross premium elected at age x_m has a net premium per unit exceeding ${}_{19}P_{x_m+1}$, or where the net premium calculated by the net preliminary term formula C(6) turns out to exceed ${}_{19}P_{x_m+1}$, requiring recalculation by C(7). In rare cases the criterion may not be valid when π_{x_m} is less than ${}_{19}P_{x_m+1}$. An example is the case given below when the policy is changed at age 45 from $20,000 life at age 70 to $50,000 life at age 60.

Let ${}^m(EI)_{x_1}A_{x_1}$ be the value of the AL benefit at the beginning of the mth status commuted to age x_1, ${}^m(EI)_{x_1}\Delta_{x_1}$ the value of statutory expenses, ${}^m(EI)_{x_1}$ the equivalent level amount of insurance, and ${}^m(EI)_{x_1}$ $\pi_{x_1}a_{x_1:\overline{20}|}$ the commuted value of net premiums. These values relate to all values in the history of the AL policy and are written below within the brackets as of time x_m. Each series is discounted by D_{x_m}/D_{x_1} to give a commuted value as of age x_1. A_{x_r} is replaced by $A_{x_r:\overline{z-x_r}|}$ if the benefit is an endowment maturing at age z. ${}^m(EI)_{x_1} = {}^m(EI)_{x_1}A_{x_1}/A_{x_1}$.

$$
{}^m(EI)_{x_1}A_{x_1} = \frac{D_{x_m}}{D_{x_1}}\left[\frac{I_{x_1}(M_{x_1}-M_{x_2})}{D_{x_m}} + \frac{I_{x_2}(M_{x_2}-M_{x_3})}{D_{x_m}} + \ldots + \frac{I_{x_{m-1}}(M_{x_{m-1}}-M_{x_m})}{D_{x_m}} + \frac{I_{x_m}M_{x_m}}{D_{x_m}}\right]; \tag{1}
$$

$$
{}^m(EI)_{x_1}\Delta_{x_1} = \frac{D_{x_m}}{D_{x_1}}\left[I_{x_1}\Delta_{x_1}\frac{D_{x_1}}{D_{x_m}} + I_{x_2}\Delta_{x_2}\frac{D_{x_2}}{D_{x_m}} + \ldots + I_{x_{m-1}}\Delta_{x_{m-1}}\frac{D_{x_{m-1}}}{D_{x_m}} + I_{x_m}\Delta_{x_m}\right]; \tag{2}
$$

$$
{}^m(EI)_{x_1}\pi_{x_1}\ddot{a}_{x_1:\overline{20}|} = \frac{D_{x_m}}{D_{x_1}}\left[\frac{I_{x_1}\pi_{x_1}(N_{x_1}-N_{x_2})}{D_{x_m}} + \frac{I_{x_2}\pi_{x_2}(N_{x_2}-N_{x_3})}{D_{x_m}} + \ldots + \frac{I_{x_{m-1}}\pi_{x_{m-1}}(N_{x_{m-1}}-N_{x_m})}{D_{x_m}} + \frac{I_{x_m}\pi_{x_m}(N_{x_m}-N_{w})}{D_{x_m}}\right]. \tag{3}
$$

Since the value of the net premiums equals the sum of the value of benefits and expense allowances, expression (3) minus expression (2) minus expression (1) equals zero. It may be noted that the net sum of all terms except the last term in the brackets of the three equations is the reserve, ${}_{\overline{x_m - x_1}}V_{x_1}$.

Rewriting the terms with this simplification and solving for the AL net premium at age x_m, we obtain

$$ {}_{\overline{x_m - x_1}}V_{x_1} + I_{x_m}\pi_{x_m}\ddot{a}_{x_m:\overline{w - x_m}|} - I_{x_m}\Delta_{x_m} - I_{x_m}A_{x_m} = 0 $$

and

$$ I_{x_m}\pi_{x_m} = \frac{I_{x_m}A_{x_m} + I_{x_m}\Delta_{x_m} - {}_{\overline{x_m - x_1}}V_{x_1}}{\ddot{a}_{x_m:\overline{w - x_m}|}} . $$

In terms of the equivalent of all values entering into the AL net premium, the equivalent twenty-payment AL net premium is

$$ {}^{m}(EI)_{x_1}\pi_{x_1} = \frac{{}^{m}(EI)_{x_1}A_{x_1} + {}^{m}(EI)_{x_1}\Delta_{x_1}}{\ddot{a}_{x_1:\overline{20}|}} . $$

The comparison of ${}^{m}(EI)_{x_1}\pi_{x_1}$ with the comparable equivalent twenty-payment life net premium, ${}^{m}(EI)_{x_1}\ {}_{19}P_{x_1+1}$, for the illustrative example is shown below:

Plan $10,000 Life @ 70 at Age 25, $20,000 Life @ 70 at Age 35, and Life @ 60 at Age 45 for	Equivalent Amount of Insurance	Equivalent 20-Pay Life Net at 25	Equivalent AL 20-Payment Premiums at Age 25	
			Goldfinger	Chapin
$40,000	$36,563	$722.60	$717.26	$721.82
50,000	45,165	892.60	887.23	894.57

When the increase in insurance at age 45 is from $20,000 to $40,000 and the plan changes from life paid up at 70 to life paid up at 60, both Goldfinger and Chapin equivalent AL premiums are less than twenty-payment life. Since there is no cutback to a ${}_{19}P_{x_1+1}$ limit in the Chapin third status net premium that enters the C(3) expense formula, the $40,000 π_{x_1} is preliminary term. While the Goldfinger equivalent AL net premium apparently qualifies as preliminary term, there is a cutback in the net premium, $P^{A}_{x_m}$, in G(14). This introduces a contradiction. The same situation applies in the change to $50,000, where the Chapin net of \$894.57 is higher than the equivalent net of \$892.60. Resolution of this error is described below.

Let the expense terms in the equivalent twenty-payment life policy be discounted to age x_1 in the same manner as the AL expense terms and be identified as ${}^{m}(EI)^{19P}_{x_1}\Delta_{x_1}$:

$$
\begin{aligned}
{}^{m}(EI)^{19P}_{x_1}\Delta_{x_1} = \frac{D_{x_m}}{D_{x_1}} \Bigg[& I_{x_1}({}_{19}P_{x_1+1} - c_{x_1}) \frac{D_{x_1}}{D_{x_m}} \\
& + (I_{x_2} - I_{x_1})({}_{19}P_{x_2+1} - c_{x_2}) \frac{D_{x_2}}{D_{x_m}} \\
& + \dots + (I_{x_m} - I_{x_{m-1}})({}_{19}P_{x_m+1} - c_{x_m}) \Bigg] .
\end{aligned}
$$

If ${}^{m}(EI)^{19P}_{x_1}\Delta_{x_1}$ is substituted for ${}^{m}(EI)_{x_1}\Delta_{x_1}$ in the formula given above for the equivalent AL net premium, the net premium becomes the equivalent CRVM twenty-payment life net premium, ${}^{m}(EI)_{x_1}\ {}_{19}P_{x_1+1}$. This is the result desired when a preliminary term net premium in status $(m - 1)$ exceeds or equals ${}_{19}P_{x_m+1}$ per unit in the mth status. The formula for the AL expense becomes

$$
\begin{aligned}
I_{x_m}\Delta_{x_m} = [{}^{m-1}(EI)^{19P}_{x_1}\Delta_{x_1} - {}^{m-1}(EI)_{x_1}\Delta_{x_1}] \frac{D_{x_1}}{D_{x_m}} \\
+ (I_{x_m} - I_{x_{m-1}})({}_{19}P_{x_m+1} - c_{x_m}) .
\end{aligned}
$$

The data for this solution may be accumulated in the record for each policy so that, in any status r,

$$
{}^{r-1}(EI)_{x_1}\Delta_{x_1} + I_{x_r}\Delta_{x_r} \frac{D_{x_r}}{D_{x_1}} = {}^{r}(EI)_{x_1}\Delta_{x_1} ;
$$

$$
{}^{r-1}(EI)^{19P}_{x_1}\Delta_{x_1} + (I_{x_r} - I_{x_{r-1}})({}_{19}P_{x_r+1} - c_{x_r}) \frac{D_{x_r}}{D_{x_1}} = {}^{r}(EI)^{19P}_{x_1}\Delta_{x_1} .
$$

Formula C(4a), which is intended to provide the above solution, understates the correct expense. It appears necessary in this one situation to employ interest and survivorship functions to provide a correct solution. This formula should replace C(4a) as given in my paper.

In the case in which the Chapin equivalent net premium exceeds the twenty-payment life equivalent net, a check figure comparing ${}^{m}(EI)^{19P}_{x_1}\Delta_{x_1}$ with ${}^{m}(EI)_{x_1}\Delta_{x_1}$ would show the latter as greater and therefore needing a correction.

Three additional formulas defining expense appear in my paper, numbered here C(3), C(4), and C(3a).

Formula C(3), reproduced below, defines expense in a status for which the net premium is preliminary term and the sum of expenses for all previous statuses may be deemed to be preliminary term.

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}}) .$$

Considering first the net premiums, $I_{x_m}\pi_{x_m}$ and $I_{x_{m-1}}\pi_{x_{m-1}}$, the increment of the net premiums between status $(m - 1)$ and status m is simply $I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}}$. This applies without respect to plan of insurance whenever $I_{x_m}\pi_{x_m} > I_{x_{m-1}}\pi_{x_{m-1}}$. Considering next the $I_{x_m}c_{x_m}$ and $I_{x_{m-1}}c_{x_{m-1}}$ terms, the increment is $(I_{x_m} - I_{x_{m-1}})c_{x_m}$ and the original amount is $I_{x_{m-1}}c_{x_{m-1}}$. These two terms may be written as $I_{x_m}c_{x_m} - I_{x_{m-1}}c_{x_{m-1}} - I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$. Combining the net premium and c terms, but leaving out $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$, reproduces C(3). If that term were included, C(3) would read:

$$I_{x_m}\Delta_{x_m} = I_{x_m}\pi_{x_m} - I_{x_{m-1}}\pi_{x_{m-1}} - (I_{x_m} - I_{x_{m-1}})c_{x_m} .$$

This agrees with G(22). The effect of dropping $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$ is to produce slightly higher reserves and lower net premiums and expenses. It was dropped for two reasons. First, the first term of the formula represents the sum of expenses and indicates that the total expense is the same as that for a preliminary term policy issued at the attained age having the same premium as the AL policy. If the C(3) expense formula is written down for statuses 1, 2, and 3, it is obvious that $I_{x_1}\Delta_{x_1} + I_{x_2}\Delta_{x_2} + I_{x_3}\Delta_{x_3}$ equals $I_{x_3}(\pi_{x_3} - c_{x_3})$ and, generally,

$$\sum_{r=1}^{m} I_{x_r}\Delta_{x_r} = I_{x_m}(\pi_{x_m} - c_{x_m}) .$$

A second reason for dropping $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$ was to develop formulas that adapt easily to continuous premiums, as illustrated below. If the reasons for dropping this term are not deemed worthwhile, the G(22) form should be used and a related adjustment made to C(6) and C(7).

Formula C(4) defines the expense in the mth status when the net AL premium per unit, π_{x_m}, exceeds ${}_{19}P_{x_m+1}$ in the mth status and $\pi_{x_{m-1}}$ exceeds ${}_{19}P_{x_{m-1}+1}$ in the $(m - 1)$st status. The formula is given as

$$\begin{aligned} I_{x_m}\Delta_{x_m} &= I_{x_m}({}_{19}P_{x_m+1} - c_{x_m}) - I_{x_{m-1}}({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}}) \\ &\qquad - I_{x_{m-1}}({}_{19}P_{x_m+1} - {}_{19}P_{x_{m-1}+1}) \\ &= (I_{x_m} - I_{x_{m-1}})\,{}_{19}P_{x_m+1} - I_{x_m}c_{x_m} + I_{x_{m-1}}c_{x_{m-1}} . \end{aligned}$$

If $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$, dropped in C(3), is added back here, the formula becomes $(I_{x_m} - I_{x_{m-1}})({}_{19}P_{x_m+1} - c_{x_m})$. In this form it agrees with the expense for the equivalent twenty-payment life premium and causes ${}^m(EI)^{19P}_{x_1}\Delta_{x_1}$ to equal ${}^m(EI)_{x_1}\Delta_{x_1}$ as long as the modified AL net premium applies.

Formula C(3a) covers the expense in the status representing a transition from a modified to a preliminary term net premium. The formula is given as

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - \sum_{r=1}^{m-1} I_{x_r}\Delta_{x_r} .$$

Mr. Goldfinger points out that the same result is more simply expressed as

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - I_{x_{m-1}}({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}}) .$$

Expense allowances are protected against duplication if the premium is decreased in one change and increased in a later change. Only when the net premium on the increased policy exceeds the highest earlier net premium will additional expense emerge. Our two papers agree on this point.

When the AL expense increments are computed from sources independent of the net premium formula, as in G(14), two problems arise. G(14) assumes that the amount and plan are known and the solution is for the net premium. If premium and amount are given and solution for plan is required, $I_{x_m}P^A_{x_m}$ cannot be found directly, which necessitates some routine of successive approximation. A similar type of solution is needed if premium and plan are known and amount is to be found.

If there is an error in the independent expense computation, the error affects the net premium and later reserves. Duplicate calculation rather than an independent check appears necessary under G(14). If the expense increment is part of the net premium calculation, as in C(6) and C(7), an error is disclosed in an independent check of the reserve when a whole life AL policy pays up, a term policy expires, or an endowment matures.

In the adaptation of AL expense and net premium formulas to continuous premiums, the slight reduction in net premiums resulting from dropping the term $I_{x_{m-1}}(c_{x_m} - c_{x_{m-1}})$ corresponds rather closely to the reductions produced by the use of curtate functions. The use of curtate functions for expense allowances with AL involves computing the expense first and introducing it into the net premium calculation. This has the disadvantages noted in connection with the same process required by G(14).

The conversion to continuous functions is accomplished by changing M_{x_r} and N_{x_r} functions to $\bar{M}_{x_r}$ and $\bar{N}_{x_r}$, changing π_{x_r} and ${}_{19}P_{x_r}$ where they appear on the right-hand side to $\bar{\pi}_{x_r}$ $\bar{D}_{x_r}/D_{x_r}$ and ${}_{19}\bar{P}_{x_r}D_{x_r}/\bar{D}_{x_r}$, and changing c_{x_r} functions to $\bar{c}_{x_r}$. The net premiums in the expense formulas are similarly treated.

The comparison of the sum of net premiums and reserves of three separate whole life policies with an AL policy providing the same benefits is illustrated below:

Age	Amount of Insurance		Total Net Premiums		Total Reserve	
	Separate Policies	AL Policy	Separate Policies	AL Policy	Separate Policies	AL Policy
25....	\$10,000	\$10,000	\$120.88	\$120.97	\$ 0	\$ 0
35....	+ 10,000	20,000	296.81	296.78	1,039.22	1,037.39
45....	+ 20,000	40,000	833.77	832.28	3,950.22	3,951.42
	40,000					
65....	\$40,000	\$40,000	\$833.77	\$832.28	\$19,539.39	\$19,554.38

In a comparison of Mr. Goldfinger's approach with mine, it is clear that we start from different premises. Mr. Goldfinger concludes that the proper expense for a new status is the expense for a new policy at the attained age for the new plan and amount less the expense for a new policy at the attained age for the plan and amount in the previous status. My premise is that the expense must recognize the past history of the policy as well as a future pattern that assumes no change. Mr. Goldfinger's expense in every case is computed independently of the AL net premium and reserve and then used with the value of the future benefit and the reserve to find the net premium. My approach is to find the net premium first, if possible, and then determine expense from one of four formulas.

I appreciate the author's study of expense allowances and the comments that have led me to propose a more desirable form of C(4), when curtate values apply, and of C(4a).

J. STANLEY HILL:

Mr. Goldfinger's approach produces different expense allowances from those produced by Mr. Chapin. Neither appears to claim a unique correctness. Rather, they demonstrate that there can be different interpretations of what may be acceptable in calculating the expense allowance when the status of an adjustable life policy is changed.

My own approach differs from both, although in certain cases it will produce the same result as Mr. Chapin's (all three approaches produce the same results in the first status). My approach is to let the present value (with interest and survivorship), at the original issue date, of all expense allowances equal the expense allowance that would have been allowed if the identical insurance amounts had been provided at the identical premiums in a conventional insurance policy.

The mathematical statement of this approach is simple, general, and readily adapted to a computer system for administering adjustable life policies.

$$I_{x_m}\Delta_{x_m} = \frac{1}{D_{x_m}}\left(\Delta^c_m D_{x_1} - \sum_{r=1}^{m-1} I_{x_r}\Delta_{x_r} D_{x_r}\right), \tag{1}$$

where Δ^c_m is the expense allowance under a conventional policy with identical insurance amounts and premiums.

$$\Delta^c_m = \min\left[\beta^{EL}_m - I^{EL}_m c_{x_1},\ I^{EL}_m({}_{19}P_{x_1+1} - c_{x_1})\right]; \tag{2}$$

β^{EL}_m is the equivalent level commissioners net premium for a conventional policy with identical insurance amounts and premiums,

$$\beta^{EL}_m = \frac{1}{N_{x_1} - N_w}\left[I_{x_m}\pi_{x_m}(N_{x_m} - N_w) + \sum_{r=1}^{m-1} I_{x_r}\pi_{x_r}(N_{x_r} - N_{x_{r+1}})\right], \tag{3}$$

and I^{EL}_m is the equivalent level amount of insurance for a conventional policy with identical insurance amounts,

$$I^{EL}_m = \frac{1}{M_{x_1} - M_z}\left[I_{x_m}(M_{x_m} - M_z) + \sum_{r=1}^{m-1} I_{x_r}(M_{x_r} - M_{x_{r+1}})\right]. \tag{4}$$

The following examples show comparative expense allowances developed by the three approaches, using 1958 CSO age-nearest-birthday curtate functions at 3 percent.

Status	Age	Plan	Amount	Goldfinger Expense	Chapin Expense	Hill Expense	Δ^c_m	β^{EL}_m	$I_{x_m}\pi_{x_m}$
1	25	Life @ 70	\$10,000	\$105.09	\$ 105.09	\$105.09	\$105.09	\$123.83	\$ 123.83
2	35	Life @ 70	20,000	161.52	155.69	135.04	203.45	239.73	308.50
3	45	Life @ 60	40,000	670.94	810.26	442.25	434.85	503.36	1,245.90
3	45	Life @ 60	50,000	964.14	1,182.20	637.78	527.15	621.78	1,655.59

Because of the interrelationship of formulas (1), (2), and (3), the simplest computer solution is an iterative one, obtaining successive approximations to the expense allowance. The convergence is very rapid if the initial trial value is taken as the expense allowance applicable when the equivalent level net premium is greater than the full preliminary term renewal net premium for a twenty-payment life for the equivalent level amount. No more than four iterations were required in calculating any of the foregoing values.

The computer program produces zero expense allowance on status changes that do not change the amount or plan—a necessary (but not sufficient) proof of its validity.

(AUTHOR'S REVIEW OF DISCUSSION)

SOLOMON GOLDFINGER:

I would like to thank Messrs. Aschenbrenner, Carroll, Chapin, and Hill for their comments. All of the discussions were stimulating and thought-provoking.

Before responding to some of the discussions, let me point out that one is free to use any reserve formula that one likes, as long as the resulting reserves exceed the statutory minimum. Thus, there is very little mention in my paper, or in my reply here, of the actual reserves that a company may decide to hold for adjustable life policies. Instead, the emphasis is on defining the statutory minimum reserves for an adjustable life policy. The relevant criteria are the definition of the CRVM and consistency with the results that are obtained by applying the CRVM to traditional policies.

Mr. Aschenbrenner takes me to task for conveying the impression that Chapin's formula in the case where the nineteen-payment life limit does not come into play is

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\beta^F_{x_m} - c_{x_m}) - I_{x_{m-1}}(\beta^F_{x_{m-1}} - c_{x_{m-1}}) , \qquad (1)$$

a formula that gives excessive expense allowances. I think he is overreacting, because the sentence in my paper immediately following the ones quoted by Aschenbrenner points out that it is clear from the rest of Chapin's paper that this is not the formula that he intended. However, if one accepts Chapin's formula C(3),

$$I_{x_m}\Delta_{x_m} = I_{x_m}(\pi_{x_m} - c_{x_m}) - I_{x_{m-1}}(\pi_{x_{m-1}} - c_{x_{m-1}})$$

when $\pi_{x_m} \leq {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} \leq {}_{19}P_{x_{m-1}+1}$, the temptation is very great to use the formula

$$I_{x_m}\Delta_{x_m} = I_{x_m}({}_{19}P_{x_m+1} - c_{x_m}) - I_{x_{m-1}}({}_{19}P_{x_{m-1}+1} - c_{x_{m-1}}) \qquad (2)$$

when $\pi_{x_m} > {}_{19}P_{x_m+1}$ and $\pi_{x_{m-1}} > {}_{19}P_{x_{m-1}+1}$. Indeed, this formula would have the property that

$$\sum_{r=1}^{m} I_{x_r}\Delta_{x_r} = I_{x_m}({}_{19}P_{x_m+1} - c_{x_m}) ,$$

which is the analog of the property that Mr. Chapin, in his discussion, finds attractive about C(3). But formula (2) above has exactly the same flaw as formula (1)—it allows an additional expense allowance for an increase in age with no change in amount or plan. Chapin himself does not use formula (2), but others using Chapin's formula C(3) might. The result would be excessive expense allowances.

Both Mr. Carroll and Mr. Aschenbrenner have derived formulas that express interesting relationships between the adjustable life net premiums and traditional net premiums. These formulas are interesting both in themselves and as a means of facilitating comparisons between Chapin's formulas and mine. Aschenbrenner's formulas demonstrate that in the case where the only change is an increase in coverage and where the nineteen-payment life limitation does not come into play, Chapin's expense allowance is generally less than mine, and less than the expense allowance that would apply to the increase in face amount if it were granted in a separate policy. Both Carroll and Aschenbrenner point out that this difference could be substantial, to the extent that many increases in face amount would provide no additional expense allowance at all! This strikes me as a rather serious flaw in Chapin's formula C(3).

Aschenbrenner feels that I have made too much of the question of whether π_{x_m} or $\beta^F_{x_m}$ should be compared to ${}_{19}P_{x_m+1}$. For a traditional policy, he is absolutely correct, since $\pi_x = \beta^F_x$ if $\beta^F_x \leq {}_{19}P_{x+1}$. However, for an adjustable life policy, the difference could be significant. The consequence of using π_{x_m} instead of $\beta^F_{x_m}$ in the nineteen-payment life test, along with Chapin's formulas, is to permit allowances that are excessive in some instances. For example, consider the following adjustable life history:

Status	Age	Plan	Amount	Goldfinger Expense	Separate Policy CRVM Expense	Chapin Expense
1.	30	Life @ 65	$30,000	$414.81	$414.81	$ 414.81
2a.	50	Life @ 65	40,000	323.07	323.07	211.06
2b.	50	Life @ 65	60,000	969.22	969.22	1,028.60

My formula yields an expense equal to that given for the separate policy under the CRVM in each case. When the increase in face amount is \$10,000, Chapin's allowance is lower than that of a separate policy, but when it is \$30,000, his allowance is higher. The reason is that the nineteen-payment life limit comes into play under my formula, while under Chapin's it does not. Chapin's formula C(3) applies in this case, and because this formula does not have a term that deals with the increase-in-age problem, the result is a higher expense allowance. Additional examples of cases for which Chapin's formulas give higher allowances than mine are given in both Chapin's and Hill's discussions.

To recapitulate: when the only change is an increase in face amount, Chapin's formulas usually yield expense allowances that are less than those that would apply to separate policies. However, in some cases, Chapin's formulas yield expense allowances that exceed those of the separate policy approach, even when the only change is an increase in face amount. The latter situation can occur because Chapin tests π_{x_m}, and not $\beta^F_{x_m}$, against ${}_{19}P_{x_m+1}$. My formulas yield allowances equal to those of a separate policy when the only change is an increase in face amount. Also, Chapin's allowance per thousand of increase in face amount varies with the size of the increase, as illustrated in the previous table.

In their discussions, Messrs. Hill and Chapin introduce a new approach to adjustable life reserves: computing the equivalent level face amount and premium and the resulting expense allowance. I was surprised to see Chapin describe his formulas as being derived from this point of view, and I could not see how his expense allowance formulas resulted from this approach. Mr. Hill's brief exposition of this approach was quite clear. However, the expense allowances that result are substantially lower than either Chapin's or mine (or the separate policy approach, where applicable.)

Hill's approach discounts all policy factors back to the original issue age of the adjustable life policy. This produces substantially lower allowances than those based on the attained age at the time of change. Once again, it is instructive to look at an increase in face amount as the typical adjustable life change. The main expense here will be a first-year commission on the increase in premium, which presumably will be based on the age at the time of increase. One would think that the focal point of the calculation of the additional expense allowance also should be the age at the time of increase. I believe it is too restrictive to provide an expense allowance for an increase in face amount that is much less than the allowance that would have been provided had the increase been accomplished through a separate policy.

Hill's approach does have some theoretical justification, however, and I would have no quarrel with the adoption of his approach if one were willing to forgo much of the expense allowance that, in my opinion, is permitted on a statutory basis. I do not think, however, that his approach should be interpreted as giving minimum statutory reserves.