TRANSACTIONS OF SOCIETY OF ACTUARIES
1980 VOL. 32

ANALYSIS OF THE DEFICIT RISK IN GROUP INSURANCE*

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ABSTRACT

The risk associated with the termination of group contracts subject to retrospective experience rating is analyzed. A formula is developed that can be used to determine the appropriate contingency charge for this risk. Numerical results are given for the data presented in a previous paper.

I. INTRODUCTION

THIS paper analyzes the deficit risk that an insurer assumes when a group insurance contract is subject to a retrospective experience-rating arrangement. The particular loss to which the deficit risk pertains is the writing off of deficits on terminating groups where such deficits are not collectible debts. As compensation for the deficit risk, insurers usually levy an arbitrary risk charge typically equal to 1 or 2 percent of premium. In this paper, we develop a method for quantifying the deficit risk and determining explicitly an appropriate risk charge level.

In his paper on experience-rating group life insurance, Bolnick [1] describes a model that simulates the loss from the deficit risk. He states that in practice it is not possible to predict a risk charge accurately because of the unpredictable nature of canceled case deficits. He recommends that insurers avoid the deficit risk by either making deficits legally collectible debts or charging an additional premium to provide stop-loss insurance at the level of the basic premium. Nevertheless, we believe that it is useful to develop a tool that an actuary can use to quantify the deficit risk accurately, provided that he is prepared to postulate rates of policy termination.

If insurers can quantify the deficit risk, they are more likely to determine an appropriate charge to cover that risk. A charge that is high may provide a stimulus for reducing the magnitude of the risk by using various risk-reducing devices, some of which we discuss in this paper.

* This research was supported by grant No. A4185 of the Natural Sciences and Engineering Research Council of Canada.

It may also provide a stimulus for the no-risk contract advocated by Bolnick, which recognizes deficits as legally collectible debts.

Under a retrospectively experience-rated contract, an accounting is performed at the end of each year to determine the dividend, if any, that is payable to the insured. In the determination of the dividend, the account is credited with premiums paid and is debited with a claim charge, an expense charge, and a profit and contingency charge. In addition, there is an adjustment for interest on any existing deficit. In this paper we ignore any profit or loss arising from the difference between expenses incurred and expenses charged, and between interest credited and interest earned.

We examine a number of ways of determining the claim charge. At one extreme is the pure accounting method, under which the claim charge equals the claims incurred. This method gives rise to the maximum deficit risk, while the other methods will tend to reduce it.

One of the methods that we examine is the aggregate stop-loss method (modified pure accounting method in [1]), under which a stop-loss level is defined and the claim charge is equal to the sum of (*a*) claims incurred up to the stop-loss level and (*b*) an amount equal to the stop-loss premium sufficient to pay expected claims in excess of the stop-loss level. The stop-loss level is a parameter that can be shifted to control the risk of deficit. As the stop-loss level increases without bound, the stop-loss method approaches the pure accounting method. If the stop-loss level is zero, the claim charge is equal to the expected claims and the deficit risk vanishes. This may be referred to as the fully pooled method.

Another method of defining the claim charge is the credibility method, under which the claim charge is a weighted average of actual and expected claims. The weight to be given to the actual claims is called the credibility factor. If the credibility factor equals 1, we have the pure accounting method. If the credibility factor equals zero, we have the fully pooled method.

It is possible to combine the aggregate stop-loss and credibility methods, in which case the claim charge is the weighted average of (*a*) actual claims up to the stop-loss level plus stop-loss premium and (*b*) the expected claims.

The deficit risk may be reduced further by requiring the maintenance of a claim fluctuation reserve. A maximum value for this reserve is agreed upon, and dividends are left on deposit with the insurer until an amount equal to the maximum value has been accumulated. Dividends will be paid only after the claim fluctuation reserve has been built up to its maximum value. Any amount in the reserve is payable to the insured in the event of contract termination.

Another risk-reducing device that we explore is that of increasing the premium. To simplify the analysis, we assume that the premium exceeds the amount of expected claims by a margin, which we refer to as the dividend loading. A larger dividend loading provides additional cushion against adverse mortality experience. We will study the effect of using a dividend strategy consisting of any combination of the various risk-reducing devices, namely: (1) aggregate stop-loss pooling, (2) credibility, (3) claim fluctuation reserve, and (4) dividend loading.

The analysis of the deficit risk is carried out using the data described by Bolnick [1]. The frequency distributions of deficits existing after n years, and of deficits written off after n years, depend on the persistency assumptions for group contracts. A range of assumptions is used to study the sensitivity of the results to changes in persistency. An interest rate assumption is also required for accumulation and present-value calculations. Finally, a formula is developed for determining a contingency charge with a present value equal to the present value of the lost deficits. The analogy to the net monthly premium for an insurance benefit becomes apparent.

II. DIVIDEND STRATEGIES

Notation

The following notation is used throughout the paper:

E = Expected claims;

DL = Dividend loading;

$P = E + DL$ = Annual premium, net of profit, expense, and contingency charges;

M = Maximum level of claim fluctuation reserve (CFR);

R_t = Level of deficit at end of tth year (a negative deficit indicates a CFR of amount $-R_t$);

D_t = Cash dividend paid at end of tth year;

C_t = Claims experienced in tth year;

CC_t = Claim charge in dividend formula in tth year;

S = Aggregate stop-loss pooling level in dividend formula;

$W(S)$ = Aggregate stop-loss premium;

k = Credibility level in dividend formula;

i = Interest rate credited on CFR balances and charged on deficits.

Claim Charge Formulas

Formulas for the claim charge for various dividend strategies are given below.

1. Pure accounting method:

$$CC_t = C_t . \tag{1}$$

2. Aggregate stop-loss method:

$$CC_t = \min \{C_t, S\} + W(S) . \tag{2}$$

3. Fully pooled method:

$$CC_t = E . \tag{3}$$

4. Credibility method:

$$CC_t = kC_t + (1 - k)E . \tag{4}$$

5. Aggregate stop-loss with credibility method:

$$CC_t = k[\min \{C_t, S\} + W(S)] + (1 - k)E . \tag{5}$$

Note that the aggregate stop-loss with credibility method is the most general of the five methods and that the first four methods are special cases of it. Thus, we can develop results for this method only, and derive results for the other methods by appropriate substitution of k and S.

Dividend Formulas

Since expenses are assumed to equal expense charges, we ignore them in the computation of dividends. As a result, the only charges made against the premium (net of profit, expense, and contingency charges) are the claim charge and interest on the previous year's deficit. The balance available, first to reduce the deficit, then to build up the claim fluctuation reserve, and finally to pay cash dividends, is

$$P - CC_t - iR_{t-1} .$$

Note that this formula makes provision for crediting interest on claim fluctuation reserve balances, that is, when R_{t-1} is negative. The deficit level at the end of the tth year is then the previous year's deficit decreased by the above balance, subject to a minimum of $-M$. Hence,

$$\begin{aligned} R_t &= \max \{R_{t-1} - P + CC_t + iR_{t-1}, -M\} \\ &= \max \{(1 + i)R_{t-1} - (P - CC_t), -M\} . \end{aligned} \tag{6}$$

The cash dividend paid at the end of the tth year is the amount by which $-M$ exceeds $(1 + i)R_{t-1} - (P - CC_t)$. Hence,

$$\begin{aligned} D_t &= -M - (1 + i)R_{t-1} + (P - CC_t) , \quad \text{if positive} \\ &= P - CC_t - M - (1 + i)R_{t-1} , \quad \text{if positive} . \end{aligned} \tag{7}$$

Note that in the case of the aggregate stop-loss with credibility method, formula (7) reduces to

$$\begin{aligned} D_t = DL - k[\min \{C_t, S\} + W(S) - E] \\ - M - (1 + i)R_{t-1} , \quad \text{if positive} . \end{aligned} \tag{8}$$

Accordingly, the cash dividend is the dividend loading less (1) credible claims in excess of expected claims, (2) the amount required to build up the claim fluctuation reserve to its maximum value, and (3) interest on the previous balance.

In subsequent sections we focus on deficit balances as given by formula (6) rather than on the actual cash dividends paid.

III. DISTRIBUTION OF DEFICIT LEVELS IN SUCCESSIVE YEARS

Let $A(x, t)$ denote the probability that, at the end of the tth year, the deficit will not exceed x. Then

$$A(x, t) = \Pr\{R_t \leq x\}\ , \qquad t = 1, 2, 3, \ldots; \quad x \geq -M\ . \tag{9}$$

Consider a group with deficit R_{t-1} at the end of $t - 1$ years. From equation (6) for $x = -M$, we see that

$$\begin{aligned} A(-M, t) &= \Pr\{R_t = -M\} \\ &= \Pr\{(1+i)R_{t-1} - (P - CC_t) \leq -M\} \\ &= \Pr\{CC_t \leq P - (1+i)R_{t-1} - M\}\ . \end{aligned} \tag{10}$$

Similarly, for $x > -M$,

$$\begin{aligned} A(x, t) &= \Pr\{R_t \leq x\} \\ &= \Pr\{(1+i)R_{t-1} - (P - CC_t) \leq x\} \\ &= \Pr\{CC_t \leq P - (1+i)R_{t-1} + x\}\ . \end{aligned} \tag{11}$$

Combining equations (10) and (11), we see that

$$A(x, t) = \Pr\{CC_t \leq P - (1+i)R_{t-1} + x\}\ , \qquad x \geq -M\ . \tag{12}$$

Claim Charge Distribution

Let $F(x)$ denote the aggregate claim distribution function for the group being studied. Bolnick ([1], p. 129) gives the aggregate claim distribution for the group of 1,050 lives studied in his paper. Then

$$F(x) = \Pr\{C_t \leq x\}\ . \tag{13}$$

Let $G(x) = \Pr\{CC_t \leq x\}$ denote the claim charge distribution. Section II provides formulas for the claim charge as a function of claims for various dividend strategies. We determine the distribution of claim charges as follows:

1. Pure accounting method:

$$G(x) = \Pr\{CC_t \leq x\} = \Pr\{C_t \leq x\} = F(x)\ . \tag{14}$$

2. Aggregate stop-loss method:

$$\begin{aligned} G(x) &= \Pr\{CC_t \le x\} \\ &= \Pr\{\min\{C_t, S\} + W(S) \le x\} \\ &= \Pr\{\min\{C_t, S\} \le x - W(S)\} \\ &= \begin{cases} \Pr\{C_t \le x - W(S)\}, & x < S + W(S) \\ 1, & x \ge S + W(S) \end{cases} \\ &= \begin{cases} F(x - W(S)), & x < S + W(S) \\ 1, & x \ge S + W(S). \end{cases} \end{aligned} \tag{15}$$

3. Fully pooled method:

$$\begin{aligned} G(x) &= \Pr\{CC_t \le x\} = \Pr\{E \le x\} \\ &= \begin{cases} 0, & x < E \\ 1, & x \ge E. \end{cases} \end{aligned} \tag{16}$$

4. Credibility method:

$$\begin{aligned} G(x) &= \Pr\{CC_t \le x\} \\ &= \Pr\{kC_t + (1-k)E \le x\} \\ &= \Pr\{C_t \le E + (x - E)/k\} \\ &= F(E + (x - E)/k). \end{aligned} \tag{17}$$

5. Aggregate stop-loss with credibility method:

$$\begin{aligned} G(x) &= \Pr\{CC_t \le x\} \\ &= \Pr\{k[\min\{C_t, S\} + W(S)] + (1-k)E \le x\} \\ &= \Pr\{\min\{C_t, S\} \le E + (x - E)/k - W(S)\} \\ &= \begin{cases} \Pr\{C_t \le E + (x-E)/k - W(S)\}, & \\ & x < k(S + W(S)) + (1-k)E \\ 1, & x \ge k(S + W(S)) + (1-k)E \end{cases} \\ &= \begin{cases} F(E + (x-E)/k - W(S)), & \\ & x < k(S + W(S)) + (1-k)E \\ 1, & x \ge k(S + W(S)) + (1-k)E. \end{cases} \end{aligned} \tag{18}$$

Distribution of Deficit Levels

We now substitute the claim charge distribution in equation (12) to obtain an expression for $A(x, t)$ in terms of the aggregate claim distribution $F(x)$. Since the aggregate stop-loss with credibility method is a generalization of the other four methods, we need only substitute equation (18) in equation (12), yielding

$$A(x, t) = G(P - (1 + i)R_{t-1} + x)$$
$$= \begin{cases} F(E + (DL - (1 + i)R_{t-1} + x)/k - W(S)) , \\ \qquad x \leq (1 + i)R_{t-1} - DL + k(S + W(S) - E) \\ 1 , \quad x > (1 + i)R_{t-1} - DL + k(S + W(S) - E) . \end{cases} \tag{19}$$

Equation (19) gives the distribution of deficits at the end of a year, given that the previous year's deficit is R_{t-1}.

Persistency

Assume that, at the end of year t at deficit level x, a group will terminate its contract with the insurer with probability $q(x, t)$. We consider only persistency rates that are independent of duration; that is, $q(x, t) = q(x)$. Let $p(x) = 1 - q(x)$ denote the persistency rate at deficit level x. Then, for a group with deficit R_{t-1} at time $t - 1$, the probability that it will have deficit R_t, not greater than x, and will terminate at the end of the tth year can be written as

$$\int_{-M}^{x} q(y)dA(y, t) . \tag{20}$$

Similarly, the probability that it will have deficit R_t, not greater than x, and will not terminate at the end of the tth year is

$$\int_{-M}^{x} p(y)dA(y, t) . \tag{21}$$

Formulas (19), (20), and (21) summarize the analysis of deficit levels for any single year. In the following section we combine the single-year analyses to obtain deficit distributions after n years.

IV. MULTIYEAR ANALYSIS

In order to determine $A(x, t)$, we resort to an approximation method using matrices and vectors. Readers not familiar with matrix algebra may wish to skip Sections IV and V and read Appendix III, in which a calculus presentation is given. To allow the matrix and vector dimensions to be finite, it is necessary to define a maximum deficit level that can arise before a group is certain to terminate its contract with the insurer.

This is a calculation expediency that should not distort the results of the analysis if the maximum deficit level is set sufficiently high.

Let R denote such a deficit level. We recommend that it be set as high as 200, 300, or 500 percent of annual premium.

We now divide the range, $[-M, R]$, of deficit levels into n intervals represented by the $n \times 1$ vector $r = (r_1 = -M, r_2, r_3, \ldots, r_n = R)^T$, where r_j is the midpoint of the jth interval.

Let $T = [t_{jk}]$ denote an $n \times n$ transition matrix, where

$$t_{jk} = \Pr\{R_t \doteqdot r_k | R_{t-1} \doteqdot r_j\} , \tag{22}$$

that is, the probability that the deficit level is in the kth interval after t years, given that it was in the jth interval after $t - 1$ years.

In Appendix I we show that

$$\begin{aligned} t_{j1} &= G(P - r_j(1+i) - M + \delta) , \\ t_{jn} &= 1 - G(P - r_j(1+i) + R - \delta) , \\ t_{jk} &= G(P - r_j(1+i) + r_k + \delta) - G(P - r_j(1+i) + r_k - \delta) , \\ & \qquad\qquad k = 2, 3, \ldots, n-1 , \end{aligned} \tag{23}$$

where $\delta = (M + R)/[2(n - 1)]$, and $G(x)$ can be evaluated by using equation (18).

Let

$$P = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & p_n \end{bmatrix}$$

be an $n \times n$ diagonal persistency matrix, where p_j represents the probability that at the end of any year a group with deficit level r_j will not terminate. Note that since we assumed that at the maximum deficit level termination is certain, we should have $p_n = 0$.

Let $Q = I - P$ denote the $n \times n$ termination matrix, where I is the $n \times n$ identity matrix.

Let $a_t = (a_{t1}, a_{t2}, \ldots, a_{tn})$ denote the $1 \times n$ vector where a_{tj} represents the probability that a group persists to the end of t years and is at that time at deficit level r_j. For a new group with initial deficit equal to zero, $a_0 = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 corresponds to deficit level zero.

Clearly, then, we have

$$
\begin{aligned}
a_1 &= a_0TP ; \\
a_2 &= a_1TP = a_0(TP)(TP) = a_0(TP)^2 ; \\
a_3 &= a_2TP = a_0(TP)^3 ; \\
&\cdot \\
&\cdot \\
&\cdot \\
a_t &= a_0(TP)^t .
\end{aligned}
\tag{24}
$$

Let $b_t = (b_{t1}, b_{t2}, \ldots, b_{tn})$ denote the $n \times 1$ vector where b_{tj} represents the probability that a group terminates at the end of t years at deficit level r_j. Then

$$
\begin{aligned}
b_1 &= a_0TQ ; \\
b_2 &= a_1TQ = a_0(TP)(TQ) ; \\
b_3 &= a_2TQ = a_0(TP)^2TQ ; \\
&\cdot \\
&\cdot \\
&\cdot \\
b_t &= a_0(TP)^{t-1}TQ .
\end{aligned}
\tag{25}
$$

The vectors a_j and b_j give us full information about the deficit risks in the jth year for $j = 1, 2, 3, \ldots$, subject to the approximations introduced by using the discrete model described above.

V. DEFICIT RISK CHARGE

The deficit risk charge (*DRC*) is part of the contingency charge built into the premium. It is charged to all groups and finances the cost of terminating groups. The criterion for establishing the *DRC* is that the present value of *DRC*'s equal the present value of termination costs. Let r^* denote the vector r modified so that all negative values are replaced by zero and so that the positive values are left unchanged.

The present value of termination costs is

$$
\begin{aligned}
PVTC &= vb_1r^* + v^2b_2r^* + v^3b_3r^* + \ldots \\
&= va_0[I + vTP + (vTP)^2 + \ldots]TQr^* \\
&= va_0(I - vTP)^{-1}TQr^* \\
&= a_0[(1 + i)I - TP]^{-1}TQr^* .
\end{aligned}
\tag{26}
$$

The present value of deficit risk charges is

$$\begin{aligned} PV\,DRC &= (v\mathbf{a}_0\mathbf{1} + v^2\mathbf{a}_1\mathbf{1} + v^3\mathbf{a}_2\mathbf{1} + \dots)\,DRC \\ &= v\mathbf{a}_0[I + vTP + (vTP)^2 + \dots]\mathbf{1}\ DRC \\ &= v\mathbf{a}_0(I - vTP)^{-1}\mathbf{1}\ DRC \\ &= \mathbf{a}_0[(1+i)I - TP]^{-1}\mathbf{1}\ DRC\,, \quad \text{where } \mathbf{1} = (1, \dots, 1)^T . \end{aligned} \tag{27}$$

Convergence of the infinite series is guaranteed by certain results of matrix calculus (see, e.g., Brinkman and Klotz [2], chap. 10). The deficit risk charge is then given by

$$DRC = \frac{\mathbf{a}_0[(1+i)I - TP]^{-1}TQ\mathbf{r}^*}{\mathbf{a}_0[(1+i)I - TP]^{-1}\mathbf{1}}\,. \tag{28}$$

VI. NUMERICAL RESULTS

Numerical values for the deficit risk charge are given in Tables 1-3 for the case of 1,050 lives considered by Bolnick ([1], p. 128). A total of 720 different combinations of credibility (k), aggregate stop-loss level (S), dividend loading (DL), and claim fluctuation reserve maximum (M) are examined. The termination rates are 1, 2, and 5 percent up to the level R (including negative values), and 100 percent at level R for Tables 1, 2, and 3, respectively. All calculations are carried out at an interest rate i of 6 percent. The number of cells used, n, is 71 in all cases.[1]

The accuracy of the technique as measured by the stability of the result for varying n is considered in Appendix II.

As can be seen from Tables 1-3, the deficit risk charge increases with increases in credibility, aggregate stop-loss level, and termination rates, and decreases with increases in dividend loading and claim fluctuation reserve.

Table 4 lists some selected results for various assumed interest rates, while Table 5 lists some selected results, for various levels of R. Note that the deficit risk charge increases as the assumed interest rate increases, and, as would be expected, decreases as R increases.

These results are based on the group described above and are not meant to be used for other groups. The results are derived in terms of the original claim distribution. Any change in claim distribution results in a change in the deficit risk charge.

[1] The computations were carried out on a PRIME 400 minicomputer in the BASIC programming language. Storage limits restricted n to 71 or less.

VII. OTHER RISK-REDUCING DEVICES

There are a number of risk-reducing devices that we have not analyzed in this paper. One such device is the individual stop-loss method. Under this method the excess on any certificate of insurance exceeding a stop-loss level is pooled. The claim charge is the aggregate of claims below the stop-loss level, plus a pooling charge. The approach in this paper can be used to analyze this method if the input is adjusted suitably to focus only on the experience-rated elements. The aggregate claim distribution should describe only the experience-rated risks (each certificate amount truncated at the stop-loss level), and the premium should be reduced by the pooling charge.

Another risk-reducing device is an averaging method for determining the claim charge. For example, the claim charge may be such that the

TABLE 1

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS, ASSUMING TERMINATION RATES OF 1 PERCENT UP TO DEFICIT LEVEL R

(R = 300 Percent of Expected Claims)*

DL	M	$k=0.25$				$k=0.50$				$k=1.0$			
S.		100	125	200	∞	100	125	200	∞	100	125	200	∞
0.	0	0.8	1.2	1.8	2.0	1.9	2.6	3.8	4.2	4.0	5.6	8.6	9.4
	10	0.5	0.9	1.5	1.7	1.6	2.2	3.4	3.8	3.7	5.2	8.0	8.9
	25	0.4	0.7	1.2	1.4	1.3	1.8	3.0	3.3	3.3	4.6	7.4	8.2
	50	0.4	0.7	1.1	1.2	1.1	1.6	2.6	2.9	2.9	4.1	6.6	7.4
1.	0	0.3	0.7	1.3	1.5	1.4	2.0	3.2	3.6	3.5	5.0	8.0	8.9
	10	0.2	0.5	1.0	1.2	1.1	1.7	2.9	3.3	3.2	4.6	7.6	8.4
	25	0.1	0.4	0.8	0.9	0.8	1.4	2.5	2.9	2.7	4.1	6.9	7.7
	50	0.1	0.4	0.7	0.8	0.7	1.2	2.1	2.4	2.3	3.6	6.1	6.9
2.	0	0.1	0.3	0.8	1.0	0.8	1.5	2.7	3.2	3.0	4.5	7.5	8.5
	10	0.1	0.2	0.6	0.8	0.6	1.2	2.4	2.8	2.6	4.1	7.1	8.0
	25	0.0	0.1	0.4	0.6	0.4	1.0	2.0	2.4	2.3	3.6	6.4	7.3
	50	0.0	0.1	0.4	0.5	0.4	0.8	1.7	2.0	1.9	3.0	5.6	6.5
5.	0	0.0	0.0	0.2	0.3	0.2	0.5	1.5	1.9	1.6	3.0	6.1	7.2
	10	0.0	0.0	0.1	0.2	0.1	0.4	1.3	1.7	1.3	2.7	5.7	6.7
	25	0.0	0.0	0.1	0.1	0.0	0.2	1.0	1.4	1.0	2.2	5.1	6.1
	50	0.0	0.0	0.0	0.1	0.0	0.2	0.8	1.1	0.8	1.8	4.4	5.3
10.	0	0.0	0.0	0.0	0.1	0.0	0.1	0.5	0.7	0.4	1.3	4.2	5.3
	10	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.6	0.3	1.1	3.8	4.9
	25	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.5	0.2	0.8	3.3	4.3
	50	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.3	0.1	0.6	2.7	3.7

* DL, M, and S also are expressed as percentages of expected claims. R = Level at which the group is certain to terminate; k = Credibility; DL = Dividend loading; S = Stop-loss level; M = Maximum claim fluctuation reserve.

aggregate claim charge for any consecutive five-year period equals the aggregate claims over the period. During the first four years the claim charge might be a weighted average of actual and expected claims. Increasing the time frame probably is equivalent to increasing the size of the group by a multiple equal to the averaging period with an adjustment for interest. In this paper we have not analyzed the effect of group size on the deficit risk charge, but we expect that there is an inverse relationship. The effect of group size enters the calculation through the aggregate claim distribution.

A risk-reducing device similar in effect to the averaging method is that of increasing the length of the accounting period. If dividends are computed at the ends of m-year intervals, the effect on the risk is equivalent to increasing the group size by a multiple of m. The method de-

TABLE 2

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS, ASSUMING TERMINATION RATES OF 2 PERCENT UP TO DEFICIT LEVEL R

(R = 300 Percent of Expected Claims)*

DL	*M*	k=0.25				k=0.50				k=1.0			
S..........		100	125	200	∞	100	125	200	∞	100	125	200	∞
0.......	0	0.9	1.3	1.9	2.1	2.0	2.8	4.0	4.4	4.3	5.9	8.9	9.8
	10	0.5	0.9	1.6	1.8	1.7	2.4	3.6	4.0	3.9	5.4	8.3	9.2
	25	0.4	0.8	1.3	1.4	1.3	1.9	3.1	3.5	3.4	4.8	7.6	8.5
	50	0.4	0.7	1.2	1.3	1.2	1.7	2.7	3.1	3.0	4.3	6.8	7.6
1.......	0	0.3	0.7	1.4	1.6	1.5	2.2	3.5	3.9	3.8	5.3	8.4	9.2
	10	0.2	0.5	1.1	1.3	1.2	1.8	3.0	3.5	3.4	4.8	7.8	8.7
	25	0.1	0.4	0.8	1.0	0.8	1.5	2.6	3.0	2.9	4.3	7.1	8.0
	50	0.1	0.4	0.7	0.9	0.7	1.3	2.2	2.6	2.4	3.8	6.3	7.1
2.......	0	0.2	0.4	1.0	1.2	0.9	1.7	2.9	3.4	3.2	4.8	7.8	8.8
	10	0.1	0.3	0.7	0.9	0.7	1.3	2.6	3.0	2.8	4.3	7.4	8.3
	25	0.1	0.2	0.5	0.7	0.5	1.1	2.2	2.6	2.4	3.8	6.7	7.5
	50	0.1	0.2	0.4	0.6	0.4	0.9	1.8	2.2	2.0	3.2	5.8	6.7
5.......	0	0.0	0.1	0.3	0.4	0.3	0.7	1.8	2.2	1.8	3.2	6.5	7.5
	10	0.0	0.0	0.2	0.3	0.1	0.5	1.5	1.9	1.5	3.0	6.0	7.0
	25	0.0	0.0	0.1	0.2	0.1	0.3	1.2	1.6	1.2	2.4	5.4	6.3
	50	0.0	0.0	0.1	0.1	0.1	0.2	0.9	1.2	0.9	2.0	4.6	5.5
10......	0	0.0	0.0	0.1	0.1	0.0	0.1	0.7	1.0	0.5	1.6	4.5	5.6
	10	0.0	0.0	0.0	0.1	0.0	0.1	0.5	0.8	0.4	1.3	4.1	5.2
	25	0.0	0.0	0.0	0.0	0.0	0.0	0.4	0.6	0.3	1.0	3.6	4.6
	50	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.2	0.8	2.9	3.9

* *DL*, *M*, and *S* also are expressed as percentages of expected claims. R = Level at which the group is certain to terminate; k = Credibility; *DL* = Dividend loading; *S* = Stop-loss level; *M* = Maximum claim fluctuation reserve.

TABLE 3

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS, ASSUMING TERMINATION RATES OF 5 PERCENT UP TO DEFICIT LEVEL R
(R = 300 Percent of Expected Claims)*

DL	M	k=0.25				k=0.50				k=1.0			
S.		100	125	200	∞	100	125	200	∞	100	125	200	∞
0.	0	1.0	1.5	2.2	2.5	2.4	3.3	4.6	5.0	4.8	6.6	9.8	10.7
	10	0.6	1.0	1.8	2.0	2.0	2.7	4.1	4.5	4.4	6.1	9.1	10.0
	25	0.5	0.8	1.5	1.7	1.5	2.2	3.5	3.9	3.9	5.4	8.4	9.2
	50	0.5	0.8	1.4	1.6	1.4	2.1	3.1	3.5	3.5	4.8	7.5	8.3
1.	0	0.5	1.0	1.7	2.0	1.9	2.6	4.1	4.5	4.4	6.0	9.3	10.2
	10	0.3	0.7	1.4	1.6	1.4	2.1	3.5	4.0	3.9	5.5	8.7	9.6
	25	0.2	0.5	1.0	1.2	1.0	1.8	3.0	3.5	3.3	4.9	7.9	8.8
	50	0.2	0.5	1.0	1.1	1.0	1.6	2.7	3.0	2.8	4.3	7.0	7.9
2.	0	0.4	0.7	1.3	1.5	1.3	2.1	3.5	4.0	3.8	5.6	8.8	9.7
	10	0.2	0.5	1.0	1.2	0.9	1.7	3.1	3.6	3.4	5.0	8.2	9.1
	25	0.1	0.3	0.7	0.9	0.7	1.4	2.6	3.0	2.9	4.4	7.4	8.3
	50	0.1	0.3	0.7	0.8	0.7	1.2	2.2	2.6	2.4	3.7	6.5	7.4
5.	0	0.0	0.2	0.6	0.8	0.7	1.1	2.4	2.9	2.5	4.0	7.4	8.5
	10	0.0	0.0	0.4	0.5	0.2	0.8	2.0	2.5	2.0	3.7	6.8	7.9
	25	0.0	0.0	0.2	0.3	0.1	0.5	1.6	2.0	1.6	3.0	6.1	7.1
	50	0.0	0.0	0.2	0.3	0.1	0.4	1.3	1.6	1.3	2.6	5.3	6.3
10.	0	0.0	0.0	0.2	0.3	0.0	0.3	1.2	1.6	1.0	2.4	5.5	6.6
	10	0.0	0.0	0.1	0.2	0.0	0.2	0.9	1.3	0.8	1.9	4.9	6.0
	25	0.0	0.0	0.0	0.1	0.0	0.1	0.7	1.0	0.5	1.5	4.3	5.4
	50	0.0	0.0	0.0	0.1	0.0	0.1	0.5	0.7	0.4	1.2	3.6	4.6

* DL, M, and S also are expressed as percentages of expected claims. R = Level at which the group is certain to terminate; k = Credibility; DL = Dividend loading; S = Stop-loss level; M = Maximum claim fluctuation reserve.

TABLE 4

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS, FOR VARIOUS ASSUMED RATES OF INTEREST

k	S	M	DL	DRC at Rate i		
				i=4%	i=6%	i=8%
0.25.	∞	50	1	0.671	0.795	0.898
1.0.	100	0	0	3.756	4.045	4.284
1.0.	∞	50	1	6.636	6.896	7.166
1.0.	100	0	2	2.543	2.965	3.219

NOTE.—(1) Termination rates of 1% up to deficit level R (R = 300% of expected claims). (2) S, M, and DL also are expressed as percentages of expected claims. R = Level at which the group is certain to terminate; k = Credibility; DL = Dividend loading; S = Stop-loss level; M = Maximum claim fluctuation reserve.

TABLE 5

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS FOR VARIOUS LEVELS OF R

k	S	M	DL	DRC at Indicated R Level		
				$R=200$	$R=300$	$R=500$
0.25	∞	50	1	0.813	0.795	0.710
1.0	100	0	0	4.453	4.045	3.787
1.0	∞	50	1	7.463	6.896	6.314
1.0	100	0	2	3.205	2.965	2.636

NOTE.—(1) Termination rates of 1% up to deficit level R. (2) S, M, and DL also are expressed as percentages of expected claims. (3) R = Level at which the group is certain to terminate; k = Credibility; DL = Dividend loading; S = Stop-loss level; M = Maximum claim fluctuation reserve.

scribed in this paper can be used by using an m-year period and an aggregate claim distribution for that m-year period.

VIII. SUMMARY

The formula for the deficit risk charge, in matrix notation, is (eq. [28])

$$DRC = \frac{a_0[(1+i)I - TP]^{-1}TQr^*}{a_0[(1+i)I - TP]^{-1}1}\,.$$

In summary, the application of the formula to a specific group requires the following steps:

1. Computation of the aggregate claim distribution.
2. Selection of a dividend strategy and an assumed interest rate.
3. Selection of the group termination rate as a function of deficit size.
4. Selection of the number of intervals to be used in the dividing up of the range of possible deficit levels.
5. Calculation of the elements t_{ik} of the matrix T according to equation (23).
6. Calculation of the persistency matrix P and its complement $Q = I - P$.
7. Calculation of the vector r^* of positive deficit levels.
8. Application of formula (28) using matrix operations.

We believe that the methodology presented in this paper can be adapted to specific insurers' situations and hope that the reader will be tempted to make his own applications.

REFERENCES

1. BOLNICK, HOWARD J. "Experience-rating Group Life Insurance," *TSA*, XXVI (1974), 123.
2. BRINKMAN, HEINRICH W., and KLOTZ, EUGENE A. *Linear Algebra and Analytic Geometry*. Reading, Mass.: Addison-Wesley Publishing Co., 1971.

APPENDIX I

Let $\delta = (R + M)/[2(n - 1)]$. Now consider the intervals $(-M - \delta, -M + \delta]$, $(-M + \delta, -M + 3\delta]$, . . . , $(R - \delta, R + \delta]$ with midpoints $r_1 = -M$, $r_2 = -M + 2\delta$, . . . , $r_n = R$. For a group with deficit r_j at time $t - 1$, the probability that the deficit is in the kth interval at time t is, from equations (11) and (13),

$$\begin{aligned} t_{jk} &= A(r_k + \delta, t) - A(r_k - \delta, t) \\ &= \Pr\{P - r_j(1 + i) + r_k - \delta < CC_t \leq P - r_j(1 + i) + r_k + \delta\} \\ &= G(P - r_j(1 + i) + r_k + \delta) - G(P - r_j(1 + i) + r_k - \delta) . \end{aligned}$$

Since $r_1 - \delta < 0$, the first-column elements reduce to

$$t_{j1} = A(-M + \delta, t) = G(P - r_j(1 + i) - M + \delta) .$$

Also, the last column should contain the probability for the interval $(R - \delta, \infty)$. For sufficiently large R, negligible error results, but for the sake of mathematical completeness the final-column probabilities are adjusted as

$$\begin{aligned} t_{jn} &= A(\infty, t) - A(r_k - \delta, t) \\ &= \Pr\{P - r_j(1 + i) + R - \delta < CC_t \leq \infty\} \\ &= 1 - G(P - r_j(1 + i) + R - \delta) . \end{aligned}$$

The above probabilities are based on the assumption that R_{t-1} is exactly r_1. In applying the matrix operations, we let the deficit levels in each interval be represented by the midpoint of the interval. The approximations of the method are a result of this substitution. Clearly, as the number of intervals n increases, the midpoint becomes more representative of the interval and the accuracy of the result increases.

APPENDIX II

The accuracy of the methodology is clearly a function of the number of intervals n. Since our computing facility did not allow n to exceed 71, we studied, for selected values of the parameters, the effect of changing the value of n. The results are given in Table 6. It would appear that the results for $n = 71$ are accurate to within 0.1 percent of expected claims. The computation could be somewhat reduced by using 60 intervals without significant loss of accuracy.

TABLE 6

DEFICIT RISK CHARGE AS A PERCENTAGE OF EXPECTED CLAIMS FOR VARIOUS VALUES OF n

k	S	M	DL	DRC for Indicated Value of n			
				$n=40$	$n=60$	$n=70$	$n=71$
0.25..	∞	50	1	0.915	0.748	0.813	0.795
1.0...	100	0	0	4.039	4.210	4.073	4.045
1.0...	∞	50	1	7.065	6.862	6.914	6.896
1.0...	100	0	2	2.942	2.884	3.008	2.965

NOTE.—(1) Termination rates of 1% up to deficit level R (R = 300% of expected claims). (2) S, M, and DL also are expressed as percentages of expected claims. (3) R = Level at which the group is certain to terminate; k = Credibility; DL = Dividend loading; S = Stop-loss level; M = Maximum claim fluctuation reserve.

APPENDIX III

Let $g(x)$ be the probability that the claim charge in any year equals x. Let $t(y, x)$ be the probability that a group with deficit y at the beginning of a year has deficit x at the end of the year. Then, according to equation (12),

$$t(x, y) = g(P - y(1 + i) + x) .$$

Let $a(x, n)$ be the probability that a group will persist for n years and have deficit x at the end of n years. Suppose that the group has deficit y at the end of $n - 1$ years. Then the probability that the group moves to deficit x and persists at the end of n years is

$$t(y, x)p(x) ,$$

where $p(x)$ is the persistency rate at deficit x. Since $a(y, n - 1)$ is the probability that the group has deficit y at the end of $n - 1$ years and y ranges from $-M$ to ∞, we have the recursive relation

$$a(x, n) = \int_{-M}^{\infty} a(y, n - 1)t(y, x)p(x)dy .$$

Let $b(x, n)$ be the probability that a group terminates at the end of n years with deficit x. If it has deficit level y at the end of $n - 1$ years, then the probability that the group moves to deficit x and terminates at the end of n years is

$$t(y, x)q(x) ,$$

where $q(x)$ is the termination rate at deficit x. Hence, we have

$$b(x, n) = \int_{-M}^{\infty} a(y, n - 1)t(y, x)q(x)dx .$$

Then the present value of termination costs is

$$PVTC = \sum_{t=1}^{\infty} v^t \int_{0}^{\infty} xb(x, t)dx \,,$$

and the present value of deficit risk charges is

$$DRC \sum_{t=0}^{\infty} v^t \int_{-M}^{\infty} a(x, t)dx \,.$$

Hence, the deficit risk charge is

$$DRC = \frac{\sum_{t=1}^{\infty} v^t \int_{0}^{\infty} xb(x, t)dx}{\sum_{t=0}^{\infty} v^t \int_{-M}^{\infty} a(x, t)dx} \,.$$

This formula is analogous to the one derived in Section IV. The advantage of the matrix method is that the summation can be obtained as a matrix inverse, greatly reducing the computations required.

DISCUSSION OF PRECEDING PAPER

HOWARD J. BOLNICK

Mr. Panjer and Mr. Mereu have once again opened the interesting question of the proper calculation of group insurance risk charges. I believe that it is quite appropriate for this issue to be discussed in the *Transactions*, and I thank the authors for their liberal references to my 1974 paper,[1] which, in part, discusses this same subject. The authors' technique and thorough treatment of the subject make this paper a valuable tool for group actuaries.

In my 1974 paper (pp. 162–65), the risk charge is identified as consisting of three elements: (1) recovery of otherwise uncollectible deficits on terminating groups, (2) collection of a reasonable rate of return on an insurer's "investment" in deficits on active groups, and (3) the possible recovery of these persistent active-group deficits. Mr. Panjer and Mr. Mereu focus on the first of these three elements. My paper takes great care to explain the potential problems in estimating deficits on terminating groups. The basic arguments (pp. 162–65 and 203–7) focus on the antiselection inherent in allowing a group with a deficit the choice of terminating without repaying that deficit, and the consequences of this choice on the calculation of an adequate risk charge. But if the actuary is prepared to postulate rates of policy termination, the methodology in this paper is quite appropriate and represents a significant advance over the simplified demonstration contained in my paper.

While the papers use the same underlying data, they take slightly different approaches to defining the aggregate and specific stop-loss, dividend loading, lapse rate, contingency reserve, deficit recovery, and credibility features of the group program being analyzed. Thus, the results are not strictly comparable. Nevertheless, I believe the risk charges are comparable.

Two observations result from an analysis of the authors' approaches. These observations have to do with the necessary choices that they made in presenting their results rather than with the propriety of their methods or results.

1. Premium is defined as $P = E + DL$, while the claim charge includes the stop-loss premium, $W(S)$. In practice, companies have a choice between charging the stop-loss premium directly to the policyholder, so that $P = E + DL + W(S)$, and eliminating $W(S)$ from the claim charge. The results

[1] Howard J. Bolnick, "Experience-rating Group Life Insurance," *TSA*, XXVI (1974), 123.

will be the same under both methods, but the interpretation of the authors' tables will be different.

2. The authors use level termination rates. In practice, insurers probably find that termination rates increase with larger deficits. It would be helpful for practicing actuaries to see results based on increasing termination rates.

Many of the actuaries reading the authors' paper would benefit from an opportunity to test their own alternative approaches to designing group programs for their impact on the risk charge. I believe that it would be a useful addition to an already valuable paper to include a listing of the current computer coding used by the authors so that the reader can easily test his own alternatives.

ALLAN BRENDER:

The termination deficit risk depends on the probability that a group will incur a deficit at a particular level and then terminate in that position. If the various probabilities are at hand, one should be able to calculate a proper charge for this risk by using standard actuarial principles. The authors have presented a clear and practical method for carrying out the calculation, for which the profession is indebted to them.

Over the past several years, the Mutual Life of Canada has developed another method for calculating the termination deficit risk charge. This method is due to K. K. von Schilling, J. D. Chapman, R. Stapleford, and R. E. Williams. It is based on the same actuarial principles as the Panjer-Mereu method, namely, that the present value of all deficit risk charges is equal to the present value of all deficit losses incurred as a result of terminations. As will be seen, the methods produce similar results. One purpose of this discussion is to describe the Mutual Life method. Some indication is also given of how the termination deficit risk charge varies with the risk characteristics of the group, the termination assumptions, the size of the group, and the maximum value of the claim fluctuation reserve. All calculations are based on the large-amount-pooling experience-rating method (referred to by the authors in Sec. VII as the individual stop-loss method). This is currently the most popular experience-rating method in use in the Canadian market.

Distribution of Total Claims

Both methods require as basic input the distribution of total claims for the group. The calculation of this distribution has long been considered a difficult problem, which has inhibited the application of risk-theoretic methods to practical insurance problems. However, an algorithm for calculating the compound Poisson distribution, the traditional model, was described in 1966 by Adelson ([1]; see also [2], p. 112). Unfortunately, this seems not to have been noticed by the actuarial community. The algorithm was independently rediscovered by R. E. Williams, A.S.A. [3]. It is easily programmed in APL and is very efficient.

Mutual Life Method

Beginning with the distribution of total claims, we can obtain the total claims for the group for each of 25,000 years. These 25,000 claim amounts are rearranged by a random permutation (in APL, using the function X ? X) and split into 2,500 ten-year groups. Within each ten-year group, an experience calculation is performed on the first year's experience, and the case is either terminated or continued, according to the deficit position and the termination assumptions. For cases that continue, a similar experience calculation is performed at the end of the second year, and so on through to the tenth year. Throughout the process, the present value of total deficit losses encountered as a result of terminations, and the present value of total premiums exposed, are recorded and summed over all 2,500 groups. The quotient of these sums is the deficit risk charge rate per unit of premium.

As can be seen from the above, this calculation uses a ten-year horizon instead of the infinite time horizon suggested by the authors. The use of 25,000 years' experience is felt to provide reasonable stability in the results without requiring unreasonable computer resources. The method is programmed in APL and runs on an IBM 370 Model 168 computer. For the sample cases described later in this discussion, central processing unit (CPU) time required for a single risk charge calculation, including calculation of the distribution of aggregate claims, ranged from 4 to 13 seconds. The method also permits use of select termination assumptions, that is, termination rates that depend on the deficit position *and* the policy year.

Implementation of the Panjer-Mereu Method

A prime consideration in programming the authors' method was that the results be comparable to those obtained from the alternative method currently in use at Mutual Life. Thus, the authors' method was modified to allow for select termination assumptions and to use only a ten-year time span. Select termination assumptions are introduced by replacing the equation

$$a_t = a_0(TP)^t \tag{24}$$

by

$$a_t = a_0 TP_1 TP_2 \ldots TP_t \tag{24'}$$

and replacing

$$\boldsymbol{b}_t = \boldsymbol{a}_0(TP)^{t-1}TQ \tag{25}$$

by

$$\boldsymbol{b}_t = Q_0TP_1TP_2 \ldots TP_{t-1}TQ_t, \tag{25'}$$

where P_i is the diagonal persistency matrix for the ith policy year and $Q_i = I - P_i$. The time span is altered by using only the first ten terms in the first line of each of formulas (26) and (27). Thus, the calculation involves a considerable number of matrix multiplications. APL workspace size and CPU time become important considerations at this point, since both storage and time requirements grow rapidly with the size of the matrices employed. After some experimentation, it was found that satisfactory results could be obtained at an acceptable computer cost if each of the intervals were taken to have width equal to one-quarter of the experience-rated expected claims. R, the maximum deficit level, was taken to be four times the experience-rated expected claims. Thus, for example, if the maximum claim fluctuation reserve, M, is equal to one-half of the experience-rated expected claims, all the matrices have dimension 19×19. All calculations shown are based on an interest rate of 8 percent.

Examples

Calculations are presented for seven groups, A–G. It should be emphasized that all are actual employee groups. Several of these cases would not be considered typical by many group actuaries. They have been included to point out that atypical groups do, in fact, arise in the normal course of business. Rules of thumb are usually derived to apply to typical groups. However, such rules often are applied by rote to groups for which they are inappropriate. This appears to be true of the deficit risk charge computation for many companies. I consider it a great virtue of the authors' method that the deficit risk charge produced thereby is tailored to the risk characteristics of the group in question. It follows that it is important for actuaries to be aware of the wide range of values the deficit risk charge can assume. For this reason, calculations for several atypical groups have been included.

Calculations are presented for each of six scales of termination rates. These scales are described in Table 1 of this discussion. Case characteristics for each of the seven cases are shown in Table 2.

Results

The basic results are shown in Tables 3A–3G. The rates shown are applied to the expected claims for the whole case, not just to the experi-

TABLE 1

TERMINATION SCALES

(In All Scales, Cases Terminate whenever Deficit Is at Least $4\times E$, where E = Expected Claims)

Scale	Years	
1	1–3	No terminations
	4–7	Case terminates if deficit $\geq 0.75\times E$
	8–10	Case terminates if deficit $\geq 0.5\times E$
2	1–2	No terminations
	3	Case terminates if deficit $\geq 3\times E$ and with probability 0.1 otherwise
	4	Case terminates if deficit $\geq 3\times E$ and with probability 0.2 otherwise
	5	Case terminates if deficit $\geq 2\times E$ and with probability 0.3 otherwise
	6	Case terminates if deficit $\geq 2\times E$ and with probability 0.4 otherwise
	7	Case terminates if deficit $\geq 2\times E$ and with probability 0.5 otherwise
	8	Case terminates if deficit $\geq 2\times E$ and with probability 0.6 otherwise
	9	Case terminates if deficit $\geq 2\times E$ and with probability 0.7 otherwise
	10	Case terminates if deficit $\geq 2\times E$ and with probability 0.8 otherwise
3	1–3	No terminations
	4–7	Case terminates if deficit $\geq 1.5\times E$
	8–10	Case terminates if deficit $\geq 1\times E$
4, 5, 6	Probability of termination in any year is q, where q=0.10 for scale 4, q=0.15 for scale 5, q=0.25 for scale 6	

TABLE 2

CASE CHARACTERISTICS FOR CASES A–G

Characteristic	Case A	Case B	Case C	Case D	Case E	Case F	Case G
Number of lives......	439	805	3,040	1,941	2,301	2,371	6,077
Maximum certificate..	$250,000	$580,000	$33,000	$200,000	$50,000	$250,000	$165,000
Average certificate....	$ 29,998	$ 59,089	$14,010	$168,201	$43,759	$ 74,494	$ 32,235
Minimum certificate...	$ 20,000	$ 9,000	$ 3,000	$ 80,000	$10,000	$ 2,000	$ 5,000
Expected claims per 1,000.............	5.0258	4.1338	3.3657	2.0954	2.6762	3.0640	2.4974

TABLE 3A

RISK CHARGES—CASE A

Pooling level	$15,000	$30,000	$50,000	$100,000	$250,000
Expected claims for experience-rated portion	$27,369	$41,196	$50,415	$ 62,435	$ 66,185
Mutual Life method:					
Scale 1	.035	.094	.142	.215	.240
Panjer-Mereu method:					
Scale 1	.051	.094	.137	.208	.242
Scale 2	.044	.083	.121	.184	.214
Scale 3	.044	.084	.122	.191	.224
Scale 4	.030	.056	.085	.135	.160
Scale 5	.036	.067	.099	.154	.180
Scale 6	.047	.085	.122	.184	.211

TABLE 3B

RISK CHARGES—CASE B

Pooling level	$ 40,000	$ 75,000	$100,000	$150,000	$580,000
Expected claims for experience-rated portion	$109,918	$167,824	$180,368	$188,274	$196,631
Mutual Life method:					
Scale 1	.045	.100	.112	.120	.128
Panjer-Mereu method:					
Scale 1	.058	.101	.114	.125	.146
Scale 2	.051	.091	.102	.112	.130
Scale 3	.047	.086	.097	.107	.127
Scale 4	.032	.059	.066	.073	.088
Scale 5	.041	.073	.082	.090	.106
Scale 6	.054	.094	.106	.115	.132

TABLE 3C

RISK CHARGES—CASE C

Pooling level	$ 8,000	$ 16,000	$ 24,000	$ 33,000
Expected claims for experience-rated portion	$72,484	$126,270	$141,856	$143,354
Mutual Life method:				
Scale 1	.000	.031	.041	.042
Panjer-Mereu method:				
Scale 1	.015	.036	.045	.046
Scale 2	.015	.037	.045	.046
Scale 3	.009	.025	.032	.033
Scale 4	.009	.021	.025	.026
Scale 5	.012	.029	.035	.035
Scale 6	.018	.041	.050	.051

TABLE 3D

RISK CHARGES—CASE D

Pooling level	$ 80,000	$100,000	$150,000	$200,000
Expected claims for experience-rated portion	$430,385	$499,958	$612,086	$684,103
Mutual Life method:				
Scale 1	.021	.029	.041	.047
Panjer-Mereu method:				
Scale 1	.038	.045	.063	.077
Scale 2	.036	.042	.060	.072
Scale 3	.029	.034	.050	.061
Scale 4	.021	.025	.035	.043
Scale 5	.028	.033	.046	.056
Scale 6	.039	.046	.064	.077

TABLE 3E

RISK CHARGES—CASE E

Pooling level	$10,000	$ 50,000
Expected claims for experience-rated portion	$58,341	$269,464
Mutual Life method:		
Scale 1	.000	.064
Panjer-Mereu method:		
Scale 1	.005	.068
Scale 2	.004	.065
Scale 3	.003	.053
Scale 4	.003	.038
Scale 5	.004	.050
Scale 6	.006	.069

TABLE 3F

RISK CHARGES—CASE F

Pooling level	$ 50,000	$100,000	$150,000	$250,000
Expected claims for experience-rated portion	$160,908	$251,166	$315,238	$395,252
Mutual Life method:				
Scale 1	.012	.053	.086	.131
Panjer-Mereu method:				
Scale 1	.031	.064	.093	.136
Scale 2	.027	.057	.083	.122
Scale 3	.024	.052	.078	.118
Scale 4	.017	.036	.053	.081
Scale 5	.022	.045	.066	.098
Scale 6	.030	.060	.086	.125

TABLE 3G

RISK CHARGES—CASE G

Pooling level	$ 40,000	$ 80,000	$120,000	$165,000
Expected claims for experience-rated portion	$410,365	$482,157	$488,177	$489,233
Mutual Life method:				
Scale 1	.017	.029	.030	.030
Panjer-Mereu method:				
Scale 1	.025	.035	.036	.036
Scale 2	.027	.036	.038	.038
Scale 3	.016	.024	.025	.026
Scale 4	.015	.021	.021	.022
Scale 5	.021	.028	.030	.030
Scale 6	.031	.041	.043	.043

ence-rated expected claims. The pooling levels shown (individual stop-loss level in the authors' terminology) are not necessarily those requested in the plan specifications. In each case, the maximum pooling level shown is the amount of the largest certificate and represents the fully experience-rated situation. Expected claims shown are expected claims on the experience-rated portion of the case only. Calculations are performed over a ten-year time span. The maximum claim fluctuation reserve is 50 percent of expected claims.

In Tables 4A–4C the calculations are based on a twenty-year time span. Comparison with Tables 3A–3C shows that the loss in accuracy due to use of the shorter ten-year period is not very significant. This is not surprising, since the effect of discounting deficits and charges for years 11–20 at an 8 percent interest rate is to minimize their contribution to the calculation.

In Tables 5A–5C the maximum claim fluctuation reserve is equal to expected claims. As can be seen, increasing the claim fluctuation reserve reduces the risk charge. However, the reduction is not nearly as great as I had expected before I saw the results.

Finally, Tables 6A–6C show charges for groups twice as large as each of groups A, B, and C. The effect of case size on the deficit risk charge is discussed in the next section.

Comments

1. COMPARISON OF METHODS

Examination of Tables 3A–3G shows the risk charges calculated by the Mutual Life method using termination scale 1 to be fairly close to those calculated by the Panjer-Mereu method. There are, however, several differences that require comment.

TABLE 4A

RISK CHARGES—CASE A
(Calculated over 20 Years)

Scale	Pooling Level				
	$15,000	$30,000	$50,000	$100,000	$250,000
1	.053	.095	.139	.212	.248
2	.044	.083	.121	.185	.214
3	.047	.087	.127	.197	.232
6	.048	.088	.126	.188	.215

TABLE 4B

RISK CHARGES—CASE B
(Calculated over 20 Years)

Scale	Pooling Level				
	$40,000	$75,000	$100,000	$150,000	$580,000
1	.059	.103	.116	.127	.148
2	.052	.091	.102	.112	.130
3	.050	.090	.102	.112	.133
6	.056	.098	.110	.119	.137

TABLE 4C

RISK CHARGES—CASE C
(Calculated over 20 Years)

Scale	Pooling Level			
	$8,000	$16,000	$24,000	$33,000
1	.014	.036	.044	.045
2	.015	.037	.045	.046
3	.010	.027	.034	.035
6	.019	.043	.052	.053

TABLE 5A

RISK CHARGES—CASE A

(Maximum Claim Fluctuation Reserve = 1 × Expected Claims)

Scale	Pooling Level				
	$15,000	$30,000	$50,000	$100,000	$250,000
1	.043	.080	.114	.176	.208
2	.040	.075	.107	.164	.192
3	.038	.073	.104	.164	.195
6	.044	.081	.113	.170	.196

TABLE 5B

RISK CHARGES—CASE B

(Maximum Claim Fluctuation Reserve = 1 × Expected Claims)

Scale	Pooling Level				
	$40,000	$75,000	$100,000	$150,000	$580,000
1	.048	.086	.096	.105	.124
2	.046	.082	.092	.101	.118
3	.041	.075	.084	.093	.111
6	.051	.089	.099	.108	.125

TABLE 5C

RISK CHARGE—CASE C

(Maximum Claim Fluctuation Reserve = 1 × Expected Claims)

Scale	Pooling Level			
	$8,000	$16,000	$24,000	$33,000
1	.013	.033	.041	.042
2	.014	.035	.043	.044
3	.009	.024	.030	.031
6	.018	.040	.049	.050

TABLE 6A

RISK CHARGE—CASE A DOUBLED

SCALE	POOLING LEVEL				
	$15,000	$30,000	$50,000	$100,000	$250,000
1	.026	.054	.081	.125	.152
2	.023	.049	.073	.112	.135
3	.018	.043	.067	.108	.133
6	.026	.053	.076	.116	.137

TABLE 6B

RISK CHARGE—CASE B DOUBLED

SCALE	POOLING LEVEL				
	$40,000	$75,000	$100,000	$150,000	$580,000
1	.030	.057	.064	.071	.086
2	.028	.054	.061	.067	.079
3	.022	.044	.050	.057	.070
6	.032	.058	.065	.071	.083

TABLE 6C

RISK CHARGE—CASE C DOUBLED

SCALE	POOLING LEVEL			
	$8,000	$16,000	$24,000	$33,000
1	.006	.018	.023	.024
2	.007	.021	.026	.027
3	.003	.011	.014	.015
6	.010	.025	.030	.031

First, the calculated rates differ more from the Panjer-Mereu rates for relatively low or very high pooling levels than for "central" levels. This is due to the manner in which the Mutual Life method is programmed. The program calculates risk charges for three predetermined pooling levels (such that either 10, 50, or 90 percent of the case is experience-rated), and a second-degree polynomial is fitted to these values. Risk charges for other pooling levels are calculated using the polynomial. A large part of the differences in calculated values at extreme pooling levels may be attributed to the use of the polynomial.

Second, the differences between the two sets of rates are greatest for cases D and F, which are large cases with many certificates for large amounts, high expected claims, and low mortality rates. Under the Panjer-Mereu method, deficit levels were partitioned into intervals having width equal to one-quarter of expected claims. For cases such as D and F, these intervals may be too large. Moreover, because mortality rates are relatively low, the distribution of aggregate claims tends to be flatter than usual. Thus, it may be inappropriate to use a maximum deficit level M as low as four times expected claims. Under the Mutual Life method, the use of considerably more than 2,500 ten-year histories might be more appropriate. Unfortunately, the usual Monte Carlo principles would require a great many more such periods to guarantee a small improvement in accuracy.

2. TERMINATION SCALES

Any method for calculating the deficit risk charge must involve rates of termination as a function of the deficit level. It is unlikely that any but the very largest of group insurers will have data available in sufficient amount for the derivation of a reliable set of rates. The choice of a termination scale clearly involves a great deal of actuarial judgment. The simplest type of scale is a flat scale in which persistency is independent of the deficit level. Scales 4, 5, and 6 are of this type. Tests show that the deficit risk charge is relatively insensitive to small changes in the termination rate, for example, from 5 to 7 percent. There appears to be a growing tendency for insured groups to put their cases on the market periodically to take advantage of the very competitive environment of group insurance today. The average case probably will not remain with its present carrier for more than seven years. Thus, if a flat termination scale is used, an annual termination rate of at least 15 percent would appear to be appropriate.

More to the point, it seems most unlikely that the decision to terminate a group case is independent of the case's deficit position. In fact, many would argue that a large deficit, with its associated reduced expectation of future experience refunds, is the strongest incentive to terminate a case. Acceptance of this line of reasoning leads us to termination scales such as scales 1, 2, and 3, in which termination rates depend on deficit levels and policy year. These scales were designed to produce differing risk charges. It is somewhat surprising to note that the deficit risk charge is not terribly sensitive to changes among these scales. The reader should also note that charges produced by the flat 25 percent scale, scale 6, are very close to those produced by scales 1, 2, and 3.

3. CASE SIZE

In Section VII the authors raise the question of the effect of case size on the deficit risk charge. Tables 6A–6C demonstrate that the risk charge drops by 40–50 percent when a particular group doubles in size. One should, however, be very cautious in drawing conclusions from these data. While large groups do tend to have more peaked distributions of aggregate claims and, consequently, present less risk to an insurer, one cannot conclude that the risk charge can be determined from case size by some simple rule. The sample calculations in Tables 3A–3G provide several examples where groups of similar size have markedly different risk charges for the same pooling level. The risk charge depends on a great many characteristics of a case, of which size is only one.

4. RISK CHARACTERISTICS

Among the risk characteristics of a particular case are case size, the size of the average certificate, the skewness of the distribution of amounts of insurance, and the level of expected mortality. While skewed distributions of amounts do increase risk, it should be borne in mind that even cases with relatively flat schedules of insurance may require significant risk charges. The reader should refer to Table 3E, bearing in mind that 1,942 of the 2,301 certificates are in the amount of $50,000 each. The level of expected mortality is very significant. For a given case, if this level is lowered, the distribution of aggregate claims tends to flatten out and the probability increases that actual experience will differ from expected.

It is difficult to measure the effect of each such characteristic on the risk charge. Moreover, the combined effect of these characteristics is not a simple combination of their separate effects. It would, therefore, appear to be inappropriate to depend on some simple formula for the calculation of the risk charge. Calculation methods such as the Panjer-Mereu method or the Mutual Life method, which recognize all characteristics of a particular case and tailor the charge to the case, are to be preferred.

Conclusion

The authors have provided us with a valuable tool in assessing a major risk in group insurance. They have also pointed the way to setting many of the common practices of group insurers on a firmer actuarial foundation. Many thanks to them for this important contribution to the literature.

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MYRON H. MARGOLIN:

Actuaries with responsibility for group insurance financial results should welcome this paper. It is an important step toward a sound methodology of pricing the fluctuation risk in group insurance. The clarity of the authors' exposition is to be especially commended.

But the paper is only a first step. The actuary who attempts to apply the methods of this paper directly to a real portfolio of group insurance risks will encounter several difficulties. One obvious problem is that of estimating appropriate persistency rates.

Some less obvious but equally significant problems relate to the claim assumptions. The authors have used an aggregate claim distribution function $F(x)$, which was borrowed from an earlier paper [1]. The underlying model presupposes that each year's actual claim charge is a random selection from a fixed, known distribution. In particular, the mean of the corresponding density function is considered to be fixed and known, presumably being equal to E, the expected claims.

Concerns about this model range from questions about certain of its specifics to a more radical questioning of its basic validity.

1. The mean of the claim density function is not and never can be known. At best, the actuary can only estimate its value. The variance of actual claims about expected claims E comprises not only the variance of the density function but also the variance of the estimated mean minus the "true" mean. The actuary can use appropriate credibility procedures to sharpen his estimate of the mean, but in general these cannot furnish completely accurate estimates.

2. The mean of the claim density function may change in time. In [4] there is a set of data showing that this mean is not constant for group life insurance. For group health it is even more obvious that the value of the mean must vary in time. Changes in its value would add another component of variance to the total variance, increasing the variance of deficit amounts.
3. The proposition that group insurance claim fluctuations are independent selections from some statistical distribution (with or without changes in the mean) has never been demonstrated. Especially in the case of group health insurance, it is reasonable to suppose that successive years' claim deviations $C_t - E_t$ and $C_{t+1} - E_{t+1}$ may not be independent.
4. A more radical view of the group insurance claim process holds that the basic model is not generally valid. When the risk process is significantly time-heterogeneous—that is, when the mean and other characteristics of the hypothetical claim distribution are not constant—the very notion of such a distribution becomes meaningless. This view is more fully discussed in [3] and [4]. The group health insurance claim process is clearly time-heterogeneous—the data in [2] evidence this clearly—and the data of [4] give a similar indication for group life.

These remarks should be construed not as an attack on this fine paper but rather as an indication of some further pitfalls to be negotiated before the paper's method can be usefully applied in practice.

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E. S. SHIU:

The authors have presented an elegant application of the Markov chain technique to the deficit risk problem in group insurance. In Appendix II they point out that the accuracy of their method depends on the number of intervals n and that their computer storage capacity limits n to a value of 71 or less. I would like to suggest a way to increase the value of n with the same computing facility.

Suppose we wish to solve for the row vector x in the matrix equation

$$x(I - B) = a_0,$$

where $B = vTP$. We can rewrite the above equation as a fixed-point equation,

$$x = xB + a_0,$$

and find x by the method of iteration (Jacobi method). Pick a row vector x_0 and compute

$$x_{n+1} = x_n B + a_0, \quad n = 0, 1, 2, \ldots .$$

The sequence of vectors $\{x_n\}$ will converge to the solution, since each eigenvalue of B, in absolute value, is less than 1. ($B = vTP$, where T is a transition matrix, $v < 1$, and $P \leq I$.) Let x denote the solution; the rate of convergence can be estimated by the inequality

$$\|x_{n+1} - x\| \leq \|x_n - x\| \, \|B\|,$$

$\| \ \|$ being the ℓ_1-norm. Since solving the matrix equation by iterations requires less memory capacity than solution by matrix inversion, we can increase the value of n.

Suppose we pick $x_0 = a_0$; then

$$\begin{aligned} x_n &= (\ldots (a_0 B + a_0) B + a_0 \ldots) B + a_0 \\ &= a_0 (B^n + \ldots + B + I). \end{aligned}$$

Thus, for a starting vector $x_0 = a_0$, x_n represents a partial sum of the Neumann series in formulas (26) and (27).

RICHARD L. VAUGHAN:

It was a real pleasure to read this paper, which attacks one of the purest and most fascinating actuarial problems in group insurance. The authors' contribution will certainly stimulate further discussion, provide an essential framework of notation and methodology, and lead to improvements in actual practice.

In their closing comment, the authors "hope that the reader will be tempted to make his own applications." I would like to suggest some possible directions.

Many applications will involve adapting the methods of the paper to particular companies' dividend formulas, especially in their handling of expenses. For example, variable expenses might be combined with C_t, producing a new random variable whose distribution could be estimated from company experience. The charges corresponding to the variable expenses could then be combined with CC_t, perhaps complicating the expressions for $G(x)$ and $A(x, t)$ but leaving the analysis otherwise unchanged. The next step, necessary for many companies, would involve recognizing such differences between expenses and expense charges as the deferral and amortization of acquisition costs. This would make the transition matrix T depend on duration, at least for the first few years; formulas (26) and (27) would then have to be summed directly for the first few terms, with the matrix-inverse expression available only for the tail of the series. Once the simplicity of formulas (26) and (27) is lost by having T depend on duration, there is no further incentive not to let P and Q, and indeed v and i, also depend on duration.

Another reason for supposing T to vary with duration is that E is an estimate of the corresponding population parameter. Each year, as E is based on more actual experience, not only is E itself likely to change, but the distribution of C_x about E will become tighter, and hence $F(x)$, $G(x)$, and T will change.

Aside from the above refinements to include expenses, and similar ones dealing with interest earnings and with unusual risk-reducing devices, there is an interesting possible generalization of the authors' technique. This involves allowing the deficit risk charge, DRC, for year $t+1$ to depend upon t and upon R_t.

The paper's assumption of *constant* DRC is consistent with a model in which offer and acceptance of the risk in exchange for the risk charge occur at time $t = 0$, and a set of persistency rates $p(x, t)$ applies thereafter.

The assumption of *variable* DRC is consistent with a model in which offer and acceptance of the risk occur at each renewal. In its purest form, this model assumes that the policyholder has available in the insurance market a set of options ranging from a nonparticipating contract with net premium E, through participating contracts with various types of pooling and levels of DL, to complete self-insurance. The policyholder can switch among these options without penalty, and DRC is the balancing item making renewal with the present insurer equivalent to the outside alternatives, except for such factors, unknown to the insurer, as the policyholder's tolerance for risk or preference for guarantees. It is also possible to fix DRC and use DL, k, or S as the balancing item. R_t is one of several parameters of the dividend formula known at the time the risk for year $t+1$ is accepted. The dividend formula has argument C_{t+1}; it produces the pair of values D_{t+1} and R_{t+1}. The fact that the parameter R_t represents prior losses or gains is immaterial. The insurer's risk is not that of lapse in an absolute deficit position R_{t+1} but rather that of lapse in a deficit position worse than R_t. It is clear that for negative values of R_t, the risk diminishes as $|R_t|$ increases; that is, the larger the claim fluctuation reserve, the greater the protection. What is more interesting is that the risk also diminishes as $|R_t|$ increases for positive values of R_t, since the dividend formula gives the insurer the opportunity for gains in the form of deficit recovery charges. For $R_t \geq E$, a rational policyholder would lapse in favor of a nonparticipating contract rather than accept $DL > 0$ and $DRC > -DL$. Under this extreme assumption that the policyholder is willing and able to lapse rather than accept an unfavorable dividend formula, the insurer could not take account of persistency rates in calculating DRC, and there would be a separate DRC for each level of the parameter R_t.

In practice, of course, the extreme assumption does not hold and there is resistance to lapsing. Perhaps each policyholder has a threshold of unfavorable anticipated dividend treatment, above which he will lapse and below which he will renew. The distribution of this threshold reflects the various other influences either encouraging or inhibiting lapse. Once a scale of deficit risk charges, possibly depending on t and R_t, is established, then each value of t and of R_t (or x) determines its own level of favorable or unfavorable anticipated dividend treatment. The proportion of thresholds that this dividend treatment exceeds determines a rate of lapse $q(x, t)$ and of persistency $p(x, t)$. Although this connection between anticipated dividend treatment and persistency rates may be tenuous, it must exist—why else would we assume $q(x, t) = 1$ for x greater than some multiple of premium?

The reluctance of policyholders to lapse permits the insurer to set his DRC formula (or, more generally, his DRC + profit + contingency + deficit amortization formula) as he wishes, to encourage a desirable mix of business. The authors' choice is to let DRC be constant, independent of t and R_t. This case leads to persistency rates $p(x, t)$ and to formulas (26), (27), and (28).

Another option is to let DRC depend on R_t in the extreme manner outlined above for the case of a completely fluid market. Then $PVTC = PVDRC$, independent of persistency rates because $TC = DRC$ at each duration.

Still other intermediate options are available. Some combination of lower DRC, or lower DL, or greater deficit forgiveness might improve persistency among cases with $R_t > 0$. Similarly, lower DRC in recognition of an accumulated CFR might improve persistency among cases with $R_t < 0$. Or DRC might be allowed to vary with duration but not with R_t. Each case would yield expressions analogous to formulas (26), (27), and (28), but with $(1, \ldots, 1)^T DRC$ in formula (27) replaced by $(DRC_1, \ldots, DRC_n)^T$ and, if DRC is also to depend on t, without the simplification involving matrix inverses.

It is natural for insurers to be concerned primarily with their overall results and to be satisfied with approximate formulas that appear to work in the aggregate. Selective forces are always present in the marketplace, however, and it seems reasonable that the most successful insurers will be those who first develop a refined model and then pull back as far as necessary in the direction of simplicity. This applies to rating methods, pool charges, reserves, expense charges, interest credits, and risk and profit charges. The authors have shown the way, for deficit risk charges, with their excellent computational model.

(AUTHORS' REVIEW OF DISCUSSION)

H. H. PANJER AND J. A. MEREU:

The five discussants address a number of interesting issues and make some significant observations.

Mr. Bolnick correctly observes that our paper focuses only on the recovery of otherwise uncollectible deficits on terminating groups. We do provide for a rate of return on deficits to the insurer through the specification of an interest rate for deficit accumulation. We believe that through the use of an infinite horizon we are bringing all deficits under the purview of the calculation.

Mr. Bolnick observes that the probability of termination is likely to be directly related to the size of the deficit. While we did employ a flat termination rate in our calculated examples, our method is more general and does allow for persistency to be separately specified for each deficit level.

We are pleased to provide, in the Appendix to this review, an APL program that can be used to calculate the deficit risk charges using our method.

Dr. Brender describes the simulation method used by one company in evaluating the deficit risk charge. He compares results obtained for several groups by using this method with corresponding results obtained by using a modification of our method. The modification involves using a ten-year time horizon rather than the infinite horizon used by the authors, as well as select persistency rates. It is gratifying to note that the results obtained with the two methods are generally very similar and that where they are different such differences are explained in the discussion.

Although we did not use select and ultimate persistency rates, our formula could be modified to incorporate them. However, Dr. Brender points out that it may be possible to use an equivalent level termination rate to develop approximate results. In that case, use of select and ultimate persistency may not be necessary. Anyone who contemplates using our formulas should consider these points.

Dr. Brender also makes the important observation that the risk charge depends upon the risk characteristics of the group in question and that it is probably inappropriate to use a simple rule of thumb to evaluate the charge. This is an even more important observation when one considers the various claim charge formulas in our paper that were not considered by Dr. Brender.

We thank Dr. Brender for the considerable effort involved in the preparation of his discussion and for his important observations. Incidentally, the method used to calculate the distribution of total claims described by Dr. Brender was also discovered in 1979 by one of the authors and is described in a more general context in a paper in this volume [1].

Mr. Margolin addresses a number of issues regarding the model used in the paper. He correctly points out that the claim distribution about an estimated expected claims value with a degree of uncertainty would have more variance than a distribution about a precisely known expected claims value. The user can modify the claim distribution that is input to our calculation procedure to reflect the uncertainty he has about the expected values.

Mr. Margolin notes that the claim density function may change in time. To the extent that the change is apparent, the user can respond by recalculating the deficit risk charge each year. He also observes that the real world may be more capricious than any model that might be constructed. This suggests that there is a limit to the amount of capriciousness that can be modeled. We see no reason why there should be such a limit.

Dr. Shiu, with commendable insight, presents an iterative technique for solving equations (26) and (27) that requires less computer storage than the technique presented in the paper. However, it may require more actual computation than the matrix method, depending upon the rate of convergence of the iterative technique. In view of Dr. Brender's observation that the result is quite stable for matrices of dimension 19×19, this recursive technique may not be necessary.

Mr. Vaughan deals with two basic questions. The first concerns the use of select and ultimate assumptions and the inclusion of an expense component. As he indicates, it is still possible to use the basic methodology in the paper by summing directly the first few terms corresponding to the select portion and then using the matrix inversion method to sum the terms over the ultimate period.

The second question raised by Mr. Vaughan deals with possible modifications in group insurance arrangements at each renewal, resulting in a deficit interchange that is dependent upon the current deficit and the prospective financing arrangement. We do not believe that any actuary has studied this type of flexible financing arrangement. This sophisticated model appears to be worthy of future research. We thank Mr. Vaughan for the thought-provoking ideas.

The authors wish to thank the discussants for their stimulating discussions.

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APPENDIX

APL PROGRAM TO COMPUTE DEFICIT RISK CHARGE

```
HIRISK;V;S1
V←⎕EX 'LIST'
⎕←'ENTER THE PROSPECT NAME'
NAME←⍞
⎕←'ENTER THE PREMIUM'
PRIN←⎕
⎕←'ENTER THE FACE AMOUNTS AND THE MORTALITY RISK FOR EACH.'
DATA←⎕
DATAM
E←+/J×THETA
⎕←'ENTER THE AGGREGATE STOP-LOSS LEVEL AS A MULTIPLE OF PREMIUM.'
⎕←'. IF NOT APPLICABLE ENTER 1000.'
SL←PRIN÷E÷⎕
⎕←'ENTER THE INDIVIDUAL STOP-LOSS LEVEL.IF NOT APPLICABLE ENTER AN '
⎕←'AMOUNT GREATER THAN THE LARGEST CERTIFICATE.'
ISL←⎕
ISLP
⎕←'ENTER THE MAXIMUM CLAIMS FLUCTUATION RESERVE AS A MULTIPLE OF'
⎕←' PREMIUM.IF YOU WISH TO DEFAULT ENTER 1000.'
M←⎕
N←25
PRM←1.05
PREM←PRM×ERI
R←3
K←1
INT←0.08
PV← 0.94 0
UNIT
MAX←⌈5000⌊10×ERI÷U
AGG
SLP
M←(1.5×RISKP×M=1000)+M×M<1000
DRC
⎕←OUT←OUTPUT
⎕←'IF YOU WISH A PRINTOUT ENTER 1,OTHERWISE 0.'
S1←⎕
→(S1≠1)/0
PRINT 'OUT'

DATAM;S1;M1;V1;V2
S1←0.5×⍴DATA
M1←(S1,2)⍴DATA
V1←M1[;1]
V2←M1[;2]
J←V1[⍋V1]
THETA←V2[⍋V1]
⎕←M1
⍝'THIS FUNCTION MASSAGES THE INPUT DATA TO GIVE A VECTOR J OF FACE'
⍝'AMOUNTS AND A VECTOR OF CORRESPONDING CLAIMS.'

ISLP;S1;S2
S1←+/J≥ISL
ERI←E
→(S1=0)/0
S2←+/(-S1)↑THETA
J←((-S1)↓J),ISL
THETA←((-S1)↓THETA),S2
ERI←+/J×THETA
⍝THIS FUNCTION ADJUSTS EXPECTED NUMBER OF CLAIMS BY FACE AMOUNT'
⍝TO RECOGNIZE INDIVIDUAL STOP-LOSS LEVELS.IT ALSO CALCULATES'
⍝EXPECTED RETAINED CLAIMS.'
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```
UNIT;V;S1;S2
V←(1↓J)-¯1↓J
S1←⌊/V
U←S1
TRY:S2←⌈/U|J
→(S2=0)/0
U←S2
→TRY
⍝THIS FUNCTION CALCULATES A FEASIBLY SIZED UNIT FOR THE AGGREGATE CLAIMS
⍝DISTRIBUTION.

AGG;I;⎕IO;S1;V1;V2
⎕IO←0
V1←0×⍳MAX
V1[0]←*-+/THETA
V2←0×⍳⍴V1
V2[J÷U]←THETA
I←0
LOOP:I←I+1
→(I=MAX)/END
S1←⌊/(I,⌈/J)
NEXT:V1[I]←(1÷I)×+/(1+⍳S1)×V2[1+⍳S1]×V1[(I-1)-⍳S1]
→LOOP
END:AGGR←+\V1
→(AGGR[MAX-1]>0.99999)/0
WARN:⎕←'AGGR TABLE TOO SHORT.'
⍝THIS FUNCTION GENERATES VALUES OF THE AGGREGATE CLAIMS DISTRIBUTION.
⍝SEE THE AGGREGATE CLAIMS DISTRIBUTION AND STOP-LOSS REINSURANCE BY
⍝HARRY H. PANJER IN THE 1980 TRANSACTIONS.

SLP;G;V1;S1
⎕IO←1
SLPR←0
S←SL×ERI
G←+\AGGR
V1←⌊1+S÷U
→(V1>⍴AGGR)/0
S1←((ERI-S)÷U)+G[V1]-AGGR[V1]×V1-S÷U
SLPR←S1×U
⍝THIS FUNCTION COMPUTES  STOP-LOSS PREMIUMS FOR A GROUP WITH
⍝AN AGGREGATE CLAIMS DISTRIBUTION GIVEN BY THE VECTOR AGG AND AT
⍝STOP-LOSS LEVELS EQUAL TO S.SEE PAPER BY PANJER IN THE 1980
⍝TRANSACTIONS.

DRC;V1;S1;S2
T
P
V1←A0+.×⌹((1+INT)×ID)-TR+.×PR
S1←V1+.×TR+.×(ID-PR)+.×RSTAR
S2←+/V1
DRCR←S1÷S2
⍝THIS FUNCTION CALCULATES THE DEFICIT RISK CHARGE FOR A GROUP.
⍝A0,T,P,AND RSTAR ARE SUBROUTINES.
⍝FOR BACKGROUND INFORMATION SEE...ANALYSIS OF THE DEFICIT RISK
⍝IN GROUP INSURANCE BY HARRY H. PANJER AND JOHN A. MEREU IN THE
⍝1980 VOLUME OF THE TRANSACTIONS OF THE SOCIETY OF ACTUARIES.

Z←A0;V1;V2;S
V1←(-M)+((⍳N)-1)×((R+M)÷(N-1))
V2←V1-0.5×(R+M)÷(N-1)
S←+/V2<0
Z←0×⍳N
Z[S]←1
⍝THIS FUNCTION GENERATES A VECTOR A0=(0,0,...,0,1,0,...,0)WHERE THE ONE
⍝CORRESPONDS TO DEFICIT LEVEL ZERO

Z←ID
Z←(N,N)⍴(((N+1)|(⍳(N*2))))=1
⍝THIS FUNCTION GENERATES AN IDENTITY MATRIX WITH N ROWS AND COLUMNS
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APPENDIX—*Continued*

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T
⎕IO←1
TR←(N,N)⍴T1⍳(N*2)
⍝T IS THE TRANSITION MATRIX IN WHICH T(I,J) IS THE PROBABILITY OF


⍝THE GROUP MOVING FROM DEFICIT LEVEL I TO DEFICIT LEVEL J IN ONE YEAR.
⍝T1 IS A SUBROUTINE.

Z←T1 X;S1;S2;S3;S4;S6;S8;S9;V1
⎕IO←1
S1←⌈X÷N
S2←Z+N×0=Z←N|X
S3←(R+M)÷2×(N−1)
V1←−M−2×S3×−1−⍳N
S4←PRM+V1[S2]+S3−V1[S1]×1+INT
S6←S4−2×S3
S8←(S2=N)+(S2≠N)×CC S4×ERI
S9←(S2≠1)×CC S6×ERI
Z←S8−S9
⍝THIS FUNCTION GENERATES A VECTOR OF THE TRANSITION MATRIX ELEMENTS.
⍝CC IS A SUBROUTINE.

Z←CC X;S1
S1←(AGG1((1÷K)×X−(ERI×1−K)+K×SLPR))
Z←(X≥(ERI×1−K)+K×S+SLPR)+S1×(X<(ERI×1−K)+K×S+SLPR)
⍝THIS FUNCTION DETERMINES THE CLAIMS CHARGE DISTRIBUTION GIVEN THE
⍝CREDIBILITY FACTOR K AND THE STOP-LOSS LEVEL S.
⍝AGG1 AND SLP ARE SUBROUTINES.


Z←AGG1 X;S1
S1←¯1+⍴AGGR
Z←AGGR[((1+⌊X÷U)×((X≥0)∧X<U×S1))+(X<0)+X≥U×S1]
Z←(X≥U×S1)+Z×(X≥0)∧X<U×S1
⍝THIS FUNCTION COMPUTES THE PROBABILITY THAT AGGREGATE CLAIMS DO NOT
⍝EXCEED X.

P;S1;V1
S1←(N−1)÷(⍴PV)−1
V1←⌈(⍳N)÷S1
PR←ID×(N,N)⍴PV[V1]
⍝THIS FUNCTION GENERATES AN N×N DIAGONAL PERSISTENCY MATRIX WHERE THE
⍝JTH ELEMENT REPRESENTS THE PROBABILITY THAT AT THE END OF ANY YEAR
⍝A GROUP IN THE JTH DEFICIT LEVEL WILL NOT TERMINATE.
⍝IF PV HAS OTHER THAN N ELEMENTS THE VECTOR IS CHANGED IN SIZE TO N.

Z←OUTPUT;V1;V;S1
S1←70
V1←S1↑⍕⎕TS[⍳3]
V1←V1,[0.5] S1↑'PROSPECT: ',NAME
V1←V1,[1] S1↑'THE PREMIUM IS',2⍕PRIN
V1←V1,[1] S1↑'THE EXPECTED CLAIMS ARE ',2⍕E
V1←V1,[1] S1↑'THE EXPECTED RETAINED CLAIMS ARE ',2⍕ER
V1←V1,[1] S1↑'THE EXPECTED STOP-LOSS CLAIMS ARE ',2⍕(E−ER)
→(SL=1000)/SKIP1
V1←V1,[1] S1↑'THE STOP-LOSS TO PREMIUM PCTG IS',0⍕100×S÷PRIN
V1←V1,[1] S1↑'THE AGGREGATE STOP-LOSS LEVEL IS ',0⍕S
SKIP1:→(ISL>⌈/DATA)/SKIP2
V1←V1,[1] S1↑'THE INDIVIDUAL STOP-LOSS LEVEL IS ',0⍕ISL
SKIP2:V1←V1,[1] S1↑'THE MAXIMUM CFR IS ',0⍕(M×ERI)
V1←V1,[1] S1↑'THE PCTG OF CFR TO PREMIUM IS ',0⍕100×M×ERI÷PRIN
Z←V1,[1] S1↑'THE DEFICIT CHARGE AS A PCT OF PREM: ',2⍕100×DRCR×ERI÷PRIN
```